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CRITICAL POTENTIALS OF THE EIGENVALUES AND EIGENVALUE GAPS OF SCHRÖDINGER OPERATORS

AHMAD EL SOUFI AND NAZIH MOUKADEM

ABSTRACT. Let M be a compact Riemannian manifold with or without boundary, and let $-\Delta$ be its Laplace-Beltrami operator. For any bounded scalar potential q , we denote by $\lambda_i(q)$ the i -th eigenvalue of the Schrödinger type operator $-\Delta + q$ acting on functions with Dirichlet or Neumann boundary conditions in case $\partial M \neq \emptyset$. We investigate critical potentials of the eigenvalues λ_i and the eigenvalue gaps $G_{ij} = \lambda_j - \lambda_i$ considered as functionals on the set of bounded potentials having a given mean value on M . We give necessary and sufficient conditions for a potential q to be critical or to be a local minimizer or a local maximizer of these functionals. For instance, we prove that a potential $q \in L^\infty(M)$ is critical for the functional λ_2 if and only if, q is smooth, $\lambda_2(q) = \lambda_3(q)$ and there exist second eigenfunctions f_1, \dots, f_k of $-\Delta + q$ such that $\sum_j f_j^2 = 1$. In particular, λ_2 (as well as any λ_i) admits no critical potentials under Dirichlet Boundary conditions. Moreover, the functional λ_2 never admits locally minimizing potentials.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Let M be a compact connected Riemannian manifold of dimension d , possibly with nonempty boundary ∂M , and let $-\Delta$ be its Laplace-Beltrami operator acting on functions with, in the case where $\partial M \neq \emptyset$, Dirichlet or Neumann boundary conditions. In all the sequel, as soon as the Neumann Laplacian will be considered, the boundary of M will be assumed to be sufficiently regular (e.g. C^1 , but weaker regularity assumptions may suffice, see [3]) in order to guarantee the compactness of the embedding $H^1(M) \hookrightarrow L^2(M)$ and, hence, the compactness of the resolvent of the Neumann Laplacian (note that it is well known, using standard arguments like in [14, p.89], that compactness results for Sobolev spaces on Euclidean domains remain valid in the Riemannian setting).

For any bounded real valued potential q on M , the Schrödinger type operator $-\Delta + q$ has compact resolvent (see [16, Theorem IV.3.17] and observe that a bounded q leads to a relatively compact operator with respect to $-\Delta$). Therefore, its spectrum consists of a nondecreasing and unbounded sequence of eigenvalues with finite multiplicities:

$$\text{Spec}(-\Delta + q) = \{\lambda_1(q) < \lambda_2(q) \leq \lambda_3(q) \leq \dots \leq \lambda_i(q) \leq \dots\}.$$

Each eigenvalue $\lambda_i(q)$ can be considered as a (continuous) function of the potential $q \in L^\infty(M)$ and there are both physical and mathematical motivations to study existence and properties of extremal potentials of the

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functionals λ_i as well as of the differences, called gaps, between them. A very rich literature is devoted to the existence and the determination of maximizing or minimizing potentials for the eigenvalues (especially the fundamental one, λ_1) and the eigenvalue gaps (especially the first one, $\lambda_2 - \lambda_1$) under various constraints often motivated by physical considerations (see, for instance, [1, 2, 4, 6, 7, 10, 11, 12, 13, 17, 19] and the references therein). Note that, since the function λ_i commutes with constant translations, that is, $\lambda_i(q + c) = \lambda_i(q) + c$, such constraints are necessary.

Our aim in this paper is to investigate critical points, including "local minimizers" and "local maximizers", of the eigenvalue functionals $q \rightarrow \lambda_i(q)$ and the eigenvalue gap functionals $q \rightarrow \lambda_j(q) - \lambda_i(q)$, the potentials q being subjected to the constraint that their mean value (or, equivalently, their integral) over M is fixed. All along this paper, the mean value of an integrable function q will be denoted \bar{q} , that is,

$$\bar{q} = \frac{1}{V(M)} \int_M q \, dv,$$

$V(M)$ and dv being respectively the Riemannian volume and the Riemannian volume element of M .

Actually, most of the results below can be extended, modulo some slight changes, to the case where this constraint is replaced by the more general one

$$\int_M F(q) \, dv = \text{constant},$$

where $F : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that $F'(x) \neq 0$ if $x \neq 0$, like $F(x) = |x|^\alpha$ or $F(x) = x|x|^{\alpha-1}$ with $\alpha \geq 1$. However, for simplicity and clarity reasons, we preferred to focus only on the mean value constraint. Therefore, we fix a constant $c \in \mathbb{R}$ and consider the functionals

$$\lambda_i : q \in L_c^\infty(M) \mapsto \lambda_i(q) \in \mathbb{R},$$

where $L_c^\infty(M) = \{q \in L^\infty(M) \mid \bar{q} = c\}$. The tangent space to $L_c^\infty(M)$ at any point q is given by

$$L_*^\infty(M) := \left\{ u \in L^\infty(M) \mid \int_M u \, dv = 0 \right\}.$$

1.1. Critical potentials of the eigenvalue functionals.

Since it is always nondegenerate, the first eigenvalue gives rise to a differentiable functional in the sense that, for any $q \in L_c^\infty(M)$ and any $u \in L_*^\infty(M)$, the function $t \mapsto \lambda_1(q + tu)$ is differentiable in t . A potential $q \in L_c^\infty(M)$ will be termed *critical* for this functional if $\left. \frac{d}{dt} \lambda_1(q + tu) \right|_{t=0} = 0$ for any $u \in L_*^\infty(M)$.

In the case of empty boundary or of Neumann boundary conditions, the constant function 1 belongs to the domain of the operator $-\Delta + q$ and one obtains, as a consequence of the min-max principle, that the constant potential c is a global maximizer of λ_1 over $L_c^\infty(M)$ (see also [6] and [13]). Constant potential c is actually the only critical one for λ_1 . On the other hand, under Dirichlet boundary conditions, the functional λ_1 admits no critical potentials in $L_c^\infty(M)$. Indeed, we have the following

Theorem 1.1. (1) Assume that either $\partial M = \emptyset$ or $\partial M \neq \emptyset$ and Neumann boundary conditions are imposed. Then, for any potential q in $L_c^\infty(M)$, we have

$$\lambda_1(q) \leq \lambda_1(c) = c,$$

where the equality holds if and only if $q = c$. Moreover, the constant potential c is the only critical one of the functional λ_1 over $L_c^\infty(M)$.

(2) Assume that $\partial M \neq \emptyset$ and that Zero Dirichlet boundary conditions are imposed. Then the functional λ_1 does not admit any critical potential in $L_c^\infty(M)$.

Higher eigenvalues are continuous but not differentiable in general. Nevertheless, perturbation theory enables us to prove that, for any function $u \in L^\infty(M)$, the function $t \mapsto \lambda_i(q + tu)$ admits left and right derivatives at $t = 0$ (see section 2.2). A generalized notion of criticality can be naturally defined as follows :

Definition 1.1. A potential q is said to be critical for the functional λ_i if, for any $u \in L_*^\infty(M)$, the left and right derivatives of $t \mapsto \lambda_i(q + tu)$ at $t = 0$ have opposite signs, that is

$$\left. \frac{d}{dt} \lambda_i(q + tu) \right|_{t=0^+} \times \left. \frac{d}{dt} \lambda_i(q + tu) \right|_{t=0^-} \leq 0.$$

It is immediate to check that q is critical for λ_i if and only if, for any $u \in L_*^\infty(M)$, one of the two following inequalities holds :

$$\lambda_i(q + tu) \leq \lambda_i(q) + o(t) \quad \text{as } t \rightarrow 0$$

or

$$\lambda_i(q + tu) \geq \lambda_i(q) + o(t) \quad \text{as } t \rightarrow 0.$$

In all the sequel, we will denote by $E_i(q)$ the eigenspace corresponding to the i -th eigenvalue $\lambda_i(q)$ whose dimension coincides with the number of indices $j \in \mathbb{N}$ such that $\lambda_j(q) = \lambda_i(q)$.

As for the first eigenvalue, the functionals λ_i , $i \geq 2$, admit no critical potentials under Dirichlet boundary conditions.

Theorem 1.2. Assume that $\partial M \neq \emptyset$ and that Zero Dirichlet boundary conditions are imposed. Then, $\forall i \in \mathbb{N}^*$, the functional λ_i does not admit any critical potential in $L_c^\infty(M)$.

Under the two remaining boundary conditions, the following theorem gives a necessary condition for a potential q to be critical for the functional λ_i . This condition is also sufficient for the indices i such that $\lambda_i(q) > \lambda_{i-1}(q)$ or $\lambda_i(q) < \lambda_{i+1}(q)$, which means that $\lambda_i(q)$ is the first one or the last one in a cluster of equal eigenvalues.

Theorem 1.3. Assume that either $\partial M = \emptyset$ or $\partial M \neq \emptyset$ and Neumann boundary conditions are imposed. Let i be a positive integer.

If $q \in L_c^\infty(M)$ is a critical potential of the functional λ_i , then q is smooth and there exists a finite family of eigenfunctions f_1, \dots, f_k in $E_i(q)$ such that $\sum_{1 \leq j \leq k} f_j^2 = 1$.

Reciprocally, if $\lambda_i(q) > \lambda_{i-1}(q)$ or $\lambda_i(q) < \lambda_{i+1}(q)$, and if there exists a family of eigenfunctions $f_1, \dots, f_k \in E_i(q)$ such that $\sum_{1 \leq j \leq k} f_j^2 = 1$, then q is a critical potential of the functional λ_i .

Note that the identity $\sum_{1 \leq j \leq k} f_j^2 = 1$, with $f_1, \dots, f_k \in E_i(q)$, immediately implies another one (that we obtain from $\Delta \sum_{1 \leq j \leq k} f_j^2 = 0$):

$$q = \lambda_i(q) - \sum_{1 \leq j \leq k} |\nabla f_j|^2,$$

from which we can deduce the smoothness of q .

Remark 1.1. 1. The identity $\sum_{1 \leq j \leq k} f_j^2 = 1$ with $-\Delta f_j + q f_j = \lambda_i(q) f_j$, means that the map $f = (f_1, \dots, f_k)$ from M to the Euclidean sphere \mathbb{S}^{k-1} is harmonic with energy density $|\nabla f|^2 = \lambda_i(q) - q$ (see [5]). Hence, a necessary (and sometime sufficient) condition for a potential q to be critical for the functional λ_i is that the function $\lambda_i(q) - q$ is the energy density of a harmonic map from M to a Euclidean sphere.

2. If one replaces the constraint on the mean value $\frac{1}{V(M)} \int_M q dv = c$ by the general constraint $\int_M F(q) dv = c$, then the necessary and sufficient condition $\sum_{1 \leq j \leq k} f_j^2 = 1$ of Theorem 1.3 becomes (even under Dirichlet boundary conditions) $\sum_{1 \leq j \leq k} f_j^2 = F'(q)$. In particular, q is a critical potential of the functional λ_1 if and only if $F'(q) \geq 0$ and $F'(q)^{\frac{1}{2}}$ is a first eigenfunction of $-\Delta + q$, see [1, 12] for a discussion of the case $F(q) = |q|^\alpha$.

Under each one of the boundary conditions we consider, a constant function can never be an eigenfunction associated to an eigenvalue $\lambda_i(q)$ with $i \geq 2$. Hence, an immediate consequence of Theorem 1.3 is the following

Corollary 1.1. *If $q \in L_c^\infty(M)$ is a critical potential of the functional λ_i with $i \geq 2$, then the eigenvalue $\lambda_i(q)$ is degenerate, that is $\lambda_i(q) = \lambda_{i-1}(q)$ or $\lambda_i(q) = \lambda_{i+1}(q)$*

If $\{f_1, \dots, f_k\}$ is an L^2 -orthonormal basis of $E_i(-\Delta)$, then the function $\sum_{1 \leq j \leq k} f_j^2$ is invariant under the isometry group of M . Indeed, for any isometry ρ of M , $\{f_1 \circ \rho, \dots, f_k \circ \rho\}$ is also an orthonormal basis of $E_i(-\Delta)$ and then, there exists a matrix $A \in O(d)$ such that $(f_1 \circ \rho, \dots, f_d \circ \rho) = A.(f_1, \dots, f_d)$. In particular, if M is homogeneous, that is, the isometry group acts transitively on M , then $\sum_{1 \leq j \leq k} f_j^2$ would be constant. Another consequence of Theorem 1.3 is then the following

Corollary 1.2. *If M is homogeneous, then constant potentials are critical for all the functionals λ_i such that $\lambda_i(-\Delta) < \lambda_{i+1}(-\Delta)$ or $\lambda_i(-\Delta) > \lambda_{i-1}(-\Delta)$.*

Recall that Euclidean spheres, projective spaces and flat tori are examples of homogeneous Riemannian spaces.

A potential $q \in L_c^\infty(M)$ is said to be a *local minimizer* (resp. *local maximizer*) of the functional λ_i (in a weak sense) if, for any $u \in L_*^\infty(M)$, the function $t \mapsto \lambda_i(q + tu)$ admits a local minimum (resp. maximum) at $t = 0$. The result of Corollary 1.1 takes the following more precise form in the case of a local minimizer or maximizer.

Theorem 1.4. *Let $q \in L_c^\infty(M)$ and $i \geq 2$.*

- (1) *If q is a local minimizer of the functional λ_i , then $\lambda_i(q) = \lambda_{i-1}(q)$.*
- (2) *If q is a local maximizer of the functional λ_i , then $\lambda_i(q) = \lambda_{i+1}(q)$.*

Since the first eigenvalue is simple, we always have $\lambda_2(q) > \lambda_1(q)$. The previous results, applied to the functional λ_2 can be summarized as follows.

Corollary 1.3. *Assume that either $\partial M = \emptyset$ or $\partial M \neq \emptyset$ and Neumann boundary conditions are imposed. A potential $q \in L_c^\infty(M)$ is critical for the functional λ_2 if and only if, q is smooth, $\lambda_2(q) = \lambda_3(q)$ and there exist eigenfunctions f_1, \dots, f_k in $E_2(q)$ such that $\sum_{1 \leq j \leq k} f_j^2 = 1$.*

Moreover, the functional λ_2 admits no local minimizers in $L_c^\infty(M)$.

In [6], Ilias and the first author have proved that, under some hypotheses on M , satisfied in particular by compact rank-one symmetric spaces, irreducible homogeneous Riemannian spaces and some flat tori, the constant potential c is a global maximizer of λ_2 over $L_c^\infty(M)$. In [8, 9], they studied the critical points of λ_i considered as a functional on the set of Riemannian metrics of fixed volume on M .

1.2. Critical potentials of the eigenvalue gaps functionals.

We consider now the eigenvalue gaps functionals $q \mapsto G_{ij}(q) = \lambda_j(q) - \lambda_i(q)$, where i and j are two distinct positive integers, and define their critical potentials as in Definition 1.1. These functionals are invariant under translations, that is $G_{ij}(q+c) = G_{ij}(q)$. Therefore, critical potentials of G_{ij} with respect to fixed mean value deformations are also critical with respect to arbitrary deformations.

Theorem 1.5. *If $q \in L_c^\infty(M)$ is a critical potential of the gap functional $G_{ij} = \lambda_j - \lambda_i$, then there exist a finite family of eigenfunctions f_1, \dots, f_k in $E_i(q)$ and a finite family of eigenfunctions g_1, \dots, g_l in $E_j(q)$, such that $\sum_{1 \leq p \leq k} f_p^2 = \sum_{1 \leq p \leq l} g_p^2$.*

Reciprocally, if $\lambda_i(q) < \lambda_{i+1}(q)$ and $\lambda_j(q) > \lambda_{j-1}(q)$, and if there exist f_1, \dots, f_k in $E_i(q)$ and g_1, \dots, g_l in $E_j(q)$ such that $\sum_{1 \leq p \leq k} f_p^2 = \sum_{1 \leq p \leq l} g_p^2$, then q is a critical potential of G_{ij} .

In the particular case of the gap between two consecutive eigenvalues, we have the following

Corollary 1.4. *A potential $q \in L_c^\infty(M)$ is critical for the gap functional $G_{i,i+1} = \lambda_{i+1} - \lambda_i$ if and only if, either $\lambda_{i+1}(q) = \lambda_i(q)$, or there exist a family of eigenfunctions f_1, \dots, f_k in $E_i(q)$ and a family of eigenfunctions g_1, \dots, g_l in $E_{i+1}(q)$, such that $\sum_{1 \leq p \leq k} f_p^2 = \sum_{1 \leq p \leq l} g_p^2$.*

Remark 1.2. *The characterization of critical potentials of G_{ij} given in Theorem 1.5 remains valid under the constraint $\int_M F(q)dv = c$.*

An immediate consequence of Theorem 1.5 is the following

Corollary 1.5. *Let $q \in L_c^\infty(M)$ be a critical potential of the gap functional $G_{ij} = \lambda_j - \lambda_i$. If $\lambda_i(q)$ (resp. $\lambda_j(q)$) is nondegenerate, then $\lambda_j(q)$ (resp. $\lambda_i(q)$) is degenerate.*

The following is an immediate consequence of the discussion above concerning homogeneous Riemannian manifolds.

Corollary 1.6. *If M is a homogeneous Riemannian manifold, then, for any positive integer i , constant potentials are critical points of the gap functional $G_{i,i+1} = \lambda_{i+1} - \lambda_i$.*

Potentials q such that $\lambda_{i+1}(q) = \lambda_i(q)$ are of course global minimizers of the gap functional $G_{i,i+1}$. These potentials are also the only local minimizers of $G_{i,i+1}$. Indeed, we have the following

Theorem 1.6. *If $q \in L_c^\infty(M)$ is a local minimizer of the gap functional $G_{ij} = \lambda_j - \lambda_i$, then, either $\lambda_i(q) = \lambda_{i+1}(q)$, or $\lambda_j(q) = \lambda_{j-1}(q)$. If q is a local maximizer of G_{ij} , then, either $\lambda_i(q) = \lambda_{i-1}(q)$, or $\lambda_j(q) = \lambda_{j+1}(q)$.*

In particular, q is a local minimizer of the gap functional $G_{i,i+1} = \lambda_{i+1} - \lambda_i$ if and only if $G_{i,i+1}(q) = 0$.

Finally, let us apply the results of this section to the first gap $G_{1,2}$.

Corollary 1.7. *A potential $q \in L_c^\infty(M)$ is critical for the gap functional $G_{1,2} = \lambda_2 - \lambda_1$ if and only if $\lambda_2(q)$ is degenerate and there exists a family of eigenfunctions g_1, \dots, g_l in $E_2(q)$ such that $\sum_{1 \leq j \leq l} g_j^2 = f^2$, where f is a basis of $E_1(q)$.*

The functional $G_{1,2}$ does not admit any local minimizer in $L_c^\infty(M)$.

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2. PROOF OF RESULTS

2.1. Variation Formula and proof of Theorem 1.1. Given on M a potential q and a function $u \in L^\infty(M)$, we consider the family of operators $-\Delta + q + tu$. Suppose that $\Lambda(t)$ is a differentiable family of eigenvalues of $-\Delta + q + tu$ and that f_t is a differentiable family of corresponding normalized eigenfunctions, that is, $\forall t$,

$$(-\Delta + q + tu)f_t = \Lambda(t)f_t,$$

and

$$\int_M f_t^2 dv = 1,$$

with $f_t|_{\partial M} = 0$ or $\frac{\partial f_t}{\partial \nu}|_{\partial M} = 0$ if $\partial M \neq \emptyset$. The following formula, giving the derivative of Λ , is already known at least in the case of Euclidean domains with Dirichlet boundary conditions.

Proposition 2.1.

$$\Lambda'(0) = \int_M u f_0^2 dv.$$

Proof. First, we have, for all t ,

$$\Lambda(t) = \Lambda(t) \int_M (f_t)^2 dv = \int_M f_t (-\Delta + q + tu) f_t dv.$$

Differentiating at $t = 0$, we get

$$\Lambda'(0) = \frac{d}{dt} \left(\int_M f_t (-\Delta + q) f_t dv + t \int_M u (f_t)^2 dv \right) \Big|_{t=0}.$$

Now, noticing that the function $\frac{d}{dt}f_t|_{t=0}$ satisfies the same boundary conditions as f_0 in case $\partial M \neq \emptyset$, and using integration by parts, we obtain

$$\begin{aligned} \frac{d}{dt} \int_M f_t(-\Delta + q)f_t dv \Big|_{t=0} &= 2 \int_M (-\Delta + q)f_0 \frac{d}{dt}f_t \Big|_{t=0} dv \\ &= 2\Lambda(0) \int_M f_0 \frac{d}{dt}f_t \Big|_{t=0} dv \\ &= \Lambda(0) \frac{d}{dt} \int_M f_t^2 dv \Big|_{t=0} = 0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \frac{d}{dt} \left(t \int_M u f_t^2 dv \right) \Big|_{t=0} &= \int_M u f_0^2 dv + \left(t \int_M u \frac{d}{dt} f_t^2 dv \right) \Big|_{t=0} \\ &= \int_M u f_0^2 v_g. \end{aligned}$$

Finally, $\Lambda'(0) = \int_M u f_0^2 dv$. \square

Proof. (of Theorem 1.1.) (i) First, let us show that constant potentials are maximizing for λ_1 . Indeed, let c be a constant potential and let q be an arbitrary one in $L_c^\infty(M)$. From the variational characterization of $\lambda_1(-\Delta + q)$ in the case $\partial M = \emptyset$ as well as in the case of Neumann boundary conditions, we get

$$\begin{aligned} \lambda_1(-\Delta + q) &= \inf_{f \in H^1(M)} \frac{\int_M (|\nabla f|^2 + qf^2) dv}{\|f\|_{L^2(M)}^2} \\ &\leq \frac{\int_M (|\nabla 1|^2 + q1^2) dv}{\|1\|_{L^2(M)}^2} = \frac{\int_M q dv}{V(M)} = c. \end{aligned}$$

Hence, $\lambda_1(q) \leq \lambda_1(c)$ and the constant potential c maximizes the functional λ_1 on $L_c^\infty(M)$. In particular, constant potentials are critical for this functional.

Now, suppose that $q \in L_c^\infty(M)$ is a critical potential for λ_1 . For any $u \in L_*^\infty(M)$, we consider a differentiable family f_t of normalized eigenfunctions corresponding to the first eigenvalue of $(-\Delta + q + tu)$ and apply the variation formula above to obtain:

$$\frac{d}{dt} \lambda_1(q + tu) \Big|_{t=0} = \int_M u f_0^2 dv.$$

Hence, $\int_M u f_0^2 dv = 0$ for any $u \in L_*^\infty(M)$, which implies that f_0 is constant on M . Since $(-\Delta + q)f_0 = qf_0 = \lambda_1(q)f_0$, the potential q must be constant on M .

(ii) Let f_0 be the first nonnegative Dirichlet eigenfunction of $-\Delta + q$ satisfying $\int_M f_0^2 dv = 1$. The function $u = V(M)f_0^2 - 1$ belongs to $L_*^\infty(M)$ and we have

$$\frac{d}{dt} \lambda_1(q + tu) \Big|_{t=0} = \int_M u f_0^2 dv = V(M) \int_M f_0^4 dv - 1 > 0,$$

where the last inequality comes from Cauchy-Schwarz inequality and the fact that f_0 is not constant (recall that $f_0|_{\partial M} = 0$). Therefore, the potential q is not critical for λ_1 . \square

2.2. Characterization of critical potentials. Let i be a positive integer and let $m \geq 1$ be the dimension of the eigenspace $E_i(q)$ associated to the eigenvalue $\lambda_i(q)$. For any function $u \in L_*^\infty(M)$, perturbation theory of unbounded self-adjoint operators (see for instance Kato's book [16]) that we apply to the one parameter family of operators $-\Delta + q + tu$, tells us that, there exists a family of m eigenfunctions $f_{1,t}, \dots, f_{m,t}$ associated with a family of m (non ordered) eigenvalues $\Lambda_1(t), \dots, \Lambda_m(t)$ of $-\Delta + q + tu$, all depending analytically in t in some interval $(-\varepsilon, \varepsilon)$, and satisfying

- $\Lambda_1(0) = \dots = \Lambda_m(0) = \lambda_i(q)$,
- $\forall t \in (-\varepsilon, \varepsilon)$, the m functions $f_{1,t}, \dots, f_{m,t}$ are orthonormal in $L^2(M)$.

From this, one can easily deduce the existence of two integers $k \leq m$ and $l \leq m$, and a small $\delta > 0$ such that

$$\lambda_i(q + tu) = \begin{cases} \Lambda_k(t) & \text{if } t \in (-\delta, 0) \\ \Lambda_l(t) & \text{if } t \in (0, \delta). \end{cases}$$

Hence, the function $t \mapsto \lambda_i(q + tu)$ admits a left sided and a right sided derivatives at $t = 0$ with

$$\left. \frac{d}{dt} \lambda_i(q + tu) \right|_{t=0^-} = \Lambda'_k(0) = \int_M u f_{k,0}^2 dv$$

and

$$\left. \frac{d}{dt} \lambda_i(q + tu) \right|_{t=0^+} = \Lambda'_l(0) = \int_M u f_{l,0}^2 dv.$$

To any function $u \in L_*^\infty(M)$ and any integer $i \in \mathbb{N}$, we associate the quadratic form Q_u^i on $E_i(q)$ defined by

$$Q_u^i(f) = \int_M u f^2 dv.$$

The corresponding symmetric linear transformation $L_u^i : E_i(q) \rightarrow E_i(q)$ is given by

$$L_u^i(f) = P_i(uf),$$

where $P_i : L^2(M) \rightarrow E_i(q)$ is the orthogonal projection of $L^2(M)$ onto $E_i(q)$.

It follows immediately that

Proposition 2.2. *If the potential q is critical for the functional λ_i , then, $\forall u \in L_*^\infty(M)$, the quadratic form $Q_u^i(f) = \int_M u f^2 dv$ is indefinite on the eigenspace $E_i(q)$.*

The following lemma enables us to establish a converse to this proposition.

Lemma 2.1. $\forall k, l \leq m$, we have

$$\int_M u f_{k,0} f_{l,0} dv = \begin{cases} 0 & \text{if } k \neq l \\ \Lambda'_k(0) & \text{if } k = l. \end{cases}$$

In other words, $\Lambda'_1(0), \dots, \Lambda'_m(0)$ are the eigenvalues of the symmetric linear transformation $L_u^i : E_i(q) \rightarrow E_i(q)$ and the functions $f_{1,0}, \dots, f_{m,0}$ constitute an orthonormal eigenbasis of L_u^i .

Proof. Differentiating at $t = 0$ the equality $(-\Delta + q + tu)f_{k,t} = \Lambda_k(t)f_{k,t}$, we obtain

$$uf_{k,0} + (-\Delta + q)\frac{d}{dt}f_{k,t}\Big|_{t=0} = \Lambda'_k(0)f_{k,0} + \Lambda_k(0)\frac{d}{dt}f_{k,t}\Big|_{t=0},$$

and then,

$$\begin{aligned} \int_M uf_{k,0}f_{l,0} dv &= \Lambda'_k(0) \int_M f_{k,0}f_{l,0} dv + \Lambda_k(0) \int_M f_{l,0}\frac{d}{dt}f_{k,t}\Big|_{t=0} dv \\ &\quad - \int_M f_{l,0}(-\Delta + q)\frac{d}{dt}f_{k,t}\Big|_{t=0} dv. \end{aligned}$$

Integration by parts gives, after noticing that $\Lambda_k(0) = \Lambda_l(0) = \lambda_i(q)$ and that the functions $\frac{d}{dt}f_{k,t}\Big|_{t=0}$ satisfy the considered boundary conditions,

$$\begin{aligned} \int_M f_{l,0}(-\Delta + q)\frac{d}{dt}f_{k,t}\Big|_{t=0} dv &= \int_M \frac{d}{dt}f_{k,t}\Big|_{t=0} (-\Delta + q)f_{l,0} dv \\ &= \Lambda_k(0) \int_M f_{l,0}\frac{d}{dt}f_{k,t}\Big|_{t=0} dv, \end{aligned}$$

and finally,

$$\int_M uf_{k,0}f_{l,0} dv = \Lambda'_k(0) \int_M f_{k,0}f_{l,0} dv = \Lambda'_k(0)\delta_{kl}.$$

□

Proposition 2.3. *Assume that $\lambda_i(q) > \lambda_{i-1}(q)$ or $\lambda_i(q) < \lambda_{i+1}(q)$. Then the following conditions are equivalent:*

- i) *the potential q is critical for λ_i*
- ii) *$\forall u \in L_*^\infty(M)$, the quadratic form $Q_u^i(f) = \int_M uf^2 dv$ is indefinite on the eigenspace $E_i(q)$.*
- iii) *$\forall u \in L_*^\infty(M)$, the linear transformation L_u^i admits eigenvalues of both signs.*

Proof. Conditions (ii) and (iii) are clearly equivalent and the fact that (i) implies (ii) was established in Proposition 2.2. Let us show that (iii) implies (i). Assume that $\lambda_i(q) > \lambda_{i-1}(q)$ and let $u \in L_*^\infty(M)$ and $\Lambda_1(t), \dots, \Lambda_m(t)$ be as above. For small t , we will have, for continuity reasons, $\forall k \leq m$, $\Lambda_k(t) > \lambda_{i-1}(q + tu)$ and then, $\lambda_i(q + tu) \leq \Lambda_k(t)$. Since $\lambda_i(q + tu) \in \{\Lambda_1(t), \dots, \Lambda_m(t)\}$, we get

$$\lambda_i(q + tu) = \min_{k \leq m} \Lambda_k(t).$$

It follows that

$$\frac{d}{dt}\lambda_i(q + tu)\Big|_{t=0^-} = \max_{k \leq m} \Lambda'_k(0)$$

and

$$\frac{d}{dt}\lambda_i(q + tu)\Big|_{t=0^+} = \min_{k \leq m} \Lambda'_k(0).$$

Thanks to Lemma 2.1, Condition (iii) implies that $\min_{k \leq m} \Lambda'_k(0) \leq 0 \leq \max_{k \leq m} \Lambda'_k(0)$ which implies the criticality of q .

The case $\lambda_i(q) < \lambda_{i+1}(q)$ can be treated in a similar manner. □

2.3. Proof of Theorems 1.2 and 1.3. Let q be a potential in $L_c^\infty(M)$. To prove Theorem 1.2 we first notice that, since $f|_{\partial M} = 0$ for any $f \in E_i(q)$, the constant function 1 does not belong to the vector space F generated in $L^2(M)$ by $\{f^2; f \in E_i(q)\}$. Hence, there exists a function u orthogonal to F and such that $\langle u, 1 \rangle_{L^2(M)} < 0$. The function $u_0 = u - \bar{u}$ belongs to $L_*^\infty(M)$ and the quadratic form $Q_{u_0}^i(f) = \int_M u_0 f^2 dv = -\bar{u} \|f\|_{L^2(M)}^2$ is positive definite on $E_i(q)$. Hence, the potential q is not critical for λ_i (see Proposition 2.2).

The proof of Theorem 1.3 follows directly from the two propositions above and the following lemma.

Lemma 2.2. *Let i be a positive integer. The two following conditions are equivalent:*

- i) $\forall u \in L_*^\infty(M)$, the quadratic form $Q_u^i(f) = \int_M u f^2 dv$ is indefinite on the eigenspace $E_i(q)$.
- ii) there exists a family of eigenfunctions f_1, \dots, f_k in $E_i(q)$ such that $\sum_{1 \leq j \leq k} f_j^2 = 1$.

Proof. To see that (i) implies (ii) we introduce the convex cone C generated in $L^2(M)$ by the set $\{f^2; f \in E_i(q)\}$, that is $C = \{\sum_{j \in J} f_j^2; f_j \in E_i(q), J \subset \mathbb{N}, J \text{ is finite}\}$. Condition (ii) is then equivalent to the fact that the constant function 1 belongs to C . Let us suppose, for a contradiction, that $1 \notin C$. Then, applying classical separation theorems (in the finite dimensional vector subspace of $L^2(M)$ generated by $\{f^2; f \in E_i(q)\}$ and 1, see [18]), we prove the existence of a function $u \in L^2(M)$ such that $\bar{u} = \frac{1}{V(M)} \int_M u \cdot 1 dv < 0$ and $\int_M u f^2 dv \geq 0$ for any $f \in C$. Hence, the function $u_0 = u - \bar{u}$ belongs to $L_*^\infty(M)$ and satisfies, $\forall f \in E_i(q)$,

$$Q_{u_0}^i(f) = \int_M u f^2 dv - \frac{1}{V(M)} \int_M u dv \int_M f^2 dv \geq -\bar{u} \|f\|_{L^2(M)}^2.$$

The quadratic form $Q_{u_0}^i$ is then positive definite which contradicts (i) (see Proposition 2.2).

Reciprocally, the existence of f_1, \dots, f_k in $E_i(q)$ satisfying $\sum_{1 \leq j \leq k} f_j^2 = 1$ implies that, $\forall u \in L_*^\infty(M)$,

$$\sum_{j \leq k} Q_u^i(f_j) = \sum_{j \leq k} \int_M u f_j^2 dv = \int_M u = 0,$$

which implies that the quadratic form Q_u^i is indefinite on $E_i(q)$. \square

Finally, let us check that the condition $\sum_{1 \leq j \leq k} f_j^2 = 1$, with $f_j \in E_i(q)$, implies that q is smooth. Indeed, since $q \in \bar{L}^\infty(M)$, we have, for any eigenfunction $f \in E_i(q)$, $\Delta f \in L^2(M)$ and then, $f \in H^{2,2}(M)$. Using standard regularity theory and Sobolev embeddings (see, for instance, [15]), we obtain by an elementary iteration, that $f \in H^{2,p}(M)$ for some $p > n$, and, then, $f \in C^1(M)$. From $\sum_{1 \leq j \leq k} f_j^2 = 1$ and $\Delta \sum_{1 \leq j \leq k} f_j^2 = 0$, we get

$$q = \lambda_i(q) - \sum_{1 \leq j \leq k} |\nabla f_j|^2,$$

which implies that q is continuous. Again, elliptic regularity theory tells us that the eigenfunctions of $-\Delta + q$ are actually smooth, and, hence, q is smooth.

2.4. Proof of Theorem 1.4. Assume that the potential q is a local minimizer of the functional λ_i on $L_c^\infty(M)$ and let us suppose for a contradiction that $\lambda_i(q) > \lambda_{i-1}(q)$. Let u be a function in $L_*^\infty(M)$ and let $\Lambda_1(t), \dots, \Lambda_m(t)$ be a family of m eigenvalues of $-\Delta + q + tu$, where m is the multiplicity of $\lambda_i(q)$, depending analytically in t and such that $\Lambda_1(0) = \dots = \Lambda_m(0) = \lambda_i(q)$. For continuity reasons, we have, for sufficiently small t and any $k \leq m$, $\Lambda_k(t) > \lambda_{i-1}(q + tu)$. Hence, $\forall k \leq m$ and $\forall t$ sufficiently small,

$$\Lambda_k(t) \geq \lambda_i(q + tu) \geq \lambda_i(q) = \Lambda_k(0).$$

Consequently, $\forall k \leq m$, $\Lambda'_k(0) = 0$. Applying Lemma 2.1 above we deduce that the symmetric linear transformation L'_u and then the quadratic form Q'_u is identically zero on the eigenspace $E_i(q)$. Therefore, $\forall u \in L_*^\infty(M)$ and $\forall f \in E_i(q)$, we have $\int_M u f^2 v_g = 0$. In conclusion, $\forall f \in E_i(q)$, f is constant on M which is impossible for $i \geq 2$. The same arguments work to prove Assertion (ii).

2.5. Proof of Theorem 1.5. Let q be a potential and let i and j be two distinct positive integers such that $\lambda_i(q) \neq \lambda_j(q)$. We denote by m (resp. n) the dimension of the eigenspace $E_i(q)$ (resp. $E_j(q)$). Given a function u in $L_*^\infty(M)$, we consider, as above, m (resp. n) $L^2(M)$ -orthonormal families of eigenfunctions $f_{1,t}, \dots, f_{m,t}$ (resp. $g_{1,t}, \dots, g_{n,t}$) associated with m (resp. n) families of eigenvalues $\Lambda_1(t), \dots, \Lambda_m(t)$ (resp. $\Gamma_1(t), \dots, \Gamma_n(t)$) of $-\Delta + q + tu$, all depending analytically in $t \in (-\varepsilon, \varepsilon)$, such that $\Lambda_1(0) = \dots = \Lambda_m(0) = \lambda_i(q)$ (resp. $\Gamma_1(0) = \dots = \Gamma_n(0) = \lambda_j(q)$). Hence, there exist four integers $k \leq m$, $k' \leq m$, $l \leq n$ and $l' \leq n$, such that

$$\begin{aligned} \frac{d}{dt}(\lambda_j - \lambda_i)(q + tu) \Big|_{t=0^-} &= \Gamma'_l(0) - \Lambda'_k(0) \\ &= \int_M u(g_{l,0}^2 - f_{k,0}^2) dv \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}(\lambda_j - \lambda_i)(q + tu) \Big|_{t=0^+} &= \Gamma'_{l'}(0) - \Lambda'_{k'}(0) \\ &= \int_M u(g_{l',0}^2 - f_{k',0}^2) dv. \end{aligned}$$

Recall that (Lemma 2.1) the eigenfunctions $f_{1,0}, \dots, f_{m,0}$ (resp. $g_{1,0}, \dots, g_{n,0}$) constitutes an $L^2(M)$ -orthonormal basis of $E_i(q)$ (resp. $E_j(q)$) which diagonalizes the quadratic form Q_u^i (resp. Q_u^j). Therefore, the family $(f_{k,0} \otimes g_{l,0})_{k \leq m, l \leq n}$ constitutes a basis of the space $E_i(q) \otimes E_j(q)$ which diagonalizes the quadratic form $S_u^{i,j}$ given by

$$\begin{aligned} S_u^{i,j}(f \otimes g) &= \|f\|_{L^2(M)}^2 Q_u^j(g) - \|g\|_{L^2(M)}^2 Q_u^i(f) \\ &= \int_M u(\|f\|_{L^2(M)}^2 g^2 - \|g\|_{L^2(M)}^2 f^2) dv. \end{aligned}$$

The corresponding eigenvalues are $(\Gamma'_l(0) - \Lambda'_k(0))_{k \leq m, l \leq n}$. The criticality of q for $\lambda_j - \lambda_i$ then implies that this quadratic form admits eigenvalues of both signs, which means that it is indefinite.

On the other hand, in the case where $\lambda_i(q) < \lambda_{i+1}(q)$ and $\lambda_j(q) > \lambda_{j-1}(q)$, we have, as in the proof of Proposition 2.3, for sufficiently small t , $\lambda_i(q+tu) = \max_{k \leq m} \Lambda_k(t)$ and $\lambda_j(q+tu) = \min_{l \leq n} \Gamma_l(t)$, which yields

$$\begin{aligned} \frac{d}{dt}(\lambda_j - \lambda_i)(q+tu) \Big|_{t=0^-} &= \max_{l \leq n} \Gamma'_l(0) - \min_{k \leq m} \Lambda'_k(0) \\ &= \max_{k \leq m, l \leq n} (\Gamma'_l(0) - \Lambda'_k(0)) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}(\lambda_j - \lambda_i)(q+tu) \Big|_{t=0^+} &= \min_{l \leq n} \Gamma'_l(0) - \max_{k \leq m} \Lambda'_k(0) \\ &= \min_{k \leq m, l \leq n} (\Gamma'_l(0) - \Lambda'_k(0)). \end{aligned}$$

One deduces the following

Proposition 2.4. *If the potential $q \in L_c^\infty(M)$ is critical for the functional $G_{ij} = \lambda_j - \lambda_i$, then, $\forall u \in L_*^\infty(M)$, the quadratic form $S_u^{i,j}$ is indefinite on $E_i(q) \otimes E_j(q)$.*

Reciprocally, if $\lambda_i(q) < \lambda_{i+1}(q)$ and $\lambda_j(q) > \lambda_{j-1}(q)$, and if, $\forall u \in L_^\infty(M)$, the quadratic form $S_u^{i,j}(g)$ is indefinite on $E_i(q) \otimes E_j(q)$, then q is a critical potential of the functional G_{ij} .*

The following lemma will complete the proof of Theorem 1.5

Lemma 2.3. *The two following conditions are equivalent:*

- i) $\forall u \in L_*^\infty(M)$, the quadratic form $S_u^{i,j}$ is indefinite on $E_i(q) \otimes E_j(q)$.
- ii) there exist a finite family of eigenfunctions f_1, \dots, f_k in $E_i(q)$ and a finite family of eigenfunctions g_1, \dots, g_l in $E_j(q)$, such that $\sum_{1 \leq p \leq k} f_p^2 = \sum_{1 \leq p \leq l} g_p^2$.

The proof of this lemma is similar to that of Lemma 2.2. Here, we consider the two convex cones C_i and C_j in $L^2(M)$ generated respectively by $\{f^2; f \in E_i(q), f \neq 0\}$ and $\{g^2; g \in E_j(q), g \neq 0\}$. Condition (ii) is then equivalent to the fact that these two cones admit a nontrivial intersection. As in the proof of Lemma 2.2, separation theorems enable us to prove that, if $C_i \cap C_j = \emptyset$, then there exists a function u such that $\int_M u f^2 dv < 0$ for any $f \in E_i(q)$, and $\int_M u g^2 dv \geq 0$ for any $f \in E_j(q)$, which implies that $S_u^{i,j}$ is positive definite on $E_i(q) \otimes E_j(q)$. Since $S_1^{i,j} = 0$, we have, $S_u^{i,j} = S_{u_0}^{i,j}$ with $u_0 = u - \bar{u} \in L_*^\infty(M)$. Proposition 2.4 enables us to conclude.

Reciprocally, assume the existence of $f_1, \dots, f_k \in E_i(q)$ and $g_1, \dots, g_l \in E_j(q)$ satisfying $\sum_{1 \leq p \leq k} f_p^2 = \sum_{1 \leq p \leq l} g_p^2$. Then, $\forall u \in L_*^\infty(M)$,

$$\sum_{1 \leq p \leq k} \sum_{1 \leq p' \leq l} S_u^{i,j}(f_p \otimes g_{p'}) = \dots = 0,$$

which implies that $S_u^{i,j}$ is indefinite on $E_i(q) \otimes E_j(q)$.

2.6. Proof of Theorem 1.6. Let q be a local minimizer of $G_{ij} = \lambda_j - \lambda_i$ and let us suppose for a contradiction that $\lambda_i(q) < \lambda_{i+1}(q)$ and $\lambda_j(q) > \lambda_{j-1}(q)$. Given a function u in $L_*^\infty(M)$, we consider, as above, m (resp. n) families of eigenvalues $\Lambda_1(t), \dots, \Lambda_m(t)$ (resp. $\Gamma_1(t), \dots, \Gamma_n(t)$) of $-\Delta + q + tu$, with $m = \dim E_i(q)$ and $n = \dim E_j(q)$, such that $\Lambda_1(0) = \dots = \Lambda_m(0) = \lambda_i(q)$ and $\Gamma_1(0) = \dots = \Gamma_n(0) = \lambda_j(q)$. As in the proof of Theorem 1.4, we will have for sufficiently small t , $\lambda_i(q + tu) = \max_{k \leq m} \Lambda_k(t)$ and $\lambda_j(q + tu) = \min_{l \leq n} \Gamma_l(t)$. Hence, $\forall k \leq m$ and $l \leq n$,

$$\begin{aligned} \Gamma_l(t) - \Lambda_k(t) &\geq \lambda_j(q + tu) - \lambda_i(q + tu) = G_{ij}(q + tu) \\ &\geq G_{ij}(q) = \Gamma_l(0) - \Lambda_k(0). \end{aligned}$$

It follows that, $\forall k \leq m$ and $l \leq n$, $\Gamma'_l(0) - \Lambda'_k(0) = 0$ and, then, the quadratic form $S_u^{i,j}$ is identically zero on $E_i(q) \otimes E_j(q)$ (recall that $\Gamma'_l(0) - \Lambda'_k(0)$ are the eigenvalues of $S_u^{i,j}$). This implies that, $\forall f \in E_i(q)$ and $\forall g \in E_j(q)$, the function $\|f\|_{L^2(M)}^2 g^2 - \|g\|_{L^2(M)}^2 f^2$ is constant equal to zero (since its integral vanishes) which is clearly impossible unless $i = j$.

REFERENCES

- [1] M. S Ashbaugh and E.M. Harrell, Maximal and minimal eigenvalues and their associated nonlinear equations. *J. Math. Phys.* **28** (1987), no. 8, 1770-1786.
- [2] M. S Ashbaugh and E.M. Harrell, and R. Svirsky, On minimal and maximal eigenvalue gaps and their causes, *Pacific J. Math.* **147** (1991), no. 1, 1-24.
- [3] V.I. Burenkov, Sobolev spaces on Domains, Teubner-Texte zur Mathematik, 137. B. G. Teubner, Stuttgart, 1988.
- [4] S. Chanillo, D. Grieser, M. Imai, K. Kurata and I. Ohnishi, Symmetry breaking and other phenomena in the optimization of eigenvalues for composite membranes, *Commun. Math. Phys.* **214** (2000) 315-337.
- [5] J. Eells and L. Lemaire, A report on harmonic maps, *Bull. London Math. Soc.* **10** (1978), no. 1, 1-68
- [6] A. El Soufi and S. Ilias, Majoration de la seconde valeur propre d'un opérateur de Schrödinger sur une variété compacte et applications, *J. Funct. Anal.* **103** (1992), no. 2, 294-316.
- [7] A. El Soufi and S. Ilias, Second eigenvalue of Schrödinger operators and mean curvature of a compact submanifold, *Commun. Math. Phys.* **208** (2000) 761-770.
- [8] A. El Soufi and S. Ilias, Riemannian manifolds admitting isometric immersions by their first eigenfunctions, *Pacific J. Math.* **195** 1 (2000) pp. 91-99.
- [9] A. El Soufi and S. Ilias, Extremal metrics for the first eigenvalue of the Laplacian in a conformal class, *Proc. Amer. Math. Soc.* **131** (2003), no. 5, 1611-1618.
- [10] Y. V. Egorov and S. Karaa, Optimisation de la première valeur propre de l'opérateur de Sturm-Liouville, *C. R. Acad. Sci. Paris Sr. I Math.* **319** (1994), no. 8, 793-798.
- [11] P. Freitas, On minimal eigenvalues of Schrödinger operators on manifolds, *Commun. Math. Phys.* **217** (2001) 375-382.
- [12] E.M. Harrell, Hamiltonian operators with maximal eigenvalues, *J. Math. Phys.* **25** (1984), no. 1, 48-51.
- [13] E.M. Harrell, On the extension of Ambarzumian's inverse spectral theorem to compact symmetric spaces. *Amer. J. Math.* **109** (1987), no. 5, 787-795.
- [14] J. Jost, Riemannian Geometry and Geometric Analysis, Second edition. Universitext. Springer-Verlag, Berlin, 1998.
- [15] J. Jost, Partial differential equations, Graduate Texts in Mathematics **214**, Springer-Verlag, New York, 2002.
- [16] T. Kato, Perturbation theory for linear operators, 2nd edition, Springer Verlag, Berlin, Heidelberg, New York, 1995.

- [17] W. Kirsch and B. Simon, Comparison theorems for the gap of Schrödinger operators, *J. Funct. Anal.* **75** (1987) 596-410.
- [18] R. T. Rockafellar : *Convex Analysis*. Princeton University Press, Princeton 1972
- [19] R. Svirsky, Maximal resonant potentials subject to p -Norm constraints, *Pac. J. Math.* **129** (1987), 357-374.

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