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# SEQUENTIAL MULTIDIMENSIONAL SPECTRAL ESTIMATION

RAMI KANHOUCHE<sup>1</sup>

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*Abstract*-By considering an empirical approximation, and a new class of operators that we will call walking operators, we construct, for any positive ND-toeplitz matrix, an infinite in all dimensions matrix, for which the inverse approximates the original matrix in its finite part. A recursive hierarchical algorithm is presented for sequential dimension spectral representation. A positive comparison in calculus cost, and numerical simulation, for 2D and 3D signals, is also presented.

*Résumé*-En considérant une approximation empirique et une nouvelle classe des opérateurs qu'on appelle les opérateurs « Walking », on arrive à construire, pour toute matrice positive ND-Toeplitz, une matrice infinie dans toutes les dimensions, et pour laquelle l'inverse; est une approximation de la matrice originale dans sa partie finie. L'algorithme hiérarchique récursif qu'on présente produit une estimation séquentielle de la puissance spectrale multidimensionnelle. On montre aussi une comparaison positive dans le coût de calcul par rapport à la méthode de Capon, et des simulations numériques pour des signaux 2D, et 3D.

## I Introduction

Lot of work has been done in the past in the domain of spectral analysis. As a basic mathematical problem giving solutions to many physical systems, it remains not a totally resolved problem in its multidimensional aspect. While the foundations of the theory were established by [11],[12], a very series of important advances in the field were achieved, especially, the connection to the information theory [13], by the introduction of the spectral maximum of entropy principle in [1]. Also the connection to an optimized Toeplitz linear system was made in [10] and [9]. In the two dimensional case, and more generally in the multidimensional case, actually, and contrary to the correlation matching, and maximum of entropy criteria achieved in the 1D case, two algebraic methods are used. Each got its advantage and disadvantage. The first method is what we can call the quarter-plan forced autoregressive filters [7], and the second is the Capon Minimum variance method [6],[15]. While the Capon estimation provide a good stable estimation; it got the disadvantage of high calculus coast, on the other hand the quarter-plan method follows an unreal model<sup>2</sup>, and for that suffers from high instability with order elevation. In this text we are proposing a method that is less costly than the Capon method, and far more stable than the quarter-plan autoregressive estimation. The work done and the proposed approximation depend highly on the previous insight achieved in [8] and [14].

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<sup>2</sup> Polynomial's roots are not forcefully inside the unit circle.

## II Multidimensional Toeplitz Characters and Spectral Block Structure

For a multidimensional signal  $x(t), t \in \mathbb{Z}^d$ , we will define the  $\gamma$  order correlation matrix,  $\gamma \in (\mathbb{Z}^+)^d$  as

$$R := E \left( X(t) (X(t))^H \right)$$

where  $X$  is defined as

$$X(t) := \begin{bmatrix} x(t_0, t_1, \dots, t_{d-1}) \\ x(t_0+1, t_1, \dots, t_{d-1}) \\ \vdots \\ x(t_0+\gamma_0-1, t_1, \dots, t_{d-1}) \\ x(t_0, t_1+1, \dots, t_{d-1}) \\ \vdots \\ x(t_0, t_1+\gamma_1-1, \dots, t_{d-1}) \\ \vdots \\ x(t_0, t_1, \dots, t_{d-1}+\gamma_{d-1}-1) \end{bmatrix}.$$

As far as this paper is concerned we will be only dealing with the case where the correlation matrix admit a toeplitz nature in all dimensions, in other words, it is  $d$  time toeplitz, so it is also adequate to represent it as an operator of a  $d$  dimension correlation signal  $c(t)$  so that

$$c(t) \in \mathbb{C}, t \in [-\gamma, \gamma], \gamma \in (\mathbb{Z}^+)^d,$$

$$c(t) \equiv \sum_{n \in \mathbb{Z}^d} x(n+t) x^*(n)$$

and we will write

$$R = R(c(\gamma)) \text{ which is of size } q = \prod_{i=0}^{d-1} \gamma_i.$$

It is also well known, and according to Kolmogorov's relation [12] that

$$c(t) = \frac{1}{(2\pi)^d} \oint_{|\mathbb{C}^d|=1} \left| \sum_{n \in \mathbb{Z}^d} x[n] e^{jw \cdot (n^T)} \right|^2 e^{-jw \cdot (t^T)} .dw.$$

By considering  $\Omega$ , a redistribution of the dimensions nesting, such that

$$\Omega \in \left\{ (\Omega_0, \Omega_1, \dots, \Omega_{d-1}) : \Omega_k \neq \Omega_l, \forall \Omega_l, 0 \leq \Omega_l \leq d-1 \right\}$$

we will define

$$q_l^\Omega := \begin{cases} \prod_{i=0}^{l-1} \gamma_{\Omega_i} & \text{if } l \neq 0 \\ 1 & \text{if } l = 0 \end{cases}$$

Of course  $\Omega$  is invertible in the sense that  $\Omega_k = l \Rightarrow \Omega^{-1}(l) = k$ .

**Definition 1.** We will define the walking operator as

$$W_{\Omega \rightarrow \Omega'}(M) : [M]_{i,j} \rightarrow [M]_{X^{\Omega \rightarrow \Omega'}(i), X^{\Omega \rightarrow \Omega'}(j)}$$

$$X^{\Omega \rightarrow \Omega'}(i_0 q_0^\Omega + i_1 q_1^\Omega + \dots + i_{d-1} q_{d-1}^\Omega) = i_0 q_{\Omega^{-1}(\Omega_1(0))}^{\Omega'} + i_1 q_{\Omega^{-1}(\Omega_1(1))}^{\Omega'} + \dots + i_{d-1} q_{\Omega^{-1}(\Omega_1(d-1))}^{\Omega'}$$

where  $[i_0, i_1, \dots, i_{d-1}] \in (\mathbb{Z}^*)^d$ ,  $0 \leq i_k \leq \gamma_{\Omega(k)}$ .

From a basic point of view, the walking operator corresponds to the convention used in the relation defining the vector  $X$ , but, as we will see later, it represents also a very important constraint on the degree of liberty of the correlation matrix, and eventually its inverse.

**Definition 2.** We call for any  $t \in (\mathbb{Z}^+)^d$ ,  $t$ -block matrix any matrix of size  $\prod_{i=0}^{d-1} t_i$ .

A  $d$ -block matrix  $M$  can be indexed by the virtue of some dimensions nesting  $\Omega$  according to

$$M \left[ i_0 q_0^\Omega + i_1 q_1^\Omega + \dots + i_{d-1} q_{d-1}^\Omega \right] \left[ j_0 q_0^\Omega + j_1 q_1^\Omega + \dots + j_{d-1} q_{d-1}^\Omega \right], \quad 0 \leq i_l \leq \gamma_{\Omega(l)}, 0 \leq j_l \leq \gamma_{\Omega(l)}.$$

**Definition 3.** A matrix admits  $(\gamma, u, \Omega)$ -toeplitz character when it is  $\gamma$ -block matrix and realize the following

$$M \left[ i_0 q_0^\Omega + i_1 q_1^\Omega + \dots + i_{d-1} q_{d-1}^\Omega \right] \left[ j_0 q_0^\Omega + j_1 q_1^\Omega + \dots + j_{d-1} q_{d-1}^\Omega \right] =$$

$$M \left[ i_0 q_0^\Omega + i_1 q_1^\Omega + (i_u - j_u) q_u^\Omega \dots + i_{d-1} q_{d-1}^\Omega \right] \left[ j_0 q_0^\Omega + j_1 q_1^\Omega + 0 \cdot q_u^\Omega + \dots + j_{d-1} q_{d-1}^\Omega \right] \quad \text{if } i_u - j_u \geq 0$$

$$M \left[ i_0 q_0^\Omega + i_1 q_1^\Omega + (0) q_u^\Omega \dots + i_{d-1} q_{d-1}^\Omega \right] \left[ j_0 q_0^\Omega + j_1 q_1^\Omega + (j_u - i_u) q_u^\Omega + \dots + j_{d-1} q_{d-1}^\Omega \right] \quad \text{if } i_u - j_u < 0$$

**Definition 4.** In the case that an  $\gamma$ -block matrix contains more than one toeplitz character, such that it is  $(\gamma, u_0, \Omega), (\gamma, u_1, \Omega) \dots (\gamma, u_{s-1}, \Omega)$ , toeplitz character, we will call it an  $(\gamma, u, \Omega)$ - $s$ -toeplitz character matrix, with  $u$  equal in that case to  $[u_0, u_1, \dots, u_{s-1}]$ .

According to **Definition 3** we can index the matrix elements using the notation

$$M \left[ i_0 q_0^\Omega + i_1 q_1^\Omega + \dots + i_{u-1} q_{u-1}^\Omega \right] \left[ j_0 q_0^\Omega + j_1 q_1^\Omega + \dots + j_{u-1} q_{u-1}^\Omega \right] (k_u) \left[ i_{u+1} q_{u+1}^\Omega + \dots + i_{d-1} q_{d-1}^\Omega \right] \left[ j_{u+1} q_{u+1}^\Omega + \dots + j_{d-1} q_{d-1}^\Omega \right]$$

To simplify the notations we will write it as

$$M^\Omega \left[ b_0^{u-1} \right] (d_u^{u+s}) \left[ b_{u+s+1}^{d-1} \right] \quad b_h^{h+y-1} \in (\mathbb{Z}^*)^{2y}, d_h^{h+y-1} \in \mathbb{Z}^y,$$

where for  $s > 0$  the matrix is  $s$ -toeplitz characters.

**Proposition 1.**  $\forall \Omega'$ , any  $(\gamma, u, \Omega)$  toeplitz character matrix can be transformed into  $(\gamma, u', \Omega')$  toeplitz character using the walking transformation  $W_{\Omega \rightarrow \Omega'}(M)$ , and we have  $u' = \Omega'^{-1}(\Omega_u)$ .

**Theorem 1.** For any full rank matrix  $(\gamma, u, \Omega)$ - $s$ -toeplitz character matrix,

$$M^\Omega \left[ b_0^{u_0-1} \right] (t_{u_0}, t_{u_1}, \dots, t_{u_{s-1}}), \quad \text{with } t_{u_i} \in [-\infty, \infty], \text{ the inverse matrix } iM \text{ is equal to}$$



$$\left[ \frac{dw}{(2\pi)^d} CC^H \right]_{i,j} = \frac{1}{(2\pi)^d} \oint_w e^{jw(i-j)} dw = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

From which, we got  $\frac{dw}{(2\pi)^d} CC^H = \mathbf{1}$ .

In the case that  $s < d$ , then we will note  $h := q_{d-s}^\Omega$  as the non-toeplitz part matrix size. As a counter part to the multidimensional Fourier transform vector in the previous definition we will define the block multidimensional Fourier transform vector as

$$F^{w_i, t} := \begin{bmatrix} \overbrace{0 \dots 0}^t & e^{-jw_i(k_\infty)^T} & \overbrace{0 \dots 0}^{h-t-1} & \dots & \overbrace{0 \dots 0}^t & e^{-jw_i(k_\infty)^T} & \overbrace{0 \dots 0}^{h-t-1} \end{bmatrix}^T,$$

$0 \leq t \leq h-1$ . By considering the Block diagonal matrix  $\lambda_s$ , which is by

definition do contain the matrix value  $\left[ \sum_{k_u} M^\Omega(t_{u_0}, t_{u_1}, \dots, t_{u_{s-1}}) e^{-jw_i(t_u)^T} \right]$ , at the

diagonal index  $w_i$ , it is simple to prove that

$$\lambda_s = \frac{dw}{(2\pi)^s} C_s (M^\Omega) C_s^H$$

where  $C_s = [F^{w_\infty, 0} \quad F^{w_\infty, 1} \quad \dots \quad F^{w_\infty, h} \quad \dots \quad F^{w_\infty, h-1} \quad F^{w_\infty, h}]$ . Next, by observing that  $C_s$  is full rank, we prove that  $\lambda_s$  is full rank also, and that proves the theorem.

### III Multidimensional Extension Approximation

Our approximation that we are proposing in this depends strongly on the previous work done in the treatment of the generalized reflection coefficients [8], and [14], to better explain this extension we start with the 1 dimensional case. For a positive definite toeplitz matrix  $M^N(l)$ ,  $l := -N+1, \dots, N-1$ , we define the polynomial  $p^{N-1}, q^{N-1}$ , as the solution for the linear system  $M^N(l)p^{N-1} = e_0$ ,  $M^N(l)q^{N-1} = e_{N-1}$ . These solutions in fact can be obtained recursively according to

$$p'^N = \begin{bmatrix} p'^{N-1} \\ 0 \end{bmatrix} + \sigma_N \begin{bmatrix} 0 \\ q'^{N-1} \end{bmatrix}.$$

Where  $v'$  is  $v$  normalized by the first value for  $p^N$ , and the last for  $q^N$ ,  $\sigma$  is the famous reflection coefficient. Putting the reflection coefficient to zero is equivalent to the existence of a positive definite matrix  $M^{N+1}(l)$ , which coincides with  $M^N(l)$ , so that  $M^{N+1}(l) = M^N(l), l = -N+1, \dots, N-1$ . By repeating the procedure until infinity. We obtain that

$$\left[ M^\infty(l) \right]^{-1} = \rho^{-1} P P^H \quad (1)$$



PROOF. Starting from  $x = 1$ ,  $R_0^{-1} := \left[ R \left( c \left( \left[ \gamma_0, \gamma_2, \dots, \gamma_{d-1} \right] \right) \right) \right]^{-1}$ , relation (3)

forwards the block-vector  $p_x$  with columns of size  $q_{d-x}^{(0,1,\dots,x-1,d-1,d-2,\dots,d-x+1,d-x)}$ , which is next applied according to the form (1). By accepting the proposed approximation, we assume that form (1) represents the inverse of ND-Toeplitz matrix  $R \left( c \left( \left[ \gamma_0, \gamma_2, \dots, \gamma_{d-x-1}, \gamma_{d-1}^\infty, \gamma_{d-2}^\infty, \dots, \gamma_{d-x}^\infty \right] \right) \right)$ , with infinite dimension according to the dimension  $d - x$ , and - by construction- coincide with the original matrix in its finite part. From that, and by **Theorem 1**, the inverse can be written according to the form (2). By Advancing  $x$ , and by **Proposition 1**, relation (4) permits us to continue the extension on the next dimension. Hence our proof is completed.

#### IV Sequential Multidimensional Spectral Estimation Algorithm

**Theorem 2** presents an analytic representation of the spectral estimation and that is by taking advantage of both the **Theorem 1** and the proposed approximation. From numerical point of view, we need more finite simple Algorithm. On the contrary of the 1 dimensional case, in the general case, there is no known finite time representation for the proposed estimation, on the other hand even in the 1 dimensional case the numerical representation of the estimation is obtained always as

$$a \text{ Fourier series of the prediction polynomial i.e. } C(j\omega) = \frac{\rho}{\left| \sum_{k=0}^{\gamma-1} p_k^\gamma e^{j\omega k} \right|^2}$$

And  $w$  is taken to a large finite number of points on the spectral circle, that is in an effort to approximate the continuous field of values.

By examining relation (2), and replacing by (3) starting from  $x = d$ , and descending in the reverse order of **Theorem 2**, we find that the spectral inverse of correlation signal  $c \left( \left[ \gamma_{d-1}^\infty, \gamma_{d-2}^\infty, \dots, \gamma_0^\infty \right] \right)$  can be obtained by a finite series of block exponential sum, with a diminishing Block size at each step. According to the previous discussion, the algorithm takes the following form:

1. Calculate the inverse of the correlation matrix,

$$R_0^{-1} := \left[ R^{(0,1,\dots,d-1)}(l_0, l_1, \dots, l_{d-1}) \right]^{-1},$$

set  $x = 1$

set

$$G(k) := R_0^{-1} \left[ i_0 q_0^\Omega + i_1 q_1^\Omega + \dots + i_{d-2} q_{d-2}^\Omega + k q_{d-1}^\Omega \right] \left[ j_0 q_0^\Omega + j_1 q_1^\Omega + \dots + j_{d-2} q_{d-2}^\Omega + 0 q_{d-1}^\Omega \right]$$

2. Apply relation (1)  $M(w_{d-1}, \dots, w_{d-x}) = \sum_{k=0}^{\gamma_x-1} G(k)(w_{d-1}, \dots, w_{d-x+1}) e^{jk w_{d-x}}$

3. Set  $G'(w_{d-1}, \dots, w_{d-x}) = M(w_{d-1}, \dots, w_{d-x}) \left[ G(0) \right]^{-1} M^H(w_{d-1}, \dots, w_{d-x})$

4. if  $(x = d)$  {



```

    stop
  }
  else {
    set
    
$$G(k)(w_{d-1}, \dots, w_{d-x}) := G' \left[ i_0 q_0^\Omega + \dots + k q_{d-x-1}^\Omega \right] \left[ j_0 q_0^\Omega + \dots + 0 q_{d-x-1}^\Omega \right] (w_{d-1}, \dots, w_{d-x})$$

  }

  set  $x = x + 1$ 
  go to 2
}

```

## V Numerical calculation cost

In this part we will try to shed the light on the numerical efficiency of the algorithm. For that, we will measure the number of operations needed to obtain the spectral estimation over an ND spectral grid with size  $C_w := [C_{w_{d-1}}, C_{w_{d-2}}, \dots, C_{w_0}]$ , where  $C_{w_i}$  is the number of points in the  $i$  dimension. In the following discussion we will omit the *Step 1* from our Calculus Cost, since that  $R_0^{-1}$ , and more precisely  $G(k), x=1$  can be obtained in different ways, like in [4], or [5], the thing that is irrelevant to the estimation method.

From the algorithm's *step 2* and *Step 3* we can write the number of operations needed at each  $t := d - x + 1$  as

$$\left[ \frac{3}{2}(q_{t-1})^3 + (q_{t-1}q_t) \right] \prod_{f=0}^{d-t} C_{w_{d-f-1}},$$

The total sum is equal to the sum from  $t = d$ , and down until  $t = 1$ , which can be expressed as

$$\sum_{t=1}^d \left( \left[ \frac{3}{2}(q_{t-1})^3 + (q_{t-1}q_t) \right] \prod_{f=0}^{d-t} C_{w_{d-f-1}} \right)$$

Figure 1 graphs the above relation in function of  $C_{w_0} = C_{w_1} = \dots = C_{w_{d-1}}$ , with values taken as  $\gamma_0 = \gamma_1 \dots = \gamma_{d-1} = 10$ , and  $d = 5$ . Also for the same values Figure 2 represents the number of operations needed by Capon Estimation, which is equal to

$(q_d)^2 \prod_{f=0}^{d-1} C_{w_f}$ . We see clearly a very good advantage using the proposed algorithm,

from a calculus coast point of view.

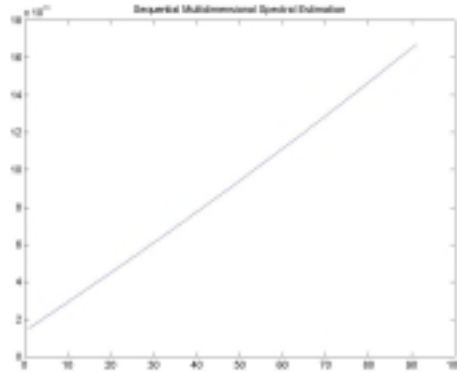


Figure 1

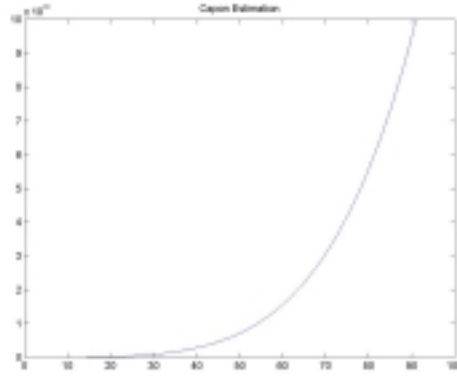


Figure 2

## VI Numerical simulations and discussion

In an effort to evaluate the proposed estimation we will present a three-dimensional test signal  $c(w_0, w_1, w_2)$ , defined in its spectral composition as zero everywhere except for  $c(0.1, 0.3, 0.7) = 1$ ,  $c(0.1, 0.6, 0.2) = 1$ ,  $c(0.6, *, *) = 1$ . Next, the signal was plunged in a white noise of variance 0.1. In our simulation we took our estimation on a grid of equal size in all dimensions,  $C_{w_i} = 10$ , and the same for the correlation matrix order, having a uniform value of  $\gamma_i = 3$ . Figure 3-Figure 22 show a series cuts in the cube for the simulated signal, with the noised spectrum on the left and the calculated estimation on the right.

In Figure 23, the graph representing the correlation matching accuracy of the proposed estimation for a two dimensional signal was also presented. The value present in the graph is the relative difference between the original correlation values

$r_{k,l}$ , and the estimated ones  $\hat{r}_{k,l}$ , that is  $\frac{|\hat{r}_{k,l} - r_{k,l}|}{r_{k,l}}$ .

From The presented Figures, and from other observed numerical simulations we found that the proposed algorithm provide a good estimation of the simulated signal, and contrary to the quarter-hyper plan filters proposed for the 2D case by [3][2], it enjoys a high degree of stability when augmenting in the Correlation matrix order  $\gamma_i$ .

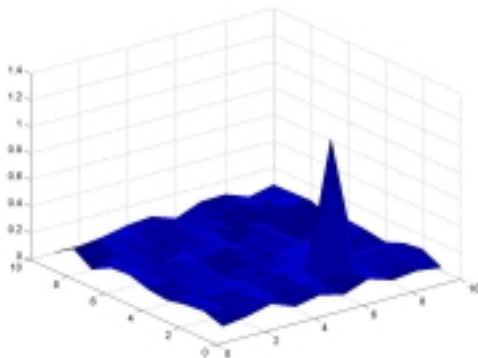


Figure 3

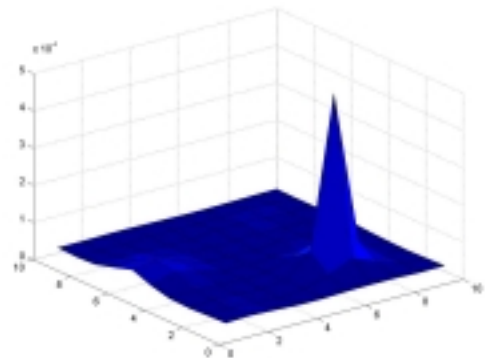


Figure 4

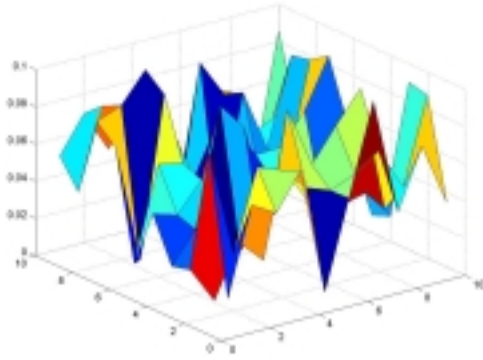


Figure 5

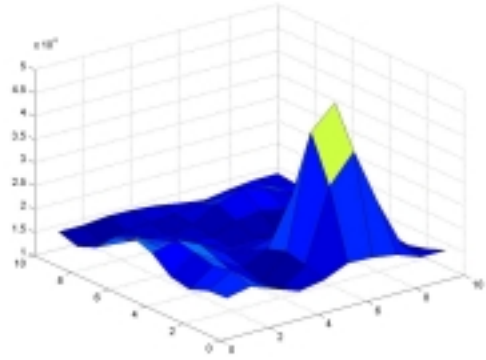


Figure 6

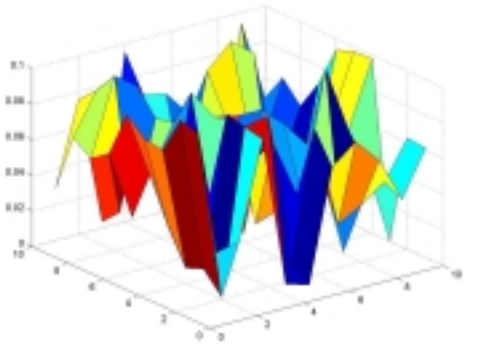


Figure 7

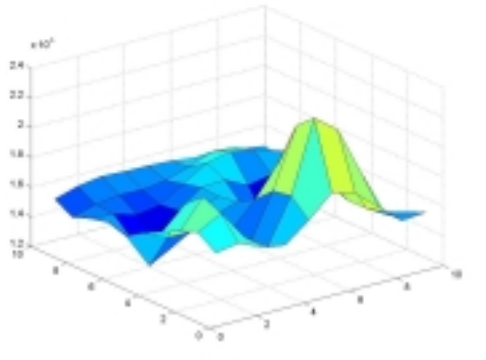


Figure 8

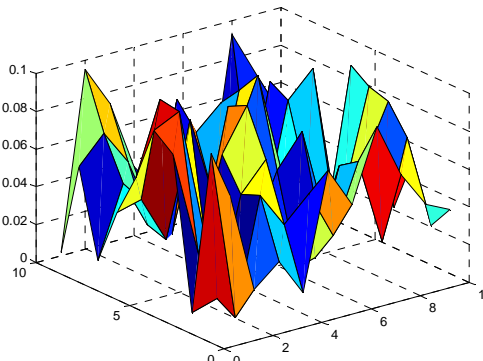


Figure 9

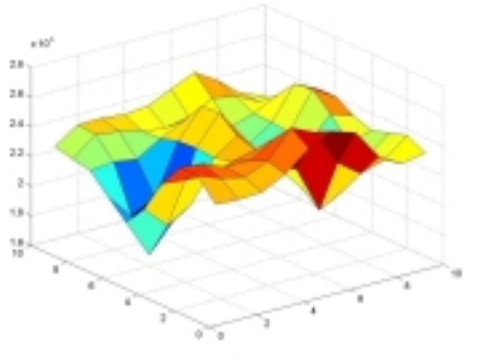


Figure 10

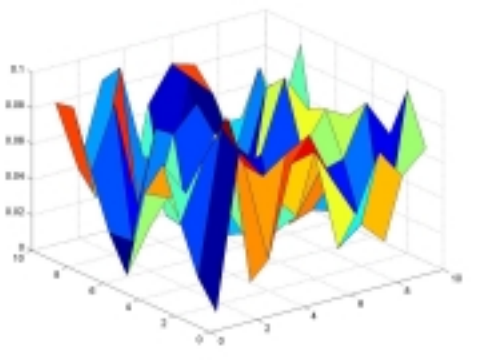


Figure 11

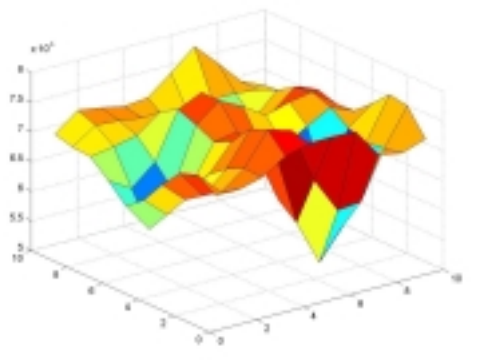


Figure 12

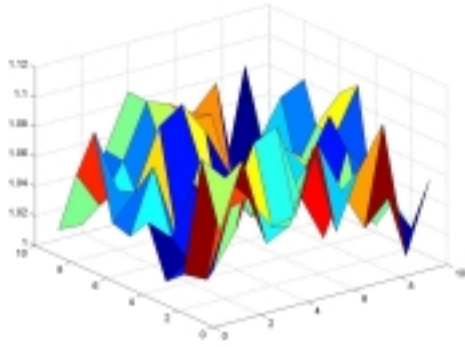


Figure 13

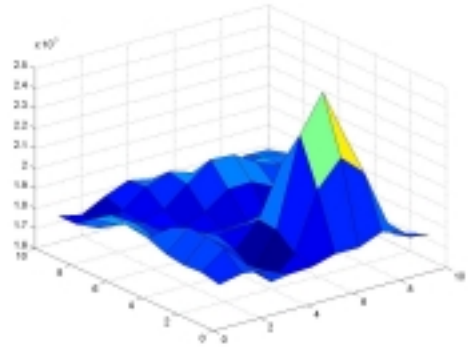


Figure 14

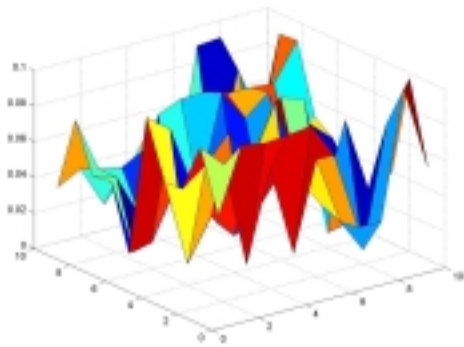


Figure 15

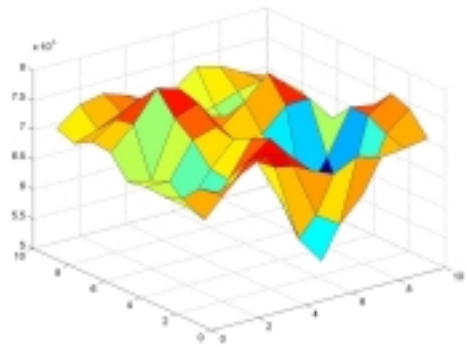


Figure 16

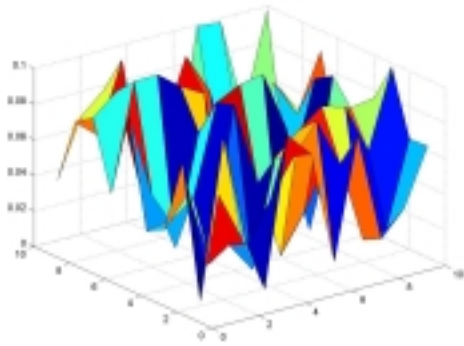


Figure 17

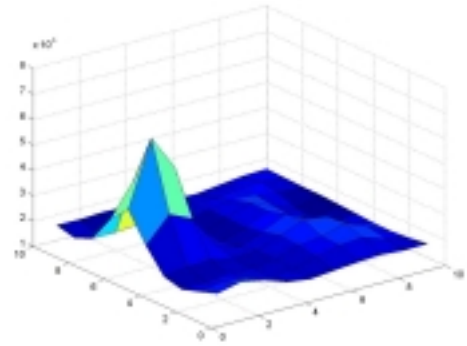


Figure 18

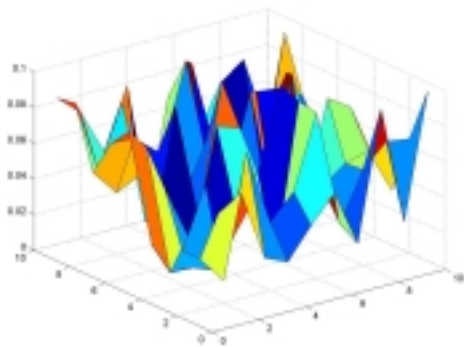


Figure 19

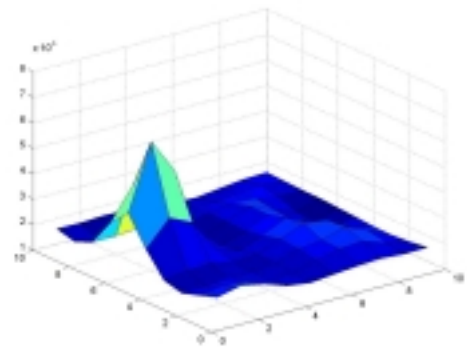


Figure 20

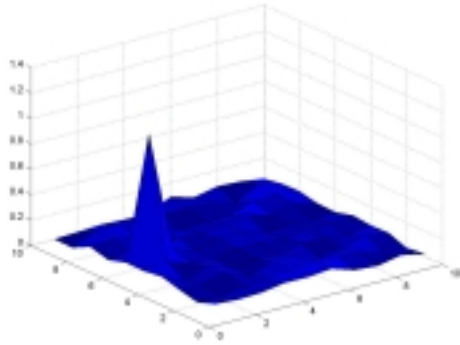


Figure 21

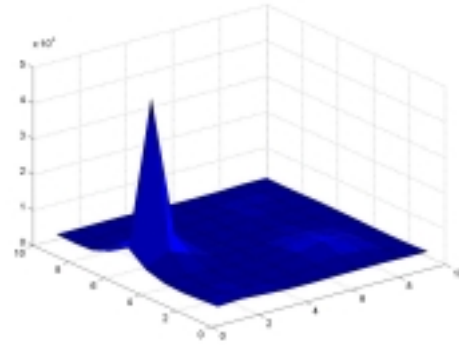


Figure 22

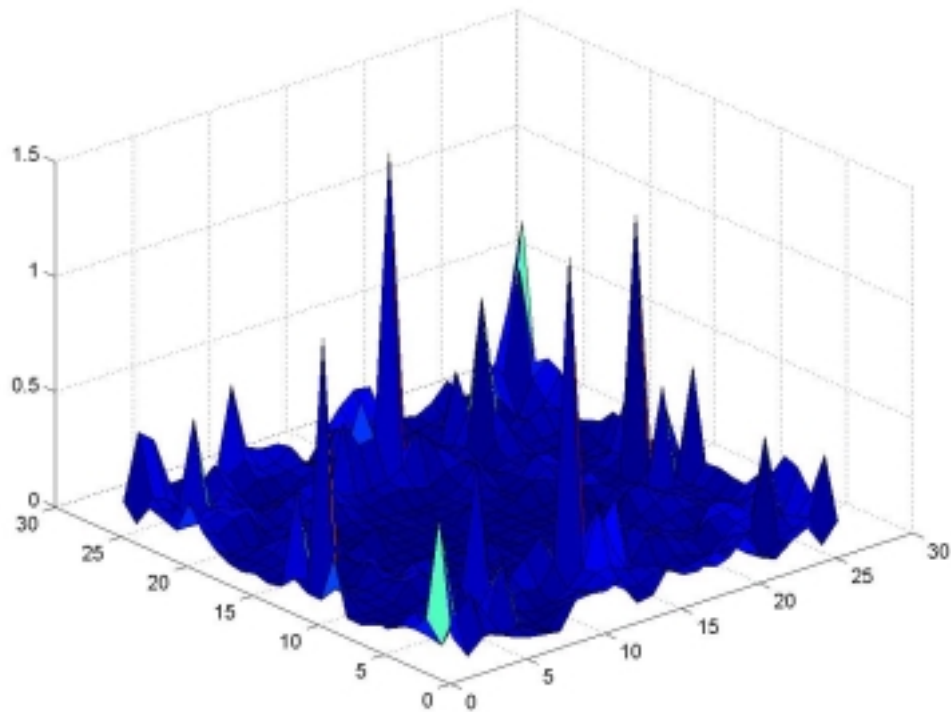


Figure 23

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