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# Stability of FD–TD schemes for Maxwell–Debye and Maxwell–Lorentz equations

Brigitte Bidégaray-Fesquet \*

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## Abstract

The stability of five finite difference–time domain (FD–TD) schemes coupling Maxwell equations to Debye or Lorentz models have been analyzed in [1], where numerical evidence for specific media have been used. We use von Neumann analysis to give necessary and sufficient stability conditions for these schemes for any medium, in accordance with the partial results of [1]. To make this approach tractable for 2D and 3D models we have developed a Computer Algebra environment which has a wider range of applicability.

## 1 Introduction

To describe the propagation of an electromagnetic wave through a dispersive medium some extensions to Maxwell equations are used. They involve time differential equations which accounts for the constitutive laws of the material that link the displacement  $\mathbf{D}$  to the electric field  $\mathbf{E}$  or equivalently the polarization  $\mathbf{P}$  to  $\mathbf{E}$ . We focus on two of these models (Debye and Lorentz models) which are addressed in [1] in view of specific applications to the interaction of an electromagnetic wave with a human body. In [1], specific values for the physical and numerical constants are chosen and numeric calculations are performed to conclude to stability or not. A survey of numerical couplings between Maxwell equations and various matter models may be found in [2], where stability conditions are given for Maxwell–Debye models but there is no proof for some of them and Maxwell–Lorentz formulae are considered too complex to be studied. In contrast with these two references, we are able to treat any medium which is described by these models. To this aim we use von Neumann analysis.

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### 1.1 Maxwell–Debye and Maxwell–Lorentz models

In our context (no magnetization) we use different formulations of Maxwell equations where Faraday equation always read

$$\partial_t \mathbf{B}(t, \mathbf{x}) = -\operatorname{curl} \mathbf{E}(t, \mathbf{x}) \quad (1.1)$$

where  $\mathbf{x} \in \mathbb{R}^N$ . On the contrary, Ampère equation may be cast using variables  $\mathbf{B}$  and  $\mathbf{D}$ :

$$\partial_t \mathbf{D}(t, \mathbf{x}) = \frac{1}{\mu_0} \operatorname{curl} \mathbf{B}(t, \mathbf{x}), \quad (1.2\text{--a})$$

or the polarization  $\mathbf{P}$ :

$$\varepsilon_0 \varepsilon_\infty \partial_t \mathbf{E}(t, \mathbf{x}) = \frac{1}{\mu_0} \operatorname{curl} \mathbf{B}(t, \mathbf{x}) - \partial_t \mathbf{P}(t, \mathbf{x}), \quad (1.2\text{--b})$$

where  $\mathbf{P}(t, \mathbf{x}) = \mathbf{D}(t, \mathbf{x}) - \varepsilon_0 \varepsilon_\infty \mathbf{E}(t, \mathbf{x})$  and  $\varepsilon_\infty$  is the relative infinite frequency permittivity. Denoting by  $\mathbf{J}$  the time derivative of  $\mathbf{P}$ , we also have

$$\varepsilon_0 \varepsilon_\infty \partial_t \mathbf{E}(t, \mathbf{x}) = \frac{1}{\mu_0} \operatorname{curl} \mathbf{B}(t, \mathbf{x}) - \mathbf{J}(t, \mathbf{x}). \quad (1.2\text{--c})$$

The system of Faraday and Ampère equations is closed by a linear constitutive law

$$\mathbf{D}(t, \mathbf{x}) = \varepsilon_0 \varepsilon_\infty \mathbf{E}(t, \mathbf{x}) + \varepsilon_0 \int_{-\infty}^t \mathbf{E}(t - \tau, \mathbf{x}) \chi(\tau) d\tau, \quad (1.3)$$

where  $\chi$  is the linear susceptibility. The discretization of the integral expression (1.3) leads to recursive schemes (see e.g. [3], [4]). However, differentiating Eq. (1.3) leads to a time differential equation for  $\mathbf{D}$  which depends on the specific form of  $\chi$ . For a Debye medium

$$t_r \partial_t \mathbf{D} + \mathbf{D} = t_r \varepsilon_0 \varepsilon_\infty \partial_t \mathbf{E} + \varepsilon_0 \varepsilon_s \mathbf{E}, \quad (1.4\text{--a})$$

where  $t_r > 0$  is the relaxation time and  $\varepsilon_s \geq \varepsilon_\infty$  is the relative static permittivity. An equivalent form of Eq. (1.4–a) using variable  $\mathbf{P}$  is

$$t_r \partial_t \mathbf{P} + \mathbf{P} = \varepsilon_0 (\varepsilon_s - \varepsilon_\infty) \mathbf{E}, \quad (1.4\text{--b})$$

which can be coupled with Ampère equation cast as (1.2–b). For a Lorentz medium with one resonant frequency  $\omega_1$ , we likewise have

$$\partial_t^2 \mathbf{D} + \nu \partial_t \mathbf{D} + \omega_1^2 \mathbf{D} = \varepsilon_0 \varepsilon_\infty \partial_t^2 \mathbf{E} + \varepsilon_0 \varepsilon_\infty \nu \partial_t \mathbf{E} + \varepsilon_0 \varepsilon_s \omega_1^2 \mathbf{E}, \quad (1.5\text{--a})$$

where  $\nu \geq 0$  is a damping coefficient, or equivalently

$$\partial_t^2 \mathbf{P} + \nu \partial_t \mathbf{P} + \omega_1^2 \mathbf{P} = \varepsilon_0 (\varepsilon_s - \varepsilon_\infty) \omega_1^2 \mathbf{E}. \quad (1.5\text{--b})$$

## 1.2 Yee-based numerical schemes

A classical and very efficient way to compute the Maxwell equations is the Yee scheme [5]. We restrict our study to existing Yee based schemes. In contrast to the recursive schemes, we are interested in direct integration schemes which are based on the finite difference–time domain (FD–TD) discretization of Eqs (1.4–a) to (1.5–b) (see [6], [7], [8]). Other space discretizations may be found in the literature in the context of Maxwell-Debye and Maxwell-Lorentz equations: see e.g. [9] for pseudo-spectral schemes or [10] for finite element–time domain (FE–TD) schemes.

The Yee scheme consists in discretizing  $\mathbf{E}$  and  $\mathbf{B}$  on staggered grids in space and time. This allows to use only centered discrete differential operators. We denote by  $h$  the space step (supposed here to be the same in all directions in the case of multi-dimensional equations) and by  $k$  the time step. In space dimension 1, we only consider the dependence in the space variable  $z$  and classically two polarizations for the field may be decoupled. For example, the transverse electric polarization only involves  $E \equiv E_x$  and  $B \equiv B_y$ . The discretized variables are  $E_j^n \simeq E(nk, jh)$  (and similar notations for  $D \equiv D_x$ ) and  $B_{j+\frac{1}{2}}^{n+\frac{1}{2}} \simeq B((n + \frac{1}{2})k, (j + \frac{1}{2})h)$ , and the Yee scheme for Maxwell equation read

$$\frac{B_{j+\frac{1}{2}}^{n+\frac{1}{2}} - B_{j+\frac{1}{2}}^{n-\frac{1}{2}}}{k} = -\frac{E_{j+1}^n - E_j^n}{h}, \quad (1.6)$$

coupled with one of the following

$$\frac{D_j^{n+1} - D_j^n}{k} = -\mu_0 \frac{B_{j+\frac{1}{2}}^{n+\frac{1}{2}} - B_{j-\frac{1}{2}}^{n+\frac{1}{2}}}{h}, \quad (1.7\text{-a})$$

$$\varepsilon_0 \varepsilon_\infty \frac{E_j^{n+1} - E_j^n}{k} = -\mu_0 \frac{B_{j+\frac{1}{2}}^{n+\frac{1}{2}} - B_{j-\frac{1}{2}}^{n+\frac{1}{2}}}{h} - \frac{P_j^{n+1} - P_j^n}{k}, \quad (1.7\text{-b})$$

$$\varepsilon_0 \varepsilon_\infty \frac{E_j^{n+1} - E_j^n}{k} = -\mu_0 \frac{B_{j+\frac{1}{2}}^{n+\frac{1}{2}} - B_{j-\frac{1}{2}}^{n+\frac{1}{2}}}{h} - J_j^{n+\frac{1}{2}}. \quad (1.7\text{-c})$$

Usual Maxwell equations consist in taking  $J_j^{n+\frac{1}{2}} \equiv 0$  in Eq. (1.7–c) or equivalently  $D_j^n = \varepsilon_0 \varepsilon_\infty E_j^n$  in Eq. (1.7–a) and leads to a stable second order scheme under a Courant–Friedrichs–Lewy (CFL) stability condition. Namely, if  $c_\infty = 1/\sqrt{\varepsilon_0 \varepsilon_\infty \mu_0}$  denotes the infinite frequency light speed, the CFL condition reads  $c_\infty k \leq h/\sqrt{N}$ .

## 1.3 Scheme naming

Since we deal with many schemes, we have to distinguish between them and name them. Numbers or names of first contributor(s) are not very meaningful and we prefer here to have a descriptive name. Our description gives both the variables used (e.g. coupling Eqs (1.2–a) and (1.4–a) uses variables  $\mathbf{B}$ ,  $\mathbf{E}$  and  $\mathbf{D}$ )

and the location of discretized variables. Space occurs only as a parameter in Debye or Lorentz equations. To avoid interpolation, the reasonable choice is to always locate variables  $\mathbf{D}$ ,  $\mathbf{P}$  and  $\mathbf{J}$  on the same space grid as  $\mathbf{E}$ . The different schemes therefore differ only in the time location of variables: in equations (1.7–a) and (1.7–b) integer times are chosen; in equation (1.7–c) half-integer times are chosen.

Our naming is the following: we separate variables at half-integer times from variables at integer times by an underscore sign, e.g. “B\_ED” when coupling (1.6) and (1.7–a). We give in Table 1 the correspondence between the article where the schemes have been derived first, the terminology of the survey [2] and our description.

Contributor(s)	Name in [2]	Description	Name in [2]	Description
	Debye		Lorentz	
Joseph et al. [7]	D-DIM 3	B_ED	L-DIM 3	B_ED
Kashiwa et al. [6]	D-DIM 2	B_EP	L-DIM 2	B_EPJ
Young [8]	D-DIM 1	BP_E	L-DIM 1	BJ_EP

Table 1: Correspondence between different namings.

## 1.4 Outline

The von Neumann stability analysis is recalled in Section 2. We also describe the sketch of our proofs which is common for all the schemes. In Section 3 three one-dimensional direct integration schemes for Debye media are presented and analyzed, pointing carefully out the physical properties needed to ensure stability and the specific cases which have to be handled separately. Numerical applications to physical media are also given. The same point of view is carried out for Lorentz media in Section 4. To address two- and three-dimensional schemes which have too many variables to compute by hand, we have developed a Computer Algebra environment based on MAPLE which is described in Section 5. Two- and three dimensional results are given in Section 6.

## 2 Principles of the von Neumann analysis

The von Neumann analysis allows to localize roots of certain classes of polynomials, which proves to be crucial here. We recall the main principles of this technique. Details and proofs of theorems may be found in [12].

### 2.1 Schur and simple von Neumann polynomials

We define two families of polynomials: Schur polynomials and simple von Neumann polynomials.

**Definition 1** A polynomial is a Schur polynomial if all its roots,  $r$ , satisfy  $|r| < 1$ .

**Definition 2** A polynomial is a simple von Neumann polynomial if all its roots,  $r$ , lie on the unit disk ( $|r| \leq 1$ ) and its roots on the unit circle are simple roots.

If a polynomial has a high degree or sophisticated coefficients, it may be difficult to locate its roots. However, there is a way to split this difficult problem into many simpler ones. For this aim, we construct a sequence of polynomials of decreasing degree. Let  $\phi$  be written as

$$\phi(z) = c_0 + c_1z + \cdots + c_pz^p,$$

where  $c_0, c_1, \dots, c_p \in \mathbb{C}$  and  $c_p \neq 0$ . We define its conjugate polynomial  $\phi^*$  by

$$\phi^*(z) = c_p^* + c_{p-1}^*z + \cdots + c_0^*z^p.$$

Given a polynomial  $\phi_0$ , we may define a sequence of polynomials

$$\phi_{m+1}(z) = \frac{\phi_m^*(0)\phi_m(z) - \phi_m(0)\phi_m^*(z)}{z}.$$

It is clear that  $\deg\phi_{m+1} < \deg\phi_m$ , if  $\phi_m \not\equiv 0$ .

**Theorem 1** A polynomial  $\phi_m$  is a Schur polynomial of exact degree  $d$  if and only if  $\phi_{m+1}$  is a Schur polynomial of exact degree  $d - 1$  and  $|\phi_m(0)| \leq |\phi_m^*(0)|$ .

**Theorem 2** A polynomial  $\phi_m$  is a simple von Neumann polynomial if and only if

- $\phi_{m+1}$  is a simple von Neumann polynomial and  $|\phi_m(0)| \leq |\phi_m^*(0)|$ ,

or

- $\phi_{m+1}$  is identically zero and  $\phi_m'$  is a Schur polynomial.

The main ingredient in the proof of both theorems is the Rouché theorem (see [11, 12]). To analyze  $\phi_0$ , at each step  $m$ , conditions should be checked (leading coefficient is non-zero,  $|\phi_m(0)| \leq |\phi_m^*(0)|$ , ...) until a definitive negative answer arises or the degree is 1.

## 2.2 Stability analysis

The models we deal with are linear models. They may therefore be analyzed in the frequency domain. In other words, any wave can be seen as a superposition of plane waves which are also solution to the equations. We can thus assume that the scheme handles a single vector-valued variable  $U_{\mathbf{j}}^n$  with spatial dependence

$$U_{\mathbf{j}}^n = U^n \exp(i\boldsymbol{\xi} \cdot \mathbf{j}),$$

where  $\boldsymbol{\xi}$  and  $\mathbf{j} \in \mathbb{R}^N$ ,  $N = 1, 2, 3$ . The amplification matrix  $G$  is the matrix such that  $U^{n+1} = GU^n$ . We assume that  $G$  does not depend on time or on  $h$  and  $k$  separately but only on the ratio  $h/k$ . Let  $\phi_0$  be the characteristic polynomial of  $G$ , then we have a sufficient stability condition.

**Theorem 3** *A sufficient stability condition is that  $\phi_0$  be a simple von Neumann polynomial.*

This condition is not necessary. A scheme is stable if and only if the sequence  $(U^n)_{n \in \mathbb{N}}$  is bounded. Since we assume that  $G$  does not depend on time,  $U^n = G^n U^0$  and stability is also the boundedness of  $(G^n)_{n \in \mathbb{N}}$ . If the eigenvalues of  $G$ , i.e. the roots  $r$  of  $\phi_0$ , lie inside the unit circle ( $|r| < 1$ ), then  $\lim_{n \rightarrow \infty} G^n = 0$  and the sequence is bounded. If any root lies outside the unit circle then  $G^n$  grows exponentially and the scheme is unstable. The intermediate case when some roots may be on the unit circle (and the others inside) may lead to different situations. Consider for example the case when  $G$  is the identity. Then  $U^n = U^0$  and the scheme is clearly stable. However there are other examples of matrices with multiple roots on the unit circle that lead either to bounded or unbounded sequences  $(G^n)_{n \in \mathbb{N}}$ . It is clearly a property of the amplification matrix and not of its characteristic polynomial. If the dimension of eigen-subspace associated to a root is equal to its multiplicity then  $G^n$  is bounded (an example for this is the identity:  $\text{Id}^n = \text{Id}$  remains bounded). In the opposite case  $G^n$  grows linearly. A paradigm for this is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

which grows linearly with iterations, and only one eigenvector can be found. Such cases (which occur for our schemes) should therefore be handled specifically.

We will stick here to locate roots in the unit circle. An other way has already been developed in e.g. [13, 14]. It consists in using a conformal map, to locate roots in the left half plane. This changes the problem in a Routh-Hurwitz problem, which reads as a list of sufficient conditions for the characteristic polynomial to be von Neumann ( $|r| \leq 1$ ). This has been implemented in the FIDE REDUCE package.

## 2.3 Sketch of proofs

In the next sections, we will not give the proofs, but only list in a table the arguments used for each situation. We describe here the general plan and give names to specific final arguments used. The detailed proofs may be found in [15] for space dimensions 1 and 2. The three-dimensional case has been performed *via* Computer Algebra simulations [16].

Usually the coupled system is given in a implicit form. The first step consists in writing it in an explicit form. This yields the amplification matrix  $G$ . Then we compute its characteristic polynomial  $\phi_0$ . In order to perform a von Neumann analysis, we compute the series  $(\phi_m)$ . In the general case, under the assumption that the stability condition cannot be better than Yee's, we can apply either Theorem 1 (*Theorem 1* argument) or Theorem 2 (*Theorem 2* argument), check estimates at each level until  $\phi_m$  is a one degree polynomial. Special cases arise

when  $\varepsilon_s = \varepsilon_\infty$ ,  $\sin(\xi/2) = 0$  or  $\pm 1$ , and sometimes for limit values of physical coefficients. In these cases, different points of view have to be considered:

- Theorem 2 has to be used instead of Theorem 1,
- Some eigenvalues lie on the unit circle (mostly  $\pm 1$  or  $\pm i$ ). Polynomial  $\phi_0$  is at best a simple von Neumann polynomial and we have to study the other eigenvalues which are roots of a lower degree polynomial (*sub-polynomial* argument),
- Some eigenvalues lie on the unit circle and are not simple. Besides the study of the other eigenvalues (*sub-polynomial* argument), we have to compare the multiplicity of the root to the number of eigenvectors found. If they are equal (stable case) this is usually checked directly on the form of matrix  $G$  ( $G$  form argument). The unstable case necessitates the computation of eigenvectors (*eigenvectors* argument).

If  $\varepsilon_s = \varepsilon_\infty$  Maxwell equations decouple from the other equations. We study this limit as a hint of how the numerical scheme may behave when  $\varepsilon_s$  is very close to  $\varepsilon_\infty$ . For the same reason, we will study the undamped oscillator ( $\nu = 0$ ) for Lorentz media.

### 3 Debye media

We address successively the three discretizations of Maxwell–Debye equations mentioned in Table 1.

#### 3.1 Debye B\_ED scheme

In [7], Joseph et al. close System (1.6)–(1.7–a) by a discretization for Eq. (1.4–a), namely

$$\varepsilon_0 \varepsilon_\infty t_r \frac{E_j^{n+1} - E_j^n}{k} + \varepsilon_0 \varepsilon_s \frac{E_j^{n+1} + E_j^n}{2} = t_r \frac{D_j^{n+1} - D_j^n}{k} + \frac{D_j^{n+1} + D_j^n}{2}.$$

The resulting system may be cast in an explicit form which handles the variable ( $^t$  denotes transposition)

$$U_j^n = (c_\infty B_{j+\frac{1}{2}}^{n-\frac{1}{2}}, E_j^n, D_j^n / \varepsilon_0 \varepsilon_\infty)^t$$

and the amplification matrix  $G$  reads

$$\begin{pmatrix} 1 & -\lambda(e^{i\xi} - 1) & 0 \\ -\frac{(1+\delta)\lambda(1-e^{-i\xi})}{1+\delta\eta_s} & \frac{(1-\delta\eta_s)+(1+\delta)\lambda^2(e^{i\xi}-2+e^{-i\xi})}{1+\delta\eta_s} & \frac{2\delta}{1+\delta\eta_s} \\ -\lambda(1-e^{-i\xi}) & \lambda^2(e^{i\xi}-2+e^{-i\xi}) & 1 \end{pmatrix}$$

where  $\lambda = c_\infty k/h$  is the CFL constant,  $\delta = k/2t_r > 0$  is the normalized time step and  $\eta_s = \varepsilon_s/\varepsilon_\infty \geq 1$  denotes the normalized static permittivity. Moreover



we define  $q = -\lambda^2(e^{i\xi} - 2 + e^{-i\xi}) = 4\lambda^2 \sin^2(\xi/2)$ . The characteristic polynomial is proportional to

$$\phi_0(Z) = [1 + \delta\eta_s]Z^3 - [3 + \delta\eta_s - (1 + \delta)q]Z^2 + [3 - \delta\eta_s - (1 - \delta)q]Z - [1 - \delta\eta_s].$$

The proof arguments are summed up in Table 2 and we deduce that the stability condition is  $q \leq 4$  if  $\varepsilon_s > \varepsilon_\infty$  and  $q < 4$  if  $\varepsilon_s = \varepsilon_\infty$ .

$q$	$\varepsilon_s$	argument	result
$]0, 4[$	$> \varepsilon_\infty$	Theorem 1	stable
$]0, 4[$	$= \varepsilon_\infty$	Theorem 2	stable
0	$\geq \varepsilon_\infty$	$G$ form	stable
4	$> \varepsilon_\infty$	Theorem 2	stable
4	$= \varepsilon_\infty$	eigenvectors	unstable

Table 2: Proof arguments and results for the Debye B\_ED and B\_EP schemes.

### 3.2 Debye B\_EP scheme

In [6], Kashiwa et al. close System (1.6)–(1.7–b) by a discretization for Eq. (1.4–b):

$$t_r \frac{P_j^{n+1} - P_j^n}{k} = -\frac{P_j^{n+1} + P_j^n}{2} + \varepsilon_0(\varepsilon_s - \varepsilon_\infty) \frac{E_j^{n+1} + E_j^n}{2}.$$

The system now handles the variable

$$U_j^n = (c_\infty B_{j+\frac{1}{2}}^{n-\frac{1}{2}}, E_j^n, P_j^n / \varepsilon_0 \varepsilon_\infty)^t$$

and the amplification matrix  $G$  reads

$$\begin{pmatrix} 1 & -\lambda(e^{i\xi} - 1) & 0 \\ -\frac{(1+\delta)\lambda(1-e^{-i\xi})}{1+\delta\eta_s} & \frac{(1-\delta\eta_s)+(1+\delta)\lambda^2(e^{i\xi}-2+e^{-i\xi})+2\delta}{1+\delta\eta_s} & \frac{2\delta}{1+\delta\eta_s} \\ -\frac{\delta(\eta_s-1)\lambda(1-e^{-i\xi})}{1+\delta\eta_s} & \frac{(\lambda^2(e^{i\xi}-2+e^{-i\xi})+2)(\eta_s-1)\delta}{1+\delta\eta_s} & \frac{1+\eta_s\delta-2\delta}{1+\delta\eta_s} \end{pmatrix}$$

and the characteristic polynomial is proportional to the same polynomial as for the B\_ED scheme. All proofs are the same except those on matrix  $G$  directly, but even in these cases the conclusions are the same. Table 2 is also valid for the B\_EP scheme.

### 3.3 Debye BP\_E scheme

In [8], Young chooses to close System (1.6)–(1.7–c) by two discretizations for Eq. (1.4–b), namely

$$\begin{aligned} t_r \frac{P_j^{n+\frac{1}{2}} - P_j^{n-\frac{1}{2}}}{k} &= -\frac{P_j^{n+\frac{1}{2}} + P_j^{n-\frac{1}{2}}}{2} + \varepsilon_0(\varepsilon_s - \varepsilon_\infty) E_j^n, \\ t_r J_j^{n+\frac{1}{2}} &= -P_j^{n+\frac{1}{2}} + \varepsilon_0(\varepsilon_s - \varepsilon_\infty) \frac{E_j^{n+1} + E_j^n}{2}. \end{aligned}$$

Although  $J_j^{n+\frac{1}{2}}$  is used for the computations, this not a genuine variable for the full system which handles the variable

$$U_j^n = (c_\infty B_{j+\frac{1}{2}}^{n-\frac{1}{2}}, E_j^n, P_j^{n-\frac{1}{2}} / \varepsilon_0 \varepsilon_\infty)^t$$

and the amplification matrix  $G$  reads

$$\begin{pmatrix} 1 & -\lambda(e^{i\xi} - 1) & 0 \\ -\frac{\lambda(1-e^{-i\xi})}{1+\delta\alpha} & \frac{1+\delta-\delta\alpha+3\delta^2\alpha-(1+\delta)q}{(1+\delta)(1+\delta\alpha)} & \frac{1-\delta}{1+\delta} \frac{2\delta}{1+\delta\alpha} \\ 0 & \frac{2\delta\alpha}{1+\delta} & \frac{1-\delta}{1+\delta} \end{pmatrix}$$

with the same notations as above and  $\alpha = \eta_s - 1 \geq 0$ . The characteristic polynomial is proportional to

$$\begin{aligned} \phi_0(Z) = & [(1+\delta\alpha)(1+\delta)]Z^3 - [3+\delta+\delta\alpha+3\delta^2\alpha-(1+\delta)q]Z^2 \\ & + [3-\delta-\delta\alpha+3\delta^2\alpha-(1-\delta)q]Z - [(1-\delta\alpha)(1-\delta)]. \end{aligned}$$

Table 3 gathers the different arguments used. The stability condition is  $q \leq 4$  and  $\delta \leq 1$  if  $\varepsilon_s > \varepsilon_\infty$  and  $q < 4$  if  $\varepsilon_s = \varepsilon_\infty$ .

$q$	$\varepsilon_s$	$\delta$	argument	result
$]0, 4]$	$> \varepsilon_\infty$	$]0, 1[$	Theorem 1	stable
$]0, 4[$	$= \varepsilon_\infty$	$> 0$	Theorem 2	stable
0	$\geq \varepsilon_\infty$	$> 0$	$G$ form	stable
$]0, 4]$	$> \varepsilon_\infty$	1	sub-polynomial	stable
4	$= \varepsilon_\infty$	$> 0$	eigenvectors	unstable

Table 3: Proof arguments and results for the Debye BP.E scheme.

### 3.4 Conclusion for one-dimensional Debye schemes

If  $\varepsilon_s > \varepsilon_\infty$ , the pure CFL condition  $q \leq 4$  is the same for both models. It is exactly the condition for Maxwell equations. However the BP.E scheme necessitates another condition,  $\delta \leq 1$ , which corresponds to a sufficient discretization of Debye equation (1.4–b). Even if we are interested here in stability properties, such conditions are to be considered to ensure equations to be correctly taken into account. Results are given in physical variables in Table 4.

To compare conditions on  $q$  and  $\delta$ , let us consider a simple physical case. We assume that a matter with  $\varepsilon_\infty = 1$  (and thus  $c_\infty \simeq 3 \times 10^8 \text{ m s}^{-1}$ ) is lighted by a wave of say wavelength 1 cm. The space step  $h$  has to be smaller than this wavelength, and therefore  $q < 4$  reads at least  $k < \frac{1}{3} \times 10^{-10} \text{ s}$ . In a Debye medium, relaxation times  $t_r$  are of the order of a picosecond (or even a nanosecond) which maybe of the same order than the previous bound. A typical example is water for which  $\varepsilon_\infty = 1.8$ ,  $\varepsilon_s = 81.0$  and  $t_r = 9.4 \times 10^{-12} \text{ s}$

Scheme	dimension 1		dimension 1	
	$\varepsilon_s > \varepsilon_\infty$		$\varepsilon_s = \varepsilon_\infty$	
B_ED or B_EP	$q \leq 4$	$k \leq h/c_\infty$	$q < 4$	$k < h/c_\infty$
BP_E	$q \leq 4, \delta \leq 1$	$k \leq \min(h/c_\infty, 2t_r)$	$q < 4$	$k < h/c_\infty$

Table 4: Stability of Debye models for  $\varepsilon_s > \varepsilon_\infty$  and  $\varepsilon_s = \varepsilon_\infty$ .

[4]. Condition  $k \leq 2t_r$  comes to  $k \leq 1.88 \times 10^{-11}$  s. Condition  $q \leq 4$  yields a similar condition if  $h = 4.2 \times 10^{-3}$  m. For a very coarse space grid and with the BP\_E scheme, the CFL condition comes from  $k \leq 2t_r$ ; otherwise, fine grid and/or B\_ED or B\_EP schemes, it comes from  $q \leq 4$ .

A quite different material is for example the 0.25-dB loaded foam given in [17] for which  $\varepsilon_\infty = 1.01$ ,  $\varepsilon_s = 1.16$  and  $t_r = 6.497 \times 10^{-10}$  s. Condition  $k \leq 2t_r$  comes to  $k \leq 1.3 \times 10^{-9}$  s and  $q \leq 4$  yields a similar condition if  $h = 3.9 \times 10^{-1}$  m. In practice  $h$  has to be smaller to describe a wave with 1 cm wavelength and the stability condition for this foam is  $q < 4$  for both schemes.

In conclusion the stability condition may depend of the material in classical applications, leading us to prefer the B\_ED scheme, when  $t_r$  is small (picosecond). The result announced in [1] was  $q \leq 4$  for the B\_ED scheme and for water, which is consistent with our result.

## 4 Lorentz media

We now address the three discretizations of Maxwell–Lorentz equations mentioned in Table 1.

Each of these schemes reads the same in the undamped ( $\nu = 0$ ) or damped ( $\nu > 0$ ) cases. However the analysis will differ greatly since  $\phi_1 \equiv 0$  for all the schemes in the undamped case.

### 4.1 Lorentz B\_ED scheme

In [7], System (1.6)–(1.7–a) is closed by a discretization for Eq. (1.5–a), namely

$$\begin{aligned} \varepsilon_0 \varepsilon_\infty \frac{E_j^{n+1} - 2E_j^n + E_j^{n-1}}{k^2} + \nu \varepsilon_0 \varepsilon_\infty \frac{E_j^{n+1} - E_j^{n-1}}{2k} + \varepsilon_0 \varepsilon_s \omega_1^2 \frac{E_j^{n+1} + E_j^{n-1}}{2} \\ = \frac{D_j^{n+1} - 2D_j^n + D_j^{n-1}}{k^2} + \nu \frac{D_j^{n+1} - D_j^{n-1}}{2k} + \omega_1^2 \frac{D_j^{n+1} + D_j^{n-1}}{2} \end{aligned}$$

The explicit version of the subsequent system does not use explicitly the value of  $D_j^{n-1}$  and therefore this system handles the variable

$$U_j^n = (c_\infty B_{j+\frac{1}{2}}^{n-\frac{1}{2}}, E_j^n, E_j^{n-1}, D_j^n / \varepsilon_0 \varepsilon_\infty)^t.$$

The amplification matrix  $G$  reads

$$\begin{pmatrix} 1 & -\lambda(e^{i\xi} - 1) & 0 & 0 \\ -\frac{2\delta\lambda(1-e^{-i\xi})}{1+\delta+\omega\eta_s} & \frac{2-q(1+\delta+\omega)}{1+\delta+\omega\eta_s} & \frac{1-\delta+\omega\eta_s}{1+\delta+\omega\eta_s} & \frac{2\omega}{1+\delta+\omega\eta_s} \\ 0 & 1 & 0 & 0 \\ -\lambda(1-e^{-i\xi}) & -q & 0 & 1 \end{pmatrix}$$

where  $\delta = \nu k/2 \geq 0$  is the new normalized time step, and  $\omega = \omega_1^2 k^2/2 > 0$  denotes the normalized squared frequency. The other notations used for the Debye model remain valid. The characteristic polynomial is proportional to

$$\begin{aligned} \phi_0(Z) = & [1 + \delta + \omega\eta_s]Z^4 - [4 + 2\delta + 2\omega\eta_s - (1 + \delta + \omega)q]Z^3 \\ & + [6 + 2\omega\eta_s - 2q]Z^2 - [4 - 2\delta + 2\omega\eta_s - (1 - \delta + \omega)q]Z + [1 - \delta + \omega\eta_s]. \end{aligned}$$

The proofs are summed up in Table 5 for the damped and the undamped case.

$q$	$\varepsilon_s$	argument	result	argument	result
		damped: $\nu > 0$		undamped: $\nu = 0$	
$]0, 2[$	$> \varepsilon_\infty$	Theorem 1	stable	Theorem 2	stable
$]0, 2]$	$= \varepsilon_\infty$	Theorem 2	stable	sub-polynomial	unstable
0	$\geq \varepsilon_\infty$	$G$ form	stable	$G$ form	stable
2	$\geq \varepsilon_\infty$	sub-polynomial	stable	sub-polynomial	stable

Table 5: Proof arguments and results for the Lorentz B\_ED scheme.

In the damped case the stability condition is  $q \leq 2$  for all  $\varepsilon_s \geq \varepsilon_\infty$ . The  $\varepsilon_s = \varepsilon_\infty$  undamped case, needs some explanation. For  $q \in ]0, 2]$ ,  $\phi_0$  may be cast as the product of two second order polynomials. The roots are two couples of conjugate complex roots of modulus 1. For the specific value  $q = 2\omega/(1 + \omega)$ , which always lies in the interval  $]0, 2]$ , the two couples degenerate in one double couple with only two eigen-directions, which is the unstable case. To avoid this instability one may think to bound  $q$  and say that the scheme is stable provided  $q \in [0, 2\omega/(1 + \omega)[$ . But if we come back to the original variables, we see that this is not an upper bound on  $k$  but rather a lower bound on  $h$ , which we surely do not want. It is therefore better to avoid using the B\_ED scheme in this very specific case,  $\varepsilon_s = \varepsilon_\infty$  and  $\nu = 0$ . A better scheme for this case is provided next.

## 4.2 Lorentz B\_EPJ scheme

In [6], Kashiwa et al. close System (1.6)–(1.7–b) by a discretization for Eq. (1.5–b), namely

$$\begin{aligned} \frac{P_j^{n+1} - P_j^n}{k} &= \frac{J_j^{n+1} + J_j^n}{2}, \\ \frac{J_j^{n+1} - J_j^n}{k} &= -\nu \frac{J_j^{n+1} + J_j^n}{2} + \omega_1^2 (\varepsilon_s - \varepsilon_\infty) \varepsilon_0 \frac{E_j^{n+1} + E_j^n}{2} - \omega_1^2 \frac{P_j^{n+1} + P_j^n}{2}. \end{aligned}$$

The explicit version of the obtained system handles the variable

$$U_j^n = (c_\infty B_{j+\frac{1}{2}}^{n-\frac{1}{2}}, E_j^n, P_j^n / \varepsilon_0 \varepsilon_\infty, k J_j^n / \varepsilon_0 \varepsilon_\infty)^t$$

and the amplification matrix  $G$  reads

$$\begin{pmatrix} 1 & -\lambda(e^{i\xi} - 1) & 0 & 0 \\ \frac{-\lambda(1-e^{-i\xi})(\Delta-\frac{1}{2}\omega\alpha)}{\Delta} & \frac{\Delta-q\Delta-(2-q)\frac{1}{2}\omega\alpha}{\Delta} & \frac{\omega}{\Delta} & \frac{-1}{\Delta} \\ \frac{-\lambda(1-e^{-i\xi})\frac{1}{2}\omega\alpha}{\Delta} & \frac{(2-q)\frac{1}{2}\omega\alpha}{\Delta} & \frac{\Delta-\omega}{\Delta} & \frac{1}{\Delta} \\ \frac{-\lambda(1-e^{-i\xi})\omega\alpha}{\Delta} & \frac{(2-q)\omega\alpha}{\Delta} & \frac{-2\omega}{\Delta} & \frac{2-\Delta}{\Delta} \end{pmatrix}$$

where together with the previously defined notations,  $\Delta = 1 + \delta + \omega\eta_s/2$ . The characteristic polynomial is proportional to

$$\begin{aligned} \phi_0(Z) = & [1 + \delta + \frac{1}{2}\omega\eta_s]Z^4 - [4 + 2\delta - (1 + \delta + \frac{1}{2}\omega)q]Z^3 \\ & + [6 - \omega\eta_s + (\omega - 2)q]Z^2 - [4 - 2\delta - (1 - \delta + \frac{1}{2}\omega)q]Z + [1 - \delta + \frac{1}{2}\omega\eta_s]. \end{aligned}$$

The proofs are summed up in Table 6. Both in the damped and undamped cases, the stability condition is  $q < 4$  which is much better than the previous scheme since we gain a factor 2 on  $k$  and we have no problem when  $\varepsilon_s = \varepsilon_\infty$  and  $\nu = 0$  as for the previous model.

$q$	$\varepsilon_s$	argument	result	argument	result
		damped: $\nu > 0$		undamped: $\nu = 0$	
$]0, 4[$	$> \varepsilon_\infty$	Theorem 1	stable	Theorem 2	stable
$]0, 4[$	$= \varepsilon_\infty$	Theorem 2	stable	Theorem 2	stable
0	$\geq \varepsilon_\infty$	G form	stable	G form	stable
4	$\geq \varepsilon_\infty$	eigenvectors	unstable	eigenvectors	unstable

Table 6: Proof arguments and results for the Lorentz B\_EPJ scheme.

### 4.3 Lorentz BJ\_EP scheme

In [8], System (1.6)–(1.7–c) is closed by a discretization for Eq. (1.5–b), namely

$$\begin{aligned} \frac{P_j^{n+1} - P_j^n}{k} &= J_j^{n+\frac{1}{2}}, \\ \frac{J_j^{n+\frac{1}{2}} - J_j^{n-\frac{1}{2}}}{k} &= -\nu \frac{J_j^{n+\frac{1}{2}} + J_j^{n-\frac{1}{2}}}{2} + \omega_1^2 (\varepsilon_s - \varepsilon_\infty) \varepsilon_0 E_j^n - \omega_1^2 P_j^n. \end{aligned}$$

The explicit version the system handles the variable

$$U_j^n = (c_\infty B_{j+\frac{1}{2}}^{n-\frac{1}{2}}, E_j^n, P_j^n / \varepsilon_0 \varepsilon_\infty, k J_j^{n-\frac{1}{2}} / \varepsilon_0 \varepsilon_\infty)^t$$

and the amplification matrix  $G$  reads

$$\begin{pmatrix} 1 & -\lambda(e^{i\xi} - 1) & 0 & 0 \\ -\lambda(1 - e^{-i\xi}) & \frac{(1-q)(1+\delta)-2\omega\alpha}{1+\delta} & \frac{2\omega}{1+\delta} & -\frac{1-\delta}{1+\delta} \\ 0 & \frac{2\omega\alpha}{1+\delta} & \frac{1+\delta-2\omega}{1+\delta} & \frac{1-\delta}{1+\delta} \\ 0 & \frac{2\omega\alpha}{1+\delta} & \frac{-2\omega}{1+\delta} & \frac{1-\delta}{1+\delta} \end{pmatrix}.$$

The characteristic polynomial is proportional to

$$\begin{aligned} \phi_0(Z) = & [1 + \delta]Z^4 - [4 + 2\delta - 2\omega\eta_s - (1 + \delta)q]Z^3 \\ & + 2[3 - 2\omega\eta_s + (\omega - 1)q]Z^2 - [4 - 2\delta - 2\omega\eta_s - (1 - \delta)q]Z + [1 - \delta]. \end{aligned}$$

The proofs are summed up in Table 7. This scheme combines three drawbacks

$q$	$\varepsilon_s$	$\omega$	argument	result
damped: $\nu > 0$				
$]0, 2[$	$> \varepsilon_\infty$	$\leq 2/(2\eta_s - 1)$	Theorem 1	stable
2	$> \varepsilon_\infty$	$< 2/(2\eta_s - 1)$		
$]0, 2[$	$= \varepsilon_\infty$	$< 2$	Theorem 2	stable
$]0, 2[$	$= \varepsilon_\infty$	$= 2$	sub-polynomial	stable
2	$> \varepsilon_\infty$	$= 2/(2\eta_s - 1)$	Theorem 2	stable
0	$\geq \varepsilon_\infty$	$\leq 2/(2\eta_s - 1)$	G form	stable
undamped: $\nu = 0$				
$]0, 2[$	$> \varepsilon_\infty$	$\leq 2/(2\eta_s - 1)$	Theorem 2	stable
2	$> \varepsilon_\infty$	$< 2/(2\eta_s - 1)$		
$]0, 2[$	$= \varepsilon_\infty$	$< 2$	eigenvectors	unstable
$]0, 2[$	$= \varepsilon_\infty$	$= 2$	Theorem 2	stable
2	$> \varepsilon_\infty$	$= 2/(2\eta_s - 1)$	eigenvectors	unstable
0	$> \varepsilon_\infty$	$\leq 2/(2\eta_s - 1)$	G form	stable
0	$= \varepsilon_\infty$	$< 2/(2\eta_s - 1)$		
0	$= \varepsilon_\infty$	$= 2/(2\eta_s - 1)$	eigenvectors	unstable

Table 7: Proof arguments and results for the Lorentz BJ-EP scheme.

we have already encountered. First as for the Debye model, there is an extra condition on the time step:  $\omega < 2/(2\eta_s - 1)$ . This will have to be compared to the condition on  $q$  for physical examples. Second, as for the Lorentz B\_ED scheme we need a twice smaller  $k$  than for raw Maxwell equations:  $q \leq 2$  instead of  $q \leq 4$ . Last, and also as for the Lorentz B\_ED scheme, the  $\varepsilon_s = \varepsilon_\infty$  and  $\nu = 0$  leads to an instability. This is exactly the same story. This time  $q = 2\omega$  leads to double couples of conjugate complex roots of modulus 1, with only two eigen-directions. If  $\omega > 1$  this value of  $q$  is however never reached. Else  $q = 2\omega$  is rather a condition on  $h$  and therefore not a proper stability condition. As for

the Lorentz B\_ED it seems better to avoid using this scheme if  $\varepsilon_s = \varepsilon_\infty$  and  $\nu = 0$ .

#### 4.4 Conclusion for one-dimensional Lorentz schemes

We summarize all our results for Lorentz schemes in Table 8. For the undamped BJ\_EP scheme if  $\varepsilon_s > \varepsilon_\infty$  the condition is slightly better since  $q = 2$  and  $\omega < 2/(2\eta_s - 1)$ , or  $q < 2$  and  $\omega = 2/(2\eta_s - 1)$  also yield stable schemes.

Contrarily to Debye materials, for which all schemes compete, the B\_EPJ scheme seems to overcome others for Lorentz material. First, there is a gain in CFL condition ( $q < 4$  is twice better as  $q \leq 2$ ), second, there are no instabilities for limiting values of the physical coefficients, and last there is no extra condition on the time step. In practice, an extra condition is however needed to account for the dynamics of the Lorentz equation, but not for stability reasons.

However we can compare the relative strength of the different conditions on  $k$  for the B\_ED and BJ\_EP schemes, and for optical waves of say wavelength  $1 \mu\text{m}$ . The values used in [1] are  $\varepsilon_\infty = 1$ ,  $\varepsilon_s = 2.25$ ,  $\omega_1 = 4 \times 10^{16} \text{ rad s}^{-1}$  and  $\nu = 0.56 \times 10^{16} \text{ rad s}^{-1}$ . Condition  $\omega \leq 2/\sqrt{2\eta_s - 1}$  comes to  $k \leq 2.7 \times 10^{-17} \text{ s}$  which is very small and corresponds to  $h = 1.13 \times 10^{-8} \text{ m}$  in the  $q < 2$  condition. This space step is more than sufficient to discretize optical waves. For such a material the extra condition imposed by the B\_ED scheme is stronger than the basic CFL condition. The B\_EPJ model is then more advisable.

Scheme		dimension 1
damped: $\nu > 0$ , and $\varepsilon_s \geq \varepsilon_\infty$		
B_ED	$q \leq 2$	$k \leq h/\sqrt{2}c_\infty$
B_EPJ	$q < 4$	$k < h/c_\infty$
BJ_EP	$q \leq 2,$ $\omega \leq 2/(2\eta_s - 1)$	$k \leq \min(h/\sqrt{2}c_\infty, 2/\omega_1\sqrt{2\eta_s - 1})$
undamped: $\nu = 0$ , and $\varepsilon_s > \varepsilon_\infty$		
B_ED	$q \leq 2$	$k \leq h/\sqrt{2}c_\infty$
B_EPJ	$q < 4$	$k < h/c_\infty$
BJ_EP	$q < 2,$ $\omega < 2/(2\eta_s - 1)$	$k < \min(h/\sqrt{2}c_\infty, 2/\omega_1\sqrt{2\eta_s - 1})$
undamped: $\nu = 0$ , and $\varepsilon_s = \varepsilon_\infty$		
B_ED	$q < 2\omega/(1 + \omega)$	condition on $h$
B_EPJ	$q < 4$	$k < h/c_\infty$
BJ_EP	$q < 2,$ $\omega < 2/(2\eta_s - 1)$	$k < h/\sqrt{2}c_\infty, 2/\omega_1\sqrt{2\eta_s - 1}$ if $\omega > 1$ condition on $h$ else

Table 8: Stability of damped and undamped Lorentz models for  $\varepsilon_s > \varepsilon_\infty$  and  $\varepsilon_s = \varepsilon_\infty$ .

In [8] there is a totally different material for which  $\varepsilon_\infty = 1.5$ ,  $\varepsilon_s = 3$ ,  $\omega_1 = 2\pi \times 5 \times 10^{10} \text{ rad s}^{-1}$  and  $\nu = 10^{10} \text{ rad s}^{-1}$  (these round values certainly refer to a model material). In this case  $\omega \leq 2/\sqrt{2\eta_s - 1}$  comes to  $k \leq 3.6 \times 10^{-12} \text{ s}$  which corresponds to  $h = 1.9 \times 10^{-3} \text{ m}$  in the  $q < 2$  condition. For this material condition  $q < 2$  is the strongest for optical waves. The B\_EPJ scheme is however more advisable, since it allows  $q < 4$  instead of  $q \leq 2$ .

The results obtained in [1] were obtained for our first cited material and for the B\_ED and BJ\_EP schemes. He observed instabilities for  $\xi > \frac{\pi}{2}$ . We note that if  $\xi \leq \frac{\pi}{2}$  then  $\sin(\xi/2) \leq 1/\sqrt{2}$  and  $q \leq 2$  instead of  $q \leq 4$ . This is exactly our result. He found also the B\_EPJ scheme to stable for  $q \leq 4$ .

## 5 Automation *via* Computer Algebra

For a 3D Lorentz medium,  $\phi_0$  is typically a 12th degree polynomial with polynomial coefficients of degree 6 in the different parameters. The above procedure becomes awful if made by hand. A Computer Algebra environment based on MAPLE has been developed with the specific aim to automate all the computational steps which may be source of errors [16]. It is still only dedicated to electromagnetic models but could be extended in the future to other applications.

The schemes are defined by four parameters:

1. the space dimension `Dim` (1, 2 or 3),
2. the polarization `Polar` (TE or TM in dimension 2),
3. the physical model `Model` (e.g. Debye),
4. the variables used `Formula` (e.g. B\_EP).

Maxwell equations have been written once and for all and have just to be “called”

```
> Faraday(Eq, Dim, Polar):
> Ampere(Eq, Dim, Polar, Formula):
```

For our applications, in the other equations space is only a parameter. Such equations are written once with no spatial dependence

```
> NewEq := tr*(P[n+1]-P[n])/dt
+ 1/2*(P[n+1]+P[n])
- eps0*(epss-epsinfini)*(E[n+1]+E[n])/2:
> CreateEq(Eq, NewEq, Dim, Polar):
```

and propagated to all the useful coordinates with the right indexes on the staggered grid, according to the space dimension and polarization.

Then change of variables are automatically performed to have dimensionless variables (specific to the model), no redundant variables (specific to the scheme)



and an explicit scheme in the frequency domain. This yields the amplification matrix  $G$ . The computation of polynomial  $\phi_0$  is then performed as well as the computational part of the von Neumann analysis: computation of the sequence of polynomials and factorizations. On these forms the user of the toolbox can easily see which are the specific cases to consider separately.

The comparisons  $|\phi_m(0)| \leq |\phi_m^*(0)|$  are the real difficult point from the computer algebra point of view. It comes to evaluate the sign of a polynomial in many variables (4 for a Lorentz medium) and of total degree of order say 6 for  $\phi_0$ , about 10 for  $\phi_1, \dots$  knowing some variables are positive (like  $\eta_s - 1$  or  $\delta$ ) and other lie within an interval (like  $q$ ). This is also automated

```
> SignCheck(phi, Z, [0 < delta, 1 < etas, 0 < q, q < 4]);
```

but sometimes MAPLE does not yield a totally explicit answer. This might lead us to migrate the whole toolbox in a C code to make use of some existing softwares specific for the solving of interval arithmetic problems.

Finally tools are defined to compare the number of eigenvectors and the multiplicity of eigenvalues in the degenerate cases.

## 6 Two- and three dimensional results

### 6.1 Equation setting

In a two-dimensional context where unknowns depend only on space variables  $x$  and  $y$ , Maxwell system may be split in two decoupled systems corresponding to the transverse electric (TE) ( $B_x, B_y, E_z$ ) and the transverse magnetic (TM) ( $B_z, E_x, E_y$ ) polarizations. In the one-dimensional case, Maxwell–Debye equations were represented by three equations and Maxwell–Lorentz by four equations. In the TE polarization, one more Faraday equation is added and we have four equations for Maxwell–Debye and five equations for Maxwell–Lorentz. In the TM polarization for the Maxwell–Debye model, one Ampère equation and one Debye equation have to be added, leading to five equations systems. For the Maxwell–Lorentz model, there are one Ampère equation and two Lorentz equations more, and the system consists of seven equations.

In the three-dimensional context, equations do not decouple any more, and systems consist in nine equations for the Maxwell–Debye schemes and twelve equations for the Maxwell–Lorentz schemes.

The principle of the stability analysis is exactly the same, but we now have larger polynomials to study. We however found out that the one-dimensional polynomials (which we denote by  $\phi_0^{1D}(Z)$ ) are a factor in two- and three-dimensional polynomials, which reduces the formal calculations. More precisely we now denote by  $h_x, h_y$  and  $h_z$  the space steps in the  $x$ -,  $y$ - and  $z$ -directions respectively and by  $q$  the quantity

$$q = q_x + q_y = 4c_\infty^2 \left( \frac{k^2}{h_x^2} \sin^2(\xi_x/2) + \frac{k^2}{h_y^2} \sin^2(\xi_y/2) \right)$$

or

$$q = q_x + q_y + q_z = 4c_\infty^2 \left( \frac{k^2}{h_x^2} \sin^2(\xi_x/2) + \frac{k^2}{h_y^2} \sin^2(\xi_y/2) + \frac{k^2}{h_z^2} \sin^2(\xi_z/2) \right)$$

according to the space dimension (recall  $q = 4c_\infty^2 \frac{k^2}{h_x^2} \sin^2(\xi_x/2)$  in 1D).

## 6.2 Results

In the two-dimensional TE polarization, the characteristic polynomial reads

$$\phi_0^{2D,TE}(Z) = [Z - 1]\phi_0^{1D}(Z),$$

for all the Maxwell–Debye and Maxwell-Lorentz schemes we study here. This could be a problem, if 1 is already a root of  $\phi_0^{1D}(Z)$ , i.e. if  $q = 0$ , but it happens that it is never a problem: minimal stable sub-spaces are always one-dimensional.

In the TM polarization, the same factorization occurs but the remaining polynomial is slightly more complicated, namely

$$\phi_0^{2D,TM}(Z) = [Z - 1]\psi_0(Z)\phi_0^{1D}(Z),$$

where  $\psi_0(Z)$  is equal to:

- Debye B\_ED and B\_EP:  $[(1 + \delta\eta_s)Z - (1 - \delta\eta_s)]$ ,
- Debye BP\_E:  $[(1 + \alpha)(1 + \delta\alpha)Z - (1 - \alpha)(1 - \delta\alpha)]$ ,
- Lorentz B\_ED:  $[(1 + \delta + \omega\eta_s)Z^2 - 2Z + (1 - \delta + \omega\eta_s)]$ ,
- Lorentz B\_EPJ:  $[(1 + \delta + \frac{1}{2}\omega\eta_s)Z^2 - (2 - \omega\eta_s)Z + (1 - \delta + \frac{1}{2}\omega\eta_s)]$ ,
- Lorentz BJ\_EP:  $[(1 + \delta)Z^2 - 2(1 - \omega\eta_s)Z + (1 - \delta)]$ .

As for the TE polarization the extra eigenvalue 1 is never a source of instability. The other extra eigenvalues always lie inside or on the unit circle (conjugate complex roots). The only problem is when modulus 1 eigenvalues are also eigenvalues of the one-dimensional polynomial. This only occurs for the Lorentz B\_ED scheme is  $\varepsilon_s = \varepsilon_\infty$ , and  $q = 2\omega/(1 + \omega)$ , which is a resonant value we have already pointed out in the undamped case for this scheme.

In the three-dimensional context, the characteristic polynomial reads

$$\phi_0^{3D}(Z) = [Z - 1]^2\psi_0(Z) (\phi_0^{1D}(Z))^2,$$

for all the schemes we have studied. In addition to extra eigenvalues 1, we have to check cases when we found out that  $\phi_0^{1D}(Z)$  is a von Neumann polynomial but not a Schur polynomial. If  $\varepsilon_s = \varepsilon_\infty$  we systematically have instabilities. Other extra instable cases depend on the scheme:  $q = 4$  for the Debye B\_ED and B\_EP schemes,  $\delta = 1$  for the Debye BP\_E scheme;  $q = 2$  for the Lorentz D\_ED and BJ\_EP schemes,  $\nu = 0$  for all Lorentz schemes.

We do not have any explanation for these factorizations. This is a property of the characteristic polynomial and not necessarily of the amplification matrix, i.e. this may occur even if variables are not decoupled. However, there is probably some underlying block-triangular structure, which is still to be found.

We shall not duplicate Tables 4 and 8 for two- and three-dimensional schemes. In the two-dimensional case, if  $h_x = h_y = h_z \equiv h$ , condition  $q \leq 4$  becomes  $k \leq h/(\sqrt{2}c_\infty)$  and condition  $q \leq 2$  becomes  $k \leq h/(2c_\infty)$  in the physical variables. Besides, Lorentz B\_ED model which was leading to a lower bound on  $h$  in the undamped case, leads also to such a bound in the damped case. These are the only differences with Tables 4 and 8.

In short, 3D conditions are also essentially the same as 1D conditions but inequalities have always to be strict and limiting (non-physical) cases  $\varepsilon_s = \varepsilon_\infty$  and  $\nu = 0$  cannot be dealt properly with a 3D model.

## 7 Conclusion

We have studied a class of FD–TD schemes for dispersive materials based on the Yee scheme for Maxwell equations and compared them from the stability point of view. This study was inspired by Petropoulos [1] who performs the same analysis but using specific values for the physical and numerical constants and using numeric routines to locate eigenvalues of the amplification matrix. Here we have general results which yield the constraint on numerical constants ( $k$  and  $h$ ) for any Debye or Lorentz material. Our results confirm those of Petropoulos.

For usual Debye media, both studied schemes are stable under the same condition as the Yee scheme for nanosecond delay materials. The B\_ED and B\_EP schemes overcome the BP\_E scheme in terms of stability condition for picosecond delay materials. Among the studied schemes for Lorentz media, the B\_EPJ scheme clearly ranks first as far as stability is concerned. Its stability condition is also that of the Yee scheme. However to take properly into account the Lorentz model, a smaller time step may have to be chosen, independently of stability issues. The two examples do not help us to deduce a general strategy to locate variables in time, in order to treat other physical models.

The Computer Algebra system that has been developed to handle the three-dimensional schemes has a much wider range of application. First it can be extended with very small effort to other electromagnetic linear models such as cold plasmas, collisionless warm plasmas or magnetic ferrites. With some more work (no computation skill but knowledge of dimensionless variables), it can be practically extended to the analysis of any linear finite-difference scheme occurring in other areas than electromagnetism.

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