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This paper is devoted to the introduction of a new class of consistent estimators of the fractal dimension of locally self-similar Gaussian processes. These estimators are based on linear combinations of empirical quantiles ($L-$statistics) of discrete variations of a sample path over a discrete grid of the interval $[0,1]$. We derive the almost sure convergence for these estimators and prove the asymptotic normality. The key-ingredient is a Bahadur representation for empirical quantiles of non-linear functions of Gaussians sequences with correlation function decreasing hyperbollically.

1 Introduction

Many naturally occuring phenomena can be effectively modelled using self-similar processes. Among the simplest models, one can consider the fractional Brownian motion introduced in the statistics community by Mandelbrot et *al.* (1968). Fractional Brownian motion can be defined as the only centered Gaussian process denoted by $(X(t))_{t\in\mathbb{R}}$, with stationary increments and with variance function, $v(\cdot)$

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defined by $v(t) = \sigma^2|t|^{2H}$ for all $t \in \mathbb{R}$. This process exhibits remarkable properties. Is is a H-self-similar process, that is for all $c > 0$

$$\{X(ct)\}_{t\in\mathbb{R}} \stackrel{d}{=} c^H \{X(t)\}_{t\in\mathbb{R}},$$

where $\stackrel{d}{=}$ means equal in finite-dimensional distributions. Autocovariance function of the fractional Brownian motion behaves like $\mathcal{O}(|k|^{2H-2})$ as $|k| \to +\infty$, so that the discretized increments of fractional Brownian motion (called fractional Gaussian noise) constitutes a short-range dependent process, when $H < 1/2$, and a long-range dependent process, when $H > 1/2$. The index H characterizes also the path regularity since the fractal dimension of fractional Brownian motion is equal to $D = 2 - H$. In this paper, we focus on a class of processes that extend the notion of self-similarity. We consider Gaussian processes $(X(t))_{t\in\mathbb{R}}$, centered, with stationary increments and such that the variance function, denoted by $v(\cdot)$, satisfies

$$v(t) = \boldsymbol{E}(X(t)^2) = \sigma^2|t|^{2H}(1 + r(t)), \qquad \text{with } r(t) = o(1) \text{ as } |t| \to 0, \tag{1}$$

for some $0 < H < 1$. Such processes are called locally self-similar (at zero) Gaussian processes. An interesting property is that their fractal dimension remains $D = 2 - H$, *e.g.* Constantine and Hall (1994).

This paper deals with estimators of the parameter H (and so on the fractal dimension). According to the context, a very large variety of estimators has been investigated. We refer the reader to Beran (1994), Coeurjolly (2000) or Bardet et *al.* (2003) for an overview of this problem. Among the most often used estimators we have: methods based on the variogram, on the log−periodogram *e.g.* Geweke and Porter-Hudak (1983) in the context of long-range dependent processes, maximum likelihood estimator (and Whittle estimator) when the model is parametric *e.g.* fractional Gaussian motion. In the precise context of locally self-similar Gaussian processes, let us cite methods based on the wavelet decomposition *e.g.* Flandrin (1992) or Stoev et *al.* (2004) and the references therein, and on discrete filtering studied by Kent and Wood (1997), Istas and Lang (1997) and Coeurjolly (2001). These two last ones rely on a specific filtering that is designed to destroy

the correlation structure of observations. More precisely for wavelet method (for example), the procedure is based, at each scale, on the estimation of the mean of the squared coefficient by empirical moment of order 2. Stoev et *al.* (2004) illustrate the fact that estimators derived from these procedures, are very sensitive to additive outliers and to non-stationary artefacts. Therefore, they propose to replace at each scale, the empirical moment of order 2, by the empirical median of the squared coefficients. This procedure for which authors assert that no theoretical result is available, is clearly more robust.

The main objective of this paper is to extend the procedure proposed by Stoev et *al.* (2004) by deriving semi-parametric estimators of the parameter H, using discrete filtering methods, for the class of processes defined by (1) and to provide convergence results. To derive convergence results of developped estimators, key-ingredient is a Bahadur representation of empirical quantiles in a certain dependence framework. Let $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ a vector of n i.i.d. random variables with cumulative distribution function F. A Bahadur representation of sample quantiles consists in the following result: assuming F'' exists and is bounded in a neighborhood of $\xi(p)$, quantile of order p, for some $0 < p < 1$, we have almost surely, as $n \to +\infty$

$$\widehat{\xi}\,(p) - \xi(p) = \frac{p - \widehat{F}\,(p)}{f(\xi(p)} + r_n,$$

with $r_n = \mathcal{O}\left(n^{-3/4}\log(n)^{1/2}\log(n)^{1/4}\right)$. This result proved by Bahadur (1966), was improved by Kiefer (1967) that obtained the rate $r_n = \mathcal{O}\left(n^{-3/4}\log\log(n)^{3/4}\right)$. Note that, assuming only that F' exists and is bounded in a neghborhood of $\xi(p)$, it can be proved *e.g.* Serfling (1980), that $r_n = \mathcal{O}\left(n^{-3/4}\log(n)^{3/4}\right)$. Extensions of above results to dependent random variables have been pursued in Sen (1972) for $\phi-$mixing variables, in Yoshihara (1995) for strongly mixing variables, and recently in Wu (2005) for short-range and long-range dependent linear processes, following works of Hesse (1990) and Ho and Hsing (1996). Our contribution is to provide a Bahadur representation for sample quantiles in another context that is for non-linear functions of Gaussian processes with correlation function decreasing hyperbollically. The bounds for r_n are obtained under a minimal assumption which is similar to the one done in Serfling (1980).

The paper is organized as follows. In Section 2, we give some basic notation and some background on discrete filtering. In Section 3, we derive semi-parametric estimators of the parameter H, when observing a unique sample path of a process defined by (1) over a discrete grid of the interval $[0,1]$ that we denote by $\boldsymbol{X}$. Estimators are based on linear combinations of empirical quantiles of $\boldsymbol{g(X^a)}$ which is a function of the series $\boldsymbol{X}$ filtered with $\boldsymbol{a}$, that is

$$\widehat{\xi}_n(\boldsymbol{p},\boldsymbol{c},\boldsymbol{g(X^a)}) = \sum_{k=1}^{K} c_k \widehat{\xi}_n(p_k, \boldsymbol{g(X^a)}).$$

Two functions are considered $g(\cdot) = |\cdot|^\alpha$ for some $\alpha > 0$ and $g(\cdot) = \log|\cdot|$, leading to two different estimators of H. Section 4 presents main results. We first establish a Bahadur representation of empirical quantiles for non-linear functions of Gaussian sequences. We apply this result to control almost surely our estimators, and to obtain, under certain conditions the asymptotic normality with rate $1/\sqrt{n}$, for all $0 < H < 1$. In Section 5 are proposed some numerical computations to compare theoretically asymptotic variances of our estimators and a simulation study. In particular, we illustrate the relative efficiency with respect to Whittle estimator and the fact that such estimators are more robust than classical ones. Finally, proofs of differents results are presented in Section 6.

2 Some notation and some background on discrete filtering

Given some random variable Y, let us denote by $F_Y(\cdot)$ its cumulative distribution function, by $\xi_Y(p)$ for some $0 < p < 1$ its theoretical quantile of order p. If $F_Y(\cdot)$ is aboslutely continuous with respect to Lebesgue measure, the probability distribution function is denoted by $f_Y(\cdot)$. The cumulative (resp. probability) distribution function of a standard Gaussian variable is denoted by $\Phi(\cdot)$ (resp. $\phi(\cdot)$). Based on the observation of a vector $\boldsymbol{Y} = (Y(1),\ldots,Y(n))$ of n random variables distributed as Y, the empirical cumulative distribution function and the

empirical quantile of order p are respectively denoted by $\widehat{F}_Y(\cdot;\boldsymbol{Y})$ and $\widehat{\xi}_Y(p;\boldsymbol{Y})$ or more easily by $\widehat{F}(\cdot;\boldsymbol{Y})$ and $\widehat{\xi}(p;\boldsymbol{Y})$. Finally, for some measurable function $g(\cdot)$, we denote by $\boldsymbol{g(Y)}$ the vector of length n with real components $g(Y(i))$, for $i=1,\ldots,n$.

Statistical model corresponds to a discretized version $\boldsymbol{X} = (X(i/n))_{i=1,\ldots,n}$ of a locally self-similar Gaussian process defined by (1). Let us now give some background discrete filtering. We denote by $\boldsymbol{a}$ a filter of length $\ell+1$ and of order $\nu \geq 1$, that is a vector of length $\ell+1$ with real components such that:

$$\sum_{q=0}^{\ell} q^j a_q = 0, \text{ for } j=0,\ldots,\nu-1 \qquad \text{and} \qquad \sum_{q=0}^{\ell} q^\nu a_q \neq 0.$$

For example, $\boldsymbol{a}=(1,-1)$ (resp. $\boldsymbol{a}=(1,-2,1)$) is a filter with order 1 (resp. 2). Let $\boldsymbol{X^a}$ be the series obtained by filtering $\boldsymbol{X}$ with $\boldsymbol{a}$, that is:

$$X^{\boldsymbol{a}}\left(\frac{i}{n}\right) = \sum_{q=0}^{\ell} a_q X\left(\frac{i-q}{n}\right) \qquad \text{for } i \geq \ell+1.$$

The following assumption is needed by different results presented hereafter:
Assumption $\mathbf{A_1}(\mathbf{k})$: for some integer $k \geq 1$, we have for $i=1,\ldots,k$

$$v^{(i)}(t) = \sigma^2 \beta(i) |t|^{2H-i} + o(|t|^{2H-i})$$

with $\beta(i) = 2H(2H-1)\ldots(2H-i+1)$.

This assumption means that the variance function $v(\cdot)$ has to be sufficiently differentiable. In particular, Assumption $\mathbf{A_1}(\mathbf{k})$ is satisfied (for every $k \geq 1$) for processes with variance function $v(t)=|t|^{2H}$, $v(t)=1-\exp(-|t|^{2H})$ or $v(t)=\log(1+|t|^{2H})$. The following Lemma asserts the change of the dependence structure when applying discrete filtering.

Lemma 2.1 (e.g. Kent and Wood (1997)) *Let $\boldsymbol{a}$ and $\boldsymbol{a'}$ be two filters of length $\ell+1$ and $\ell'+1$ and of order ν and $\nu' \geq 1$. Then we have:*

$$\begin{aligned} \boldsymbol{E}\left(X^{\boldsymbol{a}}\left(\frac{i}{n}\right) X^{\boldsymbol{a'}}\left(\frac{i+j}{n}\right)\right) &= \frac{-\sigma^2}{2} \sum_{q,q'=0}^{\ell} a_q a'_{q'} v\left(\frac{q-q'+j}{n}\right) \\ &= \gamma_n^{\boldsymbol{a},\boldsymbol{a'}}(j)\left(1+\delta_n^{\boldsymbol{a},\boldsymbol{a'}}(j)\right) \end{aligned} \tag{2}$$

with

$$\gamma_n^{\boldsymbol{a},\boldsymbol{a}'}(j) = \frac{\sigma^2}{n^{2H}}\gamma^{\boldsymbol{a},\boldsymbol{a}'}(j), \quad \gamma^{\boldsymbol{a},\boldsymbol{a}'}(j) = -\frac{1}{2}\sum_{q,q'=0}^{\ell} a_q a'_{q'}|q-q'+j|^{2H}, \tag{3}$$

and

$$\delta_n^{\boldsymbol{a},\boldsymbol{a}'}(j) = \frac{\sum_{q,q'} a_q a_{q'}|q-q'+j|^{2H} \times r\left(\frac{q-q'+j}{n}\right)}{\gamma^{\boldsymbol{a},\boldsymbol{a}'}(j)}. \tag{4}$$

Moreover, we have as $|j| \to +\infty$

$$\gamma^{\boldsymbol{a},\boldsymbol{a}'}(j) = \mathcal{O}\left(\frac{1}{|j|^{2H-\nu-\nu'}}\right). \tag{5}$$

Finally, under Assumption $\mathbf{A_1}(\nu+\nu')$*, we have, as* $n \to +\infty$

$$\delta_n^{\boldsymbol{a},\boldsymbol{a}'}(j) = o(1). \tag{6}$$

Define $\boldsymbol{Y}^{\boldsymbol{a}}$ the vector $\boldsymbol{X}^{\boldsymbol{a}}$ normalized with variance 1. The covariance between $Y^{\boldsymbol{a}}(i/n)$ and $Y^{\boldsymbol{a}'}(i+j/n)$ is denoted by $\rho_n^{\boldsymbol{a},\boldsymbol{a}'}(j)$. Under Assumption $\mathbf{A_1}(\nu+\nu')$, we have the following equivalence, as $n \to +\infty$

$$\rho_n^{\boldsymbol{a},\boldsymbol{a}'}(j) \sim \rho^{\boldsymbol{a},\boldsymbol{a}'}(j) = \frac{\gamma^{\boldsymbol{a},\boldsymbol{a}'}(j)}{\sqrt{\gamma^{\boldsymbol{a},\boldsymbol{a}}(0)\gamma^{\boldsymbol{a}',\boldsymbol{a}'}(0)}} \tag{7}$$

When $\boldsymbol{a} = \boldsymbol{a}'$, we put, for the sake of simplicity $\gamma_n^{\boldsymbol{a}}(\cdot) = \gamma_n^{\boldsymbol{a},\boldsymbol{a}}(\cdot)$, $\delta_n^{\boldsymbol{a}}(\cdot) = \delta_n^{\boldsymbol{a},\boldsymbol{a}}(\cdot)$ and $\rho_n^{\boldsymbol{a}}(\cdot) = \rho_n^{\boldsymbol{a},\boldsymbol{a}}(\cdot)$ (idem for $\gamma^{\boldsymbol{a}}(\cdot)$ and $\rho^{\boldsymbol{a}}(\cdot)$).

3 L-type estimators of H

Let $(\boldsymbol{p},\boldsymbol{c}) = (p_k, c_k)_{k=1,\ldots,K} \in ((0,1)\times\mathbb{R}^+)^K$ for an integer $K \geq 1$. Define the following statistics based on linear combinations of emprical quantiles (L-statistics):

$$\widehat{\xi}\,(\boldsymbol{p},\boldsymbol{c};\boldsymbol{X}^{\boldsymbol{a}}) = \sum_{k=1}^{K} c_k\, \widehat{\xi}\,(p_k;\boldsymbol{X}^{\boldsymbol{a}}), \tag{8}$$

where $c_k, k=1,\ldots,K$ are real numbers such that $\sum_{k=1}^K c_k = 1$. For example, this corresponds to the empirical median when $K=1, \boldsymbol{p}=1/2, \boldsymbol{c}=1$, to a mean of

quartiles when $K = 2, \boldsymbol{p} = c(1/4, 3/4), \boldsymbol{c} = (1/2, 1/2)$, and to the β trimmed-mean (for $0 < \beta < 1$) when $K = [(1-\beta)n]$ and $p_k = \beta + k/n, c_k = 1/K$ for $k = 1, \ldots, K$.

Consider the following computation: from Lemma 2.1, we have, as $n \to +\infty$

$$\widehat{\xi}\,(\boldsymbol{p}, \boldsymbol{c}; \boldsymbol{X}^{\boldsymbol{a}}) \sim \frac{\sigma^2}{n^{2H}}\, \gamma^{\boldsymbol{a}}(0) \widehat{\xi}\,(\boldsymbol{p}, \boldsymbol{c}; \boldsymbol{Y}^{\boldsymbol{a}}) \ .$$

If the observations are sufficiently decorrelated, we can hope that $\widehat{\xi}\,(\boldsymbol{p}, \boldsymbol{c}; \boldsymbol{Y}^{\boldsymbol{a}})$ converges towards a constant, as $n \to +\infty$. In itself, this result is not interesting, since two parameters remain unknown: σ^2 and H and thus, it is impossible to derive an estimator of H. This obvious remark suggests that we have to use at least two filters. Among all available filters, let us consider the sequence $(\boldsymbol{a}^m)_{m \geq 1}$ defined by

$$a_i^m \ = \begin{cases} a_j & \text{if } i = jm \\ 0 & \text{otherwise} \end{cases} \qquad\qquad \text{for } i = 0, \ldots, m\ell \ ,$$

which is nothing else that the filter $\boldsymbol{a}$ dilated m times. For example, if the filter $\boldsymbol{a} = \boldsymbol{a}^1$ corresponds to the filter $(1, -2, 1)$, then $\boldsymbol{a}^2 = (1, 0, -2, 0, 1)$, $\boldsymbol{a}^3 = (1, 0, 0, -2, 0, 0, 1), \ldots$ As noted by Kent and Wood (1997) or Istas and Lang (1997), the filter $\boldsymbol{a}^m$, of length $m\ell + 1$, is of order ν and has the following interesting property :

$$\gamma^{\boldsymbol{a}^m}(0) = m^{2H} \gamma^{\boldsymbol{a}}(0). \tag{9}$$

Our methods, that exploit the nice property (9), are based on linear combinations of empirical quantiles $\widehat{\xi}\left(\boldsymbol{p}, \boldsymbol{c}; \boldsymbol{g}(\boldsymbol{X}^{\boldsymbol{a}^m})\right)$ for two positive functions $g(\cdot)$: $g(\cdot) = |\cdot|^\alpha$ for $\alpha > 0$ and $g(\cdot) = \log|\cdot|$. The choice of positive function ensures that the statistics $\widehat{\xi}\left(\boldsymbol{p}, \boldsymbol{c}; \boldsymbol{g}(\boldsymbol{X}^{\boldsymbol{a}^m})\right)$ is strictly positive, which is important for the estimation procedure described hereafter. From (3) and (9), we have

$$\begin{aligned} \widehat{\xi}\left(\boldsymbol{p}, \boldsymbol{c}; |\boldsymbol{X}^{\boldsymbol{a}^m}|^{\boldsymbol{\alpha}}\right) \ &= \boldsymbol{E}\left((X^{\boldsymbol{a}^m}(1/n))^2\right)^{\alpha/2} \widehat{\xi}\left(\boldsymbol{p}, \boldsymbol{c}; |\boldsymbol{Y}^{\boldsymbol{a}^m}|^{\boldsymbol{\alpha}}\right) \\ &= m^{\alpha H}\ \frac{\sigma^\alpha}{n^{\alpha H}} \gamma^{\boldsymbol{a}}(0)^{\alpha/2}\ \left(1 + \delta_n^{a^m}(0)\right)^{\alpha/2} \widehat{\xi}\left(\boldsymbol{p}, \boldsymbol{c}; |\boldsymbol{Y}^{\boldsymbol{a}^m}|^{\boldsymbol{\alpha}}\right), \end{aligned} \tag{10}$$

and

$$\widehat{\xi}\left(\boldsymbol{p}, \boldsymbol{c}; \log|\boldsymbol{X}^{\boldsymbol{a}^m}|\right) \ = \frac{1}{2} \log \boldsymbol{E}(X^{\boldsymbol{a}^m}(1/n))^2 + \widehat{\xi}\left(\boldsymbol{p}, \boldsymbol{c}; \log|\boldsymbol{Y}^{\boldsymbol{a}^m}|\right)$$

$$= H\log(m) + \log\left(\frac{\sigma^2}{n^{2H}}\gamma^{\boldsymbol{a}}(0)\right) + \frac{1}{2}\log\left(1+\delta_n^{a^m}(0)\right) + \widehat{\xi}\left(\boldsymbol{p},\boldsymbol{c};\mathbf{log}\,|\boldsymbol{Y}^{\boldsymbol{a^m}}|\right). \quad (11)$$

Denote by $\kappa_H = n^{-2H}\sigma^2\gamma^{\boldsymbol{a}}(0)$ and by Y, a standard Gaussian variable. Equations (10) and (11) can be rewritten as

$$\log\widehat{\xi}\left(\boldsymbol{p},\boldsymbol{c};|\boldsymbol{X}^{\boldsymbol{a^m}}|^{\boldsymbol{\alpha}}\right) = \alpha H\log(m) + \log\left(\kappa_H^{\alpha/2}\xi_{|Y|^\alpha}(\boldsymbol{p},\boldsymbol{c})\right) + \varepsilon_m^\alpha \quad (12)$$

$$\widehat{\xi}\left(\boldsymbol{p},\boldsymbol{c};\mathbf{log}\,|\boldsymbol{X}^{\boldsymbol{a^m}}|\right) = H\log(m) + \log(\kappa_H) + \varepsilon_m^{\log} \quad (13)$$

where the random variables ε_m^α and $\varepsilon_m^{\log}$ are respectively defined by

$$\varepsilon_m^\alpha = \log\left(\frac{\widehat{\xi}\left(\boldsymbol{p},\boldsymbol{c};|\boldsymbol{Y}^{\boldsymbol{a^m}}|^{\boldsymbol{\alpha}}\right)}{\xi_{|Y|^\alpha}(\boldsymbol{p},\boldsymbol{c})}\right) + \frac{\alpha}{2}\log\left(1+\delta_n^{a^m}(0)\right) \quad (14)$$

and

$$\varepsilon_m^{\log} = \widehat{\xi}\left(\boldsymbol{p},\boldsymbol{c};\mathbf{log}\,|\boldsymbol{Y}^{\boldsymbol{a^m}}|\right) - \xi_{\log|Y|}(\boldsymbol{p},\boldsymbol{c}) + \frac{1}{2}\log\left(1+\delta_n^{a^m}(0)\right) \quad (15)$$

where, for some random variable Z

$$\xi_Z(\boldsymbol{p},\boldsymbol{c}) = \sum_{k=1}^K c_k\,\xi_Z(p_k).$$

From (12) and (13), two estimators can be defined through a simple regression of $(\log\widehat{\xi}\left(\boldsymbol{p},\boldsymbol{c};|\boldsymbol{X}^{\boldsymbol{a^m}}|^{\boldsymbol{\alpha}}\right))_{m=1,\ldots,M}$ and $(\widehat{\xi}\left(\boldsymbol{p},\boldsymbol{c};\mathbf{log}\,|\boldsymbol{X}^{\boldsymbol{a^m}}|\right))_{m=1,\ldots,M}$ on $(\log m)_{m=1,\ldots,M}$ for some $M\geq 2$. These estimators are denoted respectively by $\widehat{H}^\alpha$ and $\widehat{H}^{\log}$. By denoting $\boldsymbol{A}$ the vector of length M with components $A_m = \log m - \frac{1}{M}\sum_{m=1}^M \log(m)$, $m=1,\ldots,M$, we have explicitly:

$$\widehat{H}^\alpha = \frac{\boldsymbol{A}^T}{\alpha||\boldsymbol{A}||^2}\left(\log\widehat{\xi}\left(\boldsymbol{p},\boldsymbol{c};|\boldsymbol{X}^{\boldsymbol{a^m}}|^{\boldsymbol{\alpha}}\right)\right)_{m=1,\ldots,M}, \quad (16)$$

$$\widehat{H}^{\log} = \frac{\boldsymbol{A}^T}{||\boldsymbol{A}||^2}\left(\widehat{\xi}\left(\boldsymbol{p},\boldsymbol{c};\mathbf{log}\,|\boldsymbol{X}^{\boldsymbol{a^m}}|\right)\right)_{m=1,\ldots,M}. \quad (17)$$

We can point out that both estimators are independent of the scaling coefficient σ^2.

To simplify presentation of different results, consider the following assumption on different parameters involved in the estimation procedure

Assumption $\mathbf{A_2}$: $\boldsymbol{a}$ is a filter of order $\nu\geq 1$, α is a real strictly positive, $\boldsymbol{p}$ (resp. $\boldsymbol{c}$) is a vector of length K (for some $K\geq 1$) such that $0<p_k<1$ (resp. $c_k>0$ and $\sum_{k=1}^K c_k = 1$), M is an integer ≥ 2.

4 Main results

From previous notation, to study the convergence of $\widehat{H}_n^{\alpha}$ and $\widehat{H}_n^{\log}$, we will use the fact that

$$\widehat{H}^{\alpha} - H = \frac{\boldsymbol{A}^T}{\alpha||\boldsymbol{A}||^2}\,\boldsymbol{\varepsilon^{\alpha}} \quad \text{and} \quad \widehat{H}^{\log} - H = \frac{\boldsymbol{A}^T}{||\boldsymbol{A}||^2}\,\boldsymbol{\varepsilon^{\log}}. \tag{18}$$

where $\boldsymbol{\varepsilon^{\alpha}} = (\varepsilon_m^{\alpha})_{m=1,\ldots,M}$ and $\boldsymbol{\varepsilon^{\log}} = (\varepsilon_m^{\log})_{m=1,\ldots,M}$. Thus, it is sufficient to obtain some convergence results of empirical quantiles $\widehat{\xi}_n(p, \boldsymbol{g(Y^a)})$ for some function $g(\cdot)$ and some filter $\boldsymbol{a}$. Therefore, we first establish a Bahadur representation of empirical quantiles for some Gaussian sequences with correlation function decreasing hyperbollically.

4.1 Bahadur representation of empirical quantiles

Let us recall some important definitions on Hermite polynomials. The j-th Hermite polynomial (for $j \geq 0$) is defined for $t \in \mathbb{R}$ by

$$H_j(t) = \frac{(-1)^j}{\phi(t)}\,\frac{d^j\phi(t)}{dt^j}, \tag{19}$$

For example, the first polynomials are $H_0(t) = 1, H_1(t) = t, H_2(t) = t^2 - 1, H_3(t) = t^3 - 3t, \ldots$ The Hermite polynomials form an orthogonal system for the Gaussian measure. More precisely, by denoting Y a standard Gaussian variable, we have $\boldsymbol{E}\left(H_j(Y)H_k(Y)\right) = j!\,\delta_{j,k}$. For some measurable function $f(\cdot)$ defined on $\mathbb{R}$ such that $\boldsymbol{E}(f(Y)^2) < +\infty$, that is $f \in L^2(d\phi)$, we have the following expansion

$$f(t) = \sum_{j \geq \tau} \frac{c_j}{j!}\,H_j(t) \quad \text{with} \quad c_j = \boldsymbol{E}\left(f(Y)H_j(Y)\right).$$

The integer τ defined by $\tau = \inf\{j \geq 0, c_j \neq 0\}$, is called the Hermite rank of the function f. We also have

$$\boldsymbol{E}(f(Y)^2) = \sum_{j \geq \tau} \frac{(c_j)^2}{j!}.$$

Let us now describe assumption under which we obtain a Bahadur representation of $\widehat{\xi}\,(p; \boldsymbol{g(Y)})$.

Assumption $\mathbf{A_3}(\xi(\mathbf{p}))$: there exists U_i, $i = 1, \ldots, L$, disjoint open sets such that U_i contains a solution to the equation $g(t) = \xi_{g(Y)}(p)$ and such that g is a $\mathcal{C}^1-$diffeomorphism on $\cup_{i=1}^{L} U_i$.

Previous assumption asserts that $F'_{g(Y)}(\cdot)$ exists and is bounded in a neighborhood of $\xi_{g(Y)}(p)$. More precisely, in $\xi_{g(Y)}(p)$ we have

$$F'_{g(Y)}(\xi_{g(Y)}(p)) = f_{g(Y)}(\xi_{g(Y)}(p)) \ = \ \sum_{i=1}^{L} \frac{\phi(g_i^{-1}(t))}{g'(g_i^{-1}(t))},$$

where $g_i(\cdot)$ is the restriction of $g(\cdot)$ on U_i. Note that, Assumption $\mathbf{A_3}(\xi(\mathbf{p}))$ is the minimal hypothesis under which, in the i.i.d. case, a Bahadur representation can be obtained, see *e.g.* Serfling (1980).

Now, define, for some real u, the function $h_u(\cdot)$ by:

$$h_u(t) = \mathbf{1}_{\{g(t) \leq u\}}(t) - F_{g(Y)}(u). \tag{20}$$

We denote by $\tau(u)$ the Hermite rank of $h_u(\cdot)$. For the sake of simplicity, we put $\tau_p = \tau(\xi_{g(Y)}(p))$. For some function $g(\cdot)$ satisfying Assumption $\mathbf{A_3}(\xi(\mathbf{p}))$, we denote by

$$\overline{\tau}_p = \inf_{\gamma \in \cup_{i=1}^{L} g(U_i)} \tau(\gamma), \tag{21}$$

that is the minimal Hermite rank of $h_u(\cdot)$ for u in a neighborhood of $\xi_{g(Y)}(p)$.

Theorem 4.1 *Let $\{Y(i)\}_{i=1}^{+\infty}$ be a stationary (centered) gaussian process with variance 1, and correlation function $\rho(\cdot)$ such that, as $i \to +\infty$*

$$|\rho(i)| \sim L(i)\ i^{-\alpha} \tag{22}$$

for some $\alpha > 0$ and some slowly varying function at infinity $L(s), s \geq 0$. Then, under Assumption $\mathbf{A_3}(\xi(\mathbf{p}))$, we have almost surely, as $n \to +\infty$

$$\widehat{\xi}\,(p; \boldsymbol{g(Y)}) - \xi_{g(Y)}(p) = \frac{p - \widehat{F}\left(\xi_{g(Y)}(p); \boldsymbol{g(Y)}\right)}{f_{g(Y)}(\xi_{g(Y)}(p))} \ + \ \mathcal{O}\left(r_n(\alpha, \overline{\tau}_p)\right), \tag{23}$$

where the sequence $\left(r_n(\alpha, \overline{\tau}_p)\right)_{n \geq 1}$ is defined by

$$r_n(\alpha,\overline{\tau}_p)=\begin{cases} n^{-3/4}\log(n)^{3/4} & \text{if } \alpha\overline{\tau}_p>1,\\ n^{-3/4}\log(n)^{3/4}L_{\overline{\tau}_p}(n)^{3/4} & \text{if } \alpha\overline{\tau}_p=1,\\ n^{-1/2-\alpha\overline{\tau}_p/4}\log(n)^{\overline{\tau}_p/4+1/2}L(n)^{\overline{\tau}_p/4} & \text{if } 2/3<\alpha\overline{\tau}_p<1,\\ n^{-\alpha\overline{\tau}_p}\log(n)^{\overline{\tau}_p}L(n)^{\overline{\tau}_p} & \text{if } 0<\alpha\overline{\tau}_p\leq 2/3, \end{cases} \tag{24}$$

where for some $\tau\geq 1$, $L_\tau(n)=\sum_{|i|\leq n}|\rho(i)|^\tau$.

Note that if $L(\cdot)$ is an increasing function, $L_\tau(n)=\mathcal{O}(\log(n)L(n)^\tau)$.

Remark 4.1 *Without giving any detail here, let us precise that the behaviour of the sequence* $r_n(\cdot,\cdot)$ *is related to the characteristic (short-range or long-range dependence) of the process* $\{h_u(Y(i))\}_{i=1}^{+\infty}$ *for* u *in a neighborhood of* $\xi_{g(Y)}(p)$. *In the case* $\alpha\overline{\tau}_p>1$ *corresponding to short-range dependent processes, the result is similar to the one proved by Bahadur, see e.g. Serfling (1980), in the i.i.d. case when assuming that* $F'(\cdot)$ *exists and is bounded in a neighborhood of* $\xi(p)$. *For short-range dependent linear processes, Wu (2005) obtained a sharper bound, that is* $n^{-3/4}\log\log(n)^{3/4}$. *However, this bound was obtained under the assumption that* $F'(\cdot)$ *and* $F''(\cdot)$ *exist and are uniformly bounded. For long-range dependent processes (*$\alpha\overline{\tau}_p\leq 1$*), we can observe that the rate of convergence is always lower than* $n^{-3/4}\log(n)^{3/4}$ *and that the dominant term* $n^{-3/4}$ *is obtained when* $\alpha\overline{\tau}_p\to 1$.

Remark 4.2 *To obtain convergence results of estimators of* H, *some results are needed concerning empirical quantiles of the form* $\widehat{\xi}\left(p;\boldsymbol{g}(\boldsymbol{Y}^{\boldsymbol{a}^m}\right)$, *with* $g(\cdot)=|\cdot|$. *An obvious consequence of Lemma 6.5, is that the Hermite rank,* τ_p *of the function* $h_{\xi_{|Y|}(p)}(\cdot)$ *with* $g(\cdot)=|\cdot|$, *is given by*

$$\tau_p=\begin{cases} 2 & \text{if } p\neq 2\Phi(\sqrt{3})-1,\\ 4 & \text{if } p=2\Phi(\sqrt{3})-1. \end{cases} \tag{25}$$

Consequently, for all $0<p<1$, *we have* $\overline{\tau}_p=2$. *From Lemma 2.1, the correlation function of* $\boldsymbol{Y}^{\boldsymbol{a}^m}$ *satisfies (22) with* $\alpha=2\nu-2H$ *and* $L(\cdot)=1$. *Then by applying Theorem 4.1, we obtain that the sequence* $r_n(\cdot,\cdot)$ *is given by*

$$r_n(2\nu-2H,2)=n^{-3/4}\log(n)^{3/4},\qquad \text{if } \nu\geq 2 \tag{26}$$

and for $\nu = 1$

$$r_n(2-2H,2) = \begin{cases} n^{-3/4}\log(n)^{3/4} & \text{if } 0 < H < 3/4, \\ n^{-3/4}\log(n)^{3/2} & \text{if } H = 3/4, \\ n^{-1/2-(1-H)}\log(n) & \text{if } 3/4 < H < 5/6, \\ n^{-2(2-2H)}\log(n)^2 & \text{if } 5/6 \leq H < 1. \end{cases} \tag{27}$$

4.2 Convergence results of $\widehat{H}^\alpha$ and $\widehat{H}^{\log}$

In order to precise convergence results of $\widehat{H}^\alpha$ and $\widehat{H}^{\log}$, we make the following assumption concerning the remainder term of the variance function $v(\cdot)$.

Assumption $\mathbf{A_4}(\beta)$: there exists $\beta > 0$ such that $v(t) = \sigma^2|t|^{2H}\left(1 + \mathcal{O}(|t|^\beta\right)$, as $|t| \to 0$.

For the sake of simplicity, denote by $\widetilde{\tau} = \min_{k=1,\ldots,K} \tau_{p_k}$, where $\tau_{p_k} = \tau(\xi_{g(Y)}(p_k))$ is the Hermite rank of the function $h_{\xi(p_k)}(\cdot)$ with $g(\cdot) = |\cdot|$. From (25), we have

$$\widetilde{\tau} = \begin{cases} 2, & \text{if } K > 1 \text{ or } K = 1 \text{ and } p \neq 2\Phi(\sqrt{3}) - 1, \\ 4, & \text{if } K = 1 \text{ and } p = 2\Phi(\sqrt{3}) - 1. \end{cases} \tag{28}$$

For two sequences u_n and v_n, the notation $u_n \asymp v_n$ means that there exists two constants c_1, c_2 such that, as $n \to +\infty$, we have $c_1 v_n \leq u_n \leq c_2 v_n$.

Theorem 4.2 *Under Assumptions* $\mathbf{A_1}(\mathbf{2}\nu)$, $\mathbf{A_2}$ *and* $\mathbf{A_4}(\beta)$,

(i) we have almost surely, as $n \to +\infty$

$$\widehat{H}^\alpha - H = \begin{cases} \mathcal{O}(n^{-\beta}) + \mathcal{O}\left(n^{-1/2}\log(n)\right) & \text{if } \nu > H + \frac{1}{2\widetilde{\tau}}, \\ \mathcal{O}(n^{-\beta}) + \mathcal{O}\left(n^{-1/2}\log(n)^{3/2}\right) & \text{if } \nu = 1, H = 1 - \frac{1}{2\widetilde{\tau}}, \\ \mathcal{O}(n^{-\beta}) + \mathcal{O}\left(n^{-(1-H)\widetilde{\tau}}\log(n)^{\widetilde{\tau}/2}\right) & \text{if } \nu = 1, 1 - \frac{1}{2\widetilde{\tau}} < H < 1. \end{cases} \tag{29}$$

and $\widehat{H}^{\log} - H \asymp \widehat{H}^\alpha - H$.

(ii) the empirical mean squared errors (MSE) of $\widehat{H}^\alpha$ *and* $\widehat{H}^{\log}$ *satisfy*

$$MSE\left(\widehat{H}^\alpha - H\right) = \mathcal{O}\left(v_n(2\nu - 2H, \widetilde{\tau})\right) + \mathcal{O}\left(r_n(2\nu - 2H, 2)^2\right) + \mathcal{O}\left(n^{-2\beta}\right). \tag{30}$$

The sequence $r_n(2\nu-2H,2)$ *is given by (26) and (27) and the sequence* $v_n(\cdot,\cdot)$ *is defined by*

$$v_n(2\nu-2H,\widetilde{\tau})=\begin{cases} n^{-1} & \text{if } \nu > H+\frac{1}{2\widetilde{\tau}},\\ n^{-1}\log(n) & \text{if } \nu=1, H=1-\frac{1}{2\widetilde{\tau}},\\ n^{-2(1-H)\widetilde{\tau}} & \text{if } \nu=1, 1-\frac{1}{2\widetilde{\tau}}<H<1. \end{cases} \tag{31}$$

We also have,

$$MSE\left(\widehat{H}^{\log}-H\right)\asymp MSE\left(\widehat{H}^{\alpha}-H\right). \tag{32}$$

(iii) if the filter $\boldsymbol{a}$ *is such that* $\nu > H+1/(2\widetilde{\tau})$*, and if* $\beta > 1/2$*, then we have the following convergence in distribution, as* $n\to+\infty$

$$\sqrt{n}\left(\widehat{H}_n^{\alpha}-H\right)\longrightarrow\mathcal{N}(0,\sigma_{\alpha}^2) \qquad \text{and} \qquad \sqrt{n}\left(\widehat{H}_n^{\log}-H\right)\longrightarrow\mathcal{N}(0,\sigma_0^2), \tag{33}$$

where σ_{α}^2 *is defined for* $\alpha\geq 0$ *by*

$$\sigma_{\alpha}^2=\sum_{i\in\mathbb{Z}}\sum_{j\geq r}\frac{1}{(2j)!}\Big(\sum_{k=1}^{K}\frac{H_{2j-1}(q_k)c_k}{q_k}\,\pi_k^{\alpha}\Big)^2\boldsymbol{B}^T\underline{\boldsymbol{R}}(i,j)\boldsymbol{B}. \tag{34}$$

The vector $\boldsymbol{B}$ *is defined by* $\boldsymbol{B}=\dfrac{\boldsymbol{A}^T}{||\boldsymbol{A}||^2}$*, and the real numbers* q_k *and* π_k^{α} *are defined by*

$$q_k=\Phi^{-1}\left(\frac{1+p_k}{2}\right) \qquad \text{and} \qquad \pi_k^{\alpha}=\frac{(q_k)^{\alpha}}{\sum_{j=1}^{K}c_j(q_j)^{\alpha}}. \tag{35}$$

Finally, the matrix $\underline{\boldsymbol{R}}(i,j)$*, defined for* $i\in\mathbb{Z}$ *and* $j\geq 1$*, is a* $M\times M$ *matrix whose* (m_1,m_2) *entry is*

$$\left(\underline{\boldsymbol{R}}(i,j)\right)_{m_1,m_2}=\rho^{\boldsymbol{a^{m_1}},\boldsymbol{a^{m_2}}}(i)^{2j} \tag{36}$$

where $\rho^{\boldsymbol{a^{m_1}},\boldsymbol{a^{m_2}}}(\cdot)$ *is the correlation function defined by (7).*

Remark 4.3 *Let us discuss results (30) and (32). The first term,* $\mathcal{O}(v_n)$*, is due to the variance of the empirical cumulative distribution function. The second term,* $\mathcal{O}(r_n^2)$ *is due to the departure of* $\widehat{\xi}\,(p)-\xi(p)$ *from* $\widehat{F}\,(\xi(p))-p$*. We leave the reader to check that*

$$\mathcal{O}(r_n(2\nu-2H)^2)+\mathcal{O}(v_n(2\nu-2H,\widetilde{\tau}))=\begin{cases} \mathcal{O}(v_n(2\nu-2H,\widetilde{\tau})) & \text{if } \nu\geq H+\frac{1}{2\widetilde{\tau}},\\ \mathcal{O}(r_n(2\nu-2H)^2) & \text{if } \nu < H+\frac{1}{2\widetilde{\tau}}. \end{cases}$$

Finally, the third one, $\mathcal{O}(n^{-2\beta})$ is a bias term due to the misspecification of the variance function $v(\cdot)$ around 0.

Remark 4.4 *If $K = 1$, we have, for every $\alpha > 0$,*

$$\sigma_\alpha^2 = \sigma_0^2 = \sum_{i\in\mathbb{Z}} \sum_{j\geq 1} \frac{H_{2j-1}(q)^2}{q^2\ (2j)!}\ \boldsymbol{B}^T \underline{\boldsymbol{R}}(i,j)\boldsymbol{B},$$

If $\mathbf{A_1(2)}$ is only assumed, then, the order of the filter can not exceed 1. So, under Assumption $\mathbf{A_4}(\beta)$ with $\beta > 1/2$, we see that result (33) only holds for $0 < H < 7/8$, if $K = 1$ and $p = 2\Phi(\sqrt{3}) - 1$, and for $0 < H < 3/4$, otherwise.

Remark 4.5 *The condition $\beta > 1/2$ implying (33), ensures that the bias term is is negligible with respect to $n^{-1/2}$. Indeed, the bias term is governed by $b_n = 1/2\sum_{m=1}^M \delta_n^{\boldsymbol{a}^m}(0)$, where $\delta_n^{\boldsymbol{a}^m}(0)$ is defined by (4). And, under Assumption $\mathbf{A_4}(\beta)$, we have $b_n = \mathcal{O}(n^{-\beta})$. Let us precise, that if the bias term were known or at least estimated, the following convergence in distribution would hold, as $n \to +\infty$*

$$\sqrt{n}\left(\widehat{H}_n^\alpha - H - b_n\right) \longrightarrow \mathcal{N}(0,\sigma_\alpha^2) \quad \text{and} \quad \sqrt{n}\left(\widehat{H}_n^{\log} - H - b_n\right) \longrightarrow \mathcal{N}(0,\sigma_0^2).$$

Next Corollary asserts the link between $\widehat{H}^{\log}$ and $\widehat{H}^\alpha$.

Corollary 4.1 *Let $(\alpha_n)_{n\geq 1}$ be a sequence such that $\alpha_n \to 0$, as $n \to +\infty$. Then, under conditions of Theorem 4.2 (ii), the following convergence in distribution holds, as $n \to +\infty$*

$$\sqrt{n}\left(\widehat{H}_n^{\alpha_n} - H\right) \longrightarrow \mathcal{N}(0,\sigma_0^2). \tag{37}$$

5 Numerical computation and simulations

Estimators defined previously depend on several parameters that is the choice of the vectors $\boldsymbol{p}$ and $\boldsymbol{c}$, the filter $\boldsymbol{a}$, the number of dilated filters M and the function $g(\cdot)$ which is either $|\cdot|^\alpha$ for some $\alpha > 0$ or either $g(\cdot) = \log|\cdot|$. To have an idea on

optimal parameters, we plot the asymptoyic constant σ_α^2 defined by (34) in terms of H. Figure 1 illustrates a part of this work. In fact, we compute

$$\sum_{|i|\leq I}\sum_{j=r}^{J}\frac{1}{(2j)!}\Big(\sum_{k=1}^{K}\frac{H_{2j-1}(q_k)c_k}{q_k}\,\pi_k^{\alpha}\Big)^2\boldsymbol{B}^T\underline{\boldsymbol{R}}(i,j)\boldsymbol{B}.$$

with $I = 200$ and $J = 150$. We can propose the following general remarks:

- among all filters tested, the best one seems to be

$$\boldsymbol{a}^\star = \begin{cases} inc1 & \text{if } H < 3/4 \text{ and } K > 1, \text{ or } K = 1 \text{ and } p \neq 2\Phi(\sqrt{3}) - 1, \\ inc1 & \text{if } H < 7/8 \text{ and } K = 1, p = 2\Phi(\sqrt{3}) - 1, \\ db4 & \text{otherwise.} \end{cases}$$

  where $inc1$ (resp. $db4$) denotes the filter $(1, -1)$ (resp. a Daubechies wavelet filter with two zero moments).

- choice of M: increasing M seems to reduce the asymptotic constant σ_α^2. Obviously, a too large M increases the bias since $\widehat{\xi}\left(\boldsymbol{p}, \boldsymbol{c}; \boldsymbol{g}(\boldsymbol{X}^{\boldsymbol{a}^M})\right)$ is estimated with $N - M\ell$ observations. We advice to fix it to 5.

- the estimator based on $g(\cdot) = \log|\cdot|$ seems always to have a greater variance than the one based on the function $g(\cdot) = |\cdot|^\alpha$ for all $\alpha > 0$. We did not manage (theoretically and numerically since series defining (34) are truncated) to exhibit the optimal value of α. However, for examples considered, it should turn around the value 2.

- again, this is quite difficult (theoretically and numerically) to know which choice of $\boldsymbol{p}$ is optimal. What we observed is that, for fixed parameters $\boldsymbol{a}$, M and α, asymptotic constants are very near.

Then, we intend to conduct a short simulation study. Two locally self-similar Gaussian processes whose variance functions are $v(t) = |t|^{2H}$ (fractional Brownian motion) and $v(t) = 1 - \exp(-|t|^{2H})$ are considered. To generate sample paths discretized over a grid $[0, 1]$, we use the method of circulant matrix (see Wood and Chan (1994)), which is particularly fast, even for large sample sizes. Figure 2

presents empirical mean squared errors in terms of n, using 500 Monte-Carlo simulations for three values of the Hurst parameter ($H = 0.3, 0.5$ and 0.8). Four $L-$type estimators have been considered (based on the median, the $(2\Phi\sqrt{3}-1)-$quantile, the mean of quartiles and the 10%-trimmed mean) and are compared with a version of estimator based on quadratic variations, Coeurjolly (2001) and Whittle estimator, *e.g.* Beran (1994). We can say that $L-$type estimators are really efficient for the three values of the parameter H. We can also underline that $L-$type estimators are very competitive with classical ones.

Now, let us illustrate the fact that L$-$type estimators are more robust to additive outliers. The contaminated version of sample paths of processes $X(t)$ discretized at times i/n, that we consider, is denoted by $X^C(i/n)$ for $i = 1, \ldots, n$. We choose the following model

$$X^C(i/n) = X(i/n) + U(i)V(i), \tag{38}$$

where $U(i)$, $i = 1, \ldots, n$ are Bernoulli independent variables $\mathcal{B}(p)$, and $V(i)$, $i = 1, \ldots, n$ are independent centered Gaussian variables with variance $\sigma_C^2(i)$ such that the signal noise ratio at time i/n is equal to 20 dB. We choose $n = 1000$, $H = 0.8$ and $p = 0.005$. Figure 3 gives some examples of discretized sample paths of such processes. Table 1 summarizes a simulation of $m = 500$ replications. We observe that L$-$type estimators are unchanged and, so, are robust to the model defined by (38). This is, as expected, not the case for quadratic variations method and Whittle estimator. Indeed, concerning quadratic variations method, the estimation procedure is based on the estimation of $\boldsymbol{E}((X^{\boldsymbol{a}^m}(1/n))^2)$ by empirical mean of order 2 of $(\boldsymbol{X}^{\boldsymbol{a}^m})^2$, Coeurjolly (2001)), that is particularly sensitive to additive outliers. Bad results of Whittle estimator can be explained by the fact that maximum likelihood methods are also non-robust methods.

6 Proofs

We denote by $||\cdot||_{L2(d\phi)}$ (resp. $||\cdot||_{\ell^q}$) the norm defined by $||h||_{L^2(d\phi)} = E(h(Y)^2)^{1/2}$ for some measurable function $h(\cdot)$ (resp. $(\sum_{i\in\mathbb{Z}} |u_i|^q)^{1/2}$ for some sequence $(u_i)_{i\in\mathbb{Z}}$).

In order to simplify presentation of proofs, we use the notation $F(\cdot)$, $\xi(\cdot)$, $f(\cdot)$, $\widehat{F}(\cdot)$ and $\widehat{\xi}(\cdot)$ instead of $F_{g(Y)}(\cdot)$, $\xi_{g(Y)}(\cdot)$, $f_{g(Y)}(\cdot)$, $\widehat{F}_{g(Y)}(\cdot;\boldsymbol{g(Y)})$ and $\widehat{\xi}_{g(Y)}(\cdot;\boldsymbol{g(Y)})$ respectively. For some real x, $[x]$ denotes the integer part of x. Finally, λ denotes a generic positive constant.

6.1 Auxiliary Lemmas for proof of Theorem 4.1

Lemma 6.1 *Let $\{Y(i)\}_{i=1}^{+\infty}$ a gaussian stationary process with variance 1 and correlation function $\rho(\cdot)$ such that, as $i \to +\infty$, $|\rho(i)| \sim L(i)i^{-\alpha}$, for some $\alpha > 0$ and some slowly varying function at infinity $L(\cdot)$. Let $h(\cdot) \in L^2(d\phi)$ and denote by τ its Hermite rank. Define*

$$\overline{Y}_n = \frac{1}{n}\sum_{i=1}^{n} h(Y(i)).$$

Then, for all $\gamma > 0$, there exists a positive constant $\kappa_\gamma = \kappa_\gamma(\alpha,\tau)$ such that

$$\mathbb{P}\left(|\overline{Y}_n| \geq \kappa_\gamma y_n\right) = \mathcal{O}(n^{-\gamma}), \tag{39}$$

with

$$y_n = y_n(\alpha,\tau) = \begin{cases} n^{-1/2}\log(n)^{1/2} & \text{if} \quad \alpha\tau > 1, \\ n^{-1/2}\log(n)^{1/2}L_\tau(n)^{1/2} & \text{if} \quad \alpha\tau = 1, \\ n^{-\alpha\tau/2}\log(n)^{r/2}L(n)^{r/2} & \text{if} \quad 0 < \alpha\tau < 1. \end{cases} \tag{40}$$

where $L_\tau(n) = \sum_{|i|\leq n}|\rho(i)|^\tau$. For the case $\alpha\tau = 1$, we assume that for all $j > \tau$, the limit, $\lim_{n\to+\infty} L_\tau(n)^{-1}\sum_{|i|\leq n}|\rho(i)|^j$ exists.

Proof. Let $(y_n)_{n\geq 1}$ the sequence defined by (40). Proof is split into three parts according to the value of $\alpha\tau$.

<u>**Case $\alpha\tau < 1$:**</u> From Chebyshev's inequality, we have for all $q \geq 1$

$$\mathbb{P}\left(|\overline{Y}_n| \geq \kappa_\gamma y_n\right) \leq \frac{1}{\kappa_\gamma^{2q} y_n^{2q}}\boldsymbol{E}\Big((\overline{Y}_n)^{2q}\Big).$$

From Theorem 1 of Breuer and Major (1983) and in particular Equation (2.6), we have, as $n \to +\infty$

$$\boldsymbol{E}\Big((\overline{Y}_n)^{2q}\Big) \sim \frac{(2q)!}{2^q q!}\frac{1}{n^q}\sigma^{2q}, \quad \text{with} \quad \sigma^2 = \sum_{i\in\mathbb{Z}}\sum_{j\geq\tau}\frac{(c_j)^2}{j!}\rho(i)^j, \tag{41}$$

where c_j denotes the j-th Hermite coefficient of $h(\cdot)$. Note that $\sigma^2 \leq ||h||^2_{L^2(d\phi)}||\rho||^2_{\ell^\tau}$. Thus, for n large enough, we have

$$\mathbb{P}\left(|\overline{Y}_n| \geq \kappa_\gamma y_n\right) \leq \frac{\lambda}{n^q \varepsilon_n^{2q}} \frac{(2q)!}{2^q q!} \left(||h||^2_{L^2(d\phi)}||\rho||^2_{\ell^\tau}\kappa_\gamma^{-2}\right)^q. \tag{42}$$

From Stirling's formula, we have as $q \to +\infty$

$$\frac{(2q)!}{2^q q!} \sim \sqrt{2}\, q^q\, (2e^{-1})^q. \tag{43}$$

Then from (40) and by choosing $q = [\log(n)]$, (42) becomes

$$\mathbb{P}\left(|\overline{Y}_n| \geq \kappa_\gamma y_n\right) \leq \lambda \left(2e^{-1}||h||^2_{L^2(d\phi)}||\rho||^2_{\ell^\tau}\kappa_\gamma^{-2}\right)^{\log(n)} = \mathcal{O}(n^{-\gamma}),$$

if $\kappa_\gamma^2 > 2||h||^2_{L^2(d\phi)}||\rho||^2_{\ell^\tau} \exp(\gamma - 1)$.

Case $\alpha\tau = 1$: Using the proof of Theorem 1′ of Breuer and Major (1983), we can prove that for all $q \geq 1$

$$\begin{aligned}
\boldsymbol{E}\left((n^{1/2}L_\tau(n)^{-1/2}\overline{Y}_n)^{2q}\right) &\leq \lambda \frac{2q!}{2^q q!} \boldsymbol{E}\left((n^{1/2}L_\tau(n)^{-1/2}\overline{Y}_n)^2\right)^q \\
&\leq \lambda \frac{2q!}{2^q q!} \left(\sum_{j\geq\tau} \frac{(c_j)^2}{j!} \lim_{n\to+\infty} L_\tau(n)^{-1} \sum_{|i|\leq n} |\rho(i)|^j\right)^q \\
&\leq \lambda \frac{2q!}{2^q q!} ||h||^{2q}_{L^2(d\phi)}.
\end{aligned} \tag{44}$$

Then from Chebyshev's inequality, we have for all $q \geq 1$

$$\mathbb{P}\left(|\overline{Y}_n| \geq \kappa_\gamma y_n\right) \leq \lambda \frac{L_\tau(n)^q}{n^q \varepsilon_n^{2q}} \frac{2q!}{2^q q!} \left(||h||^2_{L^2(d\phi)}\kappa_\gamma^{-2}\right)^q.$$

Then from (40) and by choosing $q = [\log(n)]$, we obtain

$$\mathbb{P}\left(|\overline{Y}_n| \geq \kappa_\gamma y_n\right) \leq \lambda \left(2e^{-1}||h||^2_{L^2(d\phi)}\, \kappa_\gamma^{-2}\right)^{\log(n)} = \mathcal{O}(n^{-\gamma}),$$

if $\kappa_\gamma^2 > 2||h||^2_{L^2(d\phi)} \times \exp(\gamma - 1)$.

Case $\alpha\tau < 1$: Denote by k_α the lowest integer such that $k_\alpha\alpha > 1$, that is $k_\alpha = [1/\alpha] + 1$, and denote by Z_j, for $j \geq \tau$ the following random variable

$$Z_j = \frac{1}{n} \sum_{i=1}^n \frac{c_j}{j!} H_j(Y(i)).$$

Denote by $\kappa_{1,\gamma}$ and $\kappa_{2,\gamma}$ two positive constants such that $\kappa_\gamma = \max(\kappa_{1,\gamma}, \kappa_{2,\gamma})$. From triangle's inequality,

$$\mathbb{P}\left(|\overline{Y}_n| \geq \kappa_\gamma y_n\right) \leq \mathbb{P}\Big(|\overline{Y}_n - \sum_{j=\tau}^{k_\alpha - 1} Z_j| \geq \kappa_{1,\gamma} y_n\Big) + \sum_{j=\tau}^{k_\alpha - 1} \mathbb{P}\left(|Z_j| \geq \kappa_{2,\gamma} y_n\right) \tag{45}$$

Since

$$\overline{Y}_n - \sum_{j=\tau}^{k_\alpha - 1} Z_j = \frac{1}{n}\sum_{i=1}^{n}\sum_{j\geq k_\alpha} \frac{c_j}{j!} H_j(Y(i)) = \frac{1}{n}\sum_{i=1}^{n} h'(Y(i)),$$

where $h'(\cdot)$ is a function with Hermite rank k_α. Then by applying Lemma 6.1 in the case where $\alpha\tau > 1$, it comes that, for all $\gamma > 0$, there exists a constant $\kappa_{1,\gamma}$ such that, for n large enough

$$\mathbb{P}\Big(|\overline{Y}_n - \sum_{j=\tau}^{k_\alpha - 1} Z_j| \geq \kappa_{1,\gamma} y_n\Big) = \mathcal{O}(n^{-\gamma}), \tag{46}$$

Now, let $\tau \leq j < k_\alpha$ and $q \geq 1$, from Theorem 3 of Taqqu (1977), we have

$$\begin{aligned}\mathbb{P}\left(|Z_j| \geq \kappa_{2,\gamma} y_n\right) &\leq \frac{1}{\kappa_{2,\gamma}^{2q} y_n^{2q}} \left(\frac{c_j}{j!}\right)^{2q} n^{-2q}\, \boldsymbol{E}\left(\sum_{i_1,\ldots,i_{2q}} H_j(Y(i_1))\ldots H_j(Y(i_{2q}))\right) \\ &\leq \lambda \frac{L(n)^{jq}}{n^{\alpha j q}\varepsilon_n^{2q}} \left(\frac{c_j}{j!}\kappa_{2,\gamma}^{-1}\right)^{2q} \mu_{2q}, \end{aligned} \tag{47}$$

where μ_{2q} is a constant such that $\mu_{2q} \leq \left(\frac{2}{1-\alpha j}\right)^q \boldsymbol{E}\left(H_j(Y)^{2q}\right)$. It is also proved in Taqqu (1977) (p. 228), that $\boldsymbol{E}\left(H_j(Y)^{2q}\right) \sim (2jq)!/(2^{jq}(jq)!)$, as $q \to +\infty$. Thus, from Stirling's formula, we obtain as $q \to +\infty$

$$\mathbb{P}\left(|Z_j| \geq \varepsilon_n\right) \leq \lambda\, \frac{L(n)^{(j-\tau)q}}{n^{\alpha(j-\tau)q}} \log(n)^{\tau q} q^{jq} \left(\frac{2}{1-\alpha j}\left(\frac{c_j}{j!}\right)^2 \left(\frac{2j}{e}\right)^j \kappa_{2,\gamma}^{-1}\right)^q.$$

By choosing $q = [\log(n)]$, we finally obtain, as $n \to +\infty$

$$\sum_{j=\tau}^{k_\alpha - 1} \mathbb{P}\left(|Z_j| \geq \kappa_{2,\gamma} y_n\right) \leq \lambda \left(\frac{2}{1-\alpha\tau}\left(\frac{c_\tau}{\tau!}\right)^2 \left(\frac{2\tau}{e}\right)^\tau \kappa_{2,\gamma}^{-2}\right)^{\log(n)} = \mathcal{O}(n^{-\gamma}), \tag{48}$$

if $\kappa_{2,\gamma}^2 > \frac{2}{1-\alpha\tau}\left(\frac{c_\tau}{\tau!}\right)^2 (2\tau)^\tau \exp(\gamma - \tau)$. From (45), we get the result by combining (46) and (48). ■

Corollary 6.1 *Under conditions of Lemma 6.1, for all $\alpha > 0$, $j \geq 1$ and $\gamma > 0$, there exists $q = q(\gamma) \geq 1$ and $\zeta_\gamma > 0$ such that*

$$\boldsymbol{E}\left(\left\{\frac{1}{n}\sum_{i=1}^{n} H_j(Y(i))\right\}^{2q}\right) \leq \zeta_\gamma n^{-\gamma} \tag{49}$$

Proof. From (41), (44) and (47), it comes that there exists $\lambda = \lambda(q) > 0$ such that for all $q \geq 1$, we have

$$\begin{aligned}\boldsymbol{E}\left(\left\{\frac{1}{n}\sum_{i=1}^{n} H_j(Y(i))\right\}^{2q}\right) \leq \lambda(q)n^{-q} &= \lambda(q) \times \begin{cases} n^{-q} & \text{if } \alpha j > 1 \\ L_{\tau_p}(n)n^{-q} & \text{if } \alpha j = 1 \\ L(n)^{\alpha j q} n^{-\alpha j q} & \text{if } \alpha j < 1 \end{cases} \\ &= \mathcal{O}(n^{-\gamma}). \end{aligned} \tag{50}$$

Indeed, it is sufficient to choose q such that, $q > \gamma$ if $\alpha j \geq 1$ and $q > \gamma/\alpha j$ if $\alpha j < 1$. ■

Lemma 6.2 *Let $0 < p < 1$, denote by $g(\cdot)$ a function satisfying Assumption $\mathbf{A_3}(\xi(\mathbf{p}))$ and by $(x_n)_{n\geq 1}$ a sequence with real components, such that $x_n \to 0$, as $n \to +\infty$. Then, for all $j \geq 1$, there exists a positive constant $d_j = d_j(\xi(p)) < +\infty$ such that, for n large enough*

$$|c_j(\xi(p) + x_n) - c_j(\xi(p))| \leq d_j\, |x_n|. \tag{51}$$

Proof.

Let $j \geq 1$, under Assumption $\mathbf{A_3}(\xi(\mathbf{p}))$, for n large enough, $\xi(p) + x_n \in \cup_{i=1}^{L} g(U_i)$. Thus, for n large enough,

$$\begin{aligned} c_j(\xi(p) + x_n) - c_j(\xi(p)) &= \int_{\mathbb{R}} \left(h_{\xi(p)+x_n}(t) - h_{\xi(p)}(t)\right) H_j(t)\phi(t)dt \\ &= \sum_{i=1}^{L} \int_{U_i} \left(\mathbf{1}_{g_i(t)\leq \xi(p)+x_n} - \mathbf{1}_{g_i(t)\leq \xi(p)}\right) H_j(t)\phi(t)dt \\ &= \sum_{i=1}^{L} \int_{m_{i,n}}^{M_{i,n}} (-1)^j \phi^{(j)}(t)dt, \\ &= \begin{cases} \sum_{i=1}^{L} -\left(\phi(M_{i,n}) - \phi(m_{i,n})\right) & \text{if } j = 1, \\ \sum_{i=1}^{L} (-1)^j \left(\phi^{(j-1)}(M_{i,n}) - \phi^{(j-1)}(m_{i,n})\right) & \text{if } j > 1, \end{cases} \end{aligned}$$

where $g_i(\cdot)$ is the restriction of $g(\cdot)$ to U_i, and where $m_{i,n}$ (resp. $M_{i,n}$) is the minimum (resp. maximum) between $g_i^{-1}(\xi(p)+x_n)$ and $g_i^{-1}(\xi(p))$. We leave the reader to check that there exists a constant λ, such that, for n large enough

$$|c_j(\xi(p)+x_n)-c_j(\xi(p))| \leq d_j|x_n| \times \begin{cases} \sum_{i=1}^{L} \left|\phi^{(j)}(g_i^{(-1)}(u))\ (g_i^{(-1)})'(u)\right| & \text{if } j=1,2 \\ \sum_{i=1}^{L} \left|\phi^{(j-2)}(g_i^{(-1)}(u))\ (g_i^{(-1)})'(u)\right| & \text{if } j>2, \end{cases}$$

which is the result. ■

Note that (65) is valid for all $j \geq 1$ which means that for $1 \leq j < \tau_p$, we have

$$c_j(\xi(p)+x_n) \leq d_j|x_n|. \tag{52}$$

Lemma 6.3 *Under conditions of Theorem 4.1, there exists a constant denoted by $\kappa_\varepsilon = \kappa_\varepsilon(\alpha,\tau_p)$, such that, we have almost surely, as $n \to +\infty$*

$$\left|\widehat{\xi}\,(p;\boldsymbol{g(Y)}) - \xi_{g(Y)}(p)\right| \leq \varepsilon_n, \tag{53}$$

where $\varepsilon_n = \varepsilon_n(\alpha,\tau(\xi(p))) = \kappa_\varepsilon y_n(\alpha,\tau(\xi(p))$, $y_n(\cdot,\cdot)$ being defined by (40).

Proof. We have

$$\mathbb{P}\left(\left|\widehat{\xi}\,(p)-\xi(p)\right| \geq \varepsilon_n\right) = \mathbb{P}\left(\widehat{\xi}\,(p) \leq \xi(p)-\varepsilon_n\right) + \mathbb{P}\left(\widehat{\xi}\,(p) \geq \xi(p)+\varepsilon_n\right) \tag{54}$$

Using Lemma 1.1.4 (*iii*) of Serfling (1980), we have

$$\mathbb{P}\left(\widehat{\xi}\,(p) \leq \xi(p)-\varepsilon_n\right) \leq \mathbb{P}\left(\widehat{F}\,(\xi(p)-\varepsilon_n) \geq p\right), \tag{55}$$

Under Assumption $\mathbf{A_3}(\xi(\mathbf{p}))$, we have, for n large enough

$$p - F(\xi(p)-\varepsilon_n) = f(\xi(p))\varepsilon_n + o(\varepsilon_n) \geq \frac{f(\xi(p))}{2}\varepsilon_n,$$

Consequently, for n large enough, we have from (55)

$$\mathbb{P}\left(\widehat{\xi}\,(p) \leq \xi(p)-\varepsilon_n\right) \leq \mathbb{P}\left(\widehat{F}\,(\xi(p)-\varepsilon_n) - F(\xi(p)-\varepsilon_n) \geq \frac{f(\xi(p))}{2}\varepsilon_n\right) \tag{56}$$

Define $\tau_{p,n} = \tau(\xi(p)-\varepsilon_n)$, from Lemma 6.2 and from (52), we have for n large enough

$$\widehat{F}\,(\xi(p)-\varepsilon_n) - F(\xi(p)-\varepsilon_n) \geq 2\left(\widehat{F}\,(\xi(p)) - F(\xi(p))\right) + 2\varepsilon_n \sum_{j\in J_n} Z_{n,j} \tag{57}$$

where

$$J_n = \begin{cases} \{\tau_p < j \leq \tau_{p,n}\} & \text{if } \tau_{p,n} > \tau_p, \\ \emptyset & \text{if } \tau_{p,n} = \tau_p, \\ \{\tau_{p,n} \leq j < \tau_p\} & \text{if } \tau_{p,n} < \tau_p. \end{cases} \quad \text{and} \quad Z_{n,j} = \frac{1}{n}\sum_{i=1}^{n} \frac{d_j}{j!} H_j(Y(i))$$

Now, define $c_\varepsilon = \kappa_\varepsilon f(\xi(p))/4$. Let $\gamma > 0$, it comes from (49) that there exists $q \geq 1$ such that, for n large enough

$$\begin{aligned} \mathbb{P}\left(|2\varepsilon_n Z_n| \geq \frac{f(\xi(p))}{2}\varepsilon_n\right) &\leq \sum_{j\in J_n} \mathbb{P}\left(|Z_{n,j}| > c_\varepsilon\right) \\ &\leq \sum_{j\in J_n} \frac{1}{c_\varepsilon^{2q}} \boldsymbol{E}\left(Z_{n,j}^{2q}\right) = \mathcal{O}(n^{-\gamma}). \end{aligned} \tag{58}$$

Let us fix $\gamma = 2$. From (56), (57) and (58) and from Lemma 6.1 (applied to the function $h_{\xi(p)}(\cdot)$), we now obtain

$$\mathbb{P}\left(\widehat{\xi}\,(p) \leq \xi(p) - \varepsilon_n\right) \leq \mathbb{P}\left(|\widehat{F}\,(\xi(p)) - F(\xi(p))| \geq c_\varepsilon \varepsilon_n\right) + \mathcal{O}(n^{-2}) = \mathcal{O}(n^{-2}),$$

if $c_\varepsilon > \kappa_2$ that is if $\kappa_\varepsilon > 4/f(\xi(p))\kappa_2$.

Let us now focus on the second right-hand term of (54). Following the sketch of this proof, we can also obtain, for n large enough

$$\mathbb{P}\left(\widehat{\xi}\,(p) \geq \xi(p) + \varepsilon_n\right) = \mathcal{O}(n^{-2}),$$

if $\kappa_\varepsilon > 4/f(\xi(p))\kappa_2$. Thus, we obtain, for n large enough

$$\mathbb{P}\left(\left|\widehat{\xi}\,(p) - \xi(p)\right| \geq \varepsilon_n\right) = \mathcal{O}(n^{-2}),$$

which leads to the result thanks to Borel-Cantelli's Lemma. ■

Following Lemma is an analog result obtained by Bahadur in the i.i.d. framework, see Lemma E p.97 of Serfling (1980).

Lemma 6.4 *Under conditions of Theorem 4.1, denote by $\Delta(z)$ for $z \in \mathbb{R}$ the random variable, $\Delta(z) = \widehat{F}(z; \boldsymbol{g(Y)}) - F_{g(Y)}(z)$. Then, we have almost surely, as $n \to +\infty$*

$$S_n = \sup_{|x|\leq\varepsilon_n} \left|\Delta(\xi_{g(Y)}(p) + x) - \Delta(\xi_{g(Y)}(p))\right| = \mathcal{O}\left(r_n\right), \tag{59}$$

where $\varepsilon_n = \varepsilon_n(\alpha, \tau_p)$ is defined by (53) and $r_n(\alpha, \overline{\tau}_p)$ is defined by (24).

Proof. Put $\varepsilon_n = \varepsilon(\alpha, \tau_p)$ and $r_n = r_n(\alpha, \overline{\tau}_p)$. Denote by $(\beta_n)_{n\geq 1}$ and $(\eta_{b,n})_{n\geq 1}$ the two following sequences

$$\beta_n = [n^{3/4}\varepsilon_n] \quad \text{and} \quad \eta_{b,n} = \xi(p) + \varepsilon_n \frac{b}{\beta_n},$$

for $b = -\beta_n, \ldots, \beta_n$. Using the monotonicity of $F(\cdot)$ and $\widehat{F}(\cdot)$, we have,

$$S_n \leq \max_{-\beta_n \leq b \leq \beta_n} |M_{b,n}| + G_n, \tag{60}$$

where $M_{b,n} = \Delta(\eta_{b,n}) - \Delta(\xi(p))$ and

$$G_n = \max_{-\beta_n \leq b \leq \beta_n - 1} \left(F(\eta_{b+1,n}) - F(\eta_{b+1,n})\right).$$

Under Assumption $\mathbf{A_3}(\xi(\mathbf{p}))$, we have for n large enough

$$G_n \leq (\eta_{b+1,n} - \eta_{b,n}) \times \sup_{|x| \leq \varepsilon_n} f(\xi(p)) = \mathcal{O}(n^{-3/4}). \tag{61}$$

The proof is finished, if one can prove that, for all $\gamma > 0$ (and in particular $\gamma = 2$) there exists κ'_γ such that

$$\mathbb{P}\left(|M_{b,n}| \geq \kappa'_\gamma r_n\right) = \mathcal{O}(n^{-\gamma}). \tag{62}$$

Indeed, since $\beta_n = \mathcal{O}(n^{1/2+\delta})$ for all $\delta > 0$, if (62) is true, then we have

$$\begin{aligned} \mathbb{P}(\max_{-\beta_n \leq b \leq \beta_n} |M_{b,n}| \geq \theta'_2 r_n(\alpha, \tau_p)) &\leq (2\beta_n + 1)\mathbb{P}\left(|M_{b,n}| \geq \kappa'_2 r_n\right) \\ &= \mathcal{O}(n^{-3/2+\delta}). \end{aligned}$$

Thus, from Borel-Cantelli's Lemma, we have, almost surely

$$\max_{-\beta_n \leq b \leq \beta_n} |M_{b,n}| = \mathcal{O}(r_n)$$

And so, from (60) and (61),

$$S_n = \mathcal{O}(r_n) + \mathcal{O}(n^{-3/4}) = \mathcal{O}(r_n), \tag{63}$$

which is the result.

So, the rest of the proof is devoted to prove (62). For the sake of simplicity, denote by $h'_n(\cdot)$ the function $h_{\eta_{b,n}}(\cdot) - h_{\xi(p)}(\cdot)$. For n large enough, the Hermite rank of $h'_n(\cdot)$ is at least equal to $\overline{\tau}_p$, that is defined by (21). For the following, we need a bound for $||h'_n||^2_{L^2(d\phi)}$. We obtain,

$$||h'_n||^2_{L^2(d\phi)} = \boldsymbol{E}(h'_n(Y)^2) = \omega_n(1-\omega_n) \quad \text{with} \quad \omega_n = |F_{g(Y)}(\eta_{b,n}) - F_{g(Y)}(\xi(p))|.$$

As previously, we have $\omega_n = \mathcal{O}(\varepsilon_n)$ and so, there exists $\zeta > 0$, such that

$$||h'_n||^2_{L^2(d\phi)} \leq \zeta \varepsilon_n. \tag{64}$$

From now on, in order to simplify the proof, we use the following upper-bound

$$\varepsilon_n = \varepsilon_n(\alpha, \tau_p) \leq \varepsilon_n(\alpha, \overline{\tau}_p).$$

And with a slight abuse, we still denote $\varepsilon_n = \varepsilon_n(\alpha, \overline{\tau}_p)$. Note also, that from Lemma 6.2, the j-th Hermite coefficient, for some $j \geq \overline{\tau}_p$, is given by $c_j(\eta_{b,n}) - c_j(\xi(p)$. And there exists a positive constant $d_j = d_j(\xi(p))$ such that for n large enough

$$|c_j(\eta_{b,n}) - c_j(\xi(p)| \leq d_j\ \varepsilon_n \frac{|b|}{\beta_n}\ \leq\ d_j\ \varepsilon_n. \tag{65}$$

We now proceed like proof of Lemma 6.1.

Case $\alpha\overline{\tau}_\mathbf{p} > \mathbf{1}$: using Theorem 1 of Breuer and Major (1983) and (42), we can obtain for all $q \geq 1$

$$\mathbb{P}\left(|M_{b,n}| \geq \kappa'_\gamma r_n\right) \leq \lambda \frac{1}{n^q r_n^{2q}} \frac{(2q)!}{2^q q!} \frac{1}{(\kappa'_\gamma)^{2q}} ||h'_n||^{2q}_{L^2(d\phi)} ||\rho||^{2q}_{\ell^{\overline{\tau}_p}}. \tag{66}$$

As $q \to +\infty$, we get

$$\mathbb{P}\left(|M_{b,n}| \geq \kappa'_\gamma r_n\right) \leq \lambda \frac{\varepsilon_n^q}{n^q r_n^{2q}} q^q \left(2\zeta e^{-1} ||\rho||^2_{\ell^{\overline{\tau}_p}} \frac{1}{(\kappa'_\gamma)^2}\right)^q.$$

From (24), (40) (with $\tau = \overline{\tau}_p$) and by choosing $q = [\log(n)]$, we have

$$\mathbb{P}\left(|M_{b,n}| \geq \kappa'_\gamma r_n\right) \leq \lambda \left(2\zeta\kappa_\varepsilon e^{-1} ||\rho||^2_{\ell^{\overline{\tau}_p}} \frac{1}{(\kappa'_\gamma)^2}\right)^{\log(n)} = \mathcal{O}(n^{-\gamma}), \tag{67}$$

if ${\kappa'_\gamma}^2 > 2\zeta\kappa_\varepsilon ||\rho||^2_{\ell^{\tau_p}} \exp(\gamma - 1)$.

Case $\alpha\overline{\tau}_{\mathbf{p}} = \mathbf{1}$ from (44), we can obtain for all $q \geq 1$

$$\begin{aligned}\boldsymbol{E}\left(M_{b,n}^{2q}\right) &\leq \lambda \frac{(2q)!}{2^q q!} \frac{L_{\overline{\tau}_p}(n)^q}{n^q} ||h_n'||_{L^2(d\phi)}^{2q} \;\leq\; \lambda\, \zeta^q\, \frac{(2q)!}{2^q q!}\, \frac{L_{\overline{\tau}_p}(n)^q \varepsilon_n^q}{n^q} \\ &\leq \lambda\, \frac{L_{\overline{\tau}_p}(n)^q \varepsilon_n^q}{n^q}\, (2\zeta e^{-1})^q\, q^q \end{aligned} \tag{68}$$

From (24), (40) (with $\tau = \overline{\tau}_p$) and by choosing $q = [\log(n)]$, we have

$$\begin{aligned}\mathbb{P}\left(|M_{b,n}| \geq \kappa_\gamma' r_n\right) &\leq \frac{1}{{\kappa_\gamma'}^{2q} r_n^{2q}} \boldsymbol{E}\left(M_{b,n}^{2q}\right) \\ &\leq \lambda \left(2\zeta\, \kappa_\varepsilon\, e^{-1} \frac{d_{\tau_p}^2}{\tau_p!} \frac{1}{{\kappa_\gamma'}^2}\right)^{\log(n)} = \mathcal{O}(n^{-\gamma}),\end{aligned}$$

if ${\kappa_\gamma'}^2 > 2\zeta\kappa_\varepsilon d_{\tau_p}^2/\tau_p! \exp(\gamma - 1)$.

Case $\alpha\overline{\tau}_{\mathbf{p}} < \mathbf{1}$: denote by $(r_{1,n})_{n\geq 1}$ and by $(r_{2,n})_{n\geq 1}$ the two following sequences

$$r_{1,n} = n^{-1/2-\alpha\overline{\tau}_p/4} \log(n)^{\overline{\tau}_p/4+1/2} L(n)^{\overline{\tau}_p/4} \;\text{ and }\; r_{2,n} = n^{-\alpha\overline{\tau}_p} \log(n)^{\overline{\tau}_p} L(n)^{\overline{\tau}_p}. \tag{69}$$

Note that $\max(r_{1,n}, r_{2,n})$ is equal to $r_{1,n}$ when $2/3 < \alpha\overline{\tau}_p < 1$ and to $r_{2,n}$ when $0 < \alpha\overline{\tau}_p \leq 2/3$. So, in order to obtain (62) in the case $0 < \alpha\overline{\tau}_p < 1$, it is sufficient to prove that there exists κ_γ' such that, for n large enough

$$\mathbb{P}\left(|M_{b,n}| \geq \kappa_\gamma' \max(r_{1,n}, r_{2,n})\right) = \mathcal{O}(n^{-\gamma})$$

Denote by k_α the integer $[1/\alpha] + 1$, which is such that $\alpha k_\alpha > 1$, and by $Z_{j,n}$ for $\overline{\tau}_p \leq j < k_\alpha$ the random variable defined by

$$Z_{j,n} = \frac{1}{n} \sum_{i=1}^{n} \frac{c_j(\eta_{b,n}) - c_j(\xi(p)}{j!}\, H_j(Y(i)).$$

From triangle's inequality, we have

$$\mathbb{P}\left(|M_{b,n}| \geq \kappa_\gamma' \max(r_{1,n}, r_{2,n})\right) \leq \mathbb{P}(|M_{b,n} - \sum_{j=\overline{\tau}_p}^{k_\alpha - 1} Z_{j,n}| \geq \kappa_\gamma' r_{1,n}) + \sum_{j=\overline{\tau}_p}^{k_\alpha-1} \mathbb{P}\left(|Z_{j,n}| \geq \kappa_\gamma' r_{2,n}\right). \tag{70}$$

Since,

$$M_{b,n} - \sum_{j=\tau_p}^{k_\alpha-1} Z_{j,n} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j\geq k_\alpha} \frac{c_j(\eta_{b,n}) - c_j(\xi(p))}{j!} H_j(Y(i)) = \frac{1}{n} \sum_{i=1}^{n} h_n''(Y(i)),$$

where $h_n''(\cdot)$ is a function with Hermite rank k_α, such that $\alpha k_\alpha > 1$, we have from (66)

$$\mathbb{P}(|M_{b,n} - \sum_{j=\overline{\tau}_p}^{k_\alpha - 1} Z_{j,n}| \geq \kappa'_\gamma r_{1,n}) \leq \lambda \, \frac{1}{n^q r_{1,n}^{2q}} ||h'_n||^{2q}_{L^2(d\phi)} \frac{(2q)!}{2^q q!} \frac{1}{{\kappa'_\gamma}^{2q}} ||\rho||^{2q}_{\ell^{k_\alpha}} \tag{71}$$

for all $q \geq 1$. From (64), we obtain, as $q \to +\infty$

$$\mathbb{P}(|M_{b,n} - \sum_{j=\tau_p}^{k_\alpha - 1} Z_{j,n}| \geq \kappa'_\gamma r_{1,n}) \leq \lambda \frac{\varepsilon_n^q}{n^q r_{1,n}^{2q}} q^q \left(2\zeta e^{-1} ||\rho||^2_{\ell^{k_\alpha}} {\kappa'_\gamma}^{-2}\right)^q.$$

From (40) (with $\tau = \overline{\tau}_p$), (69) and by choosing $q = [\log(n)]$, we obtain

$$\mathbb{P}(|M_{b,n} - \sum_{j=\tau_p}^{k_\alpha - 1} Z_{j,n}| \geq \kappa'_\gamma r_{1,n}) \leq \lambda \left(2\zeta e^{-1} ||\rho||^2_{\ell^{k_\alpha}} \kappa_\varepsilon {\kappa'_\gamma}^{-2}\right)^{\log(n)} = \mathcal{O}(n^{-\gamma}), \tag{72}$$

if ${\kappa'_\gamma}^2 > \kappa'_{1,\gamma} = 2\zeta ||\rho||^2_{\ell^{k_\alpha}} \kappa_\varepsilon \exp(\gamma - 1)$. Now, concerning the last term of (70), from (47), we can prove, for all $\overline{\tau}_p \leq j < k_\alpha$

$$\mathbb{P}\left(Z_{j,n} \geq \kappa'_\gamma r_{2,n}\right) \leq \lambda \frac{L(n)^{jq}}{n^{\alpha jq}} \frac{1}{r_{2,n}^{2q}} \frac{1}{{\kappa'_\gamma}^{2q}} \left(\frac{c_j(\eta_{b,n}) - c_j(\xi(p))}{j!}\right)^{2q} \mu_{2q},$$

where μ_{2q} is a constant such that, as $q \to +\infty$,

$$\mu_{2q} \leq \lambda \left(\frac{2}{1 - \alpha j}\right)^q \frac{(2jq)!}{2^{jq}(jq)!}.$$

From (65), we have, as $q \to +\infty$

$$\mathbb{P}\left(Z_{j,n} \geq \kappa'_\gamma r_{2,n}\right) \leq \lambda \frac{\varepsilon_n^{2q} L(n)^{jq}}{n^{\alpha jq}} q^{jq} \left(\frac{2}{1 - \alpha j} \left(\frac{2j}{e}\right)^j d_j^2 \, {\kappa'_\gamma}^{-2}\right)^{2q}.$$

From (24), (40) (with $\tau = \overline{\tau}_p$) and by choosing $q = [\log(n)]$, we have, as $n \to +\infty$

$$\mathbb{P}\left(Z_{j,n} \geq \kappa'_\gamma r_{2,n}\right) \leq \lambda \left(\frac{\log(n) L(n)}{n^\alpha}\right)^{(j - \overline{\tau}_p)q} \left(\frac{2}{1 - \alpha j} \left(\frac{2j}{e}\right)^j d_j^2 \, \kappa_\varepsilon^2 \, {\kappa'_\gamma}^{-2}\right)^q$$

Consequently, as $n \to +\infty$, we finally obtain

$$\sum_{j=\overline{\tau}_p}^{k_\alpha - 1} \mathbb{P}\left(Z_{j,n} \geq \kappa'_\gamma r_{2,n}\right) \leq \lambda \left(\frac{2}{1 - \alpha\overline{\tau}} \left(\frac{2\overline{\tau}}{e}\right)^{\overline{\tau}} d_{\overline{\tau}}^2 \, \kappa_\varepsilon^2 \, {\kappa'_\gamma}^{-2}\right)^{\log(n)} = \mathcal{O}(n^{-\gamma}), \tag{73}$$

if ${\kappa'_\gamma}^2 > \kappa'^2_{2,\gamma} = \frac{2}{1-\alpha\overline{\tau}} \left(\frac{2\overline{\tau}}{e}\right)^{\overline{\tau}} d_{\overline{\tau}}^2 \, \kappa_\varepsilon^2 \exp(\gamma - \overline{\tau})$. Let us choose κ'_γ such that ${\kappa'_\gamma}^2 > \max(\kappa'^2_{1,\gamma}, \kappa'^2_{2,\gamma})$. Then, by combining (72) and (73), we deduce from (70) that, for every $\gamma > 0$

$$\mathbb{P}\left(|M_{b,n}| \geq \kappa'_\gamma \max(r_{1,n}, r_{2,n})\right) = \mathcal{O}(n^{-\gamma}),$$

and so, (63) is proved. ■

6.2 Proof of Theorem 4.1

Proof. From Lemma 6.3, we deduce the almost sure convergence of $\widehat{\xi}\,(p)$ towards $\xi(p)$, as $n \to +\infty$. Then, from Taylor's theorem (Young's form), we have almost surely, as $n \to +\infty$

$$F(\widehat{\xi}\,(p)) - F(\xi(p)) = f(\xi(p)) \left(\widehat{\xi}\,(p) - \xi(p)\right) + \mathcal{O}\left(\varepsilon_n(\alpha, \tau_p)^2\right),$$

Now, using Lemma 6.4, we have almost surely

$$\widehat{F}\left(\widehat{\xi}\,(p)\right) - \widehat{F}\left(\xi(p)\right) = f(\xi(p)) \left(\widehat{\xi}\,(p) - \xi(p)\right) + \mathcal{O}\left(\varepsilon_n(\alpha, \tau_p)^2\right) + \mathcal{O}\left(r_n(\alpha, \overline{\tau}_p)\right).$$

We have almost surely, see *e.g.* Serfling (1980), $\widehat{F}\left(\widehat{\xi}\,(p)\right) = p + \mathcal{O}(n^{-1})$. Thus, we finally obtain

$$\widehat{\xi}\,(p) - \xi(p) = \frac{p - \widehat{F}\left(\widehat{\xi}\,(p)\right)}{f(\xi(p))} + \mathcal{O}\left(\varepsilon_n(\alpha, \tau_p)^2\right) + \mathcal{O}\left(r_n(\alpha, \overline{\tau}_p)\right) + \mathcal{O}(n^{-1}),$$

which leads to the result by noting that $\varepsilon_n(\alpha, \tau_p)^2 = \mathcal{O}(r_n(\alpha, \overline{\tau}_p))$. ■

6.3 Auxiliary Lemma for proof of Theorem 4.2

Lemma 6.5 *Consider for $0 < p < 1$, the function $h_p(\cdot)$ given by*

$$h_p(t) = \mathbf{1}_{\{|t| \leq \xi_{|Y|}(p)\}}(t) - p, \tag{74}$$

that is the function $h_{\xi_{g(Y)}(p)}(\cdot)$ with $g(\cdot) = |\cdot|$. Then, we have for all $j \geq 1$

$$c_0^{h_p} = c_{2j+1}^{h_p} = 0 \quad \text{and} \quad c_{2j}^{h_p} = -2H_{2j-1}(q)\phi(q), \tag{75}$$

where we denote by $q = \xi_{|Y|}(p) = \Phi^{-1}\left(\frac{1+p}{2}\right)$.

Proof. Since $\mathbb{P}(|Y| \leq q) = p$ and since $h_p(\cdot)$ is even, we have $c_0^{h_p} = c_{2j+1}^{h_p} = 0$ for all $j \geq 1$. Now, from (19), it comes

$$\begin{aligned} c_{2j}^{h_p} &= \int_{\mathbb{R}} h_p(t) H_{2j}(t)\phi(t)dt = 2 \times \int_0^q H_{2j}(t)\phi(t)dt \\ &= 2 \times \left[\phi^{(2j-1)}(t)\right]_0^q = 2 \times \left[-H_{2j-1}(t)\phi(t)\right]_0^q \\ &= -2H_{2j-1}(q)\phi(q). \end{aligned}$$

■

Remark 6.1 *Let $g(\cdot) = \widetilde{g}(|\cdot|)$, where $\widetilde{g}(\cdot)$ is a strictly increasing function on $\mathbb{R}^+$, then for all $0 < p < 1$, we have*

$$\xi_{|Y|}(p) = \widetilde{g}^{-1}\left(\xi_{g(Y)}(p)\right).$$

Consequently, the functions $h_{\xi_{g(Y)}(p)}(\cdot)$ for $g(\cdot) = |\cdot|$, $g(\cdot) = |\cdot|^\alpha$ and $g(\cdot) = \log|\cdot|$ are strictly identical. And so, their Hermite decomposition is given by (75) and their Hermite rank by (25).

6.4 Proof of Theorem 4.2

Proof. (i) Define

$$b_n = \frac{1}{2}\sum_{m=1}^M B_m \delta_n^{\boldsymbol{a^m}}(0), \tag{76}$$

where $\delta_n^{\boldsymbol{a^m}}(0)$ is given by (4). From (14), (15), and (18), we have almost surely

$$\begin{aligned} \widehat{H}^\alpha - H &= \sum_{m=1}^M \frac{B_m}{\alpha}\varepsilon_m^\alpha \\ &= \sum_{m=1}^M \frac{B_m}{\alpha}\log\left(\frac{\widehat{\xi}\left(\boldsymbol{p},\boldsymbol{c};|\boldsymbol{Y}^{\boldsymbol{a^m}}|^{\boldsymbol{\alpha}}\right)}{\xi_{|Y|^\alpha}(\boldsymbol{p},\boldsymbol{c})}\right) + b_n \\ &= \sum_{m=1}^M \frac{B_m}{\alpha\xi_{|Y|^\alpha}(\boldsymbol{p},\boldsymbol{c})}\Big(\widehat{\xi}\left(\boldsymbol{p},\boldsymbol{c};|\boldsymbol{Y}^{\boldsymbol{a^m}}|^{\boldsymbol{\alpha}}\right) - \xi_{|Y|^\alpha}(\boldsymbol{p},\boldsymbol{c})\Big)(1+o(1)) + b_n. \end{aligned} \tag{77}$$

and

$$\widehat{H}^{\log} - H = \sum_{m=1}^M B_m \varepsilon_m^{\log}$$

$$= \sum_{m=1}^{M} B_m \left(\widehat{\xi}\left(\boldsymbol{p}, \boldsymbol{c}; \log |\boldsymbol{Y}^{\boldsymbol{a^m}}|\right) - \xi_{\log|Y|}\left(\boldsymbol{p}, \boldsymbol{c}\right) \right) + b_n \tag{78}$$

Under Assumption $(\mathbf{A_4}(\beta))$, we have

$$b_n = \mathcal{O}(n^{-\beta}). \tag{79}$$

Moreover, let $i, j \geq 1$, under Assumption $\mathbf{A_1}(\mathbf{2}\nu)$, we have, from Lemma 2.1

$$\boldsymbol{E}(Y^{\boldsymbol{a^m}}(i) Y^{\boldsymbol{a^m}}(i+j)) = \rho^{\boldsymbol{a^m}}(j) = \mathcal{O}(|j|^{2H-2\nu}).$$

Then, for all $m = 1, \ldots, M$ and for all $k = 1, \ldots, K$, from Lemma 6.3 and Remark 6.1, we obtain, that almost surely

$$\widehat{\xi}\left(p_k; |\boldsymbol{Y}^{\boldsymbol{a^m}}|^{\boldsymbol{\alpha}}\right) - \xi_{|Y|^\alpha}(p_k) = \mathcal{O}\left(y_n(2\nu - 2H, \tau_{p_k})\right),$$
$$\widehat{\xi}\left(p_k; \log |\boldsymbol{Y}^{\boldsymbol{a^m}}|\right) - \xi_{\log|Y|}(p_k) = \mathcal{O}\left(y_n(2\nu - 2H, \tau_{p_k})\right),$$

where the sequence $y_n(\cdot, \cdot)$ is defined by (40) with $L(\cdot) = 1$. The result (29) is obtained by combining (77), (78) and (79).

(ii) Let us apply Theorem 4.1 to the sequence $\boldsymbol{g}(\boldsymbol{Y}^{\boldsymbol{a^m}})$, for some $m = 1, \ldots, M$, with $g(\cdot) = |\cdot|$, $g(\cdot) = |\cdot|^\alpha$ and $g(\cdot) = \log|\cdot|$. For all $k = 1, \ldots, K$, we have almost surely

$$\widehat{\xi}\left(p_k; |\boldsymbol{Y}^{\boldsymbol{a^m}}|\right) - \xi_{|Y|}(p_k) = \frac{p_k - \widehat{F}\left(\xi_{|Y|}(p_k); |\boldsymbol{Y}^{\boldsymbol{a^m}}|\right)}{f_{|Y|^\alpha}(\xi_{|Y|}(p_k))} + \mathcal{O}(r_n)$$
$$\widehat{\xi}\left(p_k; |\boldsymbol{Y}^{\boldsymbol{a^m}}|^{\boldsymbol{\alpha}}\right) - \xi_{|Y|^\alpha}(p_k) = \frac{p_k - \widehat{F}\left(\xi_{|Y|^\alpha}(p_k); |\boldsymbol{Y}^{\boldsymbol{a^m}}|^{\boldsymbol{\alpha}}\right)}{f_{|Y|^\alpha}(\xi_{|Y|^\alpha}(p_k))} + \mathcal{O}(r_n)$$
$$\widehat{\xi}\left(p_k; \log |\boldsymbol{Y}^{\boldsymbol{a^m}}|\right) - \xi_{\log|Y|}(p_k) = \frac{p_k - \widehat{F}\left(\xi_{\log|Y|}(p_k); \log |\boldsymbol{Y}^{\boldsymbol{a^m}}|\right)}{f_{\log|Y|}(\xi_{\log|Y|}(p_k))} + \mathcal{O}(r_n),$$

where, for the sake of simplicity, $r_n = r_n(2\nu - 2H, \overline{\tau}_{p_k})$ defined by (26) and (27). Note that from Remark 6.1 and from (25), $\overline{\tau}_{p_k} = 2$ for all $k = 1, \ldots, K$.

With some little computation, we can obtain, almost surely

$$\widehat{\xi}\left(p_k; |\boldsymbol{Y}^{\boldsymbol{a^m}}|^{\boldsymbol{\alpha}}\right) - \xi_{|Y|^\alpha}(p_k) = \alpha \xi_{|Y|}(p_k)^{\alpha-1} \left(\widehat{\xi}\left(p_k; |\boldsymbol{Y}^{\boldsymbol{a^m}}|\right) - \xi_{|Y|}(p_k) \right) + \mathcal{O}(r_n), \tag{80}$$

and

$$\widehat{\xi}\left(p_k; \log |\boldsymbol{Y}^{\boldsymbol{a^m}}|\right) - \xi_{\log|Y|}(p_k) = \xi_{|Y|}(p_k)^{-1} \left(\widehat{\xi}\left(p_k; |\boldsymbol{Y}^{\boldsymbol{a^m}}|\right) - \xi_{|Y|}(p_k) \right) + \mathcal{O}(r_n). \tag{81}$$

From (77), (78), (80), (81) and properties of Gaussian variables, the following results hold almost surely

$$\widehat{H}^{\alpha} - H = \sum_{m=1}^{M}\sum_{k=1}^{K} \frac{B_m\, c_k}{2q_k\phi(q_k)}\pi_{k,\alpha}\left(\widehat{F}(q_k;|\boldsymbol{Y}|) - p_k\right) + \mathcal{O}(r_n) + \mathcal{O}(b_n), \tag{82}$$

and

$$\widehat{H}^{\log} - H = \sum_{m=1}^{M}\sum_{k=1}^{K} \frac{B_m\, c_k}{2q_k\phi(q_k)}\left(\widehat{F}(q_k;|\boldsymbol{Y}|) - p_k\right) + \mathcal{O}(r_n) + \mathcal{O}(b_n), \tag{83}$$

where q_k and π_k^{α} are defined by (35). Denote by $\theta_{m,k}^{\alpha}$ the following constant

$$\theta_{m,k}^{\alpha} = \frac{B_m c_k}{2q_k\phi(q_k)}\pi_k^{\alpha}.$$

Since $\pi_k^0 = 1$, (82) and (83) can be rewritten as

$$\widehat{H}^{\alpha} - H = Z_n^{\alpha} + \mathcal{O}\left(r_n(2\nu - 2H, 2)\right) + \mathcal{O}(b_n) \tag{84}$$

$$\widehat{H}^{\log} - H = Z_n^{0} + \mathcal{O}\left(r_n(2\nu - 2H, 2)\right) + \mathcal{O}(b_n), \tag{85}$$

where for $\alpha \geq 0$,

$$Z_n^{\alpha} = \sum_{m=1}^{M}\sum_{k=1}^{K} \theta_{m,k}^{\alpha}\left(\widehat{F}(q_k;|\boldsymbol{Y}|) - p_k\right). \tag{86}$$

Thus, under Assumption $\mathbf{A_4}(\beta)$, we have, as $n \to +\infty$,

$$MSE(\widehat{H}^{\alpha} - H) = \mathcal{O}\left(\boldsymbol{E}\left((Z_n^{\alpha})^2\right)\right) + \mathcal{O}\left(r_n(2\nu - 2H, \widetilde{\tau})^2\right) + \mathcal{O}(n^{-2\beta}), \tag{87}$$

$$MSE(\widehat{H}^{\log} - H) = \mathcal{O}\left(\boldsymbol{E}\left((Z_n^{0})^2\right)\right) + \mathcal{O}\left(r_n(2\nu - 2H, \widetilde{\tau})^2\right) + \mathcal{O}(n^{-2\beta}). \tag{88}$$

Now,

$$\boldsymbol{E}\left((Z_n^{\alpha})^2\right) = \frac{1}{n^2}\sum_{m_1,m_2=1}^{M}\sum_{k_1,k_2=1}^{K}\sum_{i_1,i_2=1}^{n} \theta_{m_1,k_1}^{\alpha}\theta_{m_2,k_2}^{\alpha}\boldsymbol{E}\left(h_{q_{k_1}}(Y^{\boldsymbol{a}^{m_1}}(i_1))h_{q_{k_2}}(Y^{\boldsymbol{a}^{m_2}}(i_2))\right).$$

For $k_1, k_2 = 1, \ldots, K$, $m_1, m_2 = 1, \ldots, M$ and $i_1, i_2 = 1, \ldots, n$, we have from Lemma 6.5,

$$\begin{aligned}\boldsymbol{E}\left(h_{q_{k_1}}(Y^{\boldsymbol{a}^{m_1}}(i_1))h_{q_{k_2}}(Y^{\boldsymbol{a}^{m_2}}(i_2))\right) &= \sum_{j_1 \geq \tau_{p_{k_1}}/2}\ \sum_{j_2 \geq \tau_{p_{k_2}}/2} \frac{c_{2j_1}(q_{k_1})c_{2j_2}(q_{k_2})}{(2j_1)!(2j_2)!} \\ &\qquad \times \boldsymbol{E}\left(H_{2j_1}(Y^{\boldsymbol{a}^{m_1}}(i_1))H_{2j_2}(Y^{\boldsymbol{a}^{m_2}}(i_2))\right) \\ &= \sum_{j \geq \widetilde{\tau}/2} \frac{c_{2j}(q_{k_1})c_{2j}(q_{k_2})}{(2j)!}\rho^{\boldsymbol{a}^{m_1},\boldsymbol{a}^{m_2}}(i_2 - i_1)^{2j}.\end{aligned}$$

Under Assumption $\mathbf{A_1}(\mathbf{2}\nu)$, we have from Lemma 2.1, $\rho^{\boldsymbol{a}^{m_1},\boldsymbol{a}^{m_2}}(i) = \mathcal{O}(|i|^{2H-2\nu})$. Now, we leave the reader to check that, as $n \to +\infty$

$$\frac{1}{n^2}\sum_{i_1,i_2=1}^{n} \rho^{\boldsymbol{a}^{m_1},\boldsymbol{a}^{m_2}}(i_2-i_1)^{\widetilde{\tau}} = \mathcal{O}\left(\frac{1}{n}\sum_{|i|\leq n}|i|^{(2H-2\nu)\widetilde{\tau}}\right) = \mathcal{O}\left(v_n(2\nu-2H),\widetilde{\tau})\right),$$

where the sequence $v_n(\cdot,\cdot)$ is given by (31). Thus, we have, as $n \to +\infty$, $\boldsymbol{E}\left((Z_n^\alpha)^2\right) = \mathcal{O}\left(v_n(2\nu-2H),\widetilde{\tau})\right)$, which leads to the result from (87) and (88).

(iii) Assume $\nu > H + 1/(2\widetilde{\tau})$ and $\beta > 1/2$, then from (84) and (85), the following equivalences in distribution hold

$$\sqrt{n}\left(\widehat{H}_n^\alpha - H\right) \sim \sqrt{n}\, Z_n^\alpha \quad \text{and} \quad \sqrt{n}\left(\widehat{H}_n^{\log} - H\right) \sim \sqrt{n}\, Z_n^0. \tag{89}$$

Now, decompose $Z_n^\alpha = T_n^1 + T_n^2$, where

$$T_n^1 = \frac{1}{\sqrt{n}}\sum_{m=1}^{M}\sum_{k=1}^{K}\theta_{m,k}^\alpha \sum_{i=\ell+1}^{M\ell+1} h_{q_k}(Y^{\boldsymbol{a}^{\boldsymbol{m}}}(i))$$

and

$$T_n^2 = \sqrt{n}\sum_{m=1}^{M}\sum_{k=1}^{K}\theta_{m,k}^\alpha \left\{\frac{1}{n}\sum_{i=M\ell+1}^{n} h_{q_k}(Y^{\boldsymbol{a}^{\boldsymbol{m}}}(i))\right\},$$

Clearly, T_n^1 converges to 0 in probability, as $n \to +\infty$. Therefore, we have, as $n \to +\infty$

$$Z_n^\alpha \sim \sqrt{n}\left\{\frac{1}{n}\sum_{i=M\ell+1}^{n} G^\alpha\left(Y^{\boldsymbol{a}^{\mathbf{1}}}(i),\ldots,Y^{\boldsymbol{a}^{\boldsymbol{M}}}(i)\right)\right\} \tag{90}$$

where G^α is the function from $\mathbb{R}^M$ to $\mathbb{R}$ defined for $\alpha \geq 0$ and $t_1,\ldots,t_M \in \mathbb{R}$ by:

$$G^\alpha(t_1,\ldots,t_M) = \sum_{m=1}^{M}\sum_{k=1}^{K}\theta_{m,k}\, h_{q_k}(t_m). \tag{91}$$

Denote by $\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}(i)$, the vector defined for $i = M\ell+1,\ldots,n$ by

$$\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}(i) = (Y^{\boldsymbol{a}^{\mathbf{1}}}(i),\ldots,Y^{\boldsymbol{a}^{\boldsymbol{M}}}(i)).$$

We obviously have $\boldsymbol{E}(G^\alpha(\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}(i))^2) < +\infty$. Since, for all $k = 1,\ldots,K$, the functions h_{q_k} have Hermite rank τ_{p_k}, the function G^α has Hermite rank $\widetilde{\tau}$ (see *e.g.*

Arcones (1994) for the definition of the Hermite rank of multivariate functions). Moreover under Assumption $\mathbf{A_1}(\mathbf{2}\nu)$, we have from Lemma 2.1, as $j \to +\infty$

$$\boldsymbol{E}\left(Y^{\boldsymbol{a^{m_1}}}(i)Y^{\boldsymbol{a^{m_2}}}(i+j)\right) = \rho^{\boldsymbol{a^{m_1}},\boldsymbol{a^{m_2}}}(j) = \mathcal{O}\left(|j|^{2H-2\nu}\right) \in \ell^2(\mathbb{Z}),$$

as soon as $\nu > H + 1/(2\widetilde{\tau})$. Thus, from Theorem 4 of Arcones (1994), there exists σ_α^2 (defined for $\alpha \geq 0$) such that, as $n \to +\infty$, the following convergence in distribution holds

$$Z_n^\alpha \longrightarrow \mathcal{N}(0, \sigma_\alpha^2)$$

with

$$\sigma_\alpha^2 = \sum_{i\in\mathbb{Z}} \boldsymbol{E}\left(G^\alpha\left(\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}(i')\right) G^\alpha\left(\widetilde{\boldsymbol{Y}}^{\boldsymbol{a}}(i'+i)\right)\right).$$

With previous notations, we have

$$\begin{aligned}\sigma_\alpha^2 &= \sum_{i\in\mathbb{Z}} \sum_{m_1,m_2=1}^{M} \sum_{k_1,k_2=1}^{K} \theta^\alpha_{m_1,k_1}\theta^\alpha_{m_2,k_2} \boldsymbol{E}\left(h_{p_{k_1}}(Y^{\boldsymbol{a^{m_1}}}(i'))h_{p_{k_2}}(Y^{\boldsymbol{a^{m_2}}}(i'+i))\right) \\ &= \sum_{i\in\mathbb{Z}} \sum_{m_1,m_2=1}^{M} \sum_{k_1,k_2=1}^{K} \sum_{j\geq r} \frac{c_{2j}^{h_{p_{k_1}}} c_{2j}^{h_{p_{k_2}}}}{(2j)!} \theta^\alpha_{m_1,k_1}\theta^\alpha_{m_2,k_2}\, \rho^{\boldsymbol{a^{m_1}},\boldsymbol{a^{m_2}}}(i)^{2j}. \end{aligned} \tag{92}$$

From (75), we can see that formula (92) is equivalent to (34), which ends the proof from (89). ■

6.5 Proof of Corollary 4.1

Proof. Equation (90) is still available for a sequence α_n such that $\alpha_n \to 0$ as $n \to +\infty$, that is

$$\sqrt{n}\left(\widehat{H}_n^{\alpha_n} - H\right) \sim \sqrt{n}\left\{\frac{1}{n}\sum_{i=M\ell+1}^{n} G^{\alpha_n}\left(Y^{\boldsymbol{a^1}}(i), \ldots, Y^{\boldsymbol{a^M}}(i)\right)\right\}$$

From (91) and since $\pi_k^{\alpha_n} \to 1$, as $n \to +\infty$, we have $G^{\alpha_n}(\cdot) \to G^0(\cdot)$. Therefore, the following equivalence in distribution holds, as $n \to +\infty$

$$\sqrt{n}\left(\widehat{H}_n^{\alpha_n} - H\right) \sim \sqrt{n}\left(\widehat{H}_n^{\log} - H\right),$$

which ends the proof. ■

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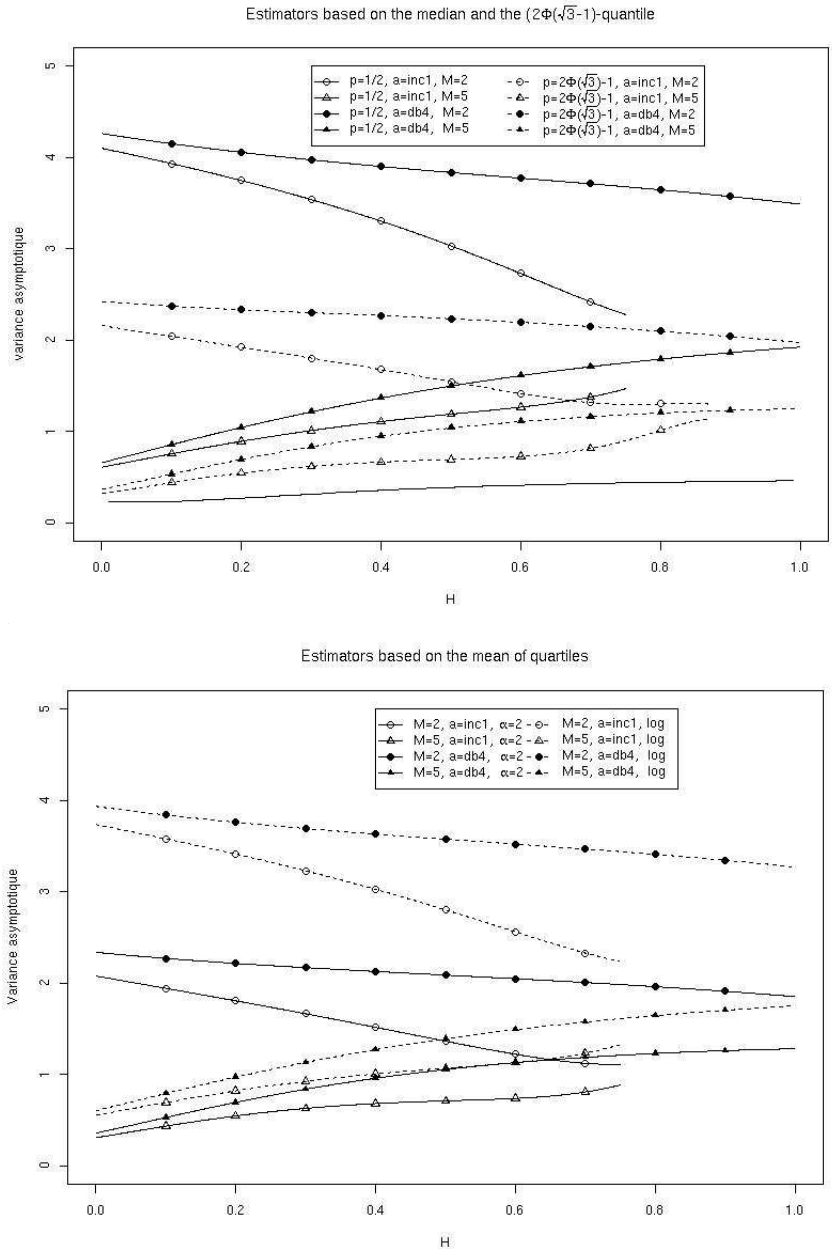

Figure 1: Constants of empirical variances in terms of H for estimators based on the median and on the $(2\Phi(\sqrt{3})-1)-$quantile (top) and on the mean of quartiles (below).

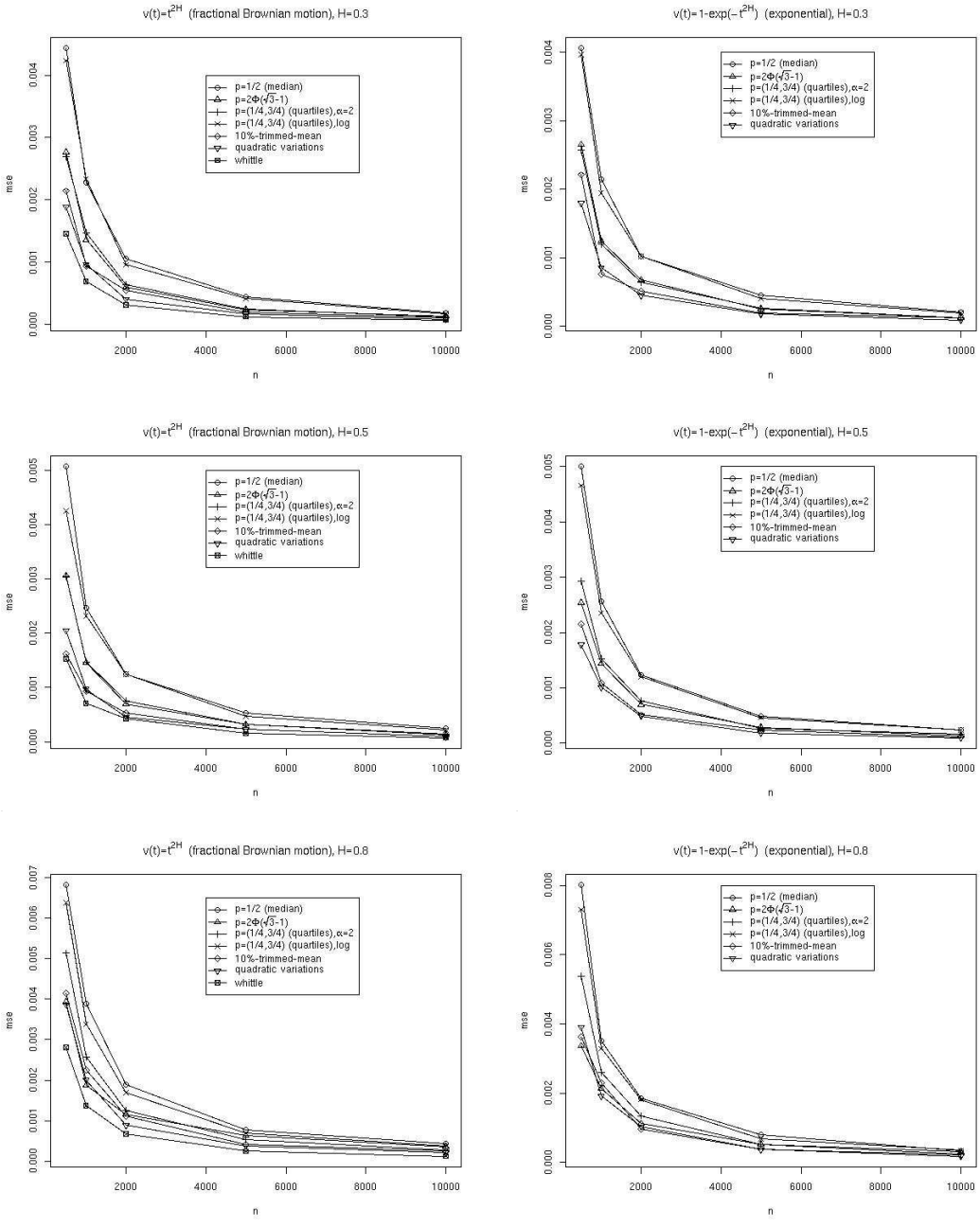

Figure 2: Empirical mean squared errrors in terms of n using 500 Monte Carlo simulations of sample paths of processes with variance function $v(t) = |t|^{2H}$, fractional Brownian motion, (left) and $v(\cdot) = 1 - \exp(-|\cdot|^{2H})$ (right). The parameter H equals 0.3 (top), 0.5 (middle) and 0.8 (below).

Estimators	$v(\cdot) = \lvert\cdot\rvert^{2H}$	$v(\cdot) = 1 - \exp(-\lvert\cdot\rvert^{2H})$
$\boldsymbol{p} = 1/2$, $\boldsymbol{c} = 1$ (median)	0.798 (0.047)	0.803 (0.045)
$\boldsymbol{p} = 2\Phi(\sqrt{3}) - 1$, $\boldsymbol{c} = 1$	0.793 (0.033)	0.789 (0.032)
$\boldsymbol{p} = (1/4, 3/4)$,$\boldsymbol{c} = (1/2, 1/2)$, $g(\cdot) = \lvert\cdot\rvert^2$	0.797 (0.040)	0.796 (0.037)
$\boldsymbol{p} = (1/4, 3/4)$,$\boldsymbol{c} = (1/2, 1/2)$, $g(\cdot) = \log\lvert\cdot\rvert$	0.798 (0.044)	0.804 (0.044)
10%-trimmed mean, $g(\cdot) = \lvert\cdot\rvert^2$	0.792 (0.037)	0.797 (0.033)
Quadratic variations method	0.329 (0.162)	0.353 (0.149)
Whittle estimator	0.519 (0.106)	0.510 (0.100)

Table 1: Empirical mean and standard deviations for $n = 1000$ and $H = 0.8$ using 500 Monte Carlo simulations of contaminated (see (38)) sample paths of processes with variance function $v(\cdot) = \lvert\cdot\rvert^{2H}$ and $v(\cdot) = 1 - \exp(-\lvert\cdot\rvert^{2H})$.

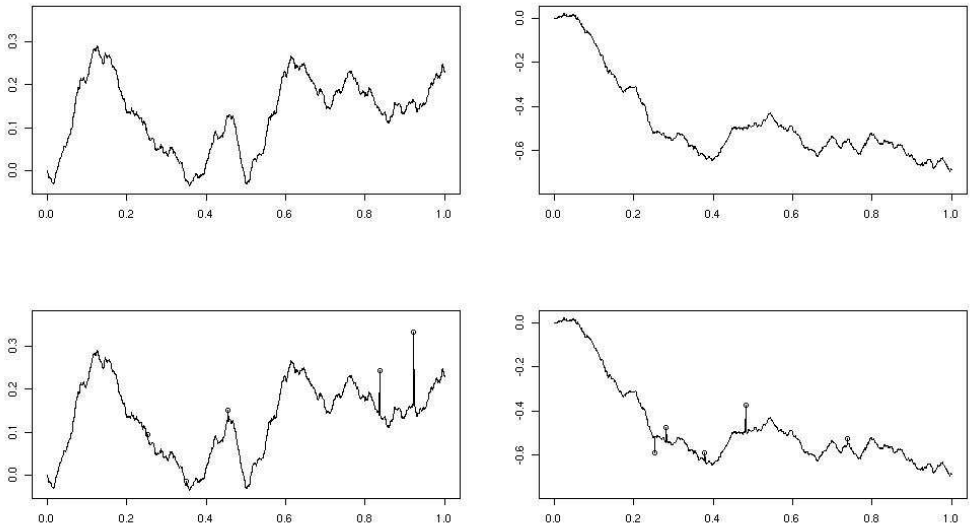

Figure 3: Example of sample path of non-contaminted (top) and contaminated (bottom), see (38), processes with variance function $v(\cdot) = \lvert\cdot\rvert^{2H}$ (left) and $v(\cdot) = 1 - \exp(-\lvert\cdot\rvert^{2H})$ (right).