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► **To cite this version:**

Jonathan Ben-Naim. Preferential and Preferential-discriminative Consequence relations. Journal of Logic and Computation, 2005, 15, number 3, pp. 263-294. 10.1093/logcom/exi013 . hal-00005232v1

HAL Id: hal-00005232

<https://hal.science/hal-00005232v1>

Submitted on 9 Jun 2005 (v1), last revised 8 Apr 2007 (v6)

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Preferential and Preferential-discriminative Consequence Relations*

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June 9, 2005

Abstract

The theory of preferential consequence relations, where copies of valuations (or states labeled by valuations) serve as the terms of the preference relation, has been investigated extensively in the classical context. The first purpose of the present paper is to extend the theory to certain three/four-valued contexts, well-known as the paraconsistent logics J_3 and \mathcal{FOUR} . We give characterizations of several families of preferential consequence relations in these two contexts. Our second and main purpose is to investigate a qualified version of preferential consequence, which we call preferential-discriminative consequence. This is defined to hold between a set of formulas Γ and formula α iff $\Gamma \mid\sim \alpha$ but $\Gamma \not\mid\sim \neg\alpha$, where $\mid\sim$ is the plain relation. We provide characterizations of several families of such relations for all of the classical, three, and four-valued contexts.

1 Introduction

The theory of *preferential consequence relations* (preferential CRs for short), where *copies of valuations* (or *states labeled by valuations*) serve as the terms of the preference relation, has been investigated extensively in the classical context, see e.g. [15, 18, 20, 21, 22, 23]. In the present paper, the author continues these investigations, but on a more general level that covers certain three/four-valued contexts, well-known as the paraconsistent logics J_3 and \mathcal{FOUR} . The motivation is that preferential CRs represent natural ways of reasoning which are useful to handle incomplete information in the classical context and both incomplete and inconsistent information in the J_3 and \mathcal{FOUR} contexts. We will illustrate this, together with the advantages and drawbacks of the different contexts, with examples in Sections 2.10.1 and 2.10.2, once the formal definitions will be made.

The first purpose of the paper is to show that some characterizations of preferential CRs, proved by Karl Schlechta in the classical context [22, 23], hold also in the J_3 and \mathcal{FOUR} contexts. The second and main purpose is to investigate a qualified version of preferential CRs, which we call *preferential-discriminative CRs*. They are defined to hold between a set of formulas Γ and formula

*A previous version on this work has been published in the Journal of Logic and Computation, 15(3):263-294, 2005. The present version contains some improvements.

α iff $\Gamma \mid\sim \alpha$ but $\Gamma \not\mid\sim \neg\alpha$, where $\mid\sim$ is the plain relation. We will provide characterizations of several families of such preferential-discriminative CRs for all of the classical, J_3 , and $FOUR$ contexts. Most of the time, our characterizations have a purely syntactic aspect (i.e. they involve only the language and some proof systems). They are helpful to answer questions like: “is that CR (usually defined syntactically) a preferential or preferential-discriminative CR of that or that family?” More generally, these characterizations can help to find relationships between preferential, preferential-discriminative, and other CRs.

The rest of the paper is organized as follows. In Section 2, we give the formal definitions, an overview of the characterizations, and examples in the classical and $FOUR$ contexts. In Sections 3 and 4, we give characterizations of several families of preferential CRs. In Section 5 and 6, we give characterizations of several families of preferential-discriminative CRs. In Section 7, we give some proof systems used in the characterizations. And, Section 8 is a conclusion summing and explaining what has been achieved.

2 Preliminaries

2.1 A general framework

We fix once and for all a set \mathcal{L} , a set \mathcal{V} , and a relation \models on $\mathcal{V} \times \mathcal{L}$. Intuitively, \mathcal{L} is a set of formulas, \mathcal{V} is a set of valuations, and \models is a satisfaction relation (i.e. $\forall x \in \mathcal{V}, \forall \alpha \in \mathcal{L}, x \models \alpha$ means that the formula α is satisfied in the valuation x , i.e. x is a model for α). We fix also once and for all a function $neg : \mathcal{L} \rightarrow \mathcal{L}$, a function $or : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$, and a function $and : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$. Intuitively, $\forall \alpha, \beta \in \mathcal{L}$, $neg(\alpha)$ is the negation of α , $or(\alpha, \beta)$ is the disjunction of α and β , and $and(\alpha, \beta)$ is the conjunction of α and β . We use these functions instead of the usual connectives: \neg , \vee , and \wedge , because we do not want nor need to assume any structure on \mathcal{L} . We emphasize that for the time being, *no* assumption is made about \mathcal{L} , \mathcal{V} , \models , neg , or , and and . A similar approach has been taken in two well-known papers, [19, 16], without anticipating any of the substantive work in this paper.

Notation 1 For all $\Gamma \subseteq \mathcal{L}$, we denote by M_Γ the set such that $M_\Gamma = \{x \in \mathcal{V} : \forall \alpha \in \Gamma, x \models \alpha\}$. Let $X \subseteq \mathcal{V}$. We denote by $T(X)$ the set such that $T(X) = \{\alpha \in \mathcal{L} : X \subseteq M_\alpha\}$. We denote by $T_d(X)$ the set such that $T_d(X) = \{\alpha \in \mathcal{L} : X \subseteq M_\alpha \text{ and } X \not\subseteq M_{neg(\alpha)}\}$. We denote by $T_c(X)$ the set such that $T_c(X) = \{\alpha \in \mathcal{L} : X \subseteq M_\alpha \text{ and } X \subseteq M_{neg(\alpha)}\}$. We denote by \mathbf{D} the set such that $\mathbf{D} = \{X \subseteq \mathcal{V} : \exists \Gamma \subseteq \mathcal{L}, M_\Gamma = X\}$. We denote by \mathbf{C} the set such that $\mathbf{C} = \{X \subseteq \mathcal{V} : \forall \alpha \in \mathcal{L}, X \subseteq M_\alpha \text{ or } X \subseteq M_{neg(\alpha)}\}$.

Intuitively, M_Γ is the set of all models for Γ , $T(X)$ is the set of all formulas satisfied in X , $T_d(X)$ is the set of all formulas “discriminatively” satisfied in X , $T_c(X)$ is the set of all formulas “complementarily” satisfied in X , \mathbf{D} is the set of every set of valuations definable by a set of formulas, and \mathbf{C} is the set of all “consistent” sets of valuations. It is worth noticing that $\forall \Gamma, \Delta \subseteq \mathcal{L}$, $M_\Gamma \cap M_\Delta = M_{\Gamma \cup \Delta}$, and thus \mathbf{D} is closed under arbitrarily intersections.

2.2 Consequence relations and extended consequence relations

Definition 2 We denote by \mathcal{P} the power set operator. We say that $\mid\sim$ is a *consequence relation* (CR for short) iff $\mid\sim$ is a relation on $\mathcal{P}(\mathcal{L}) \times \mathcal{L}$. For all $\Gamma \subseteq \mathcal{L}$, we denote by $\tilde{\Gamma}$ the set such that $\tilde{\Gamma} = \{\alpha \in \mathcal{L} : \Gamma \mid\sim \alpha\}$. We denote by \vdash the CR such that $\forall \Gamma \subseteq \mathcal{L}, \forall \alpha \in \mathcal{L}, \Gamma \vdash \alpha$ iff $M_\Gamma \subseteq M_\alpha$. For all $\Gamma \subseteq \mathcal{L}$, we denote by $\bar{\Gamma}$ the set such that $\bar{\Gamma} = \{\alpha \in \mathcal{L} : \Gamma \vdash \alpha\}$. We denote by \vdash_d the CR

such that $\forall \Gamma \subseteq \mathcal{L}, \forall \alpha \in \mathcal{L}, \Gamma \vdash_d \alpha$ iff $\Gamma \vdash \alpha$ and $\Gamma \not\vdash \text{neg}(\alpha)$. For all $\Gamma \subseteq \mathcal{L}$, we denote by $\overline{\Gamma}^d$ the set such that $\overline{\Gamma}^d = \{\alpha \in \mathcal{L} : \Gamma \vdash_d \alpha\}$.

Intuitively, \vdash is the basic CR and \vdash_d is the discriminative CR. It is worth noticing that $\forall \Gamma, \Delta \subseteq \mathcal{L}, \overline{\Gamma} = T(M_\Gamma), \overline{\Gamma}^d = T_d(M_\Gamma), M_\Gamma = M_{\overline{\Gamma}}$, and $\Gamma \subseteq \overline{\Delta}$ iff $\overline{\Gamma} \subseteq \overline{\Delta}$ iff $M_\Delta \subseteq M_\Gamma$.

Definition 3 We say that $\|\sim$ is an *extended consequence relation* (ECR for short) iff $\|\sim$ is a relation on $\mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$. We denote by \Vdash the ECR such that $\forall \Gamma, \Delta \subseteq \mathcal{L}, \Gamma \Vdash \Delta$ iff $\forall x \in M_\Gamma, \exists \delta \in \Delta, x \in M_\delta$. We denote by \Vdash_d the ECR such that $\forall \Gamma, \Delta \subseteq \mathcal{L}, \Gamma \Vdash_d \Delta$ iff $\Gamma \Vdash \Delta$ and $\exists \delta \in \Delta, \Gamma \not\vdash \text{neg}(\delta)$.

Intuitively, \Vdash is the disjunctive ECR, i.e. $\forall \Gamma, \Delta \subseteq \mathcal{L}, \Gamma \Vdash \Delta$ iff every model for Γ is a model for the disjunction (of the formulas) of Δ . Intuitively, \Vdash_d is the disjunctive-discriminative ECR, i.e. $\forall \Gamma, \Delta \subseteq \mathcal{L}, \Gamma \Vdash_d \Delta$ iff the disjunction of Δ is satisfied in every model for Γ , but *not* the negation of the disjunction of Δ . Note that $\forall \Gamma \subseteq \mathcal{L}, \forall \alpha \in \mathcal{L}$, we have $\Gamma \Vdash \{\alpha\}$ iff $\Gamma \vdash \alpha$, and we have $\Gamma \Vdash_d \{\alpha\}$ iff $\Gamma \vdash_d \alpha$. Thus, any proof system for \Vdash (resp. \Vdash_d) is a proof system for \vdash (resp. \vdash_d).

2.3 Preferential consequence relations

Preferential CRs represent natural ways of drawing conclusions which are useful to handle incomplete information in the classical context and both incomplete and inconsistent information the paraconsistent contexts. Examples will be given in Sections 2.10.1 and 2.10.2. We need to introduce *preferential structures*.

Historically, preferential structures were first introduced by Hansson (see [13]) to give a semantics for deontic logics. Then, they have been rediscovered by Shoham (see [24]) to give a semantics for nonmonotonic logics. Then, it seems that Imielinski is one of the first persons to introduce preferential structures with copies of valuations (or states labeled by valuations), see [14]. They have been used to give a more general semantics for nonmonotonic logics, see e.g. [15, 22, 23]. Note that the terminology: “copies” is used by e.g. Schlechta and the equivalent terminology: “states” is used by e.g. Kraus, Lehmann, and Magidor.

Definition 4 We say that \mathcal{Z} is preferential structure on \mathcal{V} iff \mathcal{Z} is an ordered pair $\langle \mathcal{X}, \prec \rangle$ where $\mathcal{X} \subseteq \mathcal{V} \times \mathcal{I}$, where \mathcal{I} is any set, and \prec is any relation on $\mathcal{X} \times \mathcal{X}$. Let $\mathcal{Z} = \langle \mathcal{X}, \prec \rangle$ be a preferential structure. For all $X \subseteq \mathcal{V}$, we denote by $C_{\mathcal{X}}(X)$ the set such that $C_{\mathcal{X}}(X) = \{\langle x, i \rangle \in \mathcal{X} : x \in X\}$. We denote by $\mu_{\mathcal{Z}} : \mathcal{P}(\mathcal{V}) \rightarrow \mathcal{P}(\mathcal{V})$ the function such that $\forall X \subseteq \mathcal{V}$,

$$\mu_{\mathcal{Z}}(X) = \{x \in X : \exists c \in C_{\mathcal{X}}(x), \forall c' \in C_{\mathcal{X}}(X), c' \not\prec c\}$$

The definition may become more accessible if we see it as putting $\mu_{\mathcal{Z}}(X) = C_{\mathcal{X}}^{-1}(\text{min}_{\prec}(C_{\mathcal{X}}(X)))$.

Intuitively, \mathcal{X} is a set of copies of valuations. More precisely, $\forall \langle x, i \rangle \in \mathcal{X}, \langle x, i \rangle$ is a copy of the valuation x . Intuitively, \prec is a preferential binary relation on \mathcal{X} . More precisely, $\forall c, c' \in \mathcal{X}, c \prec c'$ means that the copy c is preferred to the copy c' . Thus, \mathcal{Z} is a set of copies of valuations together with a preferential binary relation on it. Reciprocally, any set of copies of valuations together with any preferential binary relation on it can be represented by a preferential structure. Now, a question arise: what is exactly a copy of a valuation (or a state labeled by a valuation) in daily life? A beginning of answer can be found in e.g. [14, 15, 23]. But, to the author’s knowledge, it seems that this question has not been answered in a definitive and satisfactory manner. Let $X \subseteq \mathcal{V}$. Intuitively, $C_{\mathcal{X}}(X)$ is the set of every element of \mathcal{X} that is a copy of a valuation in X . And, $\mu_{\mathcal{Z}}(X)$ is the set of every valuation in X that is *most preferred* among all valuations in X .

Definition 5 Let $\mathcal{Z} = \langle \mathcal{X}, \prec \rangle$ be a preferential structure on \mathcal{V} . We say that \mathcal{Z} is transitive, irreflexive, etc. iff \prec is *transitive*, *irreflexive*, etc. We say that \mathcal{Z} is *definability preserving* (DP for short) iff $\forall X \in \mathbf{D}, \mu_{\mathcal{Z}}(X) \in \mathbf{D}$. Let μ be a function from $\mathbf{Y} \subseteq \mathcal{P}(\mathcal{V})$ to $\mathbf{Z} \subseteq \mathcal{P}(\mathcal{V})$. We say that μ is *consistency preserving* (CP for short) iff $\forall X \in \mathbf{Y} \cap \mathbf{D} \cap \mathbf{C}, \mu(X) \in \mathbf{C}$. We say that \mathcal{Z} is CP iff $\mu_{\mathcal{Z}}$ is CP. Let $\mathcal{Y} \subseteq \mathcal{X}$ and $c \in \mathcal{Y}$. We say that c is \prec -preferred in \mathcal{Y} iff $\forall c' \in \mathcal{Y}, c' \not\prec c$. Let $\mathbf{Y} \subseteq \mathcal{P}(\mathcal{V})$. We say that \mathcal{Z} is *\mathbf{Y} -smooth* (alias *\mathbf{Y} -stoppered*) iff $\forall X \in \mathbf{Y}, \forall c \in C_{\mathcal{X}}(X)$, either c is \prec -preferred in $C_{\mathcal{X}}(X)$ or there exists $c' \in C_{\mathcal{X}}(X)$ such that $c' \prec c$ and c' is \prec -preferred in $C_{\mathcal{X}}(X)$. If we say that \mathcal{Z} is smooth, we mean that \mathcal{Z} is \mathbf{D} -smooth.

Definition 6 Let $|\sim$ be a CR. We say that $|\sim$ is a preferential CR iff there exists a preferential structure \mathcal{Z} on \mathcal{V} such that $\forall \Gamma \subseteq \mathcal{L}, \tilde{\Gamma} = T(\mu_{\mathcal{Z}}(M_{\Gamma}))$. We say that $|\sim$ is smooth, DP, etc. iff $|\sim$ can be defined by a smooth, DP, etc. preferential structure on \mathcal{V} .

Intuitively, $|\sim$ is a preferential CR iff there exist a preferential structure such that $\Gamma |\sim \alpha$ iff every most preferred model for Γ is a model for α . Note that Salem Benferhat, Didier Dubois, and Henri Prade showed in [7] that there is an equivalent definition without copies, but with multiple preferential relations. And, Karl Schlechta showed by Theorem 7 below that there is an equivalent definition with the *choice functions* used in *Social Choice* (see [17] for more details about choice functions):

Theorem 7 From [22, 23]. Let $\mathbf{Y} \subseteq \mathcal{P}(\mathcal{V})$ and μ be a function from \mathbf{Y} to \mathbf{Y} . Then,

- (i) if there exists a preferential structure \mathcal{Z} on \mathcal{V} such that $\forall X \in \mathbf{Y}, \mu(X) = \mu_{\mathcal{Z}}(X)$, then $(\mu 0)$ and $(\mu 1)$ hold;
- (ii) if $(\mu 0)$ and $(\mu 1)$ hold, then there exists a transitive irreflexive preferential structure \mathcal{Z} on \mathcal{V} such that $\forall X \in \mathbf{Y}, \mu(X) = \mu_{\mathcal{Z}}(X)$;
- (iii) if there exists an \mathbf{Y} -smooth preferential structure \mathcal{Z} on \mathcal{V} such that $\forall X \in \mathbf{Y}, \mu(X) = \mu_{\mathcal{Z}}(X)$, then $(\mu 0)$, $(\mu 1)$, and $(\mu 2)$ hold;
- (iv) if \mathbf{Y} is closed under finite unions and intersections and $(\mu 0)$, $(\mu 1)$, and $(\mu 2)$ hold, then there exists a transitive irreflexive \mathbf{Y} -smooth preferential structure \mathcal{Z} on \mathcal{V} such that $\forall X \in \mathbf{Y}, \mu(X) = \mu_{\mathcal{Z}}(X)$;

where

- $(\mu 0) \forall X \in \mathbf{Y}, \mu(X) \subseteq X$;
- $(\mu 1) \forall X, Y \in \mathbf{Y}, \text{ if } X \subseteq Y, \text{ then } \mu(Y) \cap X \subseteq \mu(X)$;
- $(\mu 2) \forall X, Y \in \mathbf{Y}, \text{ if } \mu(X) \subseteq Y \subseteq X, \text{ then } \mu(X) = \mu(Y)$.

2.4 Preferential-discriminative consequence relations

They capture the idea that usually we do not want to keep contradictory conclusions.

Definition 8 Let $|\sim$ be a CR. We say that $|\sim$ is a preferential-discriminative CR iff there exists a choice function μ such that $\forall \Gamma \subseteq \mathcal{L}, \tilde{\Gamma} = T_d(\mu(M_{\Gamma}))$. We say that $|\sim$ is DP, CP, etc. iff $|\sim$ can be defined by a DP, CP, etc. choice function.

Intuitively, $|\sim$ is a preferential-discriminative CR iff there exists a preferential structure on \mathcal{V} such that $\Gamma |\sim \alpha$ iff α is satisfied in all preferred models for Γ , but not its negation.

2.5 Normal Characterizations

Some of our characterizations are said to be *normal*, whilst some others are not. Let us make this explicit. For all set A , we denote by $|A|$ the cardinality of A . Let C be a family of CRs. Intuitively, we have a normal characterization of C iff we have found some conditions that determine whether a CR $|\sim$ is in C or not, only with $\lambda \leq |\mathcal{L}|$ many sets of formulas: $\Gamma_1, \dots, \Gamma_\lambda$ and their $|\sim$ -closures: $\widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_\lambda$. This notion was introduced by Schlechta [23]. We now formalize it in Definition 9:

Definition 9 Let C be a set of CRs. We have a normal characterization of C iff we have found a (finite or infinite) cardinal $\lambda \leq |\mathcal{L}|$ and a relation Φ on $\mathcal{P}(\mathcal{L})^{2\lambda}$ such that for all CR $|\sim$,

$$|\sim \in C \text{ iff } \forall \Gamma_1, \dots, \Gamma_\lambda \subseteq \mathcal{L}, \Phi(\Gamma_1, \dots, \Gamma_\lambda, \widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_\lambda)$$

We specify that Φ is a relation in the straightforward set-theoretic sense (i.e. any set of 2λ -tuples of subsets of \mathcal{L}). Let C be a family of CRs and $A \subseteq \mathcal{P}(\mathcal{L})$. Here is an example of normal characterization: for all CR $|\sim$, $|\sim \in C$ iff $\forall \Gamma, \Delta \in A, \Gamma \cup \widetilde{\Delta} = \overline{\emptyset}$. Indeed, just take the relation Φ such that $\Phi(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6)$ iff (if $\Gamma_1, \Gamma_2 \in A$ and $\Gamma_3 = \Gamma_1 \cup \Gamma_5$, then $\Gamma_6 = \overline{\emptyset}$). Then, clearly $|\sim \in C$ iff $\forall \Gamma_1, \Gamma_2, \Gamma_3, \Phi(\Gamma_1, \Gamma_2, \Gamma_3, \widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \widetilde{\Gamma}_3)$. This example shows that normal characterizations cover, for instance, “applications of $|\sim$ ” to unions of sets, nested “applications of $|\sim$ ”, constants like \emptyset , limited domains like A , etc.

Here is now an example of non-normal characterization: for all CR $|\sim$, $|\sim \in C$ iff $\forall \Gamma, \widetilde{\Gamma} = T(\{x \in M_\Gamma : \forall \Delta, \text{ if } M_\Delta \subseteq M_\Gamma \text{ and } x \in M_\Delta, \text{ then } x \in M_{\widetilde{\Delta}}\})$. We cannot get a normal characterization because the condition depends on Δ , for $2^{|\mathcal{L}|}$ many Δ (in fact, for any Δ). And, we need that the validity of every condition depends on Δ for at most $|\mathcal{L}|$ many Δ .

2.6 Overview of the characterizations

Sometimes, to show a characterization, we will need to make *some* of the following assumptions about $\mathcal{L}, \mathcal{V}, \models, \text{neg}, \text{or}$, and *and*:

(A0) \mathbf{D} is closed under finite unions;

(A1) $\forall \alpha, \beta \in \mathcal{L}, M_{\text{or}(\alpha, \beta)} = M_\alpha \cup M_\beta$ and $M_{\text{and}(\alpha, \beta)} = M_\alpha \cap M_\beta$ and $M_{\text{neg} \circ \text{neg}(\alpha)} = M_\alpha$ and $M_{\text{neg} \circ \text{or}(\alpha, \beta)} = M_{\text{and}(\text{neg}(\alpha), \text{neg}(\beta))}$ and $M_{\text{neg} \circ \text{and}(\alpha, \beta)} = M_{\text{or}(\text{neg}(\alpha), \text{neg}(\beta))}$, where \circ is the function composition operator (i.e. $\text{neg} \circ \text{or}(\alpha, \beta) = \text{neg}(\text{or}(\alpha, \beta))$, etc.);

(A2) \mathcal{V} is finite;

(A3) $\forall \Gamma \subseteq \mathcal{L}, \forall \alpha \in \mathcal{L}$, if $\alpha \notin T(M_\Gamma)$ and $\text{neg}(\alpha) \notin T(M_\Gamma)$, then $M_\Gamma \cap M_\alpha \not\subseteq M_{\text{neg}(\alpha)}$.

We emphasize that for the time being, we do *not* make any of these assumptions. We emphasize again that to show our characterizations, we will *never* need other assumptions about $\mathcal{L}, \mathcal{V}, \models, \text{neg}, \text{or}$, and *and*. Note that (A0), (A1), (A2), and (A3) are not independent of each other. For instance,

Proposition 10 If (A1) holds, then (A0) holds too.

Proof Let $\Gamma, \Delta \subseteq \mathcal{L}$. It suffices to show $M_\Gamma \cup M_\Delta = M_\Theta$, with $\Theta = \{\text{or}(\alpha, \beta) : \alpha \in \Gamma, \beta \in \Delta\}$.
 Direction: “ \subseteq ”. Let $x \in M_\Gamma \cup M_\Delta$ and $\gamma \in \Theta$. We get $\exists \alpha \in \Gamma, \exists \beta \in \Delta, \gamma = \text{or}(\alpha, \beta)$. Thus, $x \in M_\alpha \cup M_\beta \stackrel{(A1)}{=} M_{\text{or}(\alpha, \beta)} = M_\gamma$.
 Direction: “ \supseteq ”. Suppose $\exists x, x \in M_\Theta$ and $x \notin M_\Gamma \cup M_\Delta$. Then, $\exists \alpha \in \Gamma, x \notin M_\alpha$ and $\exists \beta \in \Delta, x \notin M_\beta$, thus $x \notin M_\alpha \cup M_\beta \stackrel{(A1)}{=} M_{\text{or}(\alpha, \beta)}$, but $\text{or}(\alpha, \beta) \in \Theta$, thus $x \notin M_\Theta$, impossible. ■

We will give normal characterizations of the six following families:

- the DP preferential CRs (in Section 3);
- the smooth DP preferential CRs, under (A0) (in Section 3);
- the DP preferential-discriminative CRs, under (A1), (A2), and (A3) (in Section 5);
- the smooth DP preferential-discriminative CRs, under (A1), (A2), and (A3) (in Section 5);
- the CP DP preferential-discriminative CRs, under (A1) and (A2) (in Section 5);
- the smooth CP DP preferential-discriminative CRs, under (A1) and (A2) (in Section 5).

In addition, we will give nonnormal characterizations of the three following families:

- the preferential CRs (in Section 4);
- the preferential-discriminative CRs, under (A1), (A2), and (A3) (in Section 6);
- the CP preferential-discriminative CRs, under (A1) and (A2) (in Section 6).

Now, we are going to see in Sections 2.7, 2.8, and 2.9 that (A0), (A1), (A2), and (A3) are weak enough to hold in the classical (propositional) context, in the *FOUR* context, and in the J_3 context.

2.7 The classical context

Notation 11 We fix once and for all a set \mathcal{A} of propositional symbols. We denote by \mathcal{L}_c the classical language generated from \mathcal{A} , the constant symbols: 0,1, and the usual connectives: \neg, \vee, \wedge . We denote by \mathcal{V}_c the classical valuations of \mathcal{L}_c and by \models_c the classical satisfaction relation on $\mathcal{V}_c \times \mathcal{L}_c$.

Roughly speaking, (A0), (A1), (A2), (A3) hold in the classical context. More precisely, first (1) entails (A0), second (1) and (2) entail (A1) and (A3), and third (1) and (3) entail (A2), where

- (1) $\mathcal{L} = \mathcal{L}_c$, $\mathcal{V} = \mathcal{V}_c$, and $\models = \models_c$;
- (2) $\forall \alpha, \beta \in \mathcal{L}_c$, $neg(\alpha) = \neg\alpha$, $or(\alpha, \beta) = \alpha \vee \beta$, and $and(\alpha, \beta) = \alpha \wedge \beta$;
- (3) \mathcal{A} is finite.

2.8 The *FOUR* context

The logic *FOUR* was introduced by Belnap in [5, 6] to deal with inconsistent and incomplete information. Intensive investigations of *FOUR* were made by e.g. Ofer Arieli and Arnon Avron [1, 2, 3]. They worked with richer languages than the classical \mathcal{L}_c , containing e.g. some constants \perp, \top and some implication connective \supset first introduced by Avron in [4]. In addition proof systems for \Vdash and \vdash in the *FOUR* context have been given in [1, 2, 3]. Now, from e.g. [5, 6, 4]:

Notation 12 We denote by \mathcal{L}_4 the propositional language generated from \mathcal{A} , the constant symbols 0, 1, \perp, \top , the unary connective \neg , and the binary connectives \vee, \wedge , and \supset .

Definition 13 We say that x is a *FOUR*-valuation iff x is a function from \mathcal{L}_4 to $\{0, 1, \perp, \top\}$ such that $x(0) = 0$, $x(1) = 1$, $x(\perp) = \perp$, $x(\top) = \top$ and truth tables 1, 2, 3, and 4 hold. We denote by \mathcal{V}_4 the set of all *FOUR*-valuations. We denote by \models_4 the relation on $\mathcal{V}_4 \times \mathcal{L}_4$ such that $\forall x \in \mathcal{V}_4, \forall \alpha \in \mathcal{L}_4, x \models_4 \alpha$ iff $x(\alpha) \in \{1, \top\}$. We call \models_4 the *FOUR*-satisfaction relation.

α	$\neg\alpha$
0	1
1	0
\perp	\perp
\top	\top

Table 1.

		β			
		0	1	\perp	\top
α	0	0	1	\perp	\top
	1	1	1	1	1
	\perp	\perp	1	\perp	1
	\top	\top	1	1	\top

Table 2.

		β			
		0	1	\perp	\top
α	0	0	0	0	0
	1	0	1	\perp	\top
	\perp	0	0	\perp	0
	\top	0	\top	0	\top

Table 3.

		β			
		0	1	\perp	\top
α	0	1	1	1	1
	1	0	1	\perp	\top
	\perp	1	1	1	1
	\top	0	1	\perp	\top

Table 4.

The connective \supset was first introduced by Arnon Avron in [4]. An alternative definition is as follows:

$$x(\alpha \supset \beta) = \begin{cases} x(\beta) & \text{if } x \models_4 \alpha \\ 1 & \text{otherwise} \end{cases}$$

We found it useful to recall an intuitive meaning for *FOUR*, valid if we restrict the language to $\mathcal{L}_c \subseteq \mathcal{L}_4$. We will use this meaning in an example in Section 2.10.2. Following e.g. John Fox [12], let $x \in \mathcal{V}_4$ and $\alpha \in \mathcal{L}_c$, then

- $x(\alpha) = 0$ means: we are informed that α is false, but not informed that α is true;
- $x(\alpha) = 1$ means: we are informed that α is true, but not informed that α is false;
- $x(\alpha) = \perp$ means: we are neither informed that α is true nor informed that α is false;
- $x(\alpha) = \top$ means: we are both informed that α is true and informed that α is false.

Now consider the following *simplifying assumption*: the information about any formula can always be obtained only from the information about the propositional symbols. In other words, the initial information is the one about the propositional symbols and the information about general formulas is constructed from the initial information. Then, intuitively

- (1) We are informed that $\neg\alpha$ is true iff we are informed that α is false;
- (2) We are informed that $\neg\alpha$ is false iff we are informed that α is true;
- (3) We are informed that $\alpha \vee \beta$ is true iff we are informed that α is true or informed that β are true;
- (4) We are informed that $\alpha \vee \beta$ is false iff we are both informed that α is false and that β are false;
- (5) We are informed that $\alpha \wedge \beta$ is true iff we are both informed that α is true and that β are true;
- (6) We are informed that $\alpha \wedge \beta$ is false iff we are informed that α is false or informed that β is false.

Note that without the simplifying assumption only the left-to-right implications of (3) and (6) do not longer hold. Now, (1) and (2) are formalized in Table 1, (3) and (4) are formalized in Table 2, and (5) and (6) are formalized in Table 3. Thus, every *FOUR*-valuation represents a way to be informed about the formulas under the simplifying assumption. And clearly, every such way is represented by a *FOUR*-valuation.

Roughly speaking, (A0), (A1), (A2), but *not* (A3) hold in the *FOUR* context. Indeed, first (1) entails (A0), second (1), (2) entail (A1) but *not* (A3), and third (1), (3) entail (A2), where

(1) $\mathcal{L} = \mathcal{L}_4$, $\mathcal{V} = \mathcal{V}_4$, and $\models = \models_4$;

(2) $\forall \alpha, \beta \in \mathcal{L}_4$, $neg(\alpha) = \neg\alpha$, $or(\alpha, \beta) = \alpha \vee \beta$, and $and(\alpha, \beta) = \alpha \wedge \beta$;

(3) \mathcal{A} is finite.

It is easy to see that our assumptions (except (A3)) still hold if we restrict the language to \mathcal{L}_c .

2.8.1 A proof system

A proof system for \models in the *FOUR* context can be found in e.g. [1, 2, 3]. We call it *systemFOUR*.

Axioms:

$$[a \Rightarrow a] \Gamma, \alpha \Rightarrow \Delta, \alpha$$

$$[\neg 1 \Rightarrow] \Gamma, \neg 1 \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, 1 \quad [\Rightarrow 1]$$

$$[0 \Rightarrow] \Gamma, 0 \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \neg 0 \quad [\Rightarrow \neg 0]$$

$$[\perp \Rightarrow] \Gamma, \perp \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \top \quad [\Rightarrow \top]$$

$$[\neg \perp \Rightarrow] \Gamma, \neg \perp \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \neg \top \quad [\Rightarrow \neg \top]$$

Rules: Exchange, Contraction, and the following rules:

$$[\neg \neg \Rightarrow] \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma, \neg \neg \alpha \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \neg \neg \alpha} \quad [\Rightarrow \neg \neg]$$

$$[\wedge \Rightarrow] \frac{\Gamma, \alpha, \beta \Rightarrow \Delta}{\Gamma, \alpha \wedge \beta \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta} \quad [\Rightarrow \wedge]$$

$$[\neg \wedge \Rightarrow] \frac{\Gamma, \neg \alpha \Rightarrow \Delta \quad \Gamma, \neg \beta \Rightarrow \Delta}{\Gamma, \neg(\alpha \wedge \beta) \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \neg \alpha, \neg \beta}{\Gamma \Rightarrow \Delta, \neg(\alpha \wedge \beta)} \quad [\Rightarrow \neg \wedge]$$

$$[\vee \Rightarrow] \frac{\Gamma, \alpha \Rightarrow \Delta \quad \Gamma, \beta \Rightarrow \Delta}{\Gamma, \alpha \vee \beta \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta} \quad [\Rightarrow \vee]$$

$$[\neg \vee \Rightarrow] \frac{\Gamma, \neg \alpha, \neg \beta \Rightarrow \Delta}{\Gamma, \neg(\alpha \vee \beta) \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \neg \alpha \quad \Gamma \Rightarrow \Delta, \neg \beta}{\Gamma \Rightarrow \Delta, \neg(\alpha \vee \beta)} \quad [\Rightarrow \neg \vee]$$

$$[\supset \Rightarrow] \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma, \beta \Rightarrow \Delta}{\Gamma, \alpha \supset \beta \Rightarrow \Delta} \quad \frac{\Gamma, \alpha \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \supset \beta} \quad [\Rightarrow \supset]$$

$$[\neg \supset \Rightarrow] \frac{\Gamma, \alpha, \neg \beta \Rightarrow \Delta}{\Gamma, \neg(\alpha \supset \beta) \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \neg \beta}{\Gamma \Rightarrow \Delta, \neg(\alpha \supset \beta)} \quad [\Rightarrow \neg \supset]$$

Proposition 14 From [1, 2, 3]. If $\mathcal{L} = \mathcal{L}_4$, $\mathcal{V} = \mathcal{V}_4$, and $\models = \models_4$, then $\forall \Gamma, \Delta \subseteq \mathcal{L}_4, \Gamma \Vdash \Delta$ iff $\Gamma \Rightarrow \Delta$ is provable in *systemFOUR*.

We will use *systemFOUR* to construct a proof system for \Vdash_d in the *FOUR* context (in Section 7). Note that a single-conclusioned proof system for \vdash in the *FOUR* context has been given in [1, 2].

2.9 The J_3 context

The logic J_3 was introduced in [10] to answer a question posed in 1948 by S. Jaśkowski, who was interested in systematizing theories capable of containing contradictions, especially if they occur in dialectical reasoning. The step from informal reasoning under contradiction and formal reasoning with databases and information was done in [8] (also specialized for real database models in [9]), where another formulation of J_3 called **LF11** was introduced, and its first-order version, semantics and proof theory were studied in detail. Investigations of J_3 have also been made by Avron in e.g. [4] where richer languages than the classical \mathcal{L}_c are considered. In addition, proof systems for \Vdash and \vdash in the J_3 context can be found in e.g. [4, 10] and the chapter IX of [11]. Now, from e.g. [10]:

Notation 15 We denote by \mathcal{L}_3 the propositional language generated from \mathcal{A} , the constant symbols 0, 1, \top , the unary connective \neg , and the binary connectives \vee , \wedge , and \supset .

Definition 16 We say that x is a J_3 -valuation iff x is a function from \mathcal{L}_3 to $\{0, 1, \top\}$ such that $x(0) = 0$, $x(1) = 1$, $x(\top) = \top$, and the truth tables 5, 6, 7 and 8 hold. We denote by \mathcal{V}_3 the set of all J_3 -valuations. We denote by \models_3 the relation on $\mathcal{V}_3 \times \mathcal{L}_3$ such that $\forall x \in \mathcal{V}_3, \forall \alpha \in \mathcal{L}_3, x \models_3 \alpha$ iff $x(\alpha) \in \{1, \top\}$. We call \models_3 the J_3 -satisfaction relation.

<table style="border-collapse: collapse; margin: auto;"> <tr><th style="padding: 2px 10px;">α</th><th style="padding: 2px 10px;">$\neg\alpha$</th></tr> <tr><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">1</td></tr> <tr><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">0</td></tr> <tr><td style="padding: 2px 10px;">\top</td><td style="padding: 2px 10px;">\top</td></tr> </table> <p>Table 5.</p>	α	$\neg\alpha$	0	1	1	0	\top	\top	<table style="border-collapse: collapse; margin: auto;"> <tr><th style="padding: 2px 10px;">α</th><th colspan="3" style="padding: 2px 10px;">β</th></tr> <tr><td style="padding: 2px 10px;"></td><th style="padding: 2px 10px;">0</th><th style="padding: 2px 10px;">1</th><th style="padding: 2px 10px;">\top</th></tr> <tr><th style="padding: 2px 10px;">0</th><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">\top</td></tr> <tr><th style="padding: 2px 10px;">1</th><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">1</td></tr> <tr><th style="padding: 2px 10px;">\top</th><td style="padding: 2px 10px;">\top</td><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">\top</td></tr> </table> <p>$\alpha \vee \beta$</p>	α	β				0	1	\top	0	0	1	\top	1	1	1	1	\top	\top	1	\top	<table style="border-collapse: collapse; margin: auto;"> <tr><th style="padding: 2px 10px;">α</th><th colspan="3" style="padding: 2px 10px;">β</th></tr> <tr><td style="padding: 2px 10px;"></td><th style="padding: 2px 10px;">0</th><th style="padding: 2px 10px;">1</th><th style="padding: 2px 10px;">\top</th></tr> <tr><th style="padding: 2px 10px;">0</th><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">0</td></tr> <tr><th style="padding: 2px 10px;">1</th><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">\top</td></tr> <tr><th style="padding: 2px 10px;">\top</th><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">\top</td><td style="padding: 2px 10px;">\top</td></tr> </table> <p>$\alpha \wedge \beta$</p>	α	β				0	1	\top	0	0	0	0	1	0	1	\top	\top	0	\top	\top	<table style="border-collapse: collapse; margin: auto;"> <tr><th style="padding: 2px 10px;">α</th><th colspan="3" style="padding: 2px 10px;">β</th></tr> <tr><td style="padding: 2px 10px;"></td><th style="padding: 2px 10px;">0</th><th style="padding: 2px 10px;">1</th><th style="padding: 2px 10px;">\top</th></tr> <tr><th style="padding: 2px 10px;">0</th><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">1</td></tr> <tr><th style="padding: 2px 10px;">1</th><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">\top</td></tr> <tr><th style="padding: 2px 10px;">\top</th><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">\top</td></tr> </table> <p>$\alpha \supset \beta$</p>	α	β				0	1	\top	0	1	1	1	1	0	1	\top	\top	0	1	\top
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If we restrict the language to $\mathcal{L}_c \subseteq \mathcal{L}_3$, the same precise meaning as for *FOUR* is valid, except that the following is added to the simplifying assumption: for every propositional symbol, we are at least informed that it is true or informed that it is false.

Roughly speaking, (A0), (A1), (A2), and (A3) hold in the J_3 context. More precisely, first (1) entails (A0), second (1) and (2) entail (A1) and (A3), and third (1) and (3) entail (A2), where

- (1) $\mathcal{L} = \mathcal{L}_3$, $\mathcal{V} = \mathcal{V}_3$, and $\models = \models_3$;
- (2) $\forall \alpha, \beta \in \mathcal{L}_3, \text{neg}(\alpha) = \neg\alpha, \text{or}(\alpha, \beta) = \alpha \vee \beta, \text{and}(\alpha, \beta) = \alpha \wedge \beta$;
- (3) \mathcal{A} is finite.

It is easy to see that our assumptions still hold if we restrict the language to \mathcal{L}_c .

2.9.1 A proof system

A proof system for \Vdash in the J_3 context can be found in e.g. [4, 10] and the chapter IX of [11]. We call it *systemJ3*. It is obtained from *systemFOUR* by removing the axioms $[\perp \Rightarrow]$ and $[\neg\perp \Rightarrow]$ and by adding the axiom

$$[\Rightarrow \alpha, \neg\alpha] \Gamma \Rightarrow \Delta, \alpha, \neg\alpha$$

Proposition 17 From [4, 10] and the chapter IX of [11]. If $\mathcal{L} = \mathcal{L}_3$, $\mathcal{V} = \mathcal{V}_3$, and $\models = \models_3$, then $\forall \Gamma, \Delta \subseteq \mathcal{L}_3, \Gamma \Vdash \Delta$ iff $\Gamma \Rightarrow \Delta$ is provable in *systemJ3*.

We will use *systemJ3* to construct a proof system for \Vdash_d in the J_3 context (in Section 7).

2.10 Different contexts different drawbacks

Preferential (resp. preferential-discriminative) CRs suffer from the following drawback: some sets of formulas are rendered useless in the sense that everything (resp. nothing) can be concluded from them. A set of formulas Γ is rendered useless in two situations:

- there does not exist a model for Γ ;
- there exist models for Γ , but they are not preferred.

Note that in the *FOUR* and J_3 contexts, these two situations happen less often than in the classical context. Roughly speaking, this is due to the fact that in the *FOUR* and J_3 contexts, there are much more valuations than in the classical context. Then, it is harder to be left with no model or no preferred model. In particular, for all classically inconsistent $\Gamma \subseteq \mathcal{L}_c$, there are generally *FOUR* and J_3 models. Thus, in the *FOUR* and J_3 contexts, preferential and preferential-discriminative CRs are paraconsistent.

On the other hand, in these paraconsistent contexts, the Disjunctive Syllogism is not satisfied, unlike in the classical context. The Disjunctive Syllogism says that from α and $\neg\alpha \vee \beta$, we can infer β . Let us illustrate all of this with examples.

2.10.1 A particular preferential consequence relation in the classical context

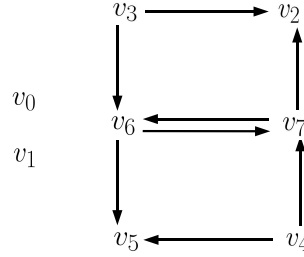
Suppose that we are in the classical context (i.e. suppose $\mathcal{L} = \mathcal{L}_c$, $\mathcal{V} = \mathcal{V}_c$, and $\models = \models_c$) and suppose $\mathcal{A} = \{r, q, p\}$. Recall that \mathcal{A} is a set of propositional symbols introduced in Section 2.7. Intuitively, r means that Nixon is a republican, q means that Nixon is a quaker, and p means that Nixon is a pacifist. Then, \mathcal{V}_c is the set of the 8 following classical valuations: $v_0, v_1, v_2, v_3, v_4, v_5, v_6$, and v_7 , which are defined in the obvious way by the following table:

	r	q	p
v_0	0	0	0
v_1	0	0	1
v_2	0	1	0
v_3	0	1	1
v_4	1	0	0
v_5	1	0	1
v_6	1	1	0
v_7	1	1	1

Now, consider the class of all republicans and the class of all quakers. Consider that a republican is normal iff he is not a pacifist and that a quaker is normal iff he is a pacifist. And, consider that a valuation x is more normal than a valuation y (from some point of view) iff there is a class C s.t.:

- Nixon is an individual of C in both x and y ;
- Nixon is normal in x ;
- Nixon is not normal in y .

Let us see this with the following graph (there is an arrow from a valuation x to a valuation y iff x is more normal than y):



This is formalized by the preferential structure \mathcal{Z} such that $\mathcal{Z} = \langle \mathcal{X}, \prec \rangle$, where $\mathcal{X} = \mathcal{V} \times \{0\}$ and \prec is the relation on $\mathcal{X} \times \mathcal{X}$ such that $\forall \langle x, 0 \rangle, \langle y, 0 \rangle \in \mathcal{X}, \langle x, 0 \rangle \prec \langle y, 0 \rangle$ iff (1) or (2) holds, where

- (1) $x \models r$ and $x \models \neg p$ and $y \models r$ and $y \not\models \neg p$;
- (2) $x \models q$ and $x \models p$ and $y \models q$ and $y \not\models p$.

Now, let $|\sim$ be the preferential CR defined by \mathcal{Z} .

Then, $|\sim$ leads us to “jump” to plausible conclusions from incomplete information and to revise previous “hasty” conclusions in the face of new and fuller information. For instance, $r \mid\sim \neg p$ and $\{r, p\} \not\mid\sim \neg p$ and $q \mid\sim p$ and $\{q, \neg p\} \not\mid\sim p$. Thus, in the classical context, preferential CRs can be useful to handle incomplete information.

But, we many sets of formulas are rendered useless because there is no preferred model. For instance, $\{q, r\} \mid\sim \alpha, \forall \alpha \in \mathcal{L}$. And, a lot of sets of formulas are rendered useless because there is just no model. For instance: $\{p, \neg p, q\} \mid\sim \alpha, \forall \alpha \in \mathcal{L}$. In other words, $|\sim$ is not paraconsistent.

2.10.2 A particular preferential consequence relation in the *FOUR* context

Suppose now that we are in the *FOUR* context (i.e. suppose $\mathcal{L} = \mathcal{L}_4, \mathcal{V} = \mathcal{V}_4$, and $\models = \models_4$) and suppose again that $\mathcal{A} = \{r, q, p\}$. Consider the same classes, etc. as in the classical context, except that this time a valuation x is considered to be more normal than a valuation y (from some point of view) iff there is a class C such that

- in both x and y , we are informed that Nixon is an individual of C ;
- in x , we are informed that Nixon is normal and not informed of the contrary;
- in y , we are not even informed that Nixon is normal.

This is formalized by the preferential structure \mathcal{Z} such that $\mathcal{Z} = \langle \mathcal{X}, \prec \rangle$, where $\mathcal{X} = \mathcal{V} \times \{0\}$ and \prec is the relation on $\mathcal{X} \times \mathcal{X}$ such that $\forall \langle x, 0 \rangle, \langle y, 0 \rangle \in \mathcal{X}, \langle x, 0 \rangle \prec \langle y, 0 \rangle$ iff (1) or (2) holds, where

- (1) $x \models r$ and $x \models \neg p$ and $x \not\models p$ and $y \models r$ and $y \not\models \neg p$;
(2) $x \models q$ and $x \models p$ and $x \not\models \neg p$ and $y \models q$ and $y \not\models p$.

Then, again $|\sim$ leads us to “jump” to plausible conclusions from incomplete information and to revise previous “hasty” conclusions in the face of new and fuller information. For instance, $r \mid\sim \neg p$ and $\{r, p\} \mid\sim \neg p$ and $q \mid\sim p$ and $\{q, \neg p\} \mid\sim p$. In addition, less sets of formulas are rendered useless because there is no preferred model. For instance, $\{q, r\} \mid\sim p$ and $\{q, r\} \mid\sim \neg p$ and $\{q, r\} \mid\sim q$ and $\{q, r\} \mid\sim \neg q$ and $\{q, r\} \mid\sim r$ and $\{q, r\} \mid\sim \neg r$. And, less sets of formulas are rendered useless because there is just no model. For instance, $\{p, \neg p, q\} \mid\sim p$ and $\{p, \neg p, q\} \mid\sim \neg p$ and $\{p, \neg p, q\} \mid\sim q$ and $\{p, \neg p, q\} \mid\sim \neg q$.

But, $|\sim$ does not satisfy the Disjunctive Syllogism. Indeed, $\{\neg r, r \vee q\} \mid\sim q$.

3 Definability preserving preferential consequence relations

In this section, we give normal characterizations of the two following families:

- the DP preferential CRs;
- the smooth DP preferential CRs, under (A0).

We emphasize that these characterizations have already been given by Karl Schlechta under the assumption that we are in the classical context, see e.g. [23]. We just show that his characterizations are valid in almost all contexts (classical, J_3 , \mathcal{FOUR} , etc.).

3.1 The necessary and sufficient conditions

Before going further, note that *for the rest of the paper* each time we write one of the following letters: $\alpha, \beta, \gamma, \Gamma, \Delta, \Theta, x, y, z, X, Y, Z, \mathbf{X}, \mathbf{Y}, \mathbf{Z}$ (possibly with exponents and/or subscripts), we assume $\alpha, \beta, \gamma \in \mathcal{L}$ and $\Gamma, \Delta, \Theta \subseteq \mathcal{L}$ and $x, y, z \in \mathcal{V}$ and $X, Y, Z \subseteq \mathcal{V}$ and $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \subseteq \mathcal{P}(\mathcal{V})$. Thus, if we write $\forall \alpha, \forall X, \exists \alpha, \exists X$, etc., we mean $\forall \alpha \in \mathcal{L}, \forall X \subseteq \mathcal{V}, \exists \alpha \in \mathcal{L}, \exists X \subseteq \mathcal{V}$, etc.

Proposition 18 Let $|\sim$ be a CR. Then,

- (i) if $|\sim$ is a DP preferential CR, then $(|\sim 0)$, $(|\sim 1)$, $(|\sim 2)$, and $(|\sim 3)$ hold;
(ii) if $(|\sim 0)$, $(|\sim 1)$, $(|\sim 2)$, $(|\sim 3)$ hold, then $|\sim$ is a transitive irreflexive DP preferential CR;
(iii) if $|\sim$ is a smooth DP preferential CR, then $(|\sim 0)$, $(|\sim 1)$, $(|\sim 2)$, $(|\sim 3)$, and $(|\sim 4)$ hold;
(iv) if (A0), $(|\sim 0)$, $(|\sim 1)$, $(|\sim 2)$, $(|\sim 3)$, and $(|\sim 4)$ hold, then $|\sim$ is a transitive irreflexive smooth DP preferential CR;

where

- $(|\sim 0) \forall \Gamma$, if $\overline{\Gamma} = \overline{\Delta}$, then $\widetilde{\Gamma} = \widetilde{\Delta}$;
 $(|\sim 1) \forall \Gamma, \overline{\widetilde{\Gamma}} = \widetilde{\Gamma}$;
 $(|\sim 2) \forall \Gamma, \Gamma \subseteq \widetilde{\Gamma}$;
 $(|\sim 3) \forall \Gamma, \Delta, \widetilde{\Gamma \cup \Delta} \subseteq \overline{\widetilde{\Gamma} \cup \Delta}$;

($|\sim 4$) $\forall \Gamma, \Delta$, if $\Gamma \subseteq \overline{\Delta} \subseteq \widetilde{\Gamma}$, then $\widetilde{\Gamma} = \widetilde{\Delta}$.

Proof For (i) and (iii), there exists a DP preferential structure \mathcal{Z} on \mathcal{V} such that $\forall \Gamma$, $\widetilde{\Gamma} = T(\mu_{\mathcal{Z}}(M_{\Gamma}))$. By Theorem 7 (i), $\mu_{\mathcal{Z}}$ satisfies $(\mu 0)$ and $(\mu 1)$. And, $\forall \Gamma$, $\mu_{\mathcal{Z}}(M_{\Gamma}) = M_{T(\mu_{\mathcal{Z}}(M_{\Gamma}))} = M_{\widetilde{\Gamma}}$, as $\mu_{\mathcal{Z}}(M_{\Gamma}) \in \mathbf{D}$. Let $\Gamma, \Delta \subseteq \mathcal{L}$. We show ($|\sim 0$). If $\overline{\Gamma} = \overline{\Delta}$, then $M_{\Gamma} = M_{\Delta}$, thus $\widetilde{\Gamma} = T(\mu_{\mathcal{Z}}(M_{\Gamma})) = T(\mu_{\mathcal{Z}}(M_{\Delta})) = \widetilde{\Delta}$. We show ($|\sim 1$). $\widetilde{\Gamma} = T(M_{T(\mu_{\mathcal{Z}}(M_{\Gamma}))}) = T(\mu_{\mathcal{Z}}(M_{\Gamma})) = \widetilde{\Gamma}$. We show ($|\sim 2$). $\Gamma \subseteq T(M_{\Gamma}) \subseteq_{(\mu 0)} T(\mu_{\mathcal{Z}}(M_{\Gamma})) = \widetilde{\Gamma}$. We show ($|\sim 3$). $M_{\Gamma \cup \Delta} \subseteq M_{\Gamma}$, thus $\mu_{\mathcal{Z}}(M_{\Gamma}) \cap M_{\Gamma \cup \Delta} \subseteq_{(\mu 1)} \mu_{\mathcal{Z}}(M_{\Gamma \cup \Delta})$, thus $\widetilde{\Gamma \cup \Delta} = T(\mu_{\mathcal{Z}}(M_{\Gamma \cup \Delta})) \subseteq T(\mu_{\mathcal{Z}}(M_{\Gamma}) \cap M_{\Gamma \cup \Delta}) =_{(\mu 0)} T(\mu_{\mathcal{Z}}(M_{\Gamma}) \cap M_{\Delta}) = T(M_{\widetilde{\Gamma}} \cap M_{\Delta}) = \widetilde{\Gamma \cup \Delta}$.

(i). Everything has been shown in the common part.

(iii). As \mathcal{Z} is smooth, by Theorem 7 (iii), $\mu_{\mathcal{Z}}$ satisfies $(\mu 2)$. We show ($|\sim 4$). If $\Gamma \subseteq \overline{\Delta} \subseteq \widetilde{\Gamma}$, then $\mu_{\mathcal{Z}}(M_{\Gamma}) = M_{\widetilde{\Gamma}} \subseteq M_{\overline{\Delta}} = M_{\Delta} \subseteq M_{\Gamma}$, thus $\widetilde{\Gamma} = T(\mu_{\mathcal{Z}}(M_{\Gamma})) =_{(\mu 2)} T(\mu_{\mathcal{Z}}(M_{\Delta})) = \widetilde{\Delta}$.

For (ii) and (iv), we have ($|\sim 0$), ($|\sim 1$), ($|\sim 2$), and ($|\sim 3$). Let $\mu : \mathbf{D} \rightarrow \mathbf{D}$ be the function such that $\forall \Gamma$, $\mu(M_{\Gamma}) = M_{\widetilde{\Gamma}}$. If $M_{\Gamma} = M_{\Delta}$, then $\overline{\Gamma} = \overline{\Delta}$, thus $\widetilde{\Gamma} =_{(|\sim 0)} \widetilde{\Delta}$, thus μ is well-defined. Moreover, $\forall \Gamma$, $\widetilde{\Gamma} =_{(|\sim 1)} \widetilde{\Gamma} = T(M_{\widetilde{\Gamma}}) = T(\mu(M_{\Gamma}))$. Let $\Gamma, \Delta \subseteq \mathcal{L}$. We show $(\mu 0)$. $\mu(M_{\Gamma}) = M_{\widetilde{\Gamma}} \subseteq_{(|\sim 2)} M_{\Gamma}$. We show $(\mu 1)$. If $M_{\Gamma} \subseteq M_{\Delta}$, then $\mu(M_{\Delta}) \cap M_{\Gamma} = M_{\widetilde{\Delta}} \cap M_{\Gamma} = M_{\widetilde{\Delta \cup \Gamma}} \subseteq_{(|\sim 3)} M_{\widetilde{\Delta \cup \Gamma}} = \mu(M_{\Delta \cup \Gamma}) = \mu(M_{\Gamma})$.

(ii). By Theorem 7 (ii), there exists a transitive irreflexive preferential structure \mathcal{Z} on \mathcal{V} s.t. $\forall \Gamma$, $\widetilde{\Gamma} = T(\mu(M_{\Gamma})) = T(\mu_{\mathcal{Z}}(M_{\Gamma}))$. In addition, \mathcal{Z} is DP, as $\forall \Gamma$, $\mu_{\mathcal{Z}}(M_{\Gamma}) = \mu(M_{\Gamma}) \in \mathbf{D}$.

(iv). We show $(\mu 2)$. $\forall \Gamma, \Delta$, if $\mu(M_{\Gamma}) \subseteq M_{\Delta} \subseteq M_{\Gamma}$, then $M_{\widetilde{\Gamma}} \subseteq M_{\Delta} \subseteq M_{\Gamma}$, thus $\Gamma \subseteq T(M_{\Gamma}) \subseteq T(M_{\Delta}) = \overline{\Delta} \subseteq T(M_{\widetilde{\Gamma}}) = \widetilde{\Gamma} =_{(|\sim 1)} \widetilde{\Gamma}$, thus $\mu(M_{\Gamma}) = M_{\widetilde{\Gamma}} =_{(|\sim 4)} M_{\widetilde{\Delta}} = \mu(M_{\Delta})$. Thus, by Theorem 7 (iv), there exists a transitive irreflexive smooth preferential structure \mathcal{Z} on \mathcal{V} such that $\forall \Gamma$, $\widetilde{\Gamma} = T(\mu(M_{\Gamma})) = T(\mu_{\mathcal{Z}}(M_{\Gamma}))$. And, \mathcal{Z} is DP, as $\forall \Gamma$, $\mu_{\mathcal{Z}}(M_{\Gamma}) = \mu(M_{\Gamma}) \in \mathbf{D}$. ■

The conditions: ($|\sim 0$), ($|\sim 1$), ($|\sim 2$), ($|\sim 3$), and ($|\sim 4$) depend only on $|\sim, \vdash$ and \mathcal{L} . As, \vdash has been defined semantically (in Section 2.2), these conditions have a semantic aspect. In parallel, as we have a proof system for \vdash in the classical, \mathcal{FOUR} , and J_3 contexts (in Section 2.8 and 2.9), the conditions have also a syntactic aspect in these contexts.

3.2 Normal characterizations

Let Φ be the relation on $\mathcal{P}(\mathcal{L})^6$ such that $\forall \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6 \subseteq \mathcal{L}$, $\Phi(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6)$ iff

- if $\overline{\Gamma_1} = \overline{\Gamma_2}$, then $\Gamma_4 = \Gamma_5$;
- $\overline{\Gamma_4} = \Gamma_4$;
- $\Gamma_1 \subseteq \Gamma_4$;
- if $\Gamma_3 = \Gamma_1 \cup \Gamma_2$, then $\Gamma_6 \subseteq \overline{\Gamma_4 \cup \Gamma_2}$.

Then, by Proposition 18 (i) and (ii), for all CR $|\sim, |\sim$ is a DP preferential CR iff $\forall \Gamma_1, \Gamma_2, \Gamma_3 \subseteq \mathcal{L}$, $\Phi(\Gamma_1, \Gamma_2, \Gamma_3, \widetilde{\Gamma_1}, \widetilde{\Gamma_2}, \widetilde{\Gamma_3})$. Let Φ' be the relation on $\mathcal{P}(\mathcal{L})^6$ such that $\forall \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6 \subseteq \mathcal{L}$, $\Phi'(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6)$ iff $\Phi(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6)$ and

$$\text{if } \Gamma_1 \subseteq \overline{\Gamma_2} \subseteq \Gamma_4, \text{ then } \Gamma_4 = \Gamma_5$$

Then, by Proposition 18 (iii) and (iv), under (A0), for all CR $|\sim$, $|\sim$ is a smooth DP preferential CR iff $\forall \Gamma_1, \Gamma_2, \Gamma_3 \subseteq \mathcal{L}$, $\Phi'(\Gamma_1, \Gamma_2, \Gamma_3, \widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \widetilde{\Gamma}_3)$.

4 Preferential consequence relations

In this section, we gave a nonnormal characterization of the family of all preferential CRs. Unfortunately, we cannot use Lemma 7 to characterize preferential CRs as we did to characterize DP preferential CRs in Proposition 18. Roughly speaking, this is due to the fact that in the proof of (i) and (iii), from $\widetilde{\Gamma} = T(\mu_{\mathcal{Z}}(M_{\Gamma}))$, we can no longer conclude $\mu_{\mathcal{Z}}(M_{\Gamma}) = M_{\widetilde{\Gamma}}$, as $\mu_{\mathcal{Z}}(M_{\Gamma})$ is no longer necessarily in \mathbf{D} . Thus, we cannot get the conditions on $|\sim$ from the conditions on μ .

Here is the Karl Schlechta's remedy. In Proposition 5.2.5 of [23], he gave for any function $\mu : \mathbf{Y} \rightarrow \mathbf{Y}$, necessary and sufficient conditions for the existence of a preferential structure \mathcal{Z} on \mathcal{V} such that $\forall X \in \mathbf{Y}$, $\mu(X)$ is the smallest element of \mathbf{Y} that contains $\mu_{\mathcal{Z}}(X)$. Note that in this proposition, it is assumed that \mathbf{Y} is closed under arbitrarily intersections and finite unions, and $\emptyset, \mathcal{V} \in \mathbf{Y}$. Then, Schlechta used Proposition 5.2.5 of [23] to give a nonnormal characterization of preferential CRs under the assumption that we are in the classical context.

Strongly inspired by Proposition 5.2.5 of [23], we will give for any function $\mu : \mathbf{Y} \rightarrow \mathbf{Y}$, necessary and sufficient conditions for the existence of a preferential structure \mathcal{Z} on \mathcal{V} such that $\forall X \in \mathbf{Y}$, $\mu(X) = M_{T(\mu_{\mathcal{Z}}(X))}$ (Lemma 21 below). Note that unlike Proposition 5.2.5 of [23], Lemma 21 has no further prerequisite. Then, by Lemma 21 we will give nonnormal characterizations of: preferential CRs (in Section 4); some families of preferential-discriminative CRs (in Section 6).

Definition 19 Let $\mathbf{Y} \subseteq \mathcal{P}(\mathcal{V})$ and $\mu : \mathbf{Y} \rightarrow \mathbf{Y}$ be a function. We denote by $\mu' : \mathbf{Y} \rightarrow \mathcal{P}(\mathcal{V})$ the function such that $\forall X \in \mathbf{Y}$,

$$\mu'(X) = \{x \in X : \forall Y \in \mathbf{Y}, \text{ if } Y \subseteq X \text{ and } x \in Y, \text{ then } x \in \mu(Y)\}$$

The lemma just below is from Karl Schlechta (Fact 5.2.2 and Proposition 5.2.4 of [23]).

Lemma 20 From [23]. Let $\mathbf{Y} \subseteq \mathcal{P}(\mathcal{V})$ and $\mu : \mathbf{Y} \rightarrow \mathbf{Y}$ be a function. Then, there exists a transitive irreflexive preferential structure \mathcal{Z} on \mathcal{V} such that $\forall X \in \mathbf{Y}$, $\mu'(X) = \mu_{\mathcal{Z}}(X)$.

Lemma 21 Let $\mathbf{Y} \subseteq \mathcal{P}(\mathcal{V})$ and $\mu : \mathbf{Y} \rightarrow \mathbf{Y}$ be a function. Then,

- (i) if there exists a preferential structure \mathcal{Z} on \mathcal{V} such that $\forall X \in \mathbf{Y}$, $\mu(X) = M_{T(\mu_{\mathcal{Z}}(X))}$, then $(\mu 3)$ holds;
- (ii) if $(\mu 3)$ holds, then there exists a transitive irreflexive preferential structure \mathcal{Z} on \mathcal{V} such that $\forall X \in \mathbf{Y}$, $\mu(X) = M_{T(\mu_{\mathcal{Z}}(X))}$;

where

$$(\mu 3) \quad \forall X \in \mathbf{Y}, \mu(X) = M_{T(\mu'(X))}.$$

Proof (i). By Theorem 7 (i), $\mu_{\mathcal{Z}}$ satisfies $(\mu 0)$ and $(\mu 1)$. We show that μ satisfies $(\mu 3)$. Let $X \in \mathbf{Y}$. Case 1: $\exists x \in \mu_{\mathcal{Z}}(X)$, $x \notin \mu'(X)$. Then, $x \in_{(\mu 0)} X$. Thus, as $x \notin \mu'(X)$, $\exists Y \in \mathbf{Y}$, $Y \subseteq X$, $x \in Y$, and $x \notin \mu(Y) = M_{T(\mu_{\mathcal{Z}}(Y))} \supseteq \mu_{\mathcal{Z}}(Y)$. However, $\mu_{\mathcal{Z}}(X) \cap Y \subseteq_{(\mu 1)} \mu_{\mathcal{Z}}(Y)$, thus $x \in \mu_{\mathcal{Z}}(Y)$, which is impossible.

Case 2: $\mu_{\mathcal{Z}}(X) \subseteq \mu'(X)$. Case 2.1: $\exists x \in \mu'(X)$, $x \notin \mu(X)$. Then, $x \in X$ and $\forall Y \in \mathbf{Y}$, if $Y \subseteq X$ and $x \in Y$, then $x \in \mu(Y)$. Thus, $x \in \mu(X)$, which is impossible. Case 2.2: $\mu'(X) \subseteq \mu(X)$. Then, $\mu(X) = M_{T(\mu_{\mathcal{Z}}(X))} \subseteq M_{T(\mu'(X))} \subseteq M_{T(\mu(X))} = M_{T(M_{T(\mu_{\mathcal{Z}}(X))})} = M_{T(\mu_{\mathcal{Z}}(X))} = \mu(X)$.

(ii). Obvious by Lemma 20 and $(\mu 3)$. \blacksquare

Proposition 22 Let $|\sim$ be a CR. Then,

- (i) if $|\sim$ is a preferential CR, then $(|\sim 5)$ holds;
- (ii) if $(|\sim 5)$ holds, then $|\sim$ is transitive irreflexive preferential CR;

where

$$(|\sim 5) \forall \Gamma, \tilde{\Gamma} = T(\{x \in M_\Gamma : \forall \Delta, \text{ if } M_\Delta \subseteq M_\Gamma \text{ and } x \in M_\Delta, \text{ then } x \in M_{\tilde{\Delta}}\}).$$

Proof (i). There is a preferential structure \mathcal{Z} on \mathcal{V} such that $\forall \Gamma, \tilde{\Gamma} = T(\mu_{\mathcal{Z}}(M_\Gamma))$. Let $\mu : \mathbf{D} \rightarrow \mathbf{D}$ be the function s.t. $\forall X \in \mathbf{D}, \mu(X) = M_{T(\mu_{\mathcal{Z}}(X))}$. By Lemma 21 (i), μ satisfies $(\mu 3)$. Moreover, $\forall \Gamma, \mu(M_\Gamma) = M_{T(\mu_{\mathcal{Z}}(M_\Gamma))} = M_{\tilde{\Gamma}}$. We show $(|\sim 5)$. $\forall \Gamma, \tilde{\Gamma} = T(\mu_{\mathcal{Z}}(M_\Gamma)) = T(M_{T(\mu_{\mathcal{Z}}(M_\Gamma))}) = T(\mu(M_\Gamma)) =_{(\mu 3)} T(M_{T(\mu'(M_\Gamma))}) = T(\mu'(M_\Gamma)) = T(\{x \in M_\Gamma : \forall Y \in \mathbf{D}, \text{ if } Y \subseteq M_\Gamma \text{ and } x \in Y, \text{ then } x \in \mu(Y)\}) = T(\{x \in M_\Gamma : \forall \Delta, \text{ if } M_\Delta \subseteq M_\Gamma \text{ and } x \in M_\Delta, \text{ then } x \in M_{\tilde{\Delta}}\})$.

(ii). Let $\mu : \mathbf{D} \rightarrow \mathbf{D}$ be the function such that $\forall \Gamma, \mu(M_\Gamma) = M_{\tilde{\Gamma}}$. If $M_\Gamma = M_\Delta$, then $\tilde{\Gamma} =_{(|\sim 5)} \tilde{\Delta}$, thus μ is well-defined. We show that μ satisfies $(\mu 3)$. $\forall \Gamma, \mu(M_\Gamma) = M_{\tilde{\Gamma}} =_{(|\sim 5)} M_{T(\{x \in M_\Gamma : \forall \Delta, \text{ if } M_\Delta \subseteq M_\Gamma \text{ and } x \in M_\Delta, \text{ then } x \in M_{\tilde{\Delta}}\})} = M_{T(\mu'(M_\Gamma))}$. Thus, by Lemma 21 (ii), there exists a transitive irreflexive preferential structure \mathcal{Z} on \mathcal{V} such that $\forall \Gamma, \tilde{\Gamma} =_{(|\sim 5)} T(M_{\tilde{\Gamma}}) = T(\mu(M_\Gamma)) = T(M_{T(\mu_{\mathcal{Z}}(M_\Gamma))}) = T(\mu_{\mathcal{Z}}(M_\Gamma))$. ■

We cannot get a normal characterization of preferential CRs from Proposition 22, because the condition depends on too many Δ . More explanations have been given in Section 2.5.

5 Definability preserving preferential-discriminative CRs

In this section, we give normal characterizations of the four following families:

- the DP preferential-discriminative CRs, under (A1), (A2), and (A3);
- the smooth DP preferential-discriminative CRs, under (A1), (A2), and (A3);
- the CP DP preferential-discriminative CRs, under (A1) and (A2);
- the smooth CP DP preferential-discriminative CRs, under (A1) and (A2).

5.1 The necessary and sufficient conditions

Definition 23 We say that Γ is consistent iff $\forall \alpha, \Gamma \not\vdash \alpha$ or $\Gamma \not\vdash \text{neg}(\alpha)$. We denote by \mathbb{N} the natural numbers including 0: $\{0, 1, 2, \dots\}$, by \mathbb{N}^+ the strictly positive natural numbers: $\{1, 2, \dots\}$, and by \mathbb{Z} the integers. For all $i, j \in \mathbb{Z}$, we denote by $[i, j]$ the set of all $k \in \mathbb{Z}$ such that $i \leq k \leq j$. In addition, for all $\beta_1, \beta_2, \dots, \beta_r \in \mathcal{L}$, when we write $or(\beta_1, \beta_2, \dots, \beta_r)$, we mean $or(or(\dots or(or(\beta_1, \beta_2), \beta_3), \dots), \beta_r))$ and when we write $or(\beta_1)$, we simply mean β_1 .

Definition 24 Let $|\sim$ be a CR, $\Gamma \subseteq \mathcal{L}$, and $i \in \mathbb{N}, i \geq 2$. Then,

- $H_1(\Gamma) \stackrel{def}{=} \{\text{neg}(\beta) : \beta \notin \tilde{\Gamma}, \beta \in \overline{\Gamma \cup \tilde{\Gamma}}, \text{ and } \text{neg}(\beta) \notin \overline{\Gamma \cup \tilde{\Gamma}}\}$;

- $H_i(\Gamma) \stackrel{def}{=} \{neg(\beta) : \begin{cases} \beta \notin \tilde{\Gamma} \text{ and} \\ \beta \in \overline{\Gamma \cup \tilde{\Gamma} \cup H_1(\Gamma) \cup \dots \cup H_{i-1}(\Gamma)} \text{ and} \\ neg(\beta) \notin \Gamma \cup \tilde{\Gamma} \cup H_1(\Gamma) \cup \dots \cup H_{i-1}(\Gamma) \end{cases}\};$
- $H(\Gamma) \stackrel{def}{=} \bigcup_{i \in \mathbb{N}^+} H_i(\Gamma).$

Note that H should be indexed by $|\sim$, but as there will never be any ambiguity, we omit it to increase readability. We can now give the representation results.

Proposition 25 Let $|\sim$ be a CR. Then,

- (i) if (A1), (A2), and (A3) hold and $|\sim$ is a DP preferential-discriminative CR, then ($|\sim$ 0), ($|\sim$ 6), ($|\sim$ 7), ($|\sim$ 8), and ($|\sim$ 9) hold;
- (ii) if (A1), (A2), ($|\sim$ 0), ($|\sim$ 6), ($|\sim$ 7), ($|\sim$ 8), and ($|\sim$ 9) hold, then $|\sim$ is a transitive irreflexive DP preferential-discriminative CR;
- (iii) if (A1), (A2), and (A3) hold and $|\sim$ is a smooth DP preferential-discriminative CR, then ($|\sim$ 0), ($|\sim$ 6), ($|\sim$ 7), ($|\sim$ 8), ($|\sim$ 9) and ($|\sim$ 10) hold;
- (iv) if (A1), (A2), ($|\sim$ 0), ($|\sim$ 6), ($|\sim$ 7), ($|\sim$ 8), ($|\sim$ 9) and ($|\sim$ 10) hold, then $|\sim$ is a transitive irreflexive smooth DP preferential-discriminative CR;
- (v) if (A1) and (A2) hold and $|\sim$ is a CP DP preferential-discriminative CR, then ($|\sim$ 0), ($|\sim$ 6), ($|\sim$ 7), ($|\sim$ 8), ($|\sim$ 9), and ($|\sim$ 11) hold;
- (vi) if (A1), (A2), ($|\sim$ 0), ($|\sim$ 6), ($|\sim$ 7), ($|\sim$ 8), ($|\sim$ 9), and ($|\sim$ 11) hold, then $|\sim$ is a transitive irreflexive CP DP preferential-discriminative CR;
- (vii) if (A1) and (A2) hold and $|\sim$ is a smooth CP DP preferential-discriminative CR, then ($|\sim$ 0), ($|\sim$ 6), ($|\sim$ 7), ($|\sim$ 8), ($|\sim$ 9), ($|\sim$ 10), and ($|\sim$ 11) hold;
- (viii) If (A1), (A2), ($|\sim$ 0), ($|\sim$ 6), ($|\sim$ 7), ($|\sim$ 8), ($|\sim$ 9), ($|\sim$ 10), and ($|\sim$ 11) hold, then $|\sim$ is a transitive irreflexive smooth CP DP preferential-discriminative CR;

where

- ($|\sim$ 6) $\forall \Gamma, \alpha, \beta$, if $\beta \notin \tilde{\Gamma}$, $\beta \in \overline{\Gamma \cup \tilde{\Gamma}}$, and $\Gamma \cup \tilde{\Gamma} \cup \{neg(\alpha)\} \subseteq \overline{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta)\}}$, then $\alpha \notin \tilde{\Gamma}$;
- ($|\sim$ 7) $\forall \Gamma, \alpha, \beta$, if $\alpha, \beta \notin \tilde{\Gamma}$, $\alpha \in \overline{\Gamma \cup \tilde{\Gamma}}$, and $\Gamma \cup \tilde{\Gamma} \cup \{\beta\} \subseteq \overline{\Gamma \cup \tilde{\Gamma} \cup \{neg(\alpha)\}}$, then $or(\alpha, \beta) \notin \tilde{\Gamma}$;
- ($|\sim$ 8) $\forall \Gamma, \alpha$, if $\alpha \in \tilde{\Gamma}$, then $neg(\alpha) \notin \overline{\Gamma \cup \tilde{\Gamma}}$;
- ($|\sim$ 9) $\forall \Gamma, \Delta$, if $\Delta \subseteq \overline{\Gamma}$, then $\Gamma \cup \tilde{\Gamma} \cup H(\Gamma) \subseteq \overline{\Delta \cup \tilde{\Delta} \cup H(\Delta) \cup \Gamma}$;
- ($|\sim$ 10) $\forall \Gamma, \Delta$, if $\Gamma \subseteq \overline{\Delta} \subseteq \overline{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)}$, then $\overline{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)} = \overline{\Delta \cup \tilde{\Delta} \cup H(\Delta)}$;
- ($|\sim$ 11) $\forall \Gamma$, if Γ is consistent, then $\tilde{\Gamma}$ is consistent, $\Gamma \subseteq \tilde{\Gamma}$, and $\overline{\tilde{\Gamma}} = \tilde{\Gamma}$.

Note that the conditions: $(|\sim 0)$, $(|\sim 6)$, $(|\sim 7)$, $(|\sim 8)$, $(|\sim 9)$, $(|\sim 10)$, and $(|\sim 11)$ depend only on $|\sim$, \vdash , \mathcal{L} , *neg*, and *or*. As \vdash has been defined semantically (in Section 2.2), these conditions have a semantic aspect. In parallel, as we have a proof system for \vdash in the classical, \mathcal{FOUR} , and J_3 contexts (Sections 2.8 and 2.9), the conditions have also a syntactic aspect in these contexts. In addition, $(|\sim 8)$ is equivalent to $(|\sim 8d)$, where

$$(|\sim 8d) \quad \forall \Gamma, \tilde{\Gamma} \subseteq \overline{\Gamma \cup \tilde{\Gamma}}^d.$$

As \vdash_d has been defined semantically (in Section 2.2), $(|\sim 8d)$ has a semantic aspect. We will give a proof system for \vdash_d and thus for \vdash in the \mathcal{FOUR} and J_3 contexts (in Section 7). Thus, $(|\sim 8d)$ has also a syntactic aspect in these contexts.

It now remains to prove Proposition 25. The proof has been relegated at the end of Section 5.1. Indeed, we first need Definitions 26 and Lemmas 27, 28, and 29 below. We now give an overview of the proofs of (i) and (ii) . The proofs of (iii) , (v) and (vii) are similar to the proof of (i) . The proofs of (iv) , (vi) , and $(viii)$ are similar to the proof of (ii) .

Overview of the proof of proposition 25 We begin with (ii) . If $(|\sim 0)$, $(|\sim 6)$, $(|\sim 7)$, $(|\sim 8)$, and $(|\sim 9)$ hold, then we can construct a function $\mu : \mathbf{D} \rightarrow \mathbf{D}$ such that $(\mu 0)$, $(\mu 1)$, and $\forall \Gamma$, $\tilde{\Gamma} = T_d(\mu(M_\Gamma))$. Then, by Theorem 7, $|\sim$ is clearly a transitive irreflexive DP preferential-discriminative CR. Recall that $(\mu 0)$ and $(\mu 1)$ have been defined in Section 3.1. We now describe μ . It suffices to take the function such that $\forall \Gamma$, $\mu(M_\Gamma) = M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)}$. Indeed, by $(|\sim 0)$, μ is well-defined, by $(|\sim 9)$, μ satisfies $(\mu 1)$, and clearly μ satisfies $(\mu 0)$. In addition, in Lemma 28 below, we show that $(|\sim 6)$, $(|\sim 7)$, and $(|\sim 8)$ entail $\forall \Gamma$, $\tilde{\Gamma} = T_d(M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)}) = T_d(\mu(M_\Gamma))$. The proofs of (iv) , (vi) and $(viii)$ are very similar. Roughly speaking, from the other properties of $|\sim$ (resp. $(|\sim 10)$ and $(|\sim 11)$), we show that μ satisfies some other properties (resp. $(\mu 2)$ and CP), from which we show that $|\sim$ belongs to the desired family (resp. smooth and CP).

We turn to (i) . If $|\sim$ is a DP preferential-discriminative CR, then there exists a DP preferential structure \mathcal{Z} on \mathcal{V} such that $\forall \Gamma$, $\tilde{\Gamma} = T_d(\mu_{\mathcal{Z}}(M_\Gamma))$. Then, $(|\sim 0)$ clearly holds. In addition, by Theorem 7, $\mu_{\mathcal{Z}}$ satisfies $(\mu 0)$ and $(\mu 1)$. Then, we show in Lemma 29 below that $(|\sim 6)$, $(|\sim 7)$, $(|\sim 8)$ hold and $\forall \Gamma$, $\mu_{\mathcal{Z}}(M_\Gamma) = M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)}$. Thus, by $(\mu 1)$, $(|\sim 9)$ also holds. The proofs of (iii) , (v) , (vii) are very similar. Roughly speaking, from the properties of $|\sim$ inherited from the family to which $|\sim$ belongs (resp. smooth and CP), we show that $\mu_{\mathcal{Z}}$ satisfies some properties (resp. $(\mu 2)$ and CP), from which we show that the desired properties of $|\sim$ hold (resp. $(|\sim 10)$ and $(|\sim 11)$).

We hope that this overview have given the reader an idea of what is shown in Lemma 28 and 29 below. Lemma 27 below is just a part of Lemma 28 that we have separated from the rest to increase readability. We now explain the utility of Definition 26 just below. The purpose of these definitions is to introduce for all Γ , a set $F(\Gamma)$, which is very close to $H(\Gamma)$. The reason is that in Lemma 28 and 29 below, we do not work directly with $H(\Gamma)$, but with $F(\Gamma)$. Let us make it more precise with the previous overview. We begin with the overview of the proof of (ii) . In fact, in Lemma 28, we first show that if $(|\sim 6)$, $(|\sim 7)$, and $(|\sim 8)$ hold, then $\forall \Gamma$, $M_{\Gamma \cup \tilde{\Gamma} \cup F(\Gamma)} = M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)}$ (this is why we said that $F(\Gamma)$ is very close to $H(\Gamma)$). Then, we show $\forall \Gamma$, $\tilde{\Gamma} = T_d(M_{\Gamma \cup \tilde{\Gamma} \cup F(\Gamma)})$. And it is only then that we conclude $\forall \Gamma$, $\tilde{\Gamma} = T_d(M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)})$. We turn to the overview of the proof of (i) . In fact, in Lemma 29 we show that $(|\sim 6)$, $(|\sim 7)$, $(|\sim 8)$ hold and that $\forall \Gamma$, $\mu_{\mathcal{Z}}(M_\Gamma) = M_{\Gamma \cup \tilde{\Gamma} \cup F(\Gamma)}$. It is only then that we conclude $\forall \Gamma$, $\mu_{\mathcal{Z}}(M_\Gamma) = M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)}$.

We hope to have given the reader an idea of the utility of $F(\Gamma)$. We confess that we do not have given any intuition behind $H(\Gamma)$ and $F(\Gamma)$, we just have explained the utility of these sets. In fact, $H(\Gamma)$ and $F(\Gamma)$ are mainly technical tools which help us to achieve the representation results (Proposition 25) and thus there is no real intuition. We now introduce $F(\Gamma)$, which requires the introduction of $M_{\tilde{\Gamma}}^i$, $M'_{\tilde{\Gamma}}$, $n_{\tilde{\Gamma}}$, $\beta_{\tilde{\Gamma}}^i$, and $\beta_{\tilde{\Gamma}}$. We also define a set $G(\Gamma)$ which is just a technical tool used in Lemma 29. We need to fix for the rest of the paper a total order $<_{\mathcal{V}}$ on \mathcal{V} . This total order is completely auxiliary and chosen arbitrarily.

Definition 26 Let (A2) holds, $|\sim$ be a CR, and $\Gamma \subseteq \mathcal{L}$. We denote by $M_{\tilde{\Gamma}}^1$ the set such that

$$M_{\tilde{\Gamma}}^1 = \{x \in M_{\Gamma \cup \tilde{\Gamma}} : \exists \beta, \beta \notin \tilde{\Gamma}, M_{\Gamma \cup \tilde{\Gamma}} \subseteq M_{\beta}, \text{ and } x \notin M_{neg(\beta)}\}$$

Let $i \in \mathbb{N}, i \geq 2$. We denote by $M_{\tilde{\Gamma}}^i$ the set such that

$$M_{\tilde{\Gamma}}^i = \{x \in M_{\Gamma \cup \tilde{\Gamma}} \setminus M_{\tilde{\Gamma}}^1 \cup \dots \cup M_{\tilde{\Gamma}}^{i-1} : \exists \beta, \beta \notin \tilde{\Gamma}, M_{\Gamma \cup \tilde{\Gamma}} \setminus M_{\tilde{\Gamma}}^1 \cup \dots \cup M_{\tilde{\Gamma}}^{i-1} \subseteq M_{\beta}, x \notin M_{neg(\beta)}\}$$

We denote by $M'_{\tilde{\Gamma}}$ the set such that

$$M'_{\tilde{\Gamma}} = \bigcup_{i \in \mathbb{N}^+} M_{\tilde{\Gamma}}^i$$

We denote by $n_{\tilde{\Gamma}}$ the cardinal such that

$$n_{\tilde{\Gamma}} = |\{i \in \mathbb{N}^+ : M_{\tilde{\Gamma}}^i \neq \emptyset\}|$$

If $M_{\tilde{\Gamma}}^1 \neq \emptyset$, then we denote by $\beta_{\tilde{\Gamma}}^1$ the element of \mathcal{L} , such that

$$\beta_{\tilde{\Gamma}}^1 = or(\beta_1, \beta_2, \dots, \beta_r)$$

where r is the cardinality of $M_{\tilde{\Gamma}}^1$ and $\forall j \in [1, r]$, β_j is an element of \mathcal{L} chosen arbitrarily such that $\beta_j \notin \tilde{\Gamma}$ and $M_{\Gamma \cup \tilde{\Gamma}} \subseteq M_{\beta_j}$ and $x_j \notin M_{neg(\beta_j)}$, where x_j is the j -th $<_{\mathcal{V}}$ -smallest element of $M_{\tilde{\Gamma}}^1$. Note that $r \geq 1$ and by (A2), r is finite, thus $\beta_{\tilde{\Gamma}}^1$ is well-defined.

Let $i \in \mathbb{N}, i \geq 2$. If $M_{\tilde{\Gamma}}^i \neq \emptyset$, then we denote by $\beta_{\tilde{\Gamma}}^i$ the element of \mathcal{L} , such that

$$\beta_{\tilde{\Gamma}}^i = or(\beta_1, \beta_2, \dots, \beta_r)$$

where r is the cardinality of $M_{\tilde{\Gamma}}^i$ and $\forall j \in [1, r]$, β_j is an element of \mathcal{L} chosen arbitrarily such that $\beta_j \notin \tilde{\Gamma}$ and $M_{\Gamma \cup \tilde{\Gamma}} \setminus M_{\tilde{\Gamma}}^1 \cup \dots \cup M_{\tilde{\Gamma}}^{i-1} \subseteq M_{\beta_j}$ and $x_j \notin M_{neg(\beta_j)}$, where x_j is the j -th $<_{\mathcal{V}}$ -smallest element of $M_{\tilde{\Gamma}}^i$. Note that $r \geq 1$ and by (A2), r is finite, thus $\beta_{\tilde{\Gamma}}^i$ is well-defined.

If $M'_{\tilde{\Gamma}} \neq \emptyset$, then we denote by $\beta_{\tilde{\Gamma}}$ the element of \mathcal{L} , such that

$$\beta_{\tilde{\Gamma}} = or(\beta_{\tilde{\Gamma}}^1, \beta_{\tilde{\Gamma}}^2, \dots, \beta_{\tilde{\Gamma}}^{n_{\tilde{\Gamma}}})$$

Note that $n_{\tilde{\Gamma}} \geq 1$. In addition, we will show in Lemma 27 that $n_{\tilde{\Gamma}}$ is finite and $\forall i \in \mathbb{N}^+, i \leq n_{\tilde{\Gamma}}, M_{\tilde{\Gamma}}^i \neq \emptyset$. Thus, $\beta_{\tilde{\Gamma}}$ is well-defined.

We denote by $F(\Gamma)$ the set such that

$$F(\Gamma) = \begin{cases} \{neg(\beta_{\tilde{\Gamma}})\} & \text{if } M'_{\tilde{\Gamma}} \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

We denote by $G(\Gamma)$ the set such that

$$G(\Gamma) = \{\alpha \in \mathcal{L} : \alpha \notin \tilde{\Gamma}, neg(\alpha) \notin \tilde{\Gamma} \text{ and } T_d(M_{\Gamma \cup \tilde{\Gamma} \cup \{\alpha\}}) \subseteq \tilde{\Gamma}\}$$

Note that these definitions should be indexed by $|\sim$, but as there will never be any ambiguity, we omit it to increase readability.

Lemma 27 Let (A2) holds, $|\sim$ be a CR, $\Gamma \subseteq \mathcal{L}$ and $i, j \in \mathbb{N}^+$. Then,

- (0) if $i \neq j$, then $M_\Gamma^i \cap M_\Gamma^j = \emptyset$;
- (1) if $M_\Gamma^i = \emptyset$, then $M_\Gamma^{i+1} = \emptyset$;
- (2) $T_d(M_{\Gamma \cup \tilde{\Gamma}}) \subseteq \tilde{\Gamma}$ iff $M_\Gamma^1 = \emptyset$;
- (3) if $i \geq 2$, then $T_d(M_{\Gamma \cup \tilde{\Gamma}} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^{i-1}) \subseteq \tilde{\Gamma}$ iff $M_\Gamma^i = \emptyset$;
- (4) n_Γ is finite;
- (5) if $i \leq n_\Gamma$, then $M_\Gamma^i \neq \emptyset$;
- (6) if $i > n_\Gamma$, then $M_\Gamma^i = \emptyset$;
- (7) if $M'_\Gamma \neq \emptyset$, then $M'_\Gamma = M_\Gamma^1 \cup \dots \cup M_\Gamma^{n_\Gamma}$;
- (8) $T_d(M_{\Gamma \cup \tilde{\Gamma}} \setminus M'_\Gamma) \subseteq \tilde{\Gamma}$.

Proof (0), (1), (2), and (3). Trivial.

(4). Obvious by (0) and (A2).

(5). Suppose $\exists i \in \mathbb{N}^+$, $M_\Gamma^i = \emptyset$ and $i \leq n_\Gamma$. By (1), $\forall j \in \mathbb{N}^+$, $j \geq i$, $M_\Gamma^j = \emptyset$. Thus, $|\{j \in \mathbb{N}^+ : M_\Gamma^j \neq \emptyset\}| < i \leq n_\Gamma$, which is impossible.

(6). Suppose $\exists i \in \mathbb{N}^+$, $M_\Gamma^i \neq \emptyset$ and $i > n_\Gamma$. By (1), $\forall j \in \mathbb{N}^+$, $j \leq i$, $M_\Gamma^j \neq \emptyset$. Thus, $|\{j \in \mathbb{N}^+ : M_\Gamma^j \neq \emptyset\}| \geq i > n_\Gamma$, which is impossible.

(7). We get $n_\Gamma \in \mathbb{N}^+$. Thus, $M'_\Gamma = M_\Gamma^1 \cup \dots \cup M_\Gamma^{n_\Gamma} \cup \dots \stackrel{(6)}{=} M_\Gamma^1 \cup \dots \cup M_\Gamma^{n_\Gamma}$.

(8). Case 1: $M'_\Gamma = \emptyset$. Then, $T_d(M_{\Gamma \cup \tilde{\Gamma}} \setminus M'_\Gamma) = T_d(M_{\Gamma \cup \tilde{\Gamma}}) \subseteq_{(2)} \tilde{\Gamma}$, as $M_\Gamma^1 = \emptyset$. Case 2: $M'_\Gamma \neq \emptyset$. $T_d(M_{\Gamma \cup \tilde{\Gamma}} \setminus M'_\Gamma) \stackrel{(7)}{=} T_d(M_{\Gamma \cup \tilde{\Gamma}} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^{n_\Gamma}) \subseteq_{(3)} \tilde{\Gamma}$, as $n_\Gamma + 1 \geq 2$ and $M_\Gamma^{n_\Gamma+1} \stackrel{(6)}{=} \emptyset$. ■

We will use Lemma 27 implicitly in the sequel. Lemmas 28 and 29 below have an interest of their own. Indeed, we will use them to characterize some families of DP preferential-discriminative CRs in this section. In addition, we will use them in Section 6 to characterize some families of preferential-discriminative CRs that are not necessarily DP. In fact, it seems that Lemmas 28 and 29 can be useful to characterize any family C of CRs such that for all CR $|\sim$, $|\sim \in C$ iff $\forall \Gamma$, $\tilde{\Gamma} = T_d(\mu(M_\Gamma))$, where $\mu : \mathbf{D} \rightarrow \mathcal{P}(\mathcal{V})$ is simply a function satisfying $(\mu 0)$.

Lemma 28 Let (A1) and (A2) hold, let $|\sim$ be a CR such that $(|\sim 6)$, $(|\sim 7)$, and $(|\sim 8)$ hold, and let $\Gamma \subseteq \mathcal{L}$. Then,

- (1) $\beta_\Gamma \notin \tilde{\Gamma}$ if $M'_\Gamma \neq \emptyset$;
- (2) $M_{\Gamma \cup \tilde{\Gamma}} \subseteq M_{\beta_\Gamma}$ if $M'_\Gamma \neq \emptyset$;
- (3) $M_{\Gamma \cup \tilde{\Gamma}} \setminus M'_\Gamma \subseteq M_{neg(\beta_\Gamma)}$ if $M'_\Gamma \neq \emptyset$;
- (4) $M'_\Gamma \cap M_{neg(\beta_\Gamma)} = \emptyset$ if $M'_\Gamma \neq \emptyset$;
- (5) $M_{\Gamma \cup \tilde{\Gamma}} \setminus M'_\Gamma = M_{\Gamma \cup \tilde{\Gamma} \cup F(\Gamma)}$;

$$(6) \quad \tilde{\Gamma} = T_d(M_{\Gamma \cup \tilde{\Gamma} \cup F(\Gamma)});$$

$$(7) \quad M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)} = M_{\Gamma \cup \tilde{\Gamma} \cup F(\Gamma)};$$

$$(8) \quad \tilde{\Gamma} = T_d(M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)}).$$

Proof (1), (2), and (4). It suffices to show by induction $\forall i \in [1, n_\Gamma]$, $p_1(i)$, $p_2(i)$, $p_3(i)$ hold, with

$$p_1(i) \quad or(\beta_\Gamma^1, \dots, \beta_\Gamma^i) \notin \tilde{\Gamma};$$

$$p_2(i) \quad M_{\Gamma \cup \tilde{\Gamma}} \subseteq M_{or(\beta_\Gamma^1, \dots, \beta_\Gamma^i)};$$

$$p_3(i) \quad (M_\Gamma^1 \cup \dots \cup M_\Gamma^i) \cap M_{neg \circ or(\beta_\Gamma^1, \dots, \beta_\Gamma^i)} = \emptyset.$$

We can pose $M_\Gamma^1 = \{x_1, \dots, x_r\}$ and $\beta_\Gamma^1 = or(\beta_1, \dots, \beta_r)$, with $r \in \mathbb{N}^+$ and $\forall j \in [1, r]$, $\beta_j \notin \tilde{\Gamma}$ and $M_{\Gamma \cup \tilde{\Gamma}} \subseteq M_{\beta_j}$ and $x_j \notin M_{neg(\beta_j)}$. By (A1), we get $p_2(1)$. We show $p_3(1)$. Let $x \in M_\Gamma^1$. We get $\exists j \in [1, r]$, $x = x_j$. Thus, $x \notin M_{neg(\beta_j)} \supseteq_{(A1)} M_{neg(\beta_\Gamma^1)}$. We show $p_1(1)$, but we relegate the proof to (*) below to increase readability.

Let $i \in [1, n_\Gamma - 1]$. Suppose $p_1(i)$, $p_2(i)$, and $p_3(i)$. We can pose $M_\Gamma^{i+1} = \{x_1, \dots, x_r\}$ and $\beta_\Gamma^{i+1} = or(\beta_1, \dots, \beta_r)$, with $r \in \mathbb{N}^+$ and $\forall j \in [1, r]$, $\beta_j \notin \tilde{\Gamma}$ and $M_{\Gamma \cup \tilde{\Gamma}} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^i \subseteq M_{\beta_j}$ and $x_j \notin M_{neg(\beta_j)}$. We show $p_2(i+1)$. We get $M_{\Gamma \cup \tilde{\Gamma}} \subseteq_{p_2(i)} M_{or(\beta_\Gamma^1, \dots, \beta_\Gamma^i)} \subseteq_{(A1)} M_{or(\beta_\Gamma^1, \dots, \beta_\Gamma^{i+1})}$. We show $p_3(i+1)$. Let $x \in M_1 \cup \dots \cup M_{i+1}$. Case 1: $x \in M_1 \cup \dots \cup M_i$. We get $x \notin_{p_3(i)} M_{neg \circ or(\beta_\Gamma^1, \dots, \beta_\Gamma^i)} \supseteq_{(A1)} M_{neg \circ or(\beta_\Gamma^1, \dots, \beta_\Gamma^{i+1})}$. Case 2: $x \in M_\Gamma^{i+1}$. We get $\exists j \in [1, r]$, $x = x_j$. Thus, $x \notin M_{neg(\beta_j)} \supseteq_{(A1)} M_{neg(\beta_\Gamma^{i+1})} \supseteq_{(A1)} M_{neg \circ or(\beta_\Gamma^1, \dots, \beta_\Gamma^{i+1})}$. We show $p_1(i+1)$. We show $or(\beta_\Gamma^1, \dots, \beta_\Gamma^i, \beta_1, \dots, \beta_r) \notin \tilde{\Gamma}$, but we relegate the proof to (**) below to increase readability. Then, we get $M_{\Gamma \cup \tilde{\Gamma}} \subseteq_{p_2(i)} M_{or(\beta_\Gamma^1, \dots, \beta_\Gamma^i)} \subseteq_{(A1)} M_{or(\beta_\Gamma^1, \dots, \beta_\Gamma^i, \beta_1, \dots, \beta_r)}$. We get also $M_{neg \circ or(\beta_\Gamma^1, \dots, \beta_\Gamma^i, \beta_1, \dots, \beta_r)} =_{(A1)} M_{neg \circ or(\beta_\Gamma^1, \dots, \beta_\Gamma^{i+1})}$. Thus, $or(\beta_\Gamma^1, \dots, \beta_\Gamma^{i+1}) \notin_{(\sim 6)} \tilde{\Gamma}$.

(*) It suffices to show by induction $\forall j \in [1, r]$, $q(j)$, where

$$q(j) \quad or(\beta_1, \dots, \beta_j) \notin \tilde{\Gamma}.$$

Clearly, $q(1)$ holds. Let $j \in [1, r-1]$. Suppose $q(j)$. We show $q(j+1)$. $M_{\Gamma \cup \tilde{\Gamma}} \subseteq_{(A1)} M_{or(\beta_1, \dots, \beta_j)}$ and $M_{\Gamma \cup \tilde{\Gamma} \cup \{neg \circ or(\beta_1, \dots, \beta_j)\}} \subseteq M_{\Gamma \cup \tilde{\Gamma}} = M_{\Gamma \cup \tilde{\Gamma} \cup \{\beta_{j+1}\}}$. Thus, $or(\beta_1, \dots, \beta_{j+1}) \notin_{q(j), (\sim 7)} \tilde{\Gamma}$.

(**) It suffices to show by induction $\forall j \in [1, r]$, $q(j)$, where

$$q(j) \quad or(\beta_\Gamma^1, \dots, \beta_\Gamma^i, \beta_1, \dots, \beta_j) \notin \tilde{\Gamma}.$$

We show $q(1)$. If $x \in M_{\Gamma \cup \tilde{\Gamma} \cup \{neg \circ or(\beta_\Gamma^1, \dots, \beta_\Gamma^i)\}}$, then $x \notin_{p_3(i)} M_\Gamma^1 \cup \dots \cup M_\Gamma^i$, thus $x \in M_{\Gamma \cup \tilde{\Gamma} \cup \{\beta_1\}}$.

Thus, $M_{\Gamma \cup \tilde{\Gamma} \cup \{neg \circ or(\beta_\Gamma^1, \dots, \beta_\Gamma^i)\}} \subseteq M_{\Gamma \cup \tilde{\Gamma} \cup \{\beta_1\}}$. Thus, $or(\beta_\Gamma^1, \dots, \beta_\Gamma^i, \beta_1) \notin_{p_1(i), p_2(i), (\sim 7)} \tilde{\Gamma}$.

Let $j \in [1, r-1]$. Suppose $q(j)$. We show $q(j+1)$. Let $x \in M_{\Gamma \cup \tilde{\Gamma} \cup \{neg \circ or(\beta_\Gamma^1, \dots, \beta_\Gamma^i, \beta_1, \dots, \beta_j)\}}$.

We get $x \in_{(A1)} M_{neg \circ or(\beta_\Gamma^1, \dots, \beta_\Gamma^i)}$, thus $x \notin_{p_3(i)} M_\Gamma^1 \cup \dots \cup M_\Gamma^i$, thus $x \in M_{\Gamma \cup \tilde{\Gamma} \cup \{\beta_{j+1}\}}$. Thus,

$M_{\Gamma \cup \tilde{\Gamma} \cup \{neg \circ or(\beta_\Gamma^1, \dots, \beta_\Gamma^i, \beta_1, \dots, \beta_j)\}} \subseteq M_{\Gamma \cup \tilde{\Gamma} \cup \{\beta_{j+1}\}}$. We get $M_{\Gamma \cup \tilde{\Gamma}} \subseteq_{p_2(i)} M_{or(\beta_\Gamma^1, \dots, \beta_\Gamma^i)} \subseteq_{(A1)}$

$M_{or(\beta_\Gamma^1, \dots, \beta_\Gamma^i, \beta_1, \dots, \beta_j)}$. Thus, $or(\beta_\Gamma^1, \dots, \beta_\Gamma^i, \beta_1, \dots, \beta_{j+1}) \notin_{q(j), (\sim 7)} \tilde{\Gamma}$.

(3). If $\exists x \in M_{\Gamma \cup \tilde{\Gamma}} \setminus M_\Gamma^i$, $x \notin M_{neg(\beta_\Gamma)}$, then $x \in_{(1), (2)} M_\Gamma^{n_\Gamma+1}$, impossible.

(5). Case 1: $M'_\Gamma \neq \emptyset$. We get $M_{\Gamma \cup \tilde{\Gamma}} \setminus M'_\Gamma \subseteq_{(3)} M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta_\Gamma)\}}$. We get $M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta_\Gamma)\}} \subseteq_{(4)} M_{\Gamma \cup \tilde{\Gamma}} \setminus M'_\Gamma$. Case 2: $M'_\Gamma = \emptyset$. We get $M_{\Gamma \cup \tilde{\Gamma}} \setminus M'_\Gamma = M_{\Gamma \cup \tilde{\Gamma}} = M_{\Gamma \cup \tilde{\Gamma} \cup F(\Gamma)}$.

(6). Direction: “ \subseteq ”. Case 1: $M'_\Gamma \neq \emptyset$. Suppose $\exists \alpha, \alpha \in \tilde{\Gamma}$ and $\alpha \notin T_d(M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta_\Gamma)\}})$. We get, $M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta_\Gamma)\}} \subseteq M_{\tilde{\Gamma}} \subseteq M_\alpha$. Thus, $M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta_\Gamma)\}} \subseteq M_{neg(\alpha)}$, thus $M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta_\Gamma)\}} \subseteq M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\alpha)\}}$. Thus, $\alpha \notin_{(1),(2),(1 \sim 6)} \tilde{\Gamma}$, which is impossible.

Case 2: $M'_\Gamma = \emptyset$. Let $\alpha \in \tilde{\Gamma}$. We get $M_{\Gamma \cup \tilde{\Gamma}} \subseteq M_{\tilde{\Gamma}} \subseteq M_\alpha$. We get $M_{\Gamma \cup \tilde{\Gamma}} \not\subseteq_{(1 \sim 8)} M_{neg(\alpha)}$. Thus, $\alpha \in T_d(M_{\Gamma \cup \tilde{\Gamma}}) = T_d(M_{\Gamma \cup \tilde{\Gamma} \cup F(\Gamma)})$.

Direction: “ \supseteq ”. Obvious by (5) and Lemma 27 (8).

(7). Direction: “ \subseteq ”. Case 1: $M'_\Gamma = \emptyset$. Case 1.1: $H_1(\Gamma) \neq \emptyset$. We get $\exists \alpha, \alpha \notin \tilde{\Gamma}, M_{\Gamma \cup \tilde{\Gamma}} \subseteq M_\alpha$, and $M_{\Gamma \cup \tilde{\Gamma}} \not\subseteq M_{neg(\alpha)}$. Thus, $\alpha \in T_d(M_{\Gamma \cup \tilde{\Gamma}}) = T_d(M_{\Gamma \cup \tilde{\Gamma} \cup \emptyset}) =_{(6)} \tilde{\Gamma}$, which is impossible. Case 1.2: $H_1(\Gamma) = \emptyset$. As $\forall i \in \mathbb{N}^+$, if $H_i(\Gamma) = \emptyset$, then $H_{i+1}(\Gamma) = \emptyset$, we get $H(\Gamma) = \emptyset = F(\Gamma)$.

Case 2: $M'_\Gamma \neq \emptyset$. As, $M'_\Gamma \subseteq M_{\Gamma \cup \tilde{\Gamma}}$, we get $M_{\Gamma \cup \tilde{\Gamma}} \not\subseteq_{(4)} M_{neg(\beta_\Gamma)}$. Thus, $neg(\beta_\Gamma) \in_{(1),(2)} H_1(\Gamma) \subseteq H(\Gamma)$. Thus, $M_{H(\Gamma)} \subseteq M_{F(\Gamma)}$.

Direction: “ \supseteq ”. Case 1: $M'_\Gamma = \emptyset$. Verbatim the same reasoning as for case 1 of the “ \subseteq ” direction. Case 2: $M'_\Gamma \neq \emptyset$. It suffices to show by induction that $\forall i \in \mathbb{N}^+, p(i)$, where

$$p(i) \quad M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta_\Gamma)\}} \subseteq M_{\Gamma \cup \tilde{\Gamma} \cup H_1(\Gamma) \cup \dots \cup H_i(\Gamma)}.$$

We show $p(1)$ by contradiction. Suppose $\exists x, x \in M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta_\Gamma)\}}$ and $x \notin M_{\Gamma \cup \tilde{\Gamma} \cup H_1(\Gamma)}$. Then, $x \notin M_{H_1(\Gamma)}$. Thus, $\exists \beta, M_{\Gamma \cup \tilde{\Gamma}} \subseteq M_\beta, \beta \notin \tilde{\Gamma}$, and $x \notin M_{neg(\beta)}$. Thus $x \in M'_\Gamma$, thus $x \in M'_\Gamma \cap M_{neg(\beta_\Gamma)}$, which is impossible by (4).

Let $i \in \mathbb{N}^+$. Suppose $p(i)$. We show $p(i+1)$ by contradiction. Suppose $\exists x, x \in M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta_\Gamma)\}}$ and $x \notin M_{\Gamma \cup \tilde{\Gamma} \cup H_1(\Gamma) \cup \dots \cup H_{i+1}(\Gamma)}$. Then, $x \in_{(5)} M_{\Gamma \cup \tilde{\Gamma}} \setminus M'_\Gamma = M_{\Gamma \cup \tilde{\Gamma}} \setminus M'_\Gamma \cup \dots \cup M_{\Gamma}^{nr}$ and $\exists j \in [1, i+1], x \notin M_{H_j(\Gamma)}$. Case 2.1: $j = 1$. As for $p(1)$, $\exists \beta, M_{\Gamma \cup \tilde{\Gamma}} \subseteq M_\beta, \beta \notin \tilde{\Gamma}$, and $x \notin M_{neg(\beta)}$. Thus $x \in M'_\Gamma \cap M_{neg(\beta_\Gamma)}$, which is impossible by (4). Case 2.2: $j > 1$. Then, $\exists \beta, M_{\Gamma \cup \tilde{\Gamma} \cup H_1(\Gamma) \cup \dots \cup H_{j-1}(\Gamma)} \subseteq M_\beta, \beta \notin \tilde{\Gamma}$, and $x \notin M_{neg(\beta)}$. But, $M_{\Gamma \cup \tilde{\Gamma}} \setminus M'_\Gamma \cup \dots \cup M_{\Gamma}^{nr} =_{(5)} M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta_\Gamma)\}} \subseteq_{p(i)} M_{\Gamma \cup \tilde{\Gamma} \cup H_1(\Gamma) \cup \dots \cup H_i(\Gamma)} \subseteq M_{\Gamma \cup \tilde{\Gamma} \cup H_1(\Gamma) \cup \dots \cup H_{j-1}(\Gamma)} \subseteq M_\beta$. Thus, $x \in M_{\Gamma}^{nr+1}$, which is impossible.

(8). Obvious by (6) and (7). ■

Lemma 29 Let (A1) and (A2) hold, $|\sim$ be a CR, and $\mu : \mathbf{D} \rightarrow \mathbf{D}$ be a function such that $(\mu 0)$ holds and $\forall \Gamma, \tilde{\Gamma} = T_d(\mu(M_\Gamma))$. Let now $\Gamma \subseteq \mathcal{L}$. Then,

(i) $(|\sim 6), (|\sim 7), (|\sim 8), (0), (1), (2)$, and (3) hold;

(ii) if (A3) holds, then (4), (5), and (6) also hold;

(iii) if μ is CP, then (6) also holds;

where

$$(0) \quad \mu(M_\Gamma) \subseteq M_{\Gamma \cup \tilde{\Gamma}};$$

$$(1) \quad M'_\Gamma \cap \mu(M_\Gamma) = \emptyset;$$

- (2) $M_{\Gamma \cup \tilde{\Gamma} \cup T_c(\mu(M_\Gamma))} = \mu(M_\Gamma)$;
(3) $M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)} = \mu(M_\Gamma)$ if $M'_\Gamma \neq \emptyset$;
(4) $M_{G(\Gamma)} = M_{T_c(\mu(M_\Gamma))}$ if $M'_\Gamma = \emptyset$;
(5) $M_{\Gamma \cup \tilde{\Gamma}} \subseteq M_{G(\Gamma)}$ if $M'_\Gamma = \emptyset$;
(6) $M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)} = \mu(M_\Gamma)$.

Proof (i). (0). Obvious by $(\mu 0)$ and if $x \in \mu(M_\Gamma)$ and $\alpha \in \tilde{\Gamma} = T_d(\mu(M_\Gamma))$, then $x \in M_\alpha$.

(~ 6). We get $\mu(M_\Gamma) \subseteq_{(0)} M_{\Gamma \cup \tilde{\Gamma}} \subseteq M_\beta$, however $\beta \notin \tilde{\Gamma} = T_d(\mu(M_\Gamma))$, thus $\mu(M_\Gamma) \subseteq M_{neg(\beta)}$, thus $\mu(M_\Gamma) \subseteq M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta)\}} \subseteq M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\alpha)\}}$, thus $\alpha \notin T_d(\mu(M_\Gamma)) = \tilde{\Gamma}$.

(~ 7). We get $\mu(M_\Gamma) \subseteq_{(0)} M_{\Gamma \cup \tilde{\Gamma}} \subseteq M_\alpha$, however $\alpha \notin T_d(\mu(M_\Gamma))$, thus $\mu(M_\Gamma) \subseteq M_{neg(\alpha)}$, thus $\mu(M_\Gamma) \subseteq M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\alpha)\}} \subseteq M_{\Gamma \cup \tilde{\Gamma} \cup \{\beta\}}$, however $\beta \notin T_d(\mu(M_\Gamma))$, thus $\mu(M_\Gamma) \subseteq M_{neg(\beta)}$. Thus, $\mu(M_\Gamma) \subseteq M_{neg(\alpha)} \cap M_{neg(\beta)} \stackrel{(A1)}{=} M_{neg \circ or(\alpha, \beta)}$, thus $or(\alpha, \beta) \notin T_d(\mu(M_\Gamma))$.

(~ 8). We get $\alpha \in T_d(\mu(M_\Gamma))$, thus $\mu(M_\Gamma) \not\subseteq M_{neg(\alpha)}$, thus $M_{\Gamma \cup \tilde{\Gamma}} \not\subseteq_{(0)} M_{neg(\alpha)}$.

(1). Case 1: $M'_\Gamma = \emptyset$. Obvious. Case 2: $M'_\Gamma \neq \emptyset$. We show $\forall i \in [1, n_\Gamma]$, $p(i)$ hold, with

$$p(i) \quad (M_\Gamma^1 \cup \dots \cup M_\Gamma^i) \cap \mu(M_\Gamma) = \emptyset.$$

We show $p(1)$ by contradiction. Suppose $\exists x \in M_\Gamma^1 \cap \mu(M_\Gamma)$. We get $\exists \beta, \beta \notin \tilde{\Gamma}, M_{\Gamma \cup \tilde{\Gamma}} \subseteq M_\beta$ and $x \notin M_{neg(\beta)}$. Thus, $\mu(M_\Gamma) \subseteq_{(0)} M_\beta$. However, $\mu(M_\Gamma) \not\subseteq M_{neg(\beta)}$. Thus, $\beta \in T_d(\mu(M_\Gamma)) = \tilde{\Gamma}$, which is impossible.

Let $i \in [1, n_\Gamma - 1]$. Suppose $p(i)$. We show $p(i+1)$. Case 1: $M_\Gamma^{i+1} \cap \mu(M_\Gamma) = \emptyset$. Obvious by $p(i)$. Case 2: $\exists x \in M_\Gamma^{i+1} \cap \mu(M_\Gamma)$. We get $\exists \beta, \beta \notin \tilde{\Gamma}, M_{\Gamma \cup \tilde{\Gamma}} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^i \subseteq M_\beta$, and $x \notin M_{neg(\beta)}$. Thus, $\mu(M_\Gamma) \subseteq_{(0), p(i)} M_{\Gamma \cup \tilde{\Gamma}} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^i \subseteq M_\beta$. However $\mu(M_\Gamma) \not\subseteq M_{neg(\beta)}$. Thus, $\beta \in T_d(\mu(M_\Gamma)) = \tilde{\Gamma}$, which is impossible.

(2). As $\mu(M_\Gamma) \in \mathbf{D}$, $\exists \Gamma', M_{\Gamma'} = \mu(M_\Gamma)$, thus $M_{T(\mu(M_\Gamma))} = M_{T(M_{\Gamma'})} = M_{\Gamma'} = \mu(M_\Gamma)$. Thus, $M_{\Gamma \cup \tilde{\Gamma} \cup T_c(\mu(M_\Gamma))} = M_{\Gamma \cup T_d(\mu(M_\Gamma)) \cup T_c(\mu(M_\Gamma))} = M_{\Gamma \cup T(\mu(M_\Gamma))} \stackrel{(\mu 0)}{=} M_{T(\mu(M_\Gamma))} = \mu(M_\Gamma)$.

(3). We show $M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta_\Gamma)\}} \subseteq \mu(M_\Gamma)$. Case 1: $\exists x, x \in M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta_\Gamma)\}}$ and $x \notin M_{T_c(\mu(M_\Gamma))}$. We get $\exists \alpha \in T_c(\mu(M_\Gamma))$, $x \notin M_\alpha$. First, by Lemma 28 (5), $x \in M_{\Gamma \cup \tilde{\Gamma}} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^{n_\Gamma}$. Second, by (0) and Lemma 28 (2), $\mu(M_\Gamma) \subseteq M_{\beta_\Gamma} \cap M_\alpha \stackrel{(A1)}{=} M_{neg \circ neg \circ and(\beta_\Gamma, \alpha)}$, thus $neg \circ and(\beta_\Gamma, \alpha) \notin T_d(\mu(M_\Gamma)) = \tilde{\Gamma}$. Third, by Lemma 28 (3), $M_{\Gamma \cup \tilde{\Gamma}} \setminus M_\Gamma^1 \cup \dots \cup M_\Gamma^{n_\Gamma} \subseteq M_{neg(\beta_\Gamma)} \subseteq_{(A1)} M_{neg \circ and(\beta_\Gamma, \alpha)}$. Fourth, $x \notin M_\alpha \supseteq_{(A1)} M_{neg \circ neg \circ and(\beta_\Gamma, \alpha)}$. Thus, $x \in M_\Gamma^{n_\Gamma+1}$, which is impossible.

Case 2: $M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta_\Gamma)\}} \subseteq M_{T_c(\mu(M_\Gamma))}$. Then, $M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta_\Gamma)\}} \subseteq M_{\Gamma \cup \tilde{\Gamma} \cup T_c(\mu(M_\Gamma))} \stackrel{(2)}{=} \mu(M_\Gamma)$.

We show $\mu(M_\Gamma) \subseteq M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta_\Gamma)\}}$. We get $\mu(M_\Gamma) \subseteq_{(0), (1)} M_{\Gamma \cup \tilde{\Gamma}} \setminus M'_\Gamma$ which is, by Lemma 28 (5), equal to $M_{\Gamma \cup \tilde{\Gamma} \cup \{neg(\beta_\Gamma)\}}$. Now, by Lemma 28 (7), we are done.

(ii). (4). Direction: “ \subseteq ”. Suppose $\exists x, x \in M_{G(\Gamma)}$ and $x \notin M_{T_c(\mu(M_\Gamma))}$. We get $\exists \delta \in T_c(\mu(M_\Gamma))$,

$x \notin M_\delta$. We show $\exists \alpha \in T_c(\mu(M_\Gamma))$, $|M_{\Gamma \cup \tilde{\Gamma} \cup \{\alpha\}}| - |\mu(M_\Gamma)| \leq -1$, but we relegate the proof to (*) below to increase readability. We then get a contradiction, as $\mu(M_\Gamma) \subseteq_{(0)} M_{\Gamma \cup \tilde{\Gamma} \cup \{\alpha\}}$.

Direction: “ \supseteq ”. Suppose $\exists x, x \in M_{T_c(\mu(M_\Gamma))}$ and $x \notin M_{G(\Gamma)}$. We show a contradiction. We get $\exists \alpha \in G(\Gamma)$, $x \notin M_\alpha$. Case 1: $neg(\alpha) \in T(M_{\Gamma \cup \tilde{\Gamma}})$. Then, $neg(\alpha) \in T_d(M_{\Gamma \cup \tilde{\Gamma}}) \cup T_c(M_{\Gamma \cup \tilde{\Gamma}})$. But, $neg(\alpha) \notin \tilde{\Gamma}$ which is, by Lemma 28 (6), equal to $T_d(M_{\Gamma \cup \tilde{\Gamma}})$, thus $neg(\alpha) \in T_c(M_{\Gamma \cup \tilde{\Gamma}})$, thus, $\alpha \in_{(A1)} T_c(M_{\Gamma \cup \tilde{\Gamma}})$, thus $\alpha \in_{(0)} T_c(\mu(M_\Gamma))$, thus $x \in M_\alpha$, which is impossible.

Case 2 : $\alpha \in T(M_{\Gamma \cup \tilde{\Gamma}})$. Similar reasoning as for case 1.

Case 3 : $neg(\alpha) \notin T(M_{\Gamma \cup \tilde{\Gamma}})$ and $\alpha \notin T(M_{\Gamma \cup \tilde{\Gamma}})$. We get $M_{\Gamma \cup \tilde{\Gamma} \cup \{\alpha\}} \not\subseteq_{(A3)} M_{neg(\alpha)}$, thus $\alpha \in T_d(M_{\Gamma \cup \tilde{\Gamma} \cup \{\alpha\}})$, thus $T_d(M_{\Gamma \cup \tilde{\Gamma} \cup \{\alpha\}}) \not\subseteq \tilde{\Gamma}$, thus $\alpha \notin G(\Gamma)$, which is impossible.

(*) It suffices to show by induction $\forall i \in \mathbb{Z}$, $i \leq |M_{\Gamma \cup \tilde{\Gamma} \cup \{\delta\}}|$, $\exists \alpha$, $p_1(\alpha)$, $p_2(\alpha)$, and $p_3(\alpha, i)$, where

$$p_1(\alpha) \quad \alpha \in T_c(\mu(M_\Gamma));$$

$$p_2(\alpha) \quad x \notin M_\alpha;$$

$$p_3(\alpha, i) \quad |M_{\Gamma \cup \tilde{\Gamma} \cup \{\alpha\}}| - |\mu(M_\Gamma)| \leq i.$$

As $|M_{\Gamma \cup \tilde{\Gamma} \cup \{\delta\}}|$ is finite by (A2), we can start with $i = |M_{\Gamma \cup \tilde{\Gamma} \cup \{\delta\}}|$. Clearly, $p_1(\delta)$, $p_2(\delta)$, and $p_3(\delta, |M_{\Gamma \cup \tilde{\Gamma} \cup \{\delta\}}|)$ hold.

Let $i \in \mathbb{Z}$, $i \leq |M_{\Gamma \cup \tilde{\Gamma} \cup \{\delta\}}|$. Suppose $\exists \alpha$, $p_1(\alpha)$, $p_2(\alpha)$, $p_3(\alpha, i)$. Case 1: $T_d(M_{\Gamma \cup \tilde{\Gamma} \cup \{\alpha\}}) \subseteq \tilde{\Gamma}$. Then, $\alpha \notin_{p_1(\alpha)} T_d(\mu(M_\Gamma)) = \tilde{\Gamma}$. We get also $neg(\alpha) \notin_{p_1(\alpha), (A1)} T_d(\mu(M_\Gamma)) = \tilde{\Gamma}$. Thus, $\alpha \in G(\Gamma)$, thus $x \in M_\alpha$, which is impossible by $p_2(\alpha)$.

Case 2: $T_d(M_{\Gamma \cup \tilde{\Gamma} \cup \{\alpha\}}) \not\subseteq \tilde{\Gamma}$. Then, $\exists \beta$, $M_{\Gamma \cup \tilde{\Gamma} \cup \{\alpha\}} \subseteq M_\beta$, $M_{\Gamma \cup \tilde{\Gamma} \cup \{\alpha\}} \not\subseteq M_{neg(\beta)}$, and $\beta \notin T_d(\mu(M_\Gamma))$. We get $\mu(M_\Gamma) \subseteq_{(0), p_1(\alpha)} M_\beta$, thus $\mu(M_\Gamma) \subseteq M_{neg(\beta)}$, as $\beta \notin T_d(\mu(M_\Gamma))$. We show $p_1(\text{and}(\alpha, neg(\beta)))$. $\mu(M_\Gamma) \subseteq_{p_1(\alpha)} M_\alpha \cap M_{neg(\beta)} =_{(A1)} M_{\text{and}(\alpha, neg(\beta))}$ and $\mu(M_\Gamma) \subseteq_{p_1(\alpha)} M_{neg(\alpha)} \subseteq_{(A1)} M_{neg \circ \text{and}(\alpha, neg(\beta))}$. We turn to $p_2(\text{and}(\alpha, neg(\beta)))$. $x \notin_{p_2(\alpha)} M_\alpha \supseteq_{(A1)} M_{\text{and}(\alpha, neg(\beta))}$. We show $p_3(\text{and}(\alpha, neg(\beta)), i - 1)$. $M_{\Gamma \cup \tilde{\Gamma} \cup \{\text{and}(\alpha, neg(\beta))\}} \subseteq_{(A1)} M_{\Gamma \cup \tilde{\Gamma} \cup \{\alpha\}}$, but $M_{\Gamma \cup \tilde{\Gamma} \cup \{\alpha\}} \not\subseteq M_{neg(\beta)} \supseteq_{(A1)} M_{\Gamma \cup \tilde{\Gamma} \cup \{\text{and}(\alpha, neg(\beta))\}}$. Thus $|M_{\Gamma \cup \tilde{\Gamma} \cup \{\text{and}(\alpha, neg(\beta))\}}| + 1 \leq |M_{\Gamma \cup \tilde{\Gamma} \cup \{\alpha\}}|$. Thus, $|M_{\Gamma \cup \tilde{\Gamma} \cup \{\text{and}(\alpha, neg(\beta))\}}| - |\mu(M_\Gamma)| \leq_{p_3(\alpha, i)} i - 1$.

(5). Suppose $\exists x, x \in M_{\Gamma \cup \tilde{\Gamma}}$ and $x \notin M_{G(\Gamma)}$. We show a contradiction. We get $\exists \alpha \in G(\Gamma)$, $x \notin M_\alpha$. Case 1: $neg(\alpha) \in T(M_{\Gamma \cup \tilde{\Gamma}})$. We get $neg(\alpha) \in T_d(M_{\Gamma \cup \tilde{\Gamma}}) \cup T_c(M_{\Gamma \cup \tilde{\Gamma}})$. We get $neg(\alpha) \notin \tilde{\Gamma}$ which is, by Lemma 28 (6), equal to $T_d(M_{\Gamma \cup \tilde{\Gamma}})$, thus $neg(\alpha) \in T_c(M_{\Gamma \cup \tilde{\Gamma}})$, thus $M_{\Gamma \cup \tilde{\Gamma}} \subseteq M_{neg \circ neg(\alpha)} =_{(A1)} M_\alpha$, thus $x \in M_\alpha$, which is impossible.

Case 2: $\alpha \in T(M_{\Gamma \cup \tilde{\Gamma}})$. Similar reasoning as for case 1.

Case 3: $neg(\alpha) \notin T(M_{\Gamma \cup \tilde{\Gamma}})$ and $\alpha \notin T(M_{\Gamma \cup \tilde{\Gamma}})$. We get $M_{\Gamma \cup \tilde{\Gamma} \cup \{\alpha\}} \not\subseteq_{(A3)} M_{neg(\alpha)}$, thus $\alpha \in T_d(M_{\Gamma \cup \tilde{\Gamma} \cup \{\alpha\}})$, thus $T_d(M_{\Gamma \cup \tilde{\Gamma} \cup \{\alpha\}}) \not\subseteq \tilde{\Gamma}$, thus $\alpha \notin G(\Gamma)$, which is impossible.

(6). Case 1: $M'_\Gamma = \emptyset$. We get, by Lemma 28 (7), $M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)} = M_{\Gamma \cup \tilde{\Gamma} \cup F(\Gamma)} = M_{\Gamma \cup \tilde{\Gamma}} =_{(5)} M_{\Gamma \cup \tilde{\Gamma} \cup G(\Gamma)} =_{(4)} M_{\Gamma \cup \tilde{\Gamma} \cup T_c(\mu(M_\Gamma))} =_{(2)} \mu(M_\Gamma)$. Case 2: $M'_\Gamma \neq \emptyset$. Obvious by (3).

(iii). Case 1: $M'_\Gamma = \emptyset$. Case 1.1: $\exists x, x \in M_{\Gamma \cup \tilde{\Gamma}}$ and $x \notin M_{T_c(\mu(M_\Gamma))}$. Case 1.1.1: Γ is not consistent. We get $\exists \alpha \in T_c(\mu(M_\Gamma))$, $x \notin M_\alpha$ and $\exists \beta$, $M_\Gamma \subseteq M_\beta$ and $M_\Gamma \subseteq M_{neg(\beta)}$. First, $M_{\Gamma \cup \tilde{\Gamma}} \subseteq M_\Gamma \subseteq_{(A1)} M_{or(\beta, neg(\alpha))}$. Second, $\mu(M_\Gamma) \subseteq_{(\mu 0)} M_{neg(\beta)} \cap M_\alpha =_{(A1)} M_{neg \circ or(\beta, neg(\alpha))}$.

thus $or(\beta, neg(\alpha)) \notin T_d(\mu(M_\Gamma)) = \tilde{\Gamma}$. Third, $x \notin M_\alpha \supseteq_{(A1)} M_{neg \circ or(\beta, neg(\alpha))}$. Thus, $x \in M_\Gamma^1 \subseteq M_\Gamma'$, which is impossible.

Case 1.1.2: Γ is consistent. Thus, $M_\Gamma \in \mathbf{D} \cap \mathbf{C}$, thus $\mu(M_\Gamma) \in \mathbf{C}$, thus $T_c(\mu(M_\Gamma)) = \emptyset$, thus $M_{T_c(\mu(M_\Gamma))} = \mathcal{V} \ni x$, which is impossible. Case 1.2: $M_{\Gamma \cup \tilde{\Gamma}} \subseteq M_{T_c(\mu(M_\Gamma))}$. We get, by Lemma 28 (7), $M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)} = M_{\Gamma \cup \tilde{\Gamma} \cup F(\Gamma)} = M_{\Gamma \cup \tilde{\Gamma}} = M_{\Gamma \cup \tilde{\Gamma} \cup T_c(\mu(M_\Gamma))} \stackrel{(2)}{=} \mu(M_\Gamma)$. Case 2: $M_\Gamma' \neq \emptyset$. Obvious by (3). ■

Now comes the **proof of Proposition 25**. Proposition 25 is stated at the beginning of Section 5.1.

Proof For (i), (iii), (v), and (vii), there exists a DP preferential structure \mathcal{Z} on \mathcal{V} such that $\forall \Gamma$, $\tilde{\Gamma} = T_d(\mu_{\mathcal{Z}}(M_\Gamma))$. Let $\mu : \mathbf{D} \rightarrow \mathbf{D}$ be the function such that $\forall X \in \mathbf{D}$, $\mu(X) = \mu_{\mathcal{Z}}(X)$. As \mathcal{Z} is DP, μ is well-defined. Moreover, $\forall \Gamma$, $\tilde{\Gamma} = T_d(\mu(M_\Gamma))$ and, by Theorem 7 (i), μ satisfies $(\mu 0)$ and $(\mu 1)$. We show $(|\sim 0)$. If $\bar{\Gamma} = \bar{\Delta}$, then $M_\Gamma = M_\Delta$, thus $\tilde{\Gamma} = T_d(\mu_{\mathcal{Z}}(M_\Gamma)) = T_d(\mu_{\mathcal{Z}}(M_\Delta)) = \tilde{\Delta}$. In addition, by Lemma 29 (i), we get $(|\sim 6)$, $(|\sim 7)$, and $(|\sim 8)$.

(i). By Lemma 29 (ii), $\forall \Gamma$, $\mu(M_\Gamma) = M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)}$, thus by $(\mu 1)$, $(|\sim 9)$ holds.

(iii). By Lemma 29 (ii), $\forall \Gamma$, $\mu(M_\Gamma) = M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)}$, thus by $(\mu 1)$, $(|\sim 9)$ holds. In addition, \mathcal{Z} is smooth. Thus by Theorem 7 (iii), μ satisfies $(\mu 2)$, thus $(|\sim 10)$ holds.

(v). \mathcal{Z} is CP, thus μ is CP. Thus, by Lemma 29 (iii), $\forall \Gamma$, $\mu(M_\Gamma) = M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)}$, thus, by $(\mu 1)$, $(|\sim 9)$ holds. We show $(|\sim 11)$. Let Γ be consistent. We get $M_\Gamma \in \mathbf{D} \cap \mathbf{C}$, thus $\mu(M_\Gamma) \in \mathbf{D} \cap \mathbf{C}$. We show $\Gamma \subseteq \tilde{\Gamma}$. Let $\alpha \in \Gamma$. We get $\mu(M_\Gamma) \subseteq_{(\mu 0)} M_\Gamma \subseteq M_\alpha$. Thus, $\mu(M_\Gamma) \not\subseteq M_{neg(\alpha)}$, thus $\alpha \in T_d(\mu(M_\Gamma)) = \tilde{\Gamma}$. We show that $\tilde{\Gamma}$ is consistent. We get $M_{\tilde{\Gamma}} = M_{T_d(\mu(M_\Gamma))} = M_{T(\mu(M_\Gamma))} = \mu(M_\Gamma) \in \mathbf{C}$. And, $\tilde{\Gamma} = T_d(\mu(M_\Gamma)) = T(\mu(M_\Gamma)) = T(M_{T(\mu(M_\Gamma))}) = T(M_{\tilde{\Gamma}}) = \tilde{\Gamma}$.

(vii). By verbatim the same proof as for (v), $\forall \Gamma$, $\mu(M_\Gamma) = M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)}$ and $(|\sim 9)$, $(|\sim 11)$ hold. In addition, \mathcal{Z} is smooth, thus, by Theorem 7 (iii), μ satisfies $(\mu 2)$, thus $(|\sim 10)$ holds too.

For (ii), (iv), (vi), and (viii), we have $(|\sim 0)$, $(|\sim 6)$, $(|\sim 7)$, $(|\sim 8)$, and $(|\sim 9)$. Let $\mu : \mathbf{D} \rightarrow \mathbf{D}$ be the function such that $\forall \Gamma$, $\mu(M_\Gamma) = M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)}$. If $M_\Gamma = M_\Delta$, then $H(\Gamma) \stackrel{(|\sim 0)}{=} H(\Delta)$, thus $\mu(M_\Gamma) = \mu(M_\Delta)$, thus μ is well-defined. By Lemma 28 (8), $\forall \Gamma$, $\tilde{\Gamma} = T_d(\mu(M_\Gamma))$. We get obviously $(\mu 0)$. And, by $(|\sim 9)$, we get $(\mu 1)$.

(ii). By Theorem 7 (ii), there exists a transitive irreflexive preferential structure \mathcal{Z} on \mathcal{V} s.t. $\forall \Gamma$, $\tilde{\Gamma} = T_d(\mu(M_\Gamma)) = T_d(\mu_{\mathcal{Z}}(M_\Gamma))$. And, \mathcal{Z} is DP, as $\forall \Gamma$, $\mu_{\mathcal{Z}}(M_\Gamma) = \mu(M_\Gamma) \in \mathbf{D}$.

(iv). By $(|\sim 10)$, we get $(\mu 2)$. By (A1) and Proposition 10, (A0) holds. Thus, by Theorem 7 (iv), there exists a transitive irreflexive smooth preferential structure \mathcal{Z} on \mathcal{V} such that $\forall \Gamma$, $\tilde{\Gamma} = T_d(\mu(M_\Gamma)) = T_d(\mu_{\mathcal{Z}}(M_\Gamma))$. In addition, \mathcal{Z} is DP, as $\forall \Gamma$, $\mu_{\mathcal{Z}}(M_\Gamma) = \mu(M_\Gamma) \in \mathbf{D}$.

(vi). We show that μ is CP. Let $M_\Gamma \in \mathbf{D} \cap \mathbf{C}$. Case 1: $H_1(\Gamma) \neq \emptyset$. Thus, $\exists \beta, \beta \notin \tilde{\Gamma}$ and $M_{\Gamma \cup \tilde{\Gamma}} \subseteq M_\beta$. As, $\Gamma \subseteq_{(|\sim 11)} \tilde{\Gamma}$, $M_{\Gamma \cup \tilde{\Gamma}} = M_{\tilde{\Gamma}}$, thus $M_{\tilde{\Gamma}} \subseteq M_\beta$, thus $\beta \in T(M_{\tilde{\Gamma}}) = \tilde{\Gamma} \stackrel{(|\sim 11)}{=} \tilde{\Gamma}$, which is impossible. Case 2: $H_1(\Gamma) = \emptyset$. Thus $H(\Gamma) = \emptyset$. Thus, $\mu(M_\Gamma) = M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)} = M_{\tilde{\Gamma}} \in_{(|\sim 11)} \mathbf{C}$. By Theorem 7 (ii), there exists a transitive irreflexive preferential structure \mathcal{Z} on \mathcal{V} such that $\forall \Gamma$, $\tilde{\Gamma} = T_d(\mu(M_\Gamma)) = T_d(\mu_{\mathcal{Z}}(M_\Gamma))$. We show that \mathcal{Z} is CP. If $X \in \mathbf{D} \cap \mathbf{C}$, then $\mu_{\mathcal{Z}}(X) = \mu(X) \in \mathbf{C}$. And, \mathcal{Z} is DP, as $\forall \Gamma$, $\mu_{\mathcal{Z}}(M_\Gamma) = \mu(M_\Gamma) \in \mathbf{D}$.

(viii). By verbatim the same proof as for (vi), μ is CP. In addition, by $(|\sim 10)$, μ satisfies $(\mu 2)$. By (A1) and Proposition 10, (A0) holds. Thus, by Theorem 7 (iv), there exists a transitive irreflexive smooth preferential structure \mathcal{Z} on \mathcal{V} such that $\forall \Gamma$, $\tilde{\Gamma} = T_d(\mu(M_\Gamma)) = T_d(\mu_{\mathcal{Z}}(M_\Gamma))$. In addition, \mathcal{Z} is CP, as if $X \in \mathbf{D} \cap \mathbf{C}$, then $\mu_{\mathcal{Z}}(X) = \mu(X) \in \mathbf{C}$. And, \mathcal{Z} is DP, as $\forall \Gamma$, $\mu_{\mathcal{Z}}(M_\Gamma) = \mu(M_\Gamma) \in \mathbf{D}$. ■

5.2 Normal characterizations

We give now in a straightforward manner, normal characterizations of the four following families: DP preferential-discriminative CRs, smooth DP preferential-discriminative CRs, CP DP preferential-discriminative CRs, and smooth CP DP preferential-discriminative CRs.

Let $\Gamma_1, \Gamma_2 \subseteq \mathcal{L}$ and $i \in \mathbb{N}, i \geq 2$. Then,

- $H_1(\Gamma_1, \Gamma_2) \stackrel{def}{=} \{neg(\beta) : \beta \notin \Gamma_2, \text{ and } \beta \in \overline{\Gamma_1 \cup \Gamma_2}, neg(\beta) \notin \overline{\Gamma_1 \cup \Gamma_2}\};$
- $H_i(\Gamma_1, \Gamma_2) \stackrel{def}{=} \{neg(\beta) : \left\{ \begin{array}{l} \beta \notin \Gamma_2 \text{ and} \\ \beta \in \overline{\Gamma_1 \cup \Gamma_2 \cup H_1(\Gamma_1, \Gamma_2) \cup \dots \cup H_{i-1}(\Gamma_1, \Gamma_2)} \text{ and} \\ neg(\beta) \notin \overline{\Gamma_1 \cup \Gamma_2 \cup H_1(\Gamma_1, \Gamma_2) \cup \dots \cup H_{i-1}(\Gamma_1, \Gamma_2)} \end{array} \right\};$
- $H(\Gamma_1, \Gamma_2) \stackrel{def}{=} \bigcup_{i \in \mathbb{N}^+} H_i(\Gamma_1, \Gamma_2).$

Let now Φ be the relation on $\mathcal{P}(\mathcal{L})^4$ such that $\forall \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Phi(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ iff

- if $\overline{\Gamma_1} = \overline{\Gamma_2}$, then $\Gamma_3 = \Gamma_4$;
- $\forall \alpha, \beta$, if $\beta \notin \Gamma_3, \beta \in \overline{\Gamma_1 \cup \Gamma_3}$ and $\Gamma_1 \cup \Gamma_3 \cup \{neg(\alpha)\} \subseteq \overline{\Gamma_1 \cup \Gamma_3 \cup \{neg(\beta)\}}$, then $\alpha \notin \Gamma_3$;
- $\forall \alpha, \beta$, if $\alpha, \beta \notin \Gamma_3, \alpha \in \overline{\Gamma_1 \cup \Gamma_3}$ and $\Gamma_1 \cup \Gamma_3 \cup \{\beta\} \subseteq \overline{\Gamma_1 \cup \Gamma_3 \cup \{neg(\alpha)\}}$, then $nor(\alpha, \beta) \notin \Gamma_3$;
- $\forall \alpha$, if $\alpha \in \Gamma_3$, then $neg(\alpha) \notin \overline{\Gamma_1 \cup \Gamma_3}$;
- if $\Gamma_2 \subseteq \overline{\Gamma_1}$, then $\Gamma_1 \cup \Gamma_3 \cup H(\Gamma_1, \Gamma_3) \subseteq \overline{\Gamma_2 \cup \Gamma_4 \cup H(\Gamma_2, \Gamma_4)} \cup \Gamma_1$.

Then, by Proposition 25 (i) and (ii), under (A1), (A2), and (A3), for all CR $|\sim, |\sim$ is a DP preferential-discriminative CR iff $\forall \Gamma_1, \Gamma_2 \subseteq \mathcal{L}, \Phi(\Gamma_1, \Gamma_2, \widetilde{\Gamma_1}, \widetilde{\Gamma_2})$. Let Φ' be the relation on $\mathcal{P}(\mathcal{L})^4$ s.t. $\forall \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Phi'(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ iff $\Phi(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ and

$$\text{if } \Gamma_1 \subseteq \overline{\Gamma_2} \subseteq \Gamma_1 \cup \Gamma_3 \cup H(\Gamma_1, \Gamma_3), \text{ then } \Gamma_1 \cup \Gamma_3 \cup H(\Gamma_1, \Gamma_3) = \Gamma_2 \cup \Gamma_4 \cup H(\Gamma_2, \Gamma_4)$$

Then, by Proposition 25 (iii) and (iv), under (A1), (A2), and (A3), for all CR $|\sim, |\sim$ is a smooth DP preferential-discriminative CR iff $\forall \Gamma_1, \Gamma_2 \subseteq \mathcal{L}, \Phi'(\Gamma_1, \Gamma_2, \widetilde{\Gamma_1}, \widetilde{\Gamma_2})$. Let Φ'' be the relation on $\mathcal{P}(\mathcal{L})^4$ s.t. $\forall \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Phi''(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ iff $\Phi(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ and

$$\text{if } \Gamma_1 \text{ is consistent, then } \Gamma_1 \subseteq \Gamma_3, \Gamma_3 \text{ is consistent, and } \overline{\Gamma_3} = \Gamma_3$$

Then, by Proposition 25 (v) and (vi), under (A1) and (A2), for all CR $|\sim, |\sim$ is a CP DP preferential-discriminative CR iff $\forall \Gamma_1, \Gamma_2 \subseteq \mathcal{L}, \Phi''(\Gamma_1, \Gamma_2, \widetilde{\Gamma_1}, \widetilde{\Gamma_2})$. Let Φ''' be the relation on $\mathcal{P}(\mathcal{L})^4$ s.t. $\forall \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Phi'''(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ iff $\Phi''(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ and $\Phi'(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$. Then, by Proposition 25 (vii) and (viii), under (A1) and (A2), for all CR $|\sim, |\sim$ is a smooth CP DP preferential-discriminative CR iff $\forall \Gamma_1, \Gamma_2 \subseteq \mathcal{L}, \Phi'''(\Gamma_1, \Gamma_2, \widetilde{\Gamma_1}, \widetilde{\Gamma_2})$.

6 Preferential-discriminative consequence relations

In this section, we give nonnormal characterizations of the two following families:

- the preferential-discriminative CRs, under (A1), (A2), and (A3);

- the CP preferential-discriminative CRs, under (A1) and (A2).

Proposition 30 Let $|\sim$ be a CR.

- (i) If (A1), (A2), and (A3) hold and $|\sim$ is a preferential-discriminative CR, then $(|\sim 0)$, $(|\sim 6)$, $(|\sim 7)$, $(|\sim 8)$, and $(|\sim 12)$ hold;
- (ii) If (A1), (A2), $(|\sim 0)$, $(|\sim 6)$, $(|\sim 7)$, $(|\sim 8)$, and $(|\sim 12)$ hold, then $|\sim$ is a transitive irreflexive preferential-discriminative CR;
- (iii) If (A1) and (A2) hold and $|\sim$ is a CP preferential-discriminative CR, then $(|\sim 0)$, $(|\sim 6)$, $(|\sim 7)$, $(|\sim 8)$, $(|\sim 11)$ and $(|\sim 12)$ hold;
- (iv) If (A1), (A2), $(|\sim 0)$, $(|\sim 6)$, $(|\sim 7)$, $(|\sim 8)$, $(|\sim 11)$ and $(|\sim 12)$ hold, then $|\sim$ is a transitive irreflexive CP preferential-discriminative CR;

where

$$(|\sim 12) \quad \forall \Gamma, M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)} = M_{T(\{x \in M_\Gamma : \forall \Delta, \text{ if } M_\Delta \subseteq M_\Gamma \text{ and } x \in M_\Delta, \text{ then } x \in M_{\Delta \cup \tilde{\Delta} \cup H(\Delta)}\})}$$

Proof For (i) and (iii), there exists a preferential structure \mathcal{Z} on \mathcal{V} such that $\forall \Gamma, \tilde{\Gamma} = T_d(\mu_{\mathcal{Z}}(M_\Gamma))$. Let $\mu : \mathbf{D} \rightarrow \mathbf{D}$ be the function such that $\forall X \in \mathbf{D}, \mu(X) = M_{T(\mu_{\mathcal{Z}}(X))}$. Then, $\forall \Gamma, \tilde{\Gamma} = T_d(\mu_{\mathcal{Z}}(M_\Gamma)) = T_d(M_{T(\mu_{\mathcal{Z}}(M_\Gamma))}) = T_d(\mu(M_\Gamma))$. We show that μ satisfies $(\mu 0)$. $\forall \Gamma, \mu(M_\Gamma) = M_{T(\mu_{\mathcal{Z}}(M_\Gamma))} \subseteq M_{T(M_\Gamma)} = M_\Gamma$, as $\mu_{\mathcal{Z}}(M_\Gamma) \subseteq M_\Gamma$. In addition, by Lemma 21 (i), μ satisfies $(\mu 3)$. Recall that $(\mu 3)$ has been defined in Section 4. We get obviously $(|\sim 0)$. And, by Lemma 29 (i), we get $(|\sim 6)$, $(|\sim 7)$, and $(|\sim 8)$.

(i). By Lemma 29 (ii), $\forall \Gamma, \mu(M_\Gamma) = M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)}$. Thus, By $(\mu 3)$, $(|\sim 12)$.

(iii). In addition, \mathcal{Z} is CP. We show that μ is CP. if $X \in \mathbf{D} \cap \mathbf{C}$, then $\mu(X) = M_{T(\mu_{\mathcal{Z}}(X))} \in \mathbf{C}$, as $\mu_{\mathcal{Z}}(X) \in \mathbf{C}$. Thus, by Lemma 29 (iii), $\mu(M_\Gamma) = M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)}$. Thus, by $(\mu 3)$, we get $(|\sim 12)$. We show $(|\sim 11)$. Verbatim the same reasoning as (v) of Proposition 25.

For (ii) and (iv), we have $(|\sim 0)$, $(|\sim 6)$, $(|\sim 7)$, $(|\sim 8)$, and $(|\sim 12)$. Let $\mu : \mathbf{D} \rightarrow \mathbf{D}$ be the function such that $\forall \Gamma, \mu(M_\Gamma) = M_{\Gamma \cup \tilde{\Gamma} \cup H(\Gamma)}$. If $M_\Gamma = M_\Delta$, then $H(\Gamma) =_{(|\sim 0)} H(\Delta)$, thus μ is well-defined. By $(|\sim 12)$, μ satisfies $(\mu 3)$. In addition, by Lemma 28 (8), $\forall \Gamma, \tilde{\Gamma} = T_d(\mu(M_\Gamma))$. Thus, by Lemma 21 (ii), there exists a transitive irreflexive preferential structure \mathcal{Z} on \mathcal{V} such that $\forall \Gamma, \tilde{\Gamma} = T_d(\mu(M_\Gamma)) = T_d(M_{T(\mu_{\mathcal{Z}}(M_\Gamma))}) = T_d(\mu_{\mathcal{Z}}(M_\Gamma))$.

(ii). Everything has been shown in the common part.

(iv). We show that μ is CP. Verbatim the same reasoning as (vi) of Proposition 25. We show that \mathcal{Z} is CP. if $X \in \mathbf{D} \cap \mathbf{C}$, then $\mu_{\mathcal{Z}}(X) \in \mathbf{C}$, as $M_{T(\mu_{\mathcal{Z}}(X))} = \mu(X) \in \mathbf{C}$. ■

We cannot get normal characterizations of these two families of preferential-discriminative CRs, from Proposition 30, because the conditions depend on Δ for too many Δ . More explanations have been given in Section 2.5.

7 Proof systems for \Vdash_d

7.1 A proof system for \Vdash_d in the *FOUR* context

For the whole Section 7.1 we assume $\mathcal{L} = \mathcal{L}_4$, $\mathcal{V} = \mathcal{V}_4$, $\models = \models_4$ and $\forall \alpha \in \mathcal{L}_4, \text{neg}(\alpha) = \neg\alpha$.

Definition 31 We say that α is a \mathcal{L}_4 -literal iff there exists a constant or propositional symbol p of \mathcal{L}_4 such that $\alpha \in \{p, \neg p\}$. For all $\alpha \in \mathcal{L}_4$, we denote by $inv_4(\alpha)$ the formula of \mathcal{L}_4 such that: $inv_4(\alpha) = \beta$ if $\exists \beta \in \mathcal{L}_4, \alpha = \neg \beta$; $inv_4(\alpha) = \neg \alpha$ otherwise. We call *systemFOURd* the proof system obtained from *systemFOUR* by adding:

Axioms:

$$\begin{array}{c} \Gamma \cup \Delta \cup \{\alpha\} \text{ is a set of } \mathcal{L}_4\text{-literals} \\ 0, \neg 1, \perp, \neg \perp \notin \Gamma \cup \{\alpha\} \\ [\alpha \Rightarrow_d \alpha] \frac{\exists \delta \in \Delta \cup \{\alpha\}, \delta \notin \{0, \neg 1, \top, \neg \top\} \text{ and } inv_4(\delta) \notin \Gamma \cup \{\alpha\}}{\Gamma, \alpha \Rightarrow_d \Delta, \alpha} \end{array}$$

$$\begin{array}{c} \Gamma \cup \Delta \text{ is a set of } \mathcal{L}_4\text{-literals} \\ 0, \neg 1, \perp, \neg \perp \notin \Gamma \\ [\Rightarrow_d \top] \frac{\exists \delta \in \Delta \setminus \{0, \neg 1, \top, \neg \top\}, inv_4(\delta) \notin \Gamma}{\Gamma \Rightarrow_d \Delta, \top} \quad \frac{0, \neg 1, \perp, \neg \perp \notin \Gamma}{\Gamma \Rightarrow_d \Delta, 1} [\Rightarrow_d 1] \end{array}$$

$$\begin{array}{c} \Gamma \cup \Delta \text{ is a set of } \mathcal{L}_4\text{-literals} \\ 0, \neg 1, \perp, \neg \perp \notin \Gamma \\ [\Rightarrow_d \neg \top] \frac{\exists \delta \in \Delta \setminus \{0, \neg 1, \top, \neg \top\}, inv_4(\delta) \notin \Gamma}{\Gamma \Rightarrow_d \Delta, \neg \top} \quad \frac{0, \neg 1, \perp, \neg \perp \notin \Gamma}{\Gamma \Rightarrow_d \Delta, \neg 0} [\Rightarrow_d \neg 0] \end{array}$$

Rules:

$$\begin{array}{c} [\Rightarrow_d \vee] \frac{\Gamma \Rightarrow_d \Delta, \alpha, \beta}{\Gamma \Rightarrow_d \Delta, \alpha \vee \beta} \quad \frac{\Gamma \Rightarrow_d \Delta, \neg \alpha, \neg \beta}{\Gamma \Rightarrow_d \Delta, \neg(\alpha \wedge \beta)} [\Rightarrow_d \neg \wedge] \\ [\alpha \vee \Rightarrow_d] \frac{\Gamma, \alpha \Rightarrow_d \Delta \quad \Gamma, \alpha \vee \beta \Rightarrow \Delta}{\Gamma, \alpha \vee \beta \Rightarrow_d \Delta} \quad \frac{\Gamma, \beta \Rightarrow_d \Delta \quad \Gamma, \alpha \vee \beta \Rightarrow \Delta}{\Gamma, \alpha \vee \beta \Rightarrow_d \Delta} [\vee \beta \Rightarrow_d] \\ [\alpha \neg \wedge \Rightarrow_d] \frac{\Gamma, \neg \alpha \Rightarrow_d \Delta \quad \Gamma, \neg(\alpha \wedge \beta) \Rightarrow \Delta}{\Gamma, \neg(\alpha \wedge \beta) \Rightarrow_d \Delta} \quad \frac{\Gamma, \neg \beta \Rightarrow_d \Delta \quad \Gamma, \neg(\alpha \wedge \beta) \Rightarrow \Delta}{\Gamma, \neg(\alpha \wedge \beta) \Rightarrow_d \Delta} [\neg \wedge \beta \Rightarrow_d] \\ [\Rightarrow_d \wedge] \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta \quad \exists \delta \in \Delta \cup \{\alpha \wedge \beta\}, \Gamma \not\Rightarrow \neg \delta}{\Gamma \Rightarrow_d \Delta, \alpha \wedge \beta} \quad \frac{\Gamma, \alpha, \beta \Rightarrow_d \Delta}{\Gamma, \alpha \wedge \beta \Rightarrow_d \Delta} [\wedge \Rightarrow_d] \\ [\Rightarrow_d \supset] \frac{\Gamma, \alpha \Rightarrow \Delta, \beta \quad \exists \delta \in \Delta \cup \{\alpha \supset \beta\}, \Gamma \not\Rightarrow \neg \delta}{\Gamma \Rightarrow_d \Delta, \alpha \supset \beta} \quad \frac{\Gamma \Rightarrow_d \Delta, \alpha}{\Gamma \Rightarrow_d \Delta, \neg \neg \alpha} [\Rightarrow_d \neg \neg] \\ [\supset \Rightarrow_d] \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma, \beta \Rightarrow \Delta \quad \exists \delta \in \Delta, \Gamma, \alpha \supset \beta \not\Rightarrow \neg \delta}{\Gamma, \alpha \supset \beta \Rightarrow_d \Delta} \quad \frac{\Gamma, \alpha \Rightarrow_d \Delta}{\Gamma, \neg \neg \alpha \Rightarrow_d \Delta} [\neg \neg \Rightarrow_d] \\ [\neg \supset \Rightarrow_d] \frac{\Gamma, \alpha, \neg \beta \Rightarrow \Delta \quad \exists \delta \in \Delta, \Gamma, \neg(\alpha \supset \beta) \not\Rightarrow \delta}{\Gamma, \neg(\alpha \supset \beta) \Rightarrow_d \Delta} \quad \frac{\Gamma, \neg \alpha, \neg \beta \Rightarrow_d \Delta}{\Gamma, \neg(\alpha \vee \beta) \Rightarrow_d \Delta} [\neg \vee \Rightarrow_d] \\ [\Rightarrow_d \neg \supset] \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \neg \beta \quad \exists \delta \in \Delta \cup \{\neg(\alpha \supset \beta)\}, \Gamma \not\Rightarrow \neg \delta}{\Gamma \Rightarrow_d \Delta, \neg(\alpha \supset \beta)} \\ [\Rightarrow_d \neg \vee] \frac{\Gamma \Rightarrow \Delta, \neg \alpha \quad \Gamma \Rightarrow \Delta, \neg \beta \quad \exists \delta \in \Delta \cup \{\neg(\alpha \vee \beta)\}, \Gamma \not\Rightarrow \neg \delta}{\Gamma \Rightarrow_d \Delta, \neg(\alpha \vee \beta)} \end{array}$$

Proposition 32

- (0) $\forall \Gamma, \Delta, \Gamma \Rightarrow \Delta$ is provable in *systemFOUR* iff $\Gamma \Rightarrow \Delta$ is provable in *systemFOURd*;
(1) $\forall \Gamma, \Delta, \Gamma \Vdash_d \Delta$ iff $\Gamma \Rightarrow_d \Delta$ is provable in *systemFOURd*.

Proof (0). Direction: “ \rightarrow ”. *systemFOUR* is a subset of *systemFOURd*. Direction: “ \leftarrow ”. The conclusions of the axioms and rules in *systemFOURd* which are not in *systemFOUR* are sequents of the form $\Gamma \Rightarrow_d \Delta$ and thus cannot be used to derive new sequents of the form $\Gamma \Rightarrow \Delta$.

(1). In the whole proof, if we write $\Gamma \Rightarrow \Delta$ or $\Gamma \Rightarrow_d \Delta$, we mean $\Gamma \Rightarrow \Delta$ or $\Gamma \Rightarrow_d \Delta$ is provable in *systemFOURd*. Direction: “ \leftarrow ”. It suffices to show by induction $\forall i \in \mathbb{N}^+, p(i)$, with $p(i) \forall \Gamma, \Delta$, if there is a proof of length i for $\Gamma \Rightarrow_d \Delta$, then $\Gamma \Vdash_d \Delta$.

We show $p(1)$. Let $\Gamma, \Delta \subseteq \mathcal{L}_4$ and A be an axiom which proves $\Gamma \Rightarrow_d \Delta$.

Case 1: $A = [\alpha \Rightarrow_d \alpha]$ or $A = [\Rightarrow_d \top]$ or $A = [\Rightarrow_d \neg \top]$. We get $\Gamma \Vdash \Delta$. In addition, $\exists \delta \in \Delta$, $inv_4(\delta) \notin \Gamma$, $inv_4(\delta)$ is a \mathcal{L}_4 -literal, and $\delta \notin \{0, \neg 1, \top, \neg \top\}$. Consider the \mathcal{L}_4 -valuation x such that $\forall \alpha \in \mathcal{A}$,

$$x(\alpha) = \begin{cases} \top & \text{if } \alpha \in \Gamma \quad \neg \alpha \in \Gamma \\ 1 & \text{if } \alpha \in \Gamma \quad \neg \alpha \notin \Gamma \\ 0 & \text{if } \alpha \notin \Gamma \quad \neg \alpha \in \Gamma \\ \perp & \text{if } \alpha \notin \Gamma \quad \neg \alpha \notin \Gamma \end{cases}$$

Then, $x \notin M_{inv_4(\delta)} = M_{\neg \delta}$. But, Γ is a set of \mathcal{L}_4 -literals and $0, \neg 1, \perp, \neg \perp \notin \Gamma$, thus $x \in M_\Gamma$. Thus $M_\Gamma \not\subseteq M_{\neg \delta}$, thus $\Gamma \not\vdash \neg \delta$.

Case 2: $A = [\Rightarrow_d 1]$. Then, $\Gamma \Vdash \Delta$. And, $0, \neg 1, \perp, \neg \perp \notin \Gamma$, thus $M_\Gamma \not\subseteq \emptyset = M_{\neg 1}$, thus $\Gamma \not\vdash \neg 1$.

Case 3: $A = [\Rightarrow_d \neg 0]$. Similar reasoning as for case 2, which ends the proof of $p(1)$.

Let $i \in \mathbb{N}^+$. Suppose $p(i)$. We show $p(i+1)$. Let $\Gamma, \Delta \subseteq \mathcal{L}_4$ and R be the last rule of a proof of length $i+1$ for $\Gamma \Rightarrow_d \Delta$.

Case 1: $R = [\Rightarrow_d \wedge]$. Then, $\exists \Delta', \alpha, \beta, \Delta = \Delta' \cup \{\alpha \vee \beta\}$, $\Gamma \Rightarrow \Delta', \alpha$, and $\Gamma \Rightarrow \Delta', \beta$. Thus, by the rule $[\Rightarrow \wedge]$, $\Gamma \Rightarrow \Delta$. Thus, by (0) and Proposition 14, $\Gamma \Vdash \Delta$. In addition, $\exists \delta \in \Delta$, $\Gamma \not\vdash \neg \delta$. Thus, by (0) and Proposition 14, $\exists \delta \in \Delta$, $\Gamma \not\vdash \neg \delta$.

Case 2: $R = [\Rightarrow_d \neg \vee]$ or $R = [\Rightarrow_d \neg \supset]$ or $R = [\neg \supset \Rightarrow_d]$ or $R = [\supset \Rightarrow_d]$ or $R = [\Rightarrow_d \supset]$. Similar reasoning as for case 1.

Case 3: $R = [\alpha \vee \Rightarrow_d]$. Then, $\Gamma \Rightarrow \Delta$, thus, by (0) and Proposition 14, $\Gamma \Vdash \Delta$. In addition, $\exists \Gamma', \alpha, \beta, \Gamma = \Gamma' \cup \{\alpha \vee \beta\}$ and there is a proof of length i for $\Gamma', \alpha \Rightarrow_d \Delta$. Thus, by $p(i)$, $\Gamma', \alpha \Vdash_d \Delta$ thus, $\exists \delta \in \Delta$, $\Gamma', \alpha \not\vdash \neg \delta$, thus $M_{\Gamma', \alpha} \not\subseteq M_{\neg \delta}$, however $M_{\Gamma', \alpha} \subseteq M_{\Gamma', \alpha \vee \beta}$, thus $M_{\Gamma', \alpha \vee \beta} \not\subseteq M_{\neg \delta}$. Thus, $\exists \delta \in \Delta$, $\Gamma \not\vdash \neg \delta$.

Case 4: $R = [\vee \beta \Rightarrow_d]$ or $R = [\alpha \neg \wedge \Rightarrow_d]$ or $R = [\neg \wedge \beta \Rightarrow_d]$. Similar reasoning as for case 3.

Case 5: $R = [\Rightarrow_d \vee]$. By $p(i)$, $\Gamma \Vdash_d \Delta', \alpha, \beta$. Thus, $\Gamma \Vdash \Delta', \alpha, \beta$, thus, by Proposition 14 and (0), $\Gamma \Rightarrow \Delta', \alpha, \beta$, thus, by the rule $[\Rightarrow \vee]$, $\Gamma \Rightarrow \Delta$, thus, by (0) and Proposition 14, $\Gamma \Vdash \Delta$. It remains to show $\exists \delta \in \Delta$, $\Gamma \not\vdash \neg \delta$. We have $\exists \delta \in \Delta' \cup \{\alpha, \beta\}$, $\Gamma \not\vdash \neg \delta$. Case 5.1: $\delta = \alpha$. Then, $M_\Gamma \not\subseteq M_{\neg \alpha} \supseteq M_{\neg(\alpha \vee \beta)}$, thus $\Gamma \not\vdash \neg(\alpha \vee \beta)$. Case 5.2: $\delta = \beta$. Similar reasoning as for case 5.1. Case 5.3: $\delta \in \Delta'$. Obvious.

Case 6: $R = [\Rightarrow_d \neg \wedge]$. Similar reasoning as for case 5.

Case 7: $R = [\neg \neg \Rightarrow_d]$ or $R = [\Rightarrow_d \neg \neg]$ or $R = [\wedge \Rightarrow_d]$ or $R = [\neg \vee \Rightarrow_d]$. Obvious.

Direction: “ \rightarrow ”. First, note that all rules in *systemFOUR* are semantically reversible. For instance, the rule $[\neg \neg \Rightarrow]$ is semantically reversible, as if $\Gamma \cup \{\neg \neg \alpha\} \Vdash \Delta$, then $\Gamma \cup \{\alpha\} \Vdash \Delta$. Now,

we denote by $c(\alpha)$, the complexity of α . More formally,

$$c(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is a } \mathcal{L}_4\text{-literal} \\ c(\beta) + 1 & \text{if } \alpha \text{ is not a } \mathcal{L}_4\text{-literal and } \exists \beta, \alpha = \neg\beta \\ \max(\{c(\beta), c(\beta')\}) + 1 & \text{if } \exists \beta, \beta', \alpha \in \{\beta \wedge \beta', \beta \vee \beta', \beta \supset \beta'\} \end{cases}$$

We denote by $c(\Gamma)$, the complexity of Γ , i.e. $c(\Gamma) = \max(\{c(\alpha) : \alpha \in \Gamma\})$. As, $\forall \Gamma, c(\Gamma) \in \mathbb{N}^+$, to show the “ \Rightarrow ” direction it suffices to show by induction that $\forall i \in \mathbb{N}^+, p(i)$, with

$$p(i) \quad \forall \Gamma, \Delta, \text{ if } \Gamma \Vdash_d \Delta \text{ and } c(\Gamma \cup \Delta) \leq i \text{ then } \Gamma \Rightarrow_d \Delta.$$

We show $p(1)$. Let $\Gamma, \Delta \subseteq \mathcal{L}_4, \Gamma \Vdash_d \Delta$ and $c(\Gamma \cup \Delta) = 1$. Then, $\Gamma \cup \Delta$ is a set of \mathcal{L}_4 -literals.

Case 1: $\Gamma \cap \{0, \neg 1, \perp, \neg \perp\} \neq \emptyset$. $M_\Gamma = \emptyset$, thus $\forall \delta \in \Delta, \Gamma \vdash \neg\delta$, thus $\Gamma \not\Vdash_d \Delta$, impossible.

Case 2: $0, \neg 1, \perp, \neg \perp \notin \Gamma$. Case 2.1: $\forall \delta \in \Delta, \text{inv}_4(\delta) \in \Gamma$ or $\delta \in \{0, \neg 1, \top, \neg \top\}$. Let $\delta \in \Delta$. If $\text{inv}_4(\delta) \in \Gamma$, then $M_\Gamma \subseteq M_{\text{inv}_4(\delta)} = M_{\neg\delta}$. Else, $\delta \in \{0, \neg 1, \top, \neg \top\}$, thus $M_\Gamma \subseteq \mathcal{L} = M_{\neg\delta}$. Thus, $\Gamma \vdash \neg\delta$. Thus, $\Gamma \not\Vdash_d \Delta$, which is impossible.

Case 2.2: $\exists \delta \in \Delta, \text{inv}_4(\delta) \notin \Gamma$ and $\delta \notin \{0, \neg 1, \top, \neg \top\}$. Case 2.2.1: $\Delta \cap \{1, \neg 0, \top, \neg \top\} \neq \emptyset$. By the rule $[\Rightarrow_d 1], [\Rightarrow_d \neg 0], [\Rightarrow_d \top],$ or $[\Rightarrow_d \neg \top], \Gamma \Rightarrow_d \Delta$.

Case 2.2.2: $1, \neg 0, \top, \neg \top \notin \Delta$. Case 2.2.2.1: $\Gamma \cap \Delta = \emptyset$. Consider the \mathcal{L}_4 -valuation x of the “ \Leftarrow ” direction. As Γ is a set of \mathcal{L}_4 -literals and $0, \neg 1, \perp, \neg \perp \notin \Gamma$, we get $x \in M_\Gamma$. Let $\delta \in \Delta$. As $\delta \notin \Gamma$, δ is a \mathcal{L}_4 -literal, and $\delta \notin \{1, \neg 0, \top, \neg \top\}$, we get $x \notin M_\delta$. Thus, $\Gamma \not\Vdash \Delta$, which is impossible.

Case 2.2.2.2: $\Gamma \cap \Delta \neq \emptyset$. By the rule $[\alpha \Rightarrow_d \alpha], \Gamma \Rightarrow_d \Delta$.

Let $i \in \mathbb{N}^+$. Suppose $p(i)$. We show $p(i+1)$. It suffices to show by induction $\forall j \in \mathbb{N}^+, q(j)$, with

$$q(j) \quad \forall \Gamma, \Delta, \text{ if } \begin{cases} \Gamma \Vdash_d \Delta \text{ and} \\ c(\Gamma \cup \Delta) \leq i+1 \text{ and} \\ |\{\alpha \in \Gamma \cup \Delta : c(\alpha) \text{ is maximal}\}| = j \end{cases} \text{ then } \Gamma \Rightarrow_d \Delta.$$

We show $q(1)$. Let $\Gamma, \Delta \subseteq \mathcal{L}_4, \Gamma \Vdash_d \Delta, c(\Gamma \cup \Delta) \leq i+1$, and $|\{\alpha \in \Gamma \cup \Delta : c(\alpha) \text{ is maximal}\}| = 1$. Let γ be the unique formula such that $c(\gamma)$ is maximal.

Case 1: γ is a \mathcal{L}_4 -literal. Obvious by $p(1)$.

Case 2: γ is not a \mathcal{L}_4 -literal. We study the sub-cases which follow. Note that the case where $\gamma = \neg\gamma'$ is slipped into the following cases $\gamma = \neg(\alpha \vee \beta), \gamma = \neg(\alpha \wedge \beta), \gamma = \neg(\alpha \supset \beta)$, and $\gamma = \neg\neg\alpha$.

Case 2.1: $\gamma = \alpha \wedge \beta \in \Delta$. Then, $\Gamma \Vdash \Delta$. We pose $\Delta = \Delta' \cup \{\alpha \wedge \beta\}$. The rule $[\Rightarrow \wedge]$ is semantically reversible, thus $\Gamma \Vdash \Delta', \alpha$ and $\Gamma \Vdash \Delta', \beta$. Thus, by Proposition 14 and (0), $\Gamma \Rightarrow \Delta', \alpha$ and $\Gamma \Rightarrow \Delta', \beta$. In addition, $\exists \delta \in \Delta, \Gamma \not\vdash \neg\delta$. Thus, by Proposition 14 and (0), $\exists \delta \in \Delta, \Gamma \not\Rightarrow \neg\delta$. Thus, by the rule $[\Rightarrow_d \wedge], \Gamma \Rightarrow_d \Delta$.

Case 2.2: $\gamma = \neg(\alpha \vee \beta) \in \Delta$ or $\gamma = \alpha \supset \beta$ or $\gamma = \neg(\alpha \supset \beta)$. Similar reasoning as for case 2.1.

Case 2.3: $\gamma = \alpha \vee \beta \in \Gamma$. Then, $\Gamma \Vdash \Delta$. Thus, by Proposition 14 and (0), $\Gamma \Rightarrow \Delta$. We pose $\Gamma = \Gamma' \cup \{\alpha \vee \beta\}$. The rule $[\vee \Rightarrow]$ is semantically reversible, thus $\Gamma', \alpha \Vdash \Delta$ and $\Gamma', \beta \Vdash \Delta$. In addition, $\exists \delta \in \Delta, \Gamma \not\vdash \neg\delta$. Thus, $M_{\Gamma', \alpha \vee \beta} \not\subseteq M_{\neg\delta}$, however $M_{\Gamma', \alpha \vee \beta} = M_{\Gamma', \alpha} \cup M_{\Gamma', \beta}$, thus $M_{\Gamma', \alpha} \not\subseteq M_{\neg\delta}$ or $M_{\Gamma', \beta} \not\subseteq M_{\neg\delta}$. Thus, $\exists \delta \in \Delta, \Gamma', \alpha \not\vdash \neg\delta$ or $\exists \delta \in \Delta, \Gamma', \beta \not\vdash \neg\delta$. Thus, $\Gamma', \alpha \Vdash_d \Delta$ or $\Gamma', \beta \Vdash_d \Delta$. However, $c(\Delta \cup \Gamma' \cup \{\alpha, \beta\}) \leq i$, thus, by $p(i)$, $\Gamma', \alpha \Rightarrow_d \Delta$ or $\Gamma', \beta \Rightarrow_d \Delta$. Thus, by the rule $[\alpha \vee \Rightarrow_d]$ or $[\vee \beta \Rightarrow_d], \Gamma \Rightarrow_d \Delta$.

Case 2.4: $\gamma = \neg(\alpha \wedge \beta) \in \Gamma$. Similar reasoning as for case 2.3.

Case 2.5: $\gamma = \alpha \vee \beta \in \Delta$. Then, $\Gamma \Vdash \Delta$. We pose $\Delta = \Delta' \cup \{\alpha \vee \beta\}$. The rule $[\Rightarrow \vee]$ is semantically reversible, thus $\Gamma \Vdash \Delta', \alpha, \beta$. We show $\exists \delta \in \Delta' \cup \{\alpha, \beta\}, \Gamma \not\vdash \neg\delta$. We have $\exists \delta \in \Delta' \cup \{\alpha \vee \beta\}, \Gamma \not\vdash \neg\delta$. Case 2.5.1: $\delta = \alpha \vee \beta$. Then, $M_\Gamma \not\subseteq M_{\neg(\alpha \vee \beta)} = M_{\neg\alpha} \cap M_{\neg\beta}$, thus $M_\Gamma \not\subseteq M_{\neg\alpha}$ or $M_\Gamma \not\subseteq M_{\neg\beta}$. Thus, $\Gamma \not\vdash \neg\alpha$ or $\Gamma \not\vdash \neg\beta$. Case 2.5.2: $\delta \in \Delta'$. Obvious. Therefore, $\Gamma \Vdash_d \Delta', \alpha, \beta$.

However, $c(\Gamma \cup \Delta' \cup \{\alpha, \beta\}) \leq i$, thus, by $p(i)$, $\Gamma \Rightarrow_d \Delta'$, α, β , thus, by the rule $[\Rightarrow_d \vee]$, $\Gamma \Rightarrow_d \Delta$.

Case 2.6: $\gamma = \neg(\alpha \wedge \beta) \in \Delta$. Similar reasoning as for case 2.5.

Case 2.7: $\gamma = \neg\neg\alpha$ or $\gamma = \alpha \wedge \beta \in \Gamma$ or $\gamma = \neg(\alpha \vee \beta) \in \Gamma$. Obvious.

Let $j \in \mathbb{N}^+$. Suppose $q(j)$. We show $q(j+1)$. Similar reasoning as for $q(1)$, except that $p(i)$ is replaced by $q(j)$ and γ is one of the formula such that $c(\gamma)$ is maximal. ■

7.2 A proof system for \Vdash_d in the J_3 context

For the whole Section 7.2 we assume $\mathcal{L} = \mathcal{L}_3$, $\mathcal{V} = \mathcal{V}_3$, $\models = \models_3$ and $\forall \alpha \in \mathcal{L}_3$, $neg(\alpha) = \neg\alpha$.

Definition 33 We say that α is a \mathcal{L}_3 -literal iff there exists a constant or propositional symbol p of \mathcal{L}_3 such that $\alpha \in \{p, \neg p\}$. For all $\alpha \in \mathcal{L}_3$, we denote by $inv_3(\alpha)$ the formula of \mathcal{L}_3 such that: $inv_3(\alpha) = \beta$ if $\exists \beta \in \mathcal{L}_3$, $\alpha = \neg\beta$; $inv_3(\alpha) = \neg\alpha$ otherwise. We call *systemJ3d* the proof system obtained from *systemJ3* by adding:

- the same axioms and rules that we have added to get *systemFOURd* from *systemFOUR*, except that: we omit \perp and $\neg\perp$ wherever they appear, we replace \mathcal{L}_4 -literal(s) by \mathcal{L}_3 -literal(s), and we replace inv_4 by inv_3 ;

$$\bullet \frac{\Gamma \cup \Delta \cup \{\alpha, \neg\alpha\} \text{ is a set of } \mathcal{L}_3\text{-literals} \quad 0, \neg 1 \notin \Gamma \quad [\Rightarrow_d \alpha, \neg\alpha] \quad \exists \delta \in \Delta \cup \{\alpha, \neg\alpha\}, inv_3(\delta) \notin \Gamma \text{ and } \delta \notin \{0, \neg 1, \top, \neg\top\}}{\Gamma \Rightarrow_d \Delta, \alpha, \neg\alpha} .$$

Proposition 34

- (0) $\forall \Gamma, \Delta, \Gamma \Rightarrow \Delta$ is provable in *systemJ3* iff $\Gamma \Rightarrow \Delta$ is provable in *systemJ3d*;
- (1) $\forall \Gamma, \Delta, \Gamma \Vdash_d \Delta$ iff $\Gamma \Rightarrow_d \Delta$ is provable in *systemJ3d*.

Proof The proofs are similar to the ones for Proposition 32, except for the parts which follow.

(1). Direction: “ \Leftarrow ”. Case 1: $R = [\alpha \Rightarrow_d \alpha]$ or $R = [\Rightarrow_d \top]$ or $R = [\Rightarrow_d \neg\top]$ or $R = [\Rightarrow_d \alpha, \neg\alpha]$. Then, $\Gamma \Vdash \Delta$. In addition, $\exists \delta \in \Delta$, $inv_3(\delta) \notin \Gamma$, $inv_3(\delta)$ is a \mathcal{L}_3 -literal, and $\delta \notin \{0, \neg 1, \top, \neg\top\}$. Consider the \mathcal{L}_3 -valuations x_1 and x_0 such that $\forall \alpha \in \mathcal{L}$,

$$x_1(\alpha) = \begin{cases} \top & \text{if } \alpha \in \Gamma \quad \neg\alpha \in \Gamma \\ 1 & \text{if } \alpha \in \Gamma \quad \neg\alpha \notin \Gamma \\ 0 & \text{if } \alpha \notin \Gamma \quad \neg\alpha \in \Gamma \\ 1 & \text{if } \alpha \notin \Gamma \quad \neg\alpha \notin \Gamma \end{cases} \quad x_0(\alpha) = \begin{cases} \top & \text{if } \alpha \in \Gamma \quad \neg\alpha \in \Gamma \\ 1 & \text{if } \alpha \in \Gamma \quad \neg\alpha \notin \Gamma \\ 0 & \text{if } \alpha \notin \Gamma \quad \neg\alpha \in \Gamma \\ 0 & \text{if } \alpha \notin \Gamma \quad \neg\alpha \notin \Gamma \end{cases}$$

Then, $x_1 \notin M_{inv_3(\delta)}$ or $x_0 \notin M_{inv_3(\delta)}$. However, Γ is a set of \mathcal{L}_3 -literals and $0, \neg 1 \notin \Gamma$, thus $x_1, x_0 \in M_\Gamma$. Thus $M_\Gamma \not\subseteq M_{inv_3(\delta)} = M_{\neg\delta}$. Thus, $\exists \delta \in \Delta, \Gamma \not\vdash \neg\delta$.

Direction: “ \rightarrow ”. We show $p(1)$. Let $\Gamma, \Delta \subseteq \mathcal{L}_3$, $\Gamma \Vdash_d \Delta$ and $c(\Gamma \cup \Delta) = 1$. Then, $\Gamma \cup \Delta$ is a set of \mathcal{L}_3 -literals.

Case 1: $\Gamma \cap \{0, \neg 1\} \neq \emptyset$, case 2: $0, \neg 1 \notin \Gamma$, and case 2.1: $\forall \delta \in \Delta$, $inv_3(\delta) \in \Gamma$ or $\delta \in \{0, \neg 1, \top, \neg\top\}$. Similar reasoning as for Proposition 32.

Case 2.2: $\exists \delta \in \Delta$, $inv_3(\delta) \notin \Gamma$ and $\delta \notin \{0, \neg 1, \top, \neg\top\}$. Case 2.2.1: $\Delta \cap \{1, \neg 0, \top, \neg\top\} \neq \emptyset$ or $\exists \alpha, \alpha, \neg\alpha \in \Delta$. By the rule $[\Rightarrow_d 1]$, $[\Rightarrow_d \neg 0]$, $[\Rightarrow_d \top]$, $[\Rightarrow_d \neg\top]$, or $[\Rightarrow_d \alpha, \neg\alpha]$, $\Gamma \Rightarrow_d \Delta$.

Case 2.2.2: $1, \neg 0, \top, \neg \top \notin \Delta$ and $\forall \alpha, \alpha \notin \Delta$ or $\neg \alpha \notin \Delta$. Case 2.2.2.1: $\Gamma \cap \Delta = \emptyset$. Consider the \mathcal{L}_3 -valuation x such that $\forall \alpha \in \mathcal{A}$

$$x(\alpha) = \begin{cases} \top & \text{if } \alpha \in \Gamma \quad \neg \alpha \in \Gamma \\ 1 & \text{if } \alpha \in \Gamma \quad \neg \alpha \notin \Gamma \\ 0 & \text{if } \alpha \notin \Gamma \quad \neg \alpha \in \Gamma \\ 0 & \text{if } \alpha \notin \Gamma \quad \neg \alpha \notin \Gamma \quad \alpha \in \Delta \quad \neg \alpha \notin \Delta \\ 1 & \text{if } \alpha \notin \Gamma \quad \neg \alpha \notin \Gamma \quad \alpha \notin \Delta \quad \neg \alpha \in \Delta \\ \top & \text{if } \alpha \notin \Gamma \quad \neg \alpha \notin \Gamma \quad \alpha \notin \Delta \quad \neg \alpha \notin \Delta \end{cases}$$

As Γ is a set of \mathcal{L}_3 -literals and $0, \neg 1 \notin \Gamma$, we get $x \in M_\Gamma$. Let $\delta \in \Delta$. As $\delta \notin \Gamma$, δ is a \mathcal{L}_3 -literal, and $\delta \notin \{1, \neg 0, \top, \neg \top\}$, we get $x \notin M_\delta$. Thus, $\Gamma \not\models \Delta$, thus $\Gamma \not\models_a \Delta$, which is impossible.

Case 2.2.2.2: $\Gamma \cap \Delta \neq \emptyset$. By the rule $[\alpha \Rightarrow_d \alpha], \Gamma \Rightarrow_d \Delta$. \blacksquare

8 Summary and future work

Preferential and preferential-discriminative CRs represent natural ways of drawing conclusions which are useful to handle incomplete information in the classical context and both incomplete and inconsistent information in the J_3 and $FOUR$ contexts. The purpose of this paper was to characterize several families of them in a general context that covers the classical, three and four-valued contexts.

Theorem 7 and Lemma 21 have been used in both the basic and the discriminative case. This suggests that they can be used to characterize some other families of CRs based on preferential structures. Similarly, Lemmas 28 and 29 have an interest of their own. Indeed, it seems that they can be used to characterize any family C of CRs such that $|\sim \in C$ iff $\forall \Gamma, \tilde{\Gamma} = T_d(\mu(M_\Gamma))$, where $\mu : \mathcal{P}(\mathcal{V}) \rightarrow \mathcal{P}(\mathcal{V})$ is simply a function satisfying $(\mu 0)$. Finally, let us say that some of the conditions that we provided are probably too complex (and ugly) to be used efficiently. Simplifying them could be the goal of a future work.

9 Acknowledgements

I owe very much to Karl Schlechta for his hints, advice, constructive criticism, and more. I acknowledge also Arnon Avron for the discussions about proof systems and the referees for their help concerning the presentation of the paper.

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