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The action functional for Moyal planes

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Abstract

Modulo some natural generalizations to noncompact spaces, we show in this letter that Moyal planes are nonunital spectral triples in the sense of Connes. The action functional of these triples is computed, and we obtain the expected result, ie the noncommutative Yang-Mills action associated with the Moyal product. In particular, we show that Moyal gauge theory naturally fit into the rigorous framework of noncommutative geometry.

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1 Introduction

In the noncommutative field theory picture, prior to quantization, fields are elements of some function algebra on some manifold endowed with a noncommutative product, such as the Moyal product. Playing this game (exchanging the pointwise product for a noncommutative one) is equivalent to consider field theory on some objects which are no longer smooth manifolds (noncommutativity destroys the concept of point). The spectral triple formalism [9, 10, 17] is exactly the mathematical and conseptual framework in which one can rigorously deal with these noncommutative spaces.

More precisely, a noncommutative Riemannian spin geometry is defined by a triplet $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, where $\mathcal{A}$ is an algebra faithfully represented on some Hilbert space $\mathcal{H}$ and $\mathcal{D}$ is an unbounded selfadjoint operator. The commutative case of a compact Riemannian spin manifold, can be recovered with $\mathcal{A} = C^\infty(M)$, $\mathcal{H}$ the space of L^2-sections of the spinor bundle and $\mathcal{D}$ the Dirac operator [17]. We may quote [24] for an approach to the semi-Riemannian case and [18, 21] for early approaches to noncompact (i.e., nonunital) noncommutative geometry.

Moyal geometry, based on spaces of functions or distributions endowed with the Moyal product, is thoroughly investigated in [15]. It has been proved that one can construct nonunital spectral triples with this product. Here we focus on the construction of the action functional [8, 11], associated with the (below defined) Moyal spectral triple. Such action is closely related to the spectral triples formalism, and allows to extract physically relevant (noncommutative) geometrical information. We will find that the action functional of the Moyal plane turns out to be the noncommutative Yang-Mills action of Moyal gauge theory, with Lagrangean $L(x) = (F^{\mu\nu} \star_\theta F_{\mu\nu})(x)$, where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + [A^\mu, A^\nu]_{\star_\theta}$.

In the first part of this letter, a definition of nonunital spectral triple will be given and discussed, then we will give basic tools and results on Moyal analysis to construct the nonunital Moyal triple. The last part will be devoted to the construction and the computation of the action functional.

2 Nonunital spectral triples

We write down axioms describing non necessarily compact non commutative spin geometries. In [15], it is described how one can arrive to them by learning from examples. In the operatorial setting, such noncommutative spaces (nonunital spectral triples) are given by the data $(\mathcal{A}, \widetilde{\mathcal{A}}, \mathcal{H}, \mathcal{D}, J, \chi)$, where $\chi = 1$ in the odd case. $\mathcal{A}$ is an a priori nonunital noncommutative algebra, $\widetilde{\mathcal{A}}$ is one (appropriately chosen) of its unitizations, and both are faithfully represented by bounded operators on an Hilbert space $\mathcal{H}$ (via a representation π). $\mathcal{D}$ is an unbounded sefladjoint operator such that each $[\mathcal{D}, a]$, for $a \in \widetilde{\mathcal{A}}$ extends to a bounded operator. The triple is said even if there is on $\mathcal{H}$ a $\mathbb{Z}_2$-grading operator χ which commutes with $\widetilde{\mathcal{A}}$ and anticommutes with $\mathcal{D}$. J is an antiunitary operator having definite commuting properties with respect to χ and $\mathcal{D}$, depending on the dimension. Those data $(\mathcal{A}, \widetilde{\mathcal{A}}, \mathcal{H}, \mathcal{D}, J, \chi)$ must moreover satisfy the following axioms:

0. *Compactness*:

   For all $a \in \mathcal{A}$ and $\lambda \notin \text{spectrum}(\mathcal{D})$, the operator $\pi(a)(\mathcal{D} - \lambda)^{-1} \in \mathcal{K}(\mathcal{H})$, where $\mathcal{K}(\mathcal{H})$ is the set of compact operators on the Hilbert space $\mathcal{H}$.

1. *Spectral dimension*:

   The spectral dimension k is the unique positive integer such that

$$\pi(a)(|\mathcal{D}| + \varepsilon)^{-1} \in \mathcal{L}^{(k,\infty)}(\mathcal{H})$$

and

$$\mathrm{Tr}_\omega(\pi(a)(|\mathcal{D}|+\varepsilon)^{-k}) < \infty,$$

and not identically zero, for all a belonging to a dense ideal of $\mathcal{A}$; k must have the same parity as the triple.

Here $\mathcal{L}^{(p,\infty)}(\mathcal{H})$ is the p-th weak-Schatten class and Tr_ω is any Dixmier trace. Recall that $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ is the natural domain of those traces.

2. *Regularity*:

   For all $a \in \widetilde{\mathcal{A}}$, the bounded operators $\pi(a)$ and $[\mathcal{D}, \pi(a)]$ lie in the smooth domain of the derivation $\delta(.) = [|\mathcal{D}|, .]$.

3. *Finiteness*:

   The algebras $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ are pre-C^*-algebras. The space $\mathcal{H}^\infty := \bigcap_{n\in\mathbb{N}} \mathrm{Dom}(\mathcal{D}^n)$ is the $\mathcal{A}$-pullback [21] of a finite projective $\widetilde{\mathcal{A}}$-module. Moreover, an $\mathcal{A}$-valued hermitian structure $(\cdot \mid \cdot)$ is implicitly defined on $\mathcal{H}^\infty$ with the noncommutative integral as follows:

$$\mathrm{Tr}_\omega\big((\pi(a)\xi \mid \eta)(|\mathcal{D}|+\varepsilon)^{-k}\big) = \langle \eta \mid \pi(a)\xi\rangle,$$

   where $a \in \widetilde{\mathcal{A}}$ and $\langle \cdot \mid \cdot \rangle$ denotes the standard inner product on $\mathcal{H}$.

4. *Reality*:

   The antiunitary operator J must define a commuting representation on $\mathcal{H}$:

$$[\pi(a), J\pi(b^*)J^{-1}] = 0$$

   for all $a, b \in \widetilde{\mathcal{A}}$. Moreover, $J^2 = \pm 1$ and $J\mathcal{D} = \pm\mathcal{D}J$, and also $J\chi = \pm\chi J$ in the even case, where the signs depend only on k mod 8 (see [10] or [17] for the table of signs).

5. *First order*:

   For all $a, b \in \widetilde{\mathcal{A}}$, we also have: $[[\mathcal{D}, \pi(a)], J\pi(b^*)J^{-1}] = 0$.

6. *Orientation*:

   There is a *Hochschild k-cycle* $\mathbf{c}$, on $\widetilde{\mathcal{A}}$ with values in $\widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}^{\mathrm{op}}$, given by $\mathbf{c} = \sum_{i\in I}(a_0^i \otimes b_0^i) \otimes a_1^i \otimes \cdots \otimes a_k^i$, where I is a finite set and such that

$$\sum_{i\in I} \pi(a_0^i) J\pi(b_0^i)^* J^{-1} [\mathcal{D}, \pi(a_1^i)] \cdots [\mathcal{D}, \pi(a_k^i)] = \chi.$$

These axioms generalize those of the unital (compact) case [10, 17]. Postulates 2, 4, 5 have not been modified from the compact case, except than in them $\widetilde{\mathcal{A}}$ has been substituted for $\mathcal{A}$ everywhere. The modifications of the axioms 0 and 1 are quite natural; they allows us to recover compact operators (recall that in the noncompact manifold case, pseudodifferential operators of strictly negative order are no longer compact). We refer to [15] on the question why in Axioms 3 and 6 we need an unitization of the algebra (a compactification of the underlying "noncommutative space").

3 The Moyal spectral triple

3.1 Basic Moyal analysis

In this part, a few basic facts on Moyal analysis are given; a complete review can be found in [15]. The starting point is the Moyal product in its integral form. In the general framework, Moyal products can be defined on $\mathbb{R}^k$ for a given $k \times k$ skew symmetric matrix Θ:

$$(f \times_\Theta g)(x) := (2\pi)^{-k} \int e^{i\xi(x-y)} f(x - \tfrac{1}{2}\Theta\xi) g(y)\, d^k y\, d^k \xi. \tag{1}$$

For technical reasons, only the nondegenerate "Darboux" case will be considerated: $k = 2N$, $\Theta = \theta S := \theta \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}$. In this case the Moyal product can be rewritten in a more symmetric way:

$$(f \star_\theta g)(x) := (\pi\theta)^{-2N} \int f(y) g(z)\, e^{\frac{2i}{\theta}(x-y)\cdot S(x-z)}\, d^{2N} y\, d^{2N} z. \tag{2}$$

For f, g lying in the Schwartz space (with its usual topology) this product is well defined, and with the complex conjugation as involution $(\mathcal{S}(\mathbb{R}^{2N}), \star_\theta)$ closes to an $*$-algebra:

Theorem 3.1. [16,26] *$\mathcal{B}_\theta := (\mathcal{S}(\mathbb{R}^{2N}), \star_\theta)$ is a nonunital, associative, involutive Fréchet algebra with a jointly continuous product.*

This algebra has some (well known) nice algebraic properties, which will be of practical use to compute the action functional. In particular, the Leibniz rule is satisfied:

$$\partial^\mu (f \star_\theta g) = \partial^\mu f \star_\theta g + f \star_\theta \partial^\mu g, \tag{3}$$

the pointwise product by coordinate functions obeys:

$$x^\mu (f \star_\theta g) = f \star_\theta (x^\mu g) + \tfrac{i\theta}{2} \frac{\partial f}{\partial (Sx)^\mu} \star_\theta g = (x^\mu f) \star_\theta g - \tfrac{i\theta}{2} f \star_\theta \frac{\partial g}{\partial (Sx)^\mu}, \tag{4}$$

the ordinary integral is a faithful trace:

$$\int (f \star_\theta g)(x)\, d^{2N} x = \int f(x) g(x)\, d^{2N} x = \int (g \star_\theta f)(x)\, d^{2N} x. \tag{5}$$

$(\mathcal{S}(\mathbb{R}^{2N}), \star_\theta)$ has an interesting countable dense subalgebra consisting of the finite linear combinations of the Wigner eigentransitions. These particular elements $\{f_{mn}\}_{m,n \in \mathbb{N}^N}$ of $\mathcal{S}(\mathbb{R}^{2N})$ are constructed from the N-dimensional harmonic oscillator theory. For $m, n \in \mathbb{N}^N$, we use the usual multiindex notation: $|m| = \sum_{j=1}^N m_j$, $m! = \prod_{j=1}^N m_j!$, $\delta_{mn} = \prod_{j=1}^N \delta_{m_j n_j}$. The $\{f_{mn}\}$ are defined by:

$$f_{mn} := \frac{1}{\sqrt{\theta^{|m|+|n|}\, m! n!}} (\bar{a})^m \star_\theta f_0 \star_\theta a^n, \tag{6}$$

where the creation and annihilation functions are respectively

$$\bar{a}_l := \tfrac{1}{\sqrt{2}}(x_l - i x_{l+N}) \quad \text{and} \quad a_l := \tfrac{1}{\sqrt{2}}(x_l + i x_{l+N}), \tag{7}$$

f_0 is the Gaussian function $f_0(x) := 2^N e^{-2H/\theta}$ associated with the N-dimensional oscillator Hamiltonian function $H(x) := \frac{1}{2}\sum_{j=1}^N (x_j^2 + x_{j+N}^2)$. The main properties of these functions are summarized in the following Lemma.

Lemma 3.2. [16,26] *Let $m,n,k,l \in \mathbb{N}^N$, then the Wigner eigentransitions satisfy: $f_{mn} \star_\theta f_{kl} = \delta_{nk} f_{ml}$, $f^*_{mn} = f_{nm}$, $\langle f_{mn}, f_{kl}\rangle = (2\pi\theta)^N \delta_{mk}\delta_{nl}$. Here $\langle .,. \rangle$ is the standard inner product of $L^2(\mathbb{R}^{2N})$, in particular $\{f_{mn}\} \subset \mathcal{S}(\mathbb{R}^{2N}) \subset L^2(\mathbb{R}^{2N})$ is an orthogonal basis.*

With these properties, the $\{f_{mn}\}$ basis allows us to define the Moyal product on larger spaces than $\mathcal{S}(\mathbb{R}^{2N})$. For instance, $f \star_\theta g \in L^2(\mathbb{R}^{2N})$ if $f, g \in L^2(\mathbb{R}^{2N})$ and $(L^2(\mathbb{R}^{2N}), \star_\theta)$ is an involutive Banach algebra. This can be shown expanding f and g with respect to the $\{f_{mn}\}$ basis and applying the Cauchy-Schwarz inequality. As a consequence, if we denote by L^θ_f the operator of left Moyal multiplication by f, acting on $L^2(\mathbb{R}^{2N})$, we have:

Lemma 3.3. *Let $f \in L^2(\mathbb{R}^{2N})$ then $\|L^\theta_f\| \leq (2\pi\theta)^{-N/2}\|f\|_2$.*

3.2 The triple

Next, a triple is given satisfying all the axioms spelled out in Section 2. Such choice can't be absolutely unique; however the analytical content of the data $(\mathcal{A}, \widetilde{\mathcal{A}}, \mathcal{H}, \mathcal{D}, J, \chi)$, is severely constrained by the axioms. The choice of the algebra is almost exclusively guided by the finiteness condition; in other words the Dirac operator tells us what the "good" algebra must be. For its unitization, there is an upper bound given by the boundness of its representation; and the orientation condition gives a lower bound because we have to be able to construct a good Hochschild cycle with it. The constraints on the algebras given by the regularity condition are relatively weak.

The Moyal spectral triple consists of the algebra $\mathcal{A}_\theta := (\mathcal{D}_{L^2}, \star_\theta)$, where $\mathcal{D}_{L^2}$ (see [22]) is the space of smooth functions in $L^2(\mathbb{R}^{2N})$ together with all of their derivatives. Using the Leibniz rule (3) and estimates similar as in Lemma 3.3, one can show that $(\mathcal{D}_{L^2}, \star_\theta)$ is an algebra with continuous product. Its unitization is chosen to be $\widetilde{\mathcal{A}}_\theta := (\mathcal{O}_0, \star_\theta)$, where $\mathcal{O}_0$ is the space of smooth functions with all their derivatives bounded. There is a natural Fréchet topology on $\mathcal{O}_0$ given by the seminorms $q_m(f) = \mathrm{Sup}\{|\partial^\alpha f|, |\alpha| \leq m\}$ —this topology actually comes from the regularity axiom (see [21]), but is not the only one to be considered [22]. It was shown some time ago [14] that $(\mathcal{O}_0, \star_\theta)$ is a unital algebra with a jointly continuous product. For both $\mathcal{A}_\theta$ and $\widetilde{\mathcal{A}}_\theta$, $\mathcal{B}_\theta := (\mathcal{S}(\mathbb{R}^{2N}), \star_\theta)$ is an essential two sided ideal.

Those algebras are faithfully represented on $\mathcal{H} := L^2(\mathbb{R}^{2N}) \otimes \mathbb{C}^{2^N}$ by $\pi^\theta(f) := L^\theta_f \otimes 1_{2^N}$. By Lemma 3.3, $\pi^\theta(f)$ is bounded for $f \in \mathcal{A}_\theta$ and for $f \in \widetilde{\mathcal{A}}_\theta$, L^θ_f is by formula (1) a zero order pseudodifferential operator (ΨDO) and hence bounded. The unbounded operator $\mathcal{D}$ is chosen to be the usual Dirac $\not{D}$ operator of the 2N-dimensional Euclidean flat space, $\not{D} = -i\partial_\mu \otimes \gamma^\mu$. Here the γ^μ are the Clifford matrices associated to $(\mathbb{R}^{2N}, \eta)$ in their irreducible 2^N-dimensional representation and η is the standard Euclidean metric on $\mathbb{R}^{2N}$. For $f \in \mathcal{A}_\theta$ or $f \in \widetilde{\mathcal{A}}_\theta$, the Leibniz rule (3) immediately shows that, $[\not{D}, \pi^\theta(f)] = -iL^\theta_{\partial_\mu f} \otimes \gamma^\mu$, which is bounded, too. As the triple is even, the grading χ is the chirality operator $\chi := \gamma_{2N+1} := 1_{\mathcal{H}_r} \otimes (-i)^N \gamma^1\gamma^2 \ldots \gamma^{2N}$, and the real structure is the charge conjugation for spinors $J := \left(1_{\mathcal{H}_r} \otimes (-i\chi)^N \prod_{\nu=0}^{N-1} \gamma^{2\nu+1}\right) \circ cc$, where cc means complex conjugation, for a representation of the γ's as in [29].

In [15] it is shown in full detail that Moyal planes are nonunital spectral triples. In particular, the compactness condition is obtained by complex interpolation between Hilbert-Schmidt and uniform norm of $L^\theta_f g(-i\nabla)$ for $f \in \mathcal{A}_\theta$, $g \in L^p(\mathbb{R}^{2N})$ and a suitable p. Indeed, the computation of the kernel of $L^\theta_f g(-i\nabla)$ and Lemma 3.3 yield $\|L^\theta_f g(-i\nabla)\|_2 = (2\pi)^{-N}\|f\|_2\|g\|_2$ and $\|L^\theta_f g(-i\nabla)\| \leq (2\pi\theta)^{-N/2}\|f\|_2\|g\|_\infty$. Because the construction of the action functional is closely related to the axiom of spectral dimension, only the latter will be tackled here. Let us then begin by the main theorem.

Theorem 3.4. *For $f \in \mathcal{S}(\mathbb{R}^{2N})$, the compact operator $\pi^\theta(f)\,(|\not{D}|+\varepsilon)^{-2N}$ lies in $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ and any of its Dixmier trace Tr_ω is independent of ε. More precisely we have,*

$$\mathrm{Tr}_\omega\big(\pi^\theta(f)\,(|\not{D}|+\varepsilon)^{-2N}\big) = \frac{2^N\,\Omega_{2N}}{2N\,(2\pi)^{2N}}\int f(x)\,d^{2N}x = \frac{1}{N!\,(2\pi)^N}\int f(x)\,d^{2N}x, \tag{8}$$

where Ω_{2N} is the hyper-area of the unit sphere in $\mathbb{R}^{2N}$.

This result is exactly the same as in the commutative (compact or not) case (see [5, 8, 15, 17]); this is the analogue of Connes' trace Theorem [8, 17] for the Moyal flat space. In particular, the Dixmier trace is independant of the deformation parameter θ. (However, as we see in the next section, the action functional will depend on θ). Because $2N$ is the biggest exponent giving a nonvanishing Dixmier trace, Theorem 3.4 yields that the spectral dimension of the Moyal plane is $2N$. Moreover, this result confirm that $Tr_\omega\big(\ .\ (|\not{D}|+\varepsilon)^{-2N}\big)$ is a good abstractly defined integral (see [9]).

Let first explain how the result (8) can be conjectured. In view of the first axiom, we only need to work with a dense ideal of $\mathcal{A}_\theta$, and $\mathcal{B}_\theta$ seems to be the most suitable (actually, we need integrable and square integrable functions). By formula (1), $\pi^\theta(f)$ with $f \in \mathcal{S}(\mathbb{R}^{2N})$ is a regularizing pseudodifferential operator with symbol $\sigma[\pi^\theta(f)](x,\xi) = f(x - \frac{\theta}{2}S\xi)$. Now the symbol formula for a product of two ΨDOs yields

$$\sigma\left[\pi^\theta(f)(|\not{D}|+\varepsilon)^{-2N}\right](x,\xi) = f(x-\tfrac{\theta}{2}S\xi)(|\xi|+\varepsilon)^{-2N}\otimes 1_{2^N}.$$

Let us define the regularized trace for a ΨDO:

$$\mathrm{Tr}_\Lambda(A) := (2\pi)^{-2N}\iint_{|\xi|\le\Lambda}\mathrm{tr}\,(\sigma[A](x,\xi))\;d^{2N}\xi\,d^{2N}x,$$

where tr is the matricial part of the trace. The Dixmier trace is heuristically linked (see [20]) with $\lim_{\Lambda\to\infty}\mathrm{Tr}_\Lambda(\cdot)/\log(\Lambda^{2N})$. Doing this computation for $A = \pi^\theta(f)\,(|\not{D}|+\varepsilon)^{-2N}$, we exactly find the right hand side of (8):

$$\lim_{\Lambda\to\infty}\frac{1}{2N\log\Lambda}\,\mathrm{Tr}_\Lambda\big(\pi^\theta(f)(|\not{D}|+\varepsilon)^{-2N}\big) = \frac{2^N\,\Omega_{2N}}{2N\,(2\pi)^{2N}}\int f(x)\,d^{2N}x.$$

However, this is far from a rigorous proof; nor do we know how to mutate it directly into one. The reader is invited to look up [15] for a way to establish it correctly, with regard to the Carey-Phillips-Sukochev work [1].

4 The action functional

Most of the interplays between the mathematical framework of noncommutative differential geometry and physics come from computation of gauge actions. These actions (functional and/or spectral) are intimately linked to the spectral triples formalism. The Connes–Lott action functional, historically the first, allows to deal with pure gauge systems (i.e., without gravity), while the spectral action unifies gravitation with other interactions. In the field of particle physics, one can recover with the action functional, in a very natural way, the full Yang–Mills–Higgs sector of the standard model (see [11, 3]). Here we compute the action functional associated with the Moyal spectral triple, and reobtain the nowadays very used noncommutative Yang–Mills action of the Moyal gauge theory [4, 12, 23, 25, 28]. So these theories naturally fit into the rigorous framework of spectral triples. For simplicity, we will only concentrate on the $U(1)$ case, the general Moyal-$U(n)$ being recovered with obvious modifications. We will first introduce the last tools we need.

4.1 The differential algebra

Because of the close relation between the action functional and the spectral dimension axiom, only $\mathcal{B}_\theta$, the dense essential ideal of $\mathcal{A}_\theta$ will be considered here. Let $\Omega^\bullet\mathcal{B}_\theta := \bigoplus_{p\in\mathbb{N}} \Omega^p\mathcal{B}_\theta$ be the universal differential graded algebra associated to $\mathcal{B}_\theta$. Recall that these graded algebra is generated by symbols:

$$\Omega^n\mathcal{B}_\theta := \mathrm{Span}\{\, f_0\, \delta f_1 \dots \delta f_n \,\},\ f_i \in \mathcal{B}_\theta, \quad \text{for} \quad i = 1, \cdots, n \quad \text{and} \quad f_0 \in \mathcal{B}_\theta \oplus \mathbb{C},$$

and relations that can be deduced from properties of the universal derivation:

$$\delta(f \star_\theta g) = \delta(f)g + f\delta(g) \quad \text{and} \quad (\delta f)^* = \delta f^*.$$

With π^θ, one can represent $\Omega^\bullet\mathcal{B}_\theta$ on $\mathcal{H}$ by:

$$\tilde{\pi}^\theta : \Omega^n\mathcal{B}_\theta \to \mathcal{L}(\mathcal{H}) : f_0\, \delta f_1 \dots \delta f_n \mapsto i^n\, \pi^\theta(f_0)\, [\not{D}, \pi^\theta(f_1)] \dots [\not{D}, \pi^\theta(f_n)],$$

here we have implicitly extended π^θ from $\mathcal{B}_\theta$ to $\mathcal{B}_\theta \oplus \mathbb{C}$. Some unpleasant accidents may happened with such construction, in particular represented forms can be the "image of zero", i.e., $\tilde{\pi}^\theta(\omega) = 0$ while $\tilde{\pi}^\theta(\delta\omega) \neq 0$. To clear out such forms, Connes has introduced a two sided ideal of $\Omega^\bullet\mathcal{B}_\theta$: Junk $:= \bigoplus_{n\in\mathbb{N}} J^n$, $J^n := J_0^n + \delta J_0^{n-1}$ where $J_0^n := \mathrm{Ker}\left(\tilde{\pi}^\theta\,|_{\Omega^n\mathcal{B}_\theta}\right)$, and finally we define,

$$\Omega_{\not{D}} := \tilde{\pi}^\theta(\Omega^\bullet\mathcal{B}_\theta)/\tilde{\pi}^\theta(\mathrm{Junk}).$$

Thanks to the $\{f_{mn}\}$ basis, the 2-Junk (the only nontrivial component needed here) is easily computable. In the next Lemma we exhibit some particular elements of $\tilde{\pi}^\theta(J^2)$, with which we are able to completely characterize $\tilde{\pi}^\theta(J^2)$.

Lemma 4.1. *For $m, n, k, l \in \mathbb{N}^N$, let $\omega_{mnkl} := f_{mk}\, \delta f_{kn} - f_{ml}\, \delta f_{ln} \in \Omega^1\mathcal{B}_\theta$ (no summation on k or l). Then*

$$\tilde{\pi}^\theta(\omega_{mnkl}) = 0 \quad \textit{and} \quad \tilde{\pi}^\theta(\delta\omega_{mnkl}) = \tfrac{2}{\theta}(|k| - |l|)\, L^\theta_{f_{mn}} \otimes 1_{2^N}.$$

Proof. Using the creation and annihilation functions (7), and adopting the notations that, for $j = 1, \dots, N$, $\partial_{a_j} = \partial/\partial a_j$ and $\bar{\partial}_{a_j} = \partial/\partial \bar{a}_j$, we may rewrite the Dirac operator as follows;

$$\not{D} = -\frac{i}{\sqrt{2}} \sum_{j=1}^N \gamma^j(\partial_{a_j} + \bar{\partial}_{a_j}) + i\gamma^{j+N}(\partial_{a_j} - \bar{\partial}_{a_j}) = -i \sum_{j=1}^N (\gamma^{a_j}\, \partial_{a_j} + \gamma^{\bar{a}_j}\, \bar{\partial}_{a_j}),$$

where $\gamma^{a_j} := \frac{1}{\sqrt{2}}(\gamma^j + i\gamma^{j+N})$ and $\gamma^{\bar{a}_j} := \frac{1}{\sqrt{2}}(\gamma^j - i\gamma^{j+N})$.

Property (4), applied to a_j and $\bar{a}_j$ respectively, yields

$$\partial_{a_j} = -\frac{1}{\theta}\, \mathrm{ad}^{\star_\theta}_{\bar{a}_j} := -\frac{1}{\theta}[\bar{a}_j, \cdot\,]_{\star_\theta}, \qquad \bar{\partial}_{a_j} = \frac{1}{\theta}\, \mathrm{ad}^{\star_\theta}_{a_j} := \frac{1}{\theta}[a_j, \cdot\,]_{\star_\theta}$$

and hence,

$$\not{D} = -\frac{i}{\theta} \sum_{j=1}^N (\mathrm{ad}^{\star_\theta}_{a_j} \otimes \gamma^{\bar{a}_j} - \mathrm{ad}^{\star_\theta}_{\bar{a}_j} \otimes \gamma^{a_j}).$$

Let $u_j := (0, 0, \dots, 1, \dots, 0)$ be the j-th standard basis vector of $\mathbb{R}^N$. From definition (6), we directly compute:

$$\bar{a}_j \star_\theta f_{mn} = \sqrt{\theta(m_j + 1)}\, f_{m+u_j,n}, \qquad f_{mn} \star_\theta \bar{a}_j = \sqrt{\theta n_j}\, f_{m,n-u_j},$$

$$a_j \star_\theta f_{mn} = \sqrt{\theta m_j}\, f_{m-u_j,n}, \qquad f_{mn} \star_\theta a_j = \sqrt{\theta(n_j + 1)}\, f_{m,n+u_j}.$$

Consequently,

$$[\not{D}, \pi^\theta(f_{mn})] = -\frac{i}{\theta}\sum_{j=1}^{N}\left(\sqrt{\theta n_j}\, L^\theta_{f_{m,n-u_j}} - \sqrt{\theta(m_j+1)}\, L^\theta_{f_{m+u_j,n}}\right)\otimes\gamma^{a_j}$$
$$+\left(\sqrt{\theta m_j}\, L^\theta_{f_{m-u_j,n}} - \sqrt{\theta(n_j+1)}\, L^\theta_{f_{m,n+u_j}}\right)\otimes\gamma^{\bar{a}_j}.$$

Finally, using the projector or nilpotent properties of the $\{f_{mn}\}$ (Lemma 3.2), the result follows by a direct (quite long) computation. □

To identify $\tilde{\pi}^\theta(J^2)$, it suffices to remark that $\tilde{\pi}^\theta(J^2) \subset \pi^\theta(\mathcal{B}_\theta)$. This follows because any $\omega \in \tilde{\pi}^\theta(J^2)$ can be written as $\omega = -\sum_{j\in I} L^\theta_{\partial_\mu f_j} L^\theta_{\partial_\nu g_j} \otimes \gamma^\mu\gamma^\nu$ where I is a finite set, and satisfies $-i\sum_{j\in I} L^\theta_{f_j\star_\theta\partial_\mu g_j}\otimes\gamma^\mu = 0$. By the Leibniz rule,

$$\omega = -\sum_{j\in I}\left(L^\theta_{\partial_\mu(f_j\star_\theta\partial_\nu g_j)} - L^\theta_{f_j\star_\theta\partial_\mu\partial_\nu g_j}\right)\otimes\gamma^\mu\gamma^\nu = \sum_{j\in I} L^\theta_{f_j\star_\theta\partial_\mu\partial_\nu g_j}\otimes\gamma^\mu\gamma^\nu$$
$$= \sum_{j\in I} L^\theta_{f_j\star_\theta\partial_\mu\partial_\nu g_j}\otimes\eta^{\mu\nu}\, 1_{2^N}.$$

Because $\{f_{mn}\}$ is a basis for $\mathcal{B}_\theta$, the previous Lemma yields $\tilde{\pi}^\theta(J^2) \supset \pi^\theta(\mathcal{B}_\theta)$, so in the end we obtain:

Proposition 4.2. $\tilde{\pi}^\theta(J^2) = \pi^\theta(\mathcal{B}_\theta) = L^\theta(\mathcal{B}_\theta)\otimes 1_{2^N}$.

4.2 The action

In a canonical way, the functional or noncommutative Yang-Mills action may be defined as:

$$YM(\alpha) := \frac{N!(2\pi)^N}{8g^2}\inf\{\, I(\eta) : \tilde{\pi}^\theta(\eta) = \alpha\,\}. \tag{9}$$

The infimum is over all $\eta \in \Omega^1\mathcal{B}_\theta$ with the same image in $\Omega^1_{\not{D}}$ of $I(\eta) := \mathrm{Tr}_\omega\big(\tilde{\pi}^\theta(F)^*\,\tilde{\pi}^\theta(F)\,(\not{D}^2+\varepsilon^2)^{-N}\big)$, where $\Omega^2\mathcal{B}_\theta \ni F = \delta\eta + \eta^2$ is the curvature of the universal connection η. If we define $\widetilde{\mathcal{H}}_n$ as the Hilbert space obtained by completion of $\tilde{\pi}^\theta(\Omega^n\mathcal{B}_\theta)$ under the scalar product

$$\langle\tilde{\pi}^\theta(\rho)\mid\tilde{\pi}^\theta(\rho')\rangle_n := \mathrm{Tr}_\omega\big(\tilde{\pi}^\theta(\rho)^*\,\tilde{\pi}^\theta(\rho')\,(\not{D}^2+\varepsilon^2)^{-N}\big),$$

for $\rho,\rho' \in \Omega^n\mathcal{B}_\theta$, and $\mathcal{H}_n := P\widetilde{\mathcal{H}}_n$ where P be the orthogonal projector on $\widetilde{\mathcal{H}}_p$ whose range is the orthogonal complement of $\tilde{\pi}^\theta(\delta J_0^{p-1})$, then one can show [11, 27]:

$$YM(\alpha) = \frac{N!(2\pi)^N}{8g^2}\langle P\tilde{\pi}^\theta(F)\mid P\tilde{\pi}^\theta(F)\rangle_2. \tag{10}$$

With Theorem 3.4 and Proposition 4.2, $YM(\alpha)$ in its form (10) is easily computable, and one gets the expected result:

Theorem 4.3. *Let $\eta = -\eta^* \in \Omega^1\mathcal{B}_\theta$. Then the Yang–Mills action $YM(\alpha)$ of the universal connection $\delta+\eta$, with $\alpha = \tilde{\pi}^\theta(\eta)$, is equal to*

$$YM(\alpha) = -\frac{1}{4g^2}\int F^{\mu\nu}\star_\theta F_{\mu\nu}(x)\, d^{2N}x,$$

where $F_{\mu\nu} := \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_{\star_\theta})$ and A_μ is defined by $\alpha = L^\theta_{A_\mu}\otimes\gamma^\mu$.

We may remark that property (5) yields: $YM(\alpha) = -\frac{1}{4g^2}\int F^{\mu\nu}(x)\,F_{\mu\nu}(x)\,d^{2N}x$, and because $F^*_{\mu\nu} = -F_{\mu\nu}$, $YM(\alpha)$ is positive definite.

5 Conclusions

Apart from the mathematical questions raised in [15] (like the computation of the Hochschild cohomology of $\mathcal{A}_\theta$ and $\widetilde{\mathcal{A}}_\theta$), there are several remaining physical tasks. Firstly, it would be desirable to extend our results to the generic Moyal product (1).

On the other hand, a quite important job is to compute the spectral action [7, 2] associated with this nonunital triple: $S(\not{D}, A, a) = \mathrm{Tr}\left[\pi^\theta(a)\phi_\Lambda\left(\not{D}_A^2\right)\right]$, where ϕ_Λ is a suitable functional with a cutoff Λ and $\not{D}_A = \not{D} + A + JAJ^{-1}$, $A \in \tilde{\pi}^\theta\left(\Omega^1\mathcal{B}_\theta\right)$. Here, it is the definition of the spectral action in the nonunital case. The mathematical problem is to compute a heat kernel (or other asymptotic [13]) expansion for a "very nonminimal" Laplace type operator, namely the sqare of the "Moyal-covariant" Dirac operator $\not{D}_A$. The treatment of quantization on Riemannian manifolds in [19] should help. Such computation will hopefully allow us to get a handle on noncommutative gravity [6].

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