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Daniel Maerten

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Positive energy-momentum theorem in asymptotically Anti-de Sitter space-times

MAERTEN DANIEL

1 The Energy-Momentum

1.1 Introduction

This paper proves a positive energy-momentum theorem under the (well known in general relativity) dominant energy condition, for orientable 3-dimensional Riemannian manifolds that are asymptotic at infinity to a standard hyperbolic slice in anti-de Sitter (AdS) space-time. Chruściel and Nagy [14] recently defined notions of mass and momentum on an asymptotically hyperbolic manifold, which generalize the analogous notions in the asymptotically flat case. Besides Chruściel and Herzlich [12] recently proved a positive mass theorem for asymptotically hyperbolic manifolds. The aim of the present paper is to extend this result in the following way: we will define a sesquilinear form Q on \mathbb{C}^4 which is related to the energy-momentum (cf. section 4) and prove under the relevant energy condition, that Q is non negative. We also give a rigidity part to our theorem: if the sesquilinear form Q is degenerate, then our 3-manifold M can be isometrically embedded in AdS.

1.2 Some definitions and notations

We consider a Lorentzian manifold N^4 and a Riemannian spacelike hypersurface M . Using geodesic coordinates along M , we shall write a neighbourhood of M in N as a subset of $] - \epsilon, \epsilon[\times M$, endowed with the metric $\gamma = -dt^2 + g_t$. The Riemannian 3-manifold M has induced metric $g_0 = g$ and second fundamental form $k := (-\frac{1}{2} \frac{d}{dt} g_t)|_{t=0}$. We assume that (M, g, k) is asymptotically hyperbolic that is to say, the metric g and the second fundamental form k are asymptotic at infinity to the metric and the second fundamental form of a standard hyperbolic slice in AdS. More precisely we adopt the following

Definition. (M, g, k) is said to be asymptotically hyperbolic if there exists some compact K , a positive number R and a homeomorphism $M \setminus K \rightarrow \mathbb{R}^3 \setminus B(0, R)$ called a chart at infinity such that in this chart we have

$$\begin{cases} e := g - b = O(e^{-\tau r}), & \partial e = O(e^{-\tau r}), & \partial^2 e = O(e^{-\tau r}), \\ k = O(e^{-\tau r}), & \partial k = O(e^{-\tau r}), \end{cases}$$

for $\tau > 3/2$ and where ∂ is taken with respect to the hyperbolic metric $b = dr^2 + \sinh^2 r g_{S^2}$ with g_{S^2} the standard metric of S^2 .

AdS space-time is merely denoted by (\tilde{N}, β) . If one considers \tilde{N} as \mathbb{R}^4 then we will write $\beta = -dt^2 + b_t$, with $b_0 = b$ the hyperbolic metric.

Remark. As we will see below, the definition of the energy-momentum only depends of the behaviour of (g, k) on $M \setminus K$. So we will assume that M and \mathbb{H}^3 are homeomorphic without any loss of generality. This also entails that $N \cong \tilde{N}$ as topological spaces. This assumption will be convenient in order to work on the whole manifold N and not only on $] - \epsilon, \epsilon[\times (M \setminus K)$. \square

The motivation for the definition of the energy-momentum comes from the study of the constraints map which by definition is

$$\begin{aligned} \Phi : \mathcal{M} \times \Gamma(S^2T^*M) &\longrightarrow C^\infty(M) \times \Gamma(T^*M) \\ (h, p) &\longmapsto \begin{pmatrix} \text{Scal}^h + (\text{tr}_h p)^2 - |p|_h^2 \\ 2(\delta_h p + \text{dtr}_h p) \end{pmatrix}, \end{aligned}$$

where \mathcal{M} is the set of Riemannian metrics on the manifold M . Let us denote by (\dot{h}, \dot{p}) an infinitesimal deformation of (h, p) . Now if we take a couple $(f, \alpha) \in C^\infty(M) \times \Gamma(T^*M)$ then we compute

$$\begin{aligned} \langle (f, \alpha), (\Phi(h + \dot{h}, p + \dot{p}) - \Phi(h, p)) \rangle &= \delta(f(\delta \dot{h} + \text{dtr} \dot{h}) + i_{\nabla f} \dot{h} - (\text{tr} \dot{h})df + 2i_\alpha \dot{p} - 2(\text{tr} \dot{p})\alpha) \\ &+ \delta(\langle p, \dot{h} \rangle \alpha + \langle h, \dot{h} \rangle i_\alpha p - 2i_{i_\alpha p} \dot{h}) \\ &+ \langle \text{d}\Phi_{(h,p)}^*(f, \alpha), (\dot{h}, \dot{p}) \rangle + Q(f, \alpha, h, p, \dot{h}, \dot{p}), \end{aligned}$$

where i_α is the metric extended to all tensors, δ is the h -divergence operator, $\text{d}\Phi_{(h,p)}^*$ is the formal adjoint of the linearized constraints map at the point (h, p) , traces are taken with respect to h and $Q(f, \alpha, h, p, \dot{h}, \dot{p})$ is a remainder which is linear with respect to (f, α) and at least quadratic with respect to (\dot{h}, \dot{p}) . Now considering the constraints map along the hyperbolic space embedded in AdS, that is to say $(h, k) = (b, 0)$ and $(\dot{h}, \dot{k}) = (g - b = e, k)$ one finds

$$\begin{aligned} \langle (f, \alpha), (\Phi(g, k) - \Phi(b, 0)) \rangle &= \delta(f(\delta e + \text{dtr} e) + i_{\nabla f} e - (\text{tr} e)df + 2i_\alpha k - 2(\text{tr} k)\alpha) \\ &+ \langle \text{d}\Phi_{(b,0)}^*(f, \alpha), (e, k) \rangle + Q(f, \alpha, b, k, e). \end{aligned}$$

As a consequence if we assume that (M, g, k) is asymptotically hyperbolic and if the function $\langle (f, \alpha), (\Phi(g, k) - \Phi(b, 0)) \rangle$ is integrable on M with respect to the measure dVol_b , then the energy-momentum \mathcal{H} can be defined as a linear form on $\text{Ker } \text{d}\Phi_{(b,0)}^*$

$$\mathcal{H} : (f, \alpha) \longmapsto \int_{S_\infty} -f(\delta_b e + \text{dtr}_b e) - i_{\nabla^b f} e + (\text{tr}_b e)df - 2i_{\alpha^\#} k + 2(\text{tr}_b k)\alpha.$$

Remark. The integrability condition $\langle (f, \alpha), (\Phi(g, k) - \Phi(b, 0)) \rangle \in L^1(M, \text{dVol}_b)$ can be replaced by the less general but more convenient condition $|\Phi(g, k) - \Phi(b, 0)| e^r \in L^1(M, \text{dVol}_b)$. Moreover the integrand in the formula of \mathcal{H} is in index notations

$$f(e_{j,i}^i - e_{i,j}^i) - f^i e_{ij} + (e_i^i) f_{,j} - 2\alpha^i k_{ij} + 2(k_i^i) \alpha_j,$$

where “,” stands for the b -derivatives and where $h^i = b^{ij}h_j$ for any tensor h . \square

The condition of integrability above and the conditions of asymptoticity are sufficient in order to make \mathcal{H} well defined and invariant under coordinate change. The sharpest conditions have been found by Chruściel and Nagy in [14].

One can show, using Moncrief argument [18], that $\text{Ker } d\Phi_{(b,0)}^* \cong \mathfrak{Kill}(N, \beta)$ where $\mathfrak{Kill}(N, \beta)$ denotes the Lie algebra of Killing vector fields on (M, β) , since the metric β satisfies the Einstein equations with a cosmological constant. The isometry group of AdS is $\text{SO}(3,2)$, and thereby $\mathfrak{Kill}(N, \beta) \cong \mathfrak{so}(3,2) \cong N_b \oplus \mathfrak{so}(3,1) \cong N_b \oplus \mathfrak{Kill}(M, b)$, where we have set $N_b = \{f \in C^\infty(M) | \text{Hess}f = fb\}$. It is well known [14], [12] that the application

$$\begin{aligned} \mathbb{R}^{3,1} &\longrightarrow N_b \\ y_k &\longmapsto x_k := y_k|_{\mathbb{H}^3} \end{aligned} \quad ,$$

(where $(y_k)_{k=0}^3$ are the standard coordinates) is an isometry, and the “mass” part of the energy-momentum is a linear form on N_b which is causal and positively oriented as soon as $\text{Scal}^g \geq -n(n-1) = \text{Scal}^b$.

Remark. It is important to notice that the decomposition $\mathfrak{so}(3,2) \cong \mathbb{R}^{3,1} \oplus \mathfrak{so}(3,1)$ does not have the same meaning than in the asymptotically Minkowski case, so that the terminology “energy” and “linear momentum” is not so relevant. Physicists may call them “global charges” and the positive energy-momentum theorem should be consequently renamed global inequalities theorem. \square

1.3 Statement of the theorems

As a matter of fact, we will see (cf. sections 2.2 and 2.4) that, given a chart at infinity, \mathcal{H} can be considered as a vector of $\mathbb{R}^{3,1} \oplus \mathfrak{sl}_2(\mathbb{C}) \cong N_b \oplus \mathfrak{so}(3,1)$ and will be denoted by $M \oplus \Xi$, where $\Xi = \begin{pmatrix} n_1 + \mathbf{i}r_1 & (n_2 - r_3) + \mathbf{i}(n_3 + r_2) \\ (n_2 + r_3) + \mathbf{i}(-n_3 + r_2) & -n_1 - \mathbf{i}r_1 \end{pmatrix}$, $M = \begin{pmatrix} m_0 + m_1 & m_2 + \mathbf{i}m_3 \\ m_2 - \mathbf{i}m_3 & m_0 + m_1 \end{pmatrix}$. The matrix M is the mass part [12] of \mathcal{H} . Then we define the matrix

$$Q = 2 \begin{pmatrix} \widehat{M} & \Xi \\ \Xi^* & M \end{pmatrix},$$

where \widehat{M} means the transposed comatrix of M .

Positive Energy-Momentum Theorem. *Let (M^3, g, k) an orientable asymptotically hyperbolic Riemannian manifold satisfying the decay conditions stated in section 1.2 and the following conditions*

- (i) $\langle (f, \alpha), (\Phi(g, k) - \Phi(b, 0)) \rangle \in L^1(M, dVol_b)$ for every $(f, \alpha) \in N_b \oplus \mathfrak{Kill}(M, b)$,
- (ii) the relative version of the dominant energy condition (cf. section 3.3) holds, that is to say $(\Phi(g, k) - \Phi(b, 0))$ is a positively oriented causal 4-vector.

Then Q is non negative.

Classical algebra results give the non negativity of each principal minors of Q which provide a set of inequalities on the coefficients of \mathcal{H} that are explicitly written in the appendix (cf. section 5).

Remark. Zhang [27] recently obtained some inequalities. It is not easy to compare them with our inequalities, but there exists some energy-momentum that satisfy Zhang's conditions and do not respect the non negativity of Q . \square

As regards the rigidity part we have the

Rigidity Theorem. *If the assumptions of the positive energy-momentum hold, and if Q is degenerate then (M, g, k) can be isometrically embedded in AdS.*

1.4 Sketch of the proofs

Analogously to the original Witten's proof of the positive mass theorem in the asymptotically flat case, this work relies on spinorial geometry [25] (recall that in dimension 3 orientable is equivalent to spin). In section 2, we make a complete study of the imaginary Killing spinors [8], [9] in AdS along the hypersurface \mathbb{H}^3 , and define some Killing invariants in order to see \mathcal{H} as an element of $N_b \oplus \mathfrak{sl}_2(\mathbb{C})$. Section 3 contains the geometric and analytical background useful to understand, following the work of Andersson and Dahl [3], the relation between the imaginary Killing spinors and the energy-momentum. We finally give in section 4 the proofs of the positive energy-momentum theorem, and the proof of the rigidity part which relies on the dominant energy condition and some relevant curvature formula in order to apply the result of [5].

2 Imaginary Killing Spinors

2.1 Tangent and Spinor bundles

In this paper, the model spaces AdS and \mathbb{H}^3 are considered as symmetric spaces:

$$\mathbb{H}^3 = \mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2) \hookrightarrow \mathrm{AdS} = \mathrm{Spin}_0(3, 2)/\mathrm{Spin}_0(3, 1),$$

with $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$ and $\mathrm{SL}(2, \mathbb{C}) \cong \mathrm{Spin}_0(3, 1)$.

The spinor bundle of AdS is $\Sigma_{\mathrm{AdS}} = \mathrm{Spin}_0(3, 2) \times_{\tilde{\rho}} \mathbb{C}^4$, where $\mathrm{Spin}_0(3, 2)$ is the bundle of the $\mathrm{Spin}_0(3, 1)$ -frames in AdS, and $\tilde{\rho}$ is the standard representation of $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{C}^4 \cong \mathbb{C}^2 \oplus \overline{\mathbb{C}^2}$. In other words

$$\begin{aligned} \tilde{\rho} : \mathrm{SL}(2, \mathbb{C}) &\longrightarrow \mathrm{M}_4(\mathbb{C}) \\ \tilde{g} &\longmapsto \begin{pmatrix} \tilde{g} & 0 \\ 0 & (\tilde{g}^*)^{-1} \end{pmatrix}, \end{aligned}$$

where $A^* = {}^t \overline{A}$, $A \in \mathrm{M}_2(\mathbb{C})$. When we restrict this bundle to the hypersurface \mathbb{H}^3 we have $\Sigma = \mathrm{SL}(2, \mathbb{C}) \times_{\tilde{\rho}|_{\mathrm{SU}(2)}} \mathbb{C}^4$.

Proposition. Σ and $\mathbb{H}^3 \times \mathbb{C}^4$ are isomorphic thanks to the following trivialisation:

$$T : \quad \begin{array}{ccc} \Sigma & \longrightarrow & \mathbb{H}^3 \times \mathbb{C}^4 \\ \{\tilde{e}, w\} & \longmapsto & ([\tilde{e}], \tilde{\rho}(\tilde{e})w) \end{array} ,$$

where $\{\tilde{e}, w\}$ denotes the class of $(\tilde{e}, w) \in \text{SL}(2, \mathbb{C}) \times \mathbb{C}^4$ in Σ , and $[\tilde{e}]$ denotes the class of $\tilde{e} \in \text{SL}(2, \mathbb{C})$ in $\mathbb{H}^3 = \text{SL}(2, \mathbb{C})/\text{SU}(2)$.

The construction of \mathbb{T}_{AdS} , the tangent bundle of AdS, is quite similar to the construction of the spinor bundle. Still noticing that the principal bundle of $\text{SO}_0(3, 1)$ -frames in AdS is isomorphic to $\text{SO}_0(3, 2)$, we write $\mathbb{T}_{AdS} = \text{SO}_0(3, 2) \times_{\rho} \mathbb{R}^4$, where ρ is the standard representation of $\text{SO}_0(3, 1)$ on \mathbb{R}^4 . By restriction to the hypersurface \mathbb{H}^3 , we obtain $\mathbb{T} = \text{SO}_0(3, 1) \times_{\rho|_{\text{SO}(3)}} \mathbb{R}^4$, where $\text{SO}(3)$ is by definition the isotropy group of f_0 if $(f_k)_{k=0}^3$ denotes the canonical basis of \mathbb{R}^4 .

Proposition. \mathbb{T} and $\mathbb{H}^3 \times \mathbb{R}^4$ are isomorphic thanks to the following trivialisation:

$$T : \quad \begin{array}{ccc} \mathbb{T} & \longrightarrow & \mathbb{H}^3 \times \mathbb{R}^4 \\ \{e, u\} & \longmapsto & ([e], \rho(e)u) \end{array} ,$$

where $\{e, u\}$ denotes the class of $(e, u) \in \text{SO}_0(3, 1) \times \mathbb{R}^4$ in \mathbb{T} , and $[e]$ denotes the class of $e \in \text{SO}_0(3, 1)$ in $\mathbb{H}^3 = \text{SO}_0(3, 1)/\text{SO}(3)$.

2.2 Algebraic structures

We are going to define the Clifford action on Σ , in the same way as in [19]. To this end, we denote by (\mathbb{R}^4, q) the Minkowski space-time of signature (3,1), where $q = -dy_0^2 + dy_1^2 + dy_2^2 + dy_3^2$. This space is isometric to a subspace of $M_2(\mathbb{C})$ via

$$\Lambda : \quad \begin{array}{ccc} (\mathbb{R}^4, q) & \longrightarrow & \mathfrak{M} := (\{A \in M_2(\mathbb{C})/A^* = A\}, -\det) \\ y = (y_i)_{i=0}^3 & \longmapsto & \begin{pmatrix} y_0 + y_1 & y_2 + iy_3 \\ y_2 - iy_3 & y_0 - y_1 \end{pmatrix} \end{array} .$$

We have thus the following real vector space isomorphisms:

$$\begin{aligned} M_2(\mathbb{C}) &\cong \mathfrak{su}(2) \oplus \mathfrak{M} \\ \mathfrak{sl}_2(\mathbb{C}) &\cong \mathfrak{su}(2) \oplus (\mathfrak{M} \cap \mathfrak{sl}_2(\mathbb{C})) \\ &\cong \mathfrak{su}(2) \oplus \mathfrak{G}, \end{aligned}$$

and $\mathfrak{G} \cong \mathbb{R}^3$. In order to make the value of the sectional curvature of \mathbb{H}^3 equal to -1, when we consider $\mathbb{H}^3 = \text{SL}(2, \mathbb{C})/\text{SU}(2)$ as a symmetric space, we have to consider \mathbb{R}^4 endowed with $4q$ and not q , and consequently the embedding of the Clifford algebra $Cl_{3,1}$ in $M_4(\mathbb{C})$ becomes

$$\Theta : X \in \mathfrak{M} \longmapsto \begin{pmatrix} 0 & 2X \\ 2\widehat{X} & 0 \end{pmatrix},$$

where \widehat{X} means the transposed comatrix of X .

It will be convenient to see \mathbb{T} as $\mathrm{SL}(2, \mathbb{C}) \times_{\mu} \mathfrak{M}$, where μ is the universal covering of $\mathrm{SO}_0(3, 1)$ by $\mathrm{SL}(2, \mathbb{C})$, which is given by:

$$\begin{aligned} \mu : \mathrm{SL}(2, \mathbb{C}) &\longrightarrow \mathrm{SO}_0(3, 1) \\ \tilde{g} &\longmapsto (\tilde{g} : X \in \mathfrak{M} \mapsto \tilde{g}X\tilde{g}^*) \end{aligned} .$$

We can now define the Clifford action. Let us take $e \in \mathrm{SO}_0(3, 1)$ and $\tilde{e} \in \mathrm{SL}(2, \mathbb{C})$ such that $e = \mu(\tilde{e})$. A vector $X = X[e]$ tangent at the point $[e] = [\tilde{e}] \in \mathbb{H}^3$, is a class $\{e, u\} \in \mathbb{T}$. A spinor $\sigma = \sigma[\tilde{e}]$ at the same point is likewise a class $\{\tilde{e}, w\} \in \Sigma$. The result of the Clifford action of X on σ is the spinor $(X \cdot \sigma)[\tilde{e}] = \{e, u\} \cdot \{\tilde{e}, w\} = \{\tilde{e}, \Theta(u)w\}$. We define a sesquilinear inner product (not definite positive) (\cdot, \cdot) on $\mathbb{C}^4 \cong \mathbb{C}^2 \oplus \overline{\mathbb{C}^2}$ as in [19] $(\xi, \eta) := \langle \xi_1, \eta_2 \rangle_{\mathbb{C}^2} + \langle \xi_2, \eta_1 \rangle_{\mathbb{C}^2}$, where $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{C}^4$ and where $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$ is the standard Hermitian product on \mathbb{C}^2 . This induces a sesquilinear product on Σ by $(\{\tilde{e}, \xi\}, \{\tilde{e}, \eta\}) := (\xi, \eta)$. In the same way we define a scalar product on Σ setting

$$\begin{aligned} \langle \{\tilde{e}, \xi\}, \{\tilde{e}, \eta\} \rangle &:= \left(\frac{1}{2} f_0 \cdot \{\tilde{e}, \xi\}, \{\tilde{e}, \eta\} \right) \\ &= \left(\{\tilde{e}, \frac{1}{2} \Theta(f_0) \xi\}, \{\tilde{e}, \eta\} \right) \\ &= \langle \xi, \eta \rangle_{\mathbb{C}^4}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^4}$ denotes the standard Hermitian product on \mathbb{C}^4 .

Since $\mathrm{SL}(2, \mathbb{C})$ is the 2-sheeted covering of $\mathrm{SO}_0(3, 1)$, there exists a natural (left) action of $\mathrm{SL}(2, \mathbb{C})$ on Σ which is derived from the natural (left) action of $\mathrm{SO}_0(3, 1)$ on \mathbb{T} : the action of the group of the isometries of AdS preserving the slice \mathbb{H}^3 that is $\tilde{g} * \{e, u\} = \{\tilde{g}e, u\}$, with $\tilde{g} \in \mathrm{SL}(2, \mathbb{C})$ and $\sigma[\tilde{e}] = \{\tilde{e}, w\}$ a spinor at $[\tilde{e}]$. To have the action on a section $\sigma \in \Gamma(\Sigma)$ we set as usual $(\tilde{g} * \sigma)[\tilde{e}] = \tilde{g} * \sigma(\tilde{g}^{-1}\tilde{e})$.

2.3 The Killing equation

Definition. *The Killing equation on a spinor field $\tau \in \Gamma(\Sigma)$ is*

$$D_X \tau + \frac{i}{2} X \cdot \tau = 0 \quad \forall X \in \Gamma(T\mathbb{H}^3),$$

where D denotes the Levi-Civita connection of AdS. Such a spinor τ is called a imaginary Killing spinor.

Remark. Notice that this equation is neither the Killing equation in AdS nor in \mathbb{H}^3 , but the Killing equation in AdS along \mathbb{H}^3 . \square

The aim of this section is to solve explicetely our Killing equation. As a matter of fact, representation theory provides us good candidates for the imaginary Killing spinors. Thanks to Schur's lemma, we have an isomorphism

$$\begin{aligned} \mathbb{C}^2 &\longrightarrow \mathrm{Hom}^{\mathrm{SU}(2)}(\mathbb{C}^2, \mathbb{C}^2 \oplus \mathbb{C}^2) \\ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &\longmapsto \begin{pmatrix} z_1 I_2 \\ z_2 I_2 \end{pmatrix}, \end{aligned}$$

We are now considering two families of spinors which are derived from representation theory. To this end, we will denote $w \otimes z \in \mathbb{C}^2 \otimes \text{Hom}^{\text{SU}(2)}(\mathbb{C}^2, \mathbb{C}^2 \oplus \mathbb{C}^2)$ thanks to the isomorphism above.

Definition. Let $w \otimes z \in \mathbb{C}^2 \otimes \mathbb{C}^2$ and set $\sigma_{w \otimes z}^{-1}[\tilde{g}] = \{\tilde{g}, z(\tilde{g}^{-1}w)\}$, $\sigma_{w \otimes z}^*[\tilde{g}] = \{\tilde{g}, z(\tilde{g}^*w)\}$.

Let us consider a spinor field $\tau \in \Gamma(\Sigma)$ and a vector field $X \in \Gamma(\mathbb{T})$ tangent to \mathbb{H}^3 . We can write $\tau[\tilde{g}] = \{\tilde{g}, v(\tilde{g})\}$ and $X[g] = \{g, \zeta(g)\}$, where $v : \mathbb{H}^3 \rightarrow \mathbb{C}^4$ and $\zeta : \mathbb{H}^3 \rightarrow \mathfrak{G}$ are respectively $\text{SO}(3)$ and $\text{SU}(2)$ -equivariant functions. We can now differentiate τ in the direction of X and write down

$$(D_X \tau)[\tilde{g}] = \{\tilde{g}, v_*(X)_{[\tilde{g}]} + \tilde{\rho}_* \circ s^* \theta(\zeta)_{[\tilde{g}]} v[\tilde{g}]\},$$

where θ is the connection 1-form of the bundle of $\text{SL}(2, \mathbb{C})$ -frames, restricted to \mathbb{H}^3 . If one remembers that θ is only the projection on the first factor in the decomposition $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{su}(2) \oplus \mathfrak{G}$, we can conclude that $\tilde{\rho}_* \circ s^* \theta(\zeta)_{[\tilde{g}]} v[\tilde{g}]$ vanishes. Besides we will apply this formula to spinors in $\{\sigma_{w \otimes z}^{-1}, \sigma_{u \otimes z}^*, w, u \in \mathbb{C}^2\}$ so that we can only derive at the point $\tilde{g} = 1$ unity in $\text{SL}(2, \mathbb{C})$ since we have the

Proposition. The set $\{\sigma_{w \otimes z}^{-1}, \sigma_{u \otimes z}^*, w, u \in \mathbb{C}^2\}$ is stable under the $\text{SL}(2, \mathbb{C})$ action. More precisely for every $\tilde{e} \in \text{SL}(2, \mathbb{C})$ we have $\tilde{e} * \sigma_{w \otimes z}^{-1} = \sigma_{\tilde{e}w \otimes z}^{-1}$ and $\tilde{e} * \sigma_{u \otimes z}^* = \sigma_{(\tilde{e}^*)^{-1}u \otimes z}^*$.

We obtain

$$\begin{cases} (D_X \sigma_{w \otimes z}^{-1})[1] &= \{1, -z(\zeta w)\} \\ (D_X \sigma_{u \otimes z}^*)[1] &= \{1, z(\zeta u)\} \end{cases},$$

where $\zeta = \zeta(1)$. We also compute the Clifford action of X on $\sigma_{w \otimes z}^{-1}, \sigma_{u \otimes z}^*$ at the point 1:

$$\begin{cases} X \cdot \sigma_{w \otimes z}^{-1}[1] &= \{1, \Theta(\zeta)z(w)\} \\ X \cdot \sigma_{u \otimes z}^*[1] &= \{1, \Theta(\zeta)z(u)\} \end{cases}.$$

We must precise $\Theta|_{\mathfrak{G}} : \zeta \mapsto \begin{pmatrix} 0 & 2\zeta \\ -2\zeta & 0 \end{pmatrix}$, and if we introduce the sections $\sigma_{w \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix}}^{-1}$ and $\sigma_{w \otimes \begin{pmatrix} 1 \\ i \end{pmatrix}}^*$, for any $w \in \mathbb{C}^2$, we have on one hand

$$\begin{cases} -\frac{i}{2} X \cdot \sigma_{w \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix}}^{-1}[1] &= -i \{1, -i\zeta w \oplus -\zeta w\} = \{1, -\zeta w \oplus i\zeta w\} \\ -\frac{i}{2} X \cdot \sigma_{w \otimes \begin{pmatrix} 1 \\ i \end{pmatrix}}^*[1] &= -i \{1, i\zeta u \oplus -\zeta u\} = \{1, \zeta u \oplus i\zeta u\} \end{cases},$$

and on the other hand

$$\begin{cases} \left(D_X \sigma_{w \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix}}^{-1} \right) [1] &= \{1, -\zeta w \oplus i\zeta w\} \\ \left(D_X \sigma_{w \otimes \begin{pmatrix} 1 \\ i \end{pmatrix}}^* \right) [1] &= \{1, \zeta u \oplus i\zeta u\} \end{cases}.$$

Since $\left\{ \sigma_{w \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix}}^{-1} + \sigma_{u \otimes \begin{pmatrix} 1 \\ i \end{pmatrix}}^* / w, u \in \mathbb{C}^2 \right\}$ is a 4-dimensional complex vector space, we obviously obtain the

Proposition. *The space of imaginary Killing spinors denoted by $IKS(\Sigma)$ is generated by*

$$\left\{ \sigma_{w \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix}}^{-1}, \sigma_{u \otimes \begin{pmatrix} 1 \\ i \end{pmatrix}}^*, w, u \in \mathbb{C}^2 \right\}.$$

2.4 Killing Invariants

Let σ an imaginary Killing spinor and set $V_\sigma := \langle \sigma, \sigma \rangle$ which is a function on \mathbb{H}^3 , and if e_0 denotes a unit normal of \mathbb{H}^3 in AdS $\alpha_\sigma(Y) := \langle Y \cdot e_0 \cdot \sigma, \sigma \rangle$ which is a real 1-form on \mathbb{H}^3 . Our goal is to find an application from $IKS(\Sigma) \longrightarrow (\mathfrak{M} \oplus \mathfrak{sl}_2(\mathbb{C}))^*$. Throughout this section, a general $\sigma \in IKS(\Sigma)$ will be denoted by $\sigma = \sigma_{w \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix}}^{-1} + \sigma_{u \otimes \begin{pmatrix} 1 \\ i \end{pmatrix}}^*$, where $w, u \in \mathbb{C}^2$.

The functions V_σ

We compute the functions V_σ which are by definition

$$\begin{aligned} V_\sigma[\tilde{g}] &= \left| \sigma_{w \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix}}^{-1}[\tilde{g}] \right|_{\mathbb{C}^4}^2 + \left| \sigma_{u \otimes \begin{pmatrix} 1 \\ i \end{pmatrix}}^*[\tilde{g}] \right|_{\mathbb{C}^4}^2 + 2\Re e \left(\left\langle \sigma_{w \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix}}^{-1}[\tilde{g}], \sigma_{u \otimes \begin{pmatrix} 1 \\ i \end{pmatrix}}^*[\tilde{g}] \right\rangle_{\mathbb{C}^4} \right) \\ &= 2|\tilde{g}^{-1}w|_{\mathbb{C}^2}^2 + 2|\tilde{g}^*u|_{\mathbb{C}^2}^2 \end{aligned}$$

Remark. $\sigma_{w \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix}}^{-1}$ and $\sigma_{u \otimes \begin{pmatrix} 1 \\ i \end{pmatrix}}^*$ are orthogonal spinors for every $u, w \in \mathbb{C}^2$. □

If $\tilde{g} \in \text{SL}(2, \mathbb{C})$, the corresponding base point is $\tilde{g}\tilde{g}^* \in \mathbb{H}^3 \subset \mathfrak{M} \cong \mathbb{R}^{3,1}$ whose coordinates are given by $(x_k)_{k=0}^3 = \Lambda^{-1}(\tilde{g}\tilde{g}^*)$.

Proposition. V_σ is a causal element of N_b .

PROOF. Let $U = \begin{pmatrix} u_1 \\ -w_2 \end{pmatrix} \in \mathbb{C}^2, V = \begin{pmatrix} u_2 \\ w_1 \end{pmatrix} \in \mathbb{C}^2$. We notice that

$$V_\sigma[\tilde{g}] = x_0(|U|^2 + |V|^2) + x_1(|U|^2 - |V|^2) + 2x_2\Re e(\langle U, V \rangle) - 2x_3\Im m(\langle U, V \rangle),$$

so that the norm of V_σ is $|V_\sigma[\tilde{g}]|^2 = 4(|\langle U, V \rangle|^2 - |U|^2|V|^2) \leq 0$, thanks to the Cauchy-Schwarz inequality for the standard Hermitian form on \mathbb{C}^2 . □

More conceptually we see that $V_\sigma[\tilde{g}] = 2(w^*\widehat{W}w + u^*Wu)$, where we have set $W := \tilde{g}\tilde{g}^* \in \mathbb{H}^3 \subset \mathfrak{M}$. Thereby we can define by extension an application

$$\begin{aligned} \mathbb{C}^2 \oplus \mathbb{C}^2 &\longrightarrow \mathfrak{M}^* \\ w \oplus u &\longmapsto \left(V_{w \oplus u} : W \mapsto 2(w^*\widehat{W}w + u^*Wu) \right). \end{aligned}$$

The 1-forms α_σ

The positively oriented unit normal of \mathbb{H}^3 in AdS is given by $e_0[\tilde{g}] = \{\tilde{g}, \frac{1}{2}\mu(\tilde{g})I_2\}$ and for any $\xi \in \mathfrak{G}$ satisfying $-\det\xi = 1$ we set $X^\xi[\tilde{g}] = \{\tilde{g}, \frac{1}{2}\mu(\tilde{g})\xi\}$. Just remember that $\alpha_\sigma(X^\xi)_{[\tilde{g}]} := \langle X^\xi \cdot e_0 \cdot \sigma, \sigma \rangle_{[\tilde{g}]}$. As we suppose that $\sigma \in IK\Sigma(\Sigma)$, we can easily compute the first derivative of α_σ

$$D_{X^\eta}\alpha_\sigma(X^\xi)_{[\tilde{g}]} = \frac{\mathbf{i}}{2} \langle (X^\eta \cdot X^\xi - X^\xi \cdot X^\eta) \cdot e_0 \cdot \sigma, \sigma \rangle_{[\tilde{g}]},$$

which is a real skew symmetric 2-form and hence α_σ is a Killing form on \mathbb{H}^3 . From now on we set $\alpha_\sigma = (\alpha_\sigma)_1$ and $D\alpha_\sigma = (D\alpha_\sigma)_1$, that we will write as function of $w \oplus u$. After some computations we find

$$\begin{cases} \alpha_\sigma(\xi) & = 2(w^*\xi u + u^*\xi w) \\ D\alpha_\sigma(\eta, \xi) & = (w^*(\xi\eta - \eta\xi)u - u^*(\xi\eta - \eta\xi)w) \end{cases}.$$

We have to notice that $\xi\eta - \eta\xi \in \mathfrak{i}\mathfrak{G}$ so that $D\alpha_\sigma$ is naturally a linear form on $\mathfrak{i}\mathfrak{G}$. As a consequence we define, thanks to the Killing 1-form α_σ , the following application

$$\begin{aligned} \mathbb{C}^2 \oplus \mathbb{C}^2 &\longrightarrow \mathfrak{sl}_2(\mathbb{C})^{*\mathbb{R}} \\ w \oplus u &\longmapsto (\alpha_{w \oplus u} : \xi \mapsto 2(w^*\xi u + u^*\xi^* w)), \end{aligned}$$

where $^{*\mathbb{R}}$ stands for the duality with respect to the reals. Now taking also the function V_σ into account we define

$$\begin{aligned} \mathcal{K} : IK\Sigma(\Sigma) \cong \mathbb{C}^2 \oplus \mathbb{C}^2 &\longrightarrow (\mathfrak{M} \oplus \mathfrak{sl}_2(\mathbb{C}))^{*\mathbb{R}} \\ w \oplus u &\longmapsto \mathcal{K}_{w \oplus u} := (V_{w \oplus u} \oplus \alpha_{w \oplus u}), \end{aligned}$$

Proposition. *The application \mathcal{K} is $\mathrm{SL}(2, \mathbb{C})$ -equivariant. More precisely, for every $\tilde{e} \in \mathrm{SL}(2, \mathbb{C})$*

$$\mathcal{K}_{\tilde{e}^*(w \oplus u)} = (V_{w \oplus u} \circ \mu(\tilde{e}^{-1}), \alpha_{w \oplus u} \circ \mathrm{Ad}(\tilde{e}^*)).$$

PROOF. We must compute for every $W \in \mathfrak{M}$ and $\xi \in \mathfrak{sl}_2(\mathbb{C})$

$$\begin{aligned} \mathcal{K}_{\tilde{e}^*(w \oplus u)}(W, \xi) &= \mathcal{K}_{\tilde{e}w \oplus (\tilde{e}^*)^{-1}u}(W, \xi) \\ &= 2(w^*\tilde{e}^*W\tilde{e}w + u^*\tilde{e}^{-1}W(\tilde{e}^*)^{-1}u + w^*\tilde{e}^*\xi(\tilde{e}^*)^{-1}u + u^*\tilde{e}^{-1}\xi^*\tilde{e}w) \\ &= V_{w \oplus u} \circ \mu(\tilde{e}^{-1})(W) + \alpha_{w \oplus u} \circ \mathrm{Ad}(\tilde{e}^*)(\xi). \end{aligned}$$

□

Remark. The norm of imaginary Killing spinors

Classical considerations on Lie algebras show that $\mathfrak{so}(3, 2)$ endowed with its Killing form, is isometric to $(\mathfrak{M}, -\det) \oplus (\mathfrak{sl}_2(\mathbb{C}), -\Re(\det))$ which is a 10-dimensional real vector space of signature (6,4). The norm of $\mathcal{K}(w \oplus u)$ with respect to the Killing form is, up to a multiplicative and positive constant $|\mathcal{K}(w \oplus u)|^2 = |\langle U, V \rangle|^2 - |U|^2|V|^2 + \Re(\chi^2)$, where we have set $\chi = \overline{u_1}w_1 + \overline{u_2}w_2$. Besides, if $V_{w \oplus u}$ is isotropic in \mathfrak{M} then $\alpha_{w \oplus u}$ and $\mathcal{K}(w \oplus u)$ are also isotropic respectively in $\mathfrak{sl}_2(\mathbb{C})^*$ and $(\mathfrak{M} \oplus \mathfrak{sl}_2(\mathbb{C}))^*$. Indeed the equality case in the Cauchy-Schwarz inequality occurs if and only if U and V satisfy $\det_{\mathbb{C}^2}(U, V) = \bar{\chi} = 0$. □

3 Analytical and geometric background

The energy-momentum functional \mathcal{H} is a linear form on $(\mathfrak{M}, -\det) \oplus (\mathfrak{sl}_2(\mathbb{C}), -\Re(\det))$, and we will see below that \mathcal{H} is non negative when it is composed with the application \mathcal{K} .

3.1 Connections and curvatures

$\nabla, \bar{\nabla}$ denote respectively the Levi-Civita connections of γ and g . Let us take a spinor field $\psi \in \Gamma(\Sigma)$ and a vector field $X \in \Gamma(TM)$, then

$$\begin{cases} \nabla_X \psi &= \bar{\nabla}_X \psi - \frac{1}{2}k(X) \cdot e_0 \cdot \psi \\ \langle k(X), Y \rangle_\gamma &= \langle \nabla_X Y, e_0 \rangle_\gamma \end{cases} .$$

In these formulae \cdot denotes the Clifford action with respect to the metric γ , and $e_0 = \partial_t$. We will use different notations when we have to make the difference between the Clifford action with respect to the metric γ or β . We define the Killing connection by

$$\widehat{\nabla}_X \psi = \nabla_X \psi + \frac{\mathbf{i}}{2} X \cdot \psi \quad .$$

The spinors which are parallel with respect to $\widehat{\nabla}$ are the γ -imaginary Killing spinors. Of course we can define in the same way, the connection \widehat{D} thanks to the Levi-Civita connections D, \bar{D} of respectively β and b . R, \widehat{R} the respective curvatures of ∇ and $\widehat{\nabla}$ are related by the formula

$$\widehat{R}_{X,Y} = R_{X,Y} - \frac{1}{4}(X \cdot Y - Y \cdot X) \cdot \quad ,$$

where we use the convention of [17] for the curvature.

3.2 Bochner-Lichnerowicz and integration formulae

From now on $(e_k)_{k=0}^3$ is an orthonormal basis at the point with respect to the metric γ . We define the Dirac-Witten operators

$$\mathfrak{D}\psi = \sum_{k=1}^n e_k \cdot \nabla_{e_k} \psi, \quad \widehat{\mathfrak{D}}\psi = \sum_{k=1}^n e_k \cdot \widehat{\nabla}_{e_k} \psi,$$

where $n = 3$ is the dimension of the standard hyperbolic slice.

Lemma. (*Bochner-Lichnerowicz-Witten formula*)

$$\widehat{\mathfrak{D}}^* \widehat{\mathfrak{D}} = \widehat{\nabla}^* \widehat{\nabla} + \widehat{\mathfrak{R}},$$

where $\widehat{\mathfrak{R}} := \frac{1}{4}(\text{Scal}^\gamma + n(n-1) + 4\text{Ric}^\gamma(e_0, e_0) + 2e_0 \cdot \text{Ric}^\gamma(e_0))$.

PROOF. The Dirac-Witten operator \mathfrak{D} is clearly formally self adjoint, and we have the classical Bochner-Lichnerowicz formula (cf. [16],[19] for instance) $\mathfrak{D}^*\mathfrak{D} = \mathfrak{D}^2 = \nabla^*\nabla + \mathfrak{R}$, where $\mathfrak{R} := \frac{1}{4}(\text{Scal}^\gamma + 4\text{Ric}^\gamma(e_0, e_0) + 2e_0 \cdot \text{Ric}^\gamma(e_0))$. We also know that $\widehat{\mathfrak{D}} = \mathfrak{D} - \mathbf{i}_{\frac{n}{2}}$ and so we get

$$\widehat{\mathfrak{D}}^*\widehat{\mathfrak{D}} = \nabla^*\nabla + \mathfrak{R} + \frac{n^2}{4},$$

but finally remarking that $\widehat{\nabla}^*\widehat{\nabla} = \nabla^*\nabla + \frac{n}{4}$ we obtain our formula. \square

We derive an integration formula from the Bochner-Lichnerowicz-Witten identity considering the 1-form θ on M defined by $\theta(X) = \left\langle \widehat{\nabla}_X\psi + X \cdot \widehat{\mathfrak{D}}\psi, \psi \right\rangle_\gamma$, where ψ is a spinor field. Straightforward computations lead to the following g -divergence formula

$$\text{div}\theta = \left\langle \widehat{\mathfrak{D}}\psi, \widehat{\mathfrak{D}}\psi \right\rangle_\gamma - \left\langle \widehat{\mathfrak{R}}\psi, \psi \right\rangle_\gamma - \left\langle \widehat{\nabla}\psi, \widehat{\nabla}\psi \right\rangle_\gamma.$$

Let S_r the g -geodesic sphere of radius r and centered in a point of M . The radius r is supposed to be as large as necessary. We denote by M_r the interior domain of S_r and ν_r the (pointing outside) unit normal. Integrating our divergence formula over M_r , we get

$$\int_{M_r} \left| \widehat{\mathfrak{D}}\psi \right|_\gamma^2 = \int_{M_r} \left(\left| \widehat{\nabla}\psi \right|_\gamma^2 + \left\langle \widehat{\mathfrak{R}}\psi, \psi \right\rangle_\gamma \right) - \int_{S_r} \left\langle \widehat{\nabla}_{\nu_r}\psi + \nu_r \cdot \widehat{\mathfrak{D}}\psi, \psi \right\rangle_\gamma \text{dVol}_{S_r}.$$

3.3 Dominant Energy Condition

Let us consider the Einstein tensor $G = \text{Ric}^\gamma - \frac{1}{2}\text{Scal}^\gamma\gamma$ with respect to the metric γ . The dominant energy condition [23] says that the speed of energy flow of matter is always less than the speed of light. More precisely, for every positively oriented time-like vector field v , the energy-momentum current of density of matter $-G(v, \cdot)^\sharp$ must be time-like or null, with the same orientation as v . The assumption we make in order to prove the positive energy-momentum theorem is a relative version of the dominant energy condition: $-\left(G - \frac{n(n-1)}{2}\gamma\right)(e_0)$ is a positively oriented time-like or null vector along M . Some easy computations give

$$\begin{aligned} \text{Scal}^\gamma &= 2(G(e_0, e_0) - \text{Ric}^\gamma(e_0, e_0)) \\ e_0 \cdot \text{Ric}^\gamma(e_0) &= e_0 \cdot G_{|TM}(e_0) - \text{Ric}^\gamma(e_0, e_0), \end{aligned}$$

where $G_{|TM}(e_0) = \sum_{k=1}^3 G(e_0, e_k)e_k$. Thereby

$$\begin{aligned} \widehat{\mathfrak{R}} &= \frac{1}{4} \left(2G(e_0, e_0) + (n(n-1)) + 2e_0 \cdot G_{|TM}(e_0) \right) \cdot \\ &= \frac{1}{2} \left(\left(G(e_0, e_0) + \frac{n(n-1)}{2} \right) e_0 - G_{|TM}(e_0) \right) \cdot e_0 \cdot \\ &= \frac{1}{2} \left(\left(G(e_0, e_0) - \frac{n(n-1)}{2}\gamma(e_0, e_0) \right) e_0 - G_{|TM}(e_0) \right) \cdot e_0 \cdot \\ &= -\frac{1}{2} \left(G - \frac{n(n-1)}{2}\gamma \right) (e_0) \cdot e_0 \cdot \quad . \end{aligned}$$

Our assumption gives the non negativity of the spinorial endomorphism $\widehat{\mathfrak{R}}$ that is to say $\langle \widehat{\mathfrak{R}}\psi, \psi \rangle \geq 0$ for every spinor field ψ .

Remark. We can express the dominant energy condition in terms of the constraints as in section 1.3 since

$$-\left(G - \frac{n(n-1)}{2}\gamma\right)(e_0) = \frac{1}{2}(\Phi(g, k) - \Phi(b, 0)).$$

□

3.4 Spinorial gauge

In the same way as Andersson and Dahl [3], but in a Lorentzian situation, we compare spinors in Σ (along M) with respect to the two different metrics β and γ . We suppose that both metrics are written in Gaussian coordinates $\beta = -dt^2 + g_t$, $\gamma = -dt^2 + b_t$. We define the spinorial gauge $\mathcal{A} \in \Gamma(\text{End}(\mathbb{T}))$ with the relations

$$\begin{cases} \gamma(\mathcal{A}X, \mathcal{A}Y) = \beta(X, Y) \\ \gamma(\mathcal{A}X, Y) = \gamma(X, \mathcal{A}Y) \end{cases},$$

where \mathbb{T} is TN restricted to M . The first relation says that \mathcal{A} sends β -orthonormal frames on γ -orthonormal frames whereas the second one means that the endomorphism \mathcal{A} is symmetric. We notice that these relations are only satisfied along $M = \{t = 0\}$ and can also be written in the following way

$$\begin{cases} \mathcal{A}e_0 = e_0 \\ g(\mathcal{A}X, \mathcal{A}Y) = b(X, Y) \\ g(\mathcal{A}X, Y) = g(X, \mathcal{A}Y) \end{cases}.$$

Consequently \mathcal{A} is an application $\text{P}_{\text{SO}_0(3,1)}(\beta)|_M \longrightarrow \text{P}_{\text{SO}_0(3,1)}(\gamma)|_M$, which can be covered by an application still denoted $\mathcal{A} : \text{P}_{\text{Spin}_0(3,1)}(\beta)|_M \longrightarrow \text{P}_{\text{Spin}_0(3,1)}(\gamma)|_M$. This application carries β -spinors on γ -spinors so that we have the compatibility relation about the Clifford actions of β and γ

$$\mathcal{A}(X \cdot_{\beta} \sigma) = (\mathcal{A}X) \cdot_{\gamma} (\mathcal{A}\sigma),$$

for every $X \in \Gamma(\mathbb{T})$, $\sigma \in \Gamma(\Sigma)$ and where $\cdot_{\beta}, \cdot_{\gamma}$ denotes the Clifford actions respectively of β and γ . We define a new connection $\widetilde{\nabla}X = \mathcal{A}(\overline{D}\mathcal{A}^{-1}X)$. It is easy to check that $\widetilde{\nabla}$ is g -metric and has torsion $\widetilde{T}(X, Y) = -((\overline{D}_X\mathcal{A})\mathcal{A}^{-1}Y - (\overline{D}_Y\mathcal{A})\mathcal{A}^{-1}X)$. We extract some formulae for later use

$$2g\left(\widetilde{\nabla}_X Y - \overline{\nabla}_X Y, Z\right) = g\left(\widetilde{T}(X, Y), Z\right) - g\left(\widetilde{T}(X, Z), Y\right) - g\left(\widetilde{T}(Y, Z), X\right).$$

Now we intend to compare the connexions $\overline{\nabla}$ and $\widetilde{\nabla}$ on Σ . $(\sigma_s)_s$ denotes the spinorial frame corresponding to the orthonormal frame $(e_k)_{k=0}^3$, and $\overline{\omega}, \widetilde{\omega}$ are the connection 1-forms respectively of $\overline{\nabla}$ and $\widetilde{\nabla}$

$$\begin{aligned} \overline{\omega}_{ij} &= g(\overline{\nabla}e_i, e_j) \\ \widetilde{\omega}_{ij} &= g(\widetilde{\nabla}e_i, e_j), \end{aligned}$$

and if we take a general spinor $\varphi = \varphi^s \sigma_s$, their derivatives are given by

$$\begin{aligned}\bar{\nabla}\varphi &= d\varphi^s \otimes \sigma_s + \frac{1}{2} \sum_{i<j} \bar{\omega}_{ij} \otimes e_i \cdot_\gamma e_j \cdot_\gamma \varphi \\ \tilde{\nabla}\varphi &= d\varphi^s \otimes \sigma_s + \frac{1}{2} \sum_{i<j} \tilde{\omega}_{ij} \otimes e_i \cdot_\gamma e_j \cdot_\gamma \varphi,\end{aligned}$$

and as a consequence

$$(\bar{\nabla} - \tilde{\nabla})\varphi = \frac{1}{4} \sum_{i,j=0}^3 (\bar{\omega}_{ij} - \tilde{\omega}_{ij}) \otimes e_i \cdot_\gamma e_j \cdot_\gamma \varphi \quad .$$

3.5 Energy-momentum and imaginary Killing spinors

Remember that

$$\int_{M_r} \left| \widehat{\mathfrak{D}}\psi \right|_\gamma^2 = \int_{M_r} \left(\left| \widehat{\nabla}\psi \right|_\gamma^2 + \left\langle \widehat{\mathfrak{R}}\psi, \psi \right\rangle_\gamma \right) - \int_{S_r} \left\langle \widehat{\nabla}_{\mathcal{A}\nu_r} \psi + \mathcal{A}\nu_r \cdot_\gamma \widehat{\mathfrak{D}}\psi, \psi \right\rangle_\gamma \, d\text{Vol}_{S_r} \quad .$$

Remark. ν_r denotes the b -normal of S_r , $e_0 = \partial_t$ and we set $e_1 = \mathcal{A}\nu_r$. □

We have to work on the expression $\left\langle \widehat{\nabla}_{\mathcal{A}\nu_r} \psi + \mathcal{A}\nu_r \cdot_\gamma \widehat{\mathfrak{D}}\psi, \psi \right\rangle_\gamma$ in order to get the integrand used to compute the energy-momentum.

Lemma.

$$\widehat{\nabla}_{\mathcal{A}\nu_r} \psi + \mathcal{A}\nu_r \cdot_\gamma \widehat{\mathfrak{D}}\psi = \mathcal{A}\nu_r \cdot_\gamma \left(\sum_{k=2}^3 e_k \cdot_\gamma \widehat{\nabla}_{e_k} \right) \psi.$$

PROOF. We only have to notice that $e_1 \cdot_\gamma e_1 \cdot_\gamma \widehat{\nabla}_{e_1} = -\widehat{\nabla}_{e_1} = -\widehat{\nabla}_{\mathcal{A}\nu_r}$. □

From now on we work on

$$\left\langle \mathcal{A}\nu_r \cdot_\gamma \left(\sum_{k=2}^3 e_k \cdot_\gamma \widehat{\nabla}_{e_k} \right) \cdot_\gamma \varphi, \varphi \right\rangle_\gamma.$$

Let us take σ a β -imaginary Killing spinor, that is to say a spinor field solution of $\widehat{D}_X \sigma = D_X \sigma + \frac{i}{2} X \cdot_\beta \sigma = 0$, for every vector field $X \in \Gamma(TM)$. Consider f a smooth cutoff function which is 0 on M except on a relatively compact neighbourhood of the infinity boundary of M where $f \equiv 1$. Then we have

$$\begin{aligned}\widehat{\nabla}_X(f\mathcal{A}\sigma) &= df(X)\mathcal{A}\sigma + f\widehat{\nabla}_X(\mathcal{A}\sigma) \\ &= df(X)\mathcal{A}\sigma + f(\bar{\nabla}_X - \tilde{\nabla}_X)(\mathcal{A}\sigma) + f \left(\tilde{\nabla}_X + \frac{i}{2} X \cdot_\gamma - \frac{1}{2} k(X) \cdot_\gamma e_0 \cdot_\gamma \right) (\mathcal{A}\sigma),\end{aligned}$$

but since $\widehat{\nabla}_X(\mathcal{A}\sigma) = \mathcal{A}\overline{D}_X\sigma = -\frac{i}{2}\mathcal{A}(X \cdot_\beta \sigma) = -\frac{i}{2}(\mathcal{A}X) \cdot_\gamma (\mathcal{A}\sigma)$, we obtain

$$\widehat{\nabla}_X(f\mathcal{A}\sigma) = df(X)\mathcal{A}\sigma + f(\overline{\nabla}_X - \widetilde{\nabla}_X)(\mathcal{A}\sigma) - \frac{1}{2}f(k(X) \cdot_\gamma e_0 + \mathbf{i}(\mathcal{A} - Id)X) \cdot_\gamma (\mathcal{A}\sigma),$$

that we restrict to the neighbourhood where $f \equiv 1$

$$\widehat{\nabla}_X(\mathcal{A}\sigma) = (\overline{\nabla}_X - \widetilde{\nabla}_X)(\mathcal{A}\sigma) - \frac{1}{2}(k(X) \cdot_\gamma e_0 + \mathbf{i}(\mathcal{A} - Id)X) \cdot_\gamma (\mathcal{A}\sigma).$$

As a consequence our boundary term becomes for r great enough (summation convention $k \in \{2, 3\}$)

$$\left\langle \mathcal{A}\nu_r \cdot_\gamma e_k \cdot_\gamma \left((\overline{\nabla}_{e_k} - \widetilde{\nabla}_{e_k}) - \frac{1}{2}(k(e_k) \cdot_\gamma e_0 + \mathbf{i}(\mathcal{A} - Id)e_k) \cdot_\gamma \right) (\mathcal{A}\sigma), \mathcal{A}\sigma \right\rangle_\gamma.$$

We will estimate this boundary term in several steps. From the decay assumptions stated section in 1.2, the gauge is supposed to be of the form $\mathcal{A} = Id + B + O(|B|^2)$, where B has the same decay to 0 as $e = g - b$. In the following $(\epsilon_k = \mathcal{A}^{-1}e_k)_{k=0}^3$ is a β -orthonormal frame.

We begin with the easiest term

$$\begin{aligned} \langle \mathcal{A}\nu_r \cdot_\gamma e_k \cdot_\gamma k(e_k) \cdot_\gamma e_0 \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma \rangle_\gamma &= \langle \mathcal{A}\nu_r \cdot_\gamma \mathcal{A}\epsilon_k \cdot_\gamma k(\mathcal{A}\epsilon_k) \cdot_\gamma \mathcal{A}\epsilon_0 \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma \rangle_\gamma \\ &= \langle \nu_r \cdot_\beta \epsilon_k \cdot_\beta \mathcal{A}^{-1} \circ k \circ \mathcal{A}(\epsilon_k) \cdot_\beta \epsilon_0 \cdot_\beta \sigma, \sigma \rangle_\beta. \end{aligned}$$

But we note that $\mathcal{A}^{-1} \circ k \circ \mathcal{A} = k - B \circ k + k \circ B + O(|B|^2)$. Now B has the same decay as k so $B \circ k + k \circ B = O(|B|^2)$, terms that we can neglect if r is great enough. We conclude that $\mathcal{A}^{-1} \circ k \circ \mathcal{A} \approx k$. Moreover

$$\begin{aligned} \nu_r \cdot_\beta \sum_{k=2}^3 \epsilon_k \cdot_\beta k(\epsilon_k) &= \epsilon_1 \cdot_\beta \left(\sum_{k=1}^3 \epsilon_k \cdot_\beta k(\epsilon_k) - \epsilon_1 \cdot_\beta k(\epsilon_1) \right) \\ &= k(\nu_r) - (\text{tr}_b k)\nu_r, \end{aligned}$$

which implies

$$\begin{aligned} \langle \mathcal{A}\nu_r \cdot_\gamma e_k \cdot_\gamma k(e_k) \cdot_\gamma e_0 \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma \rangle_\gamma &\approx \langle k(\nu_r) - (\text{tr}_b k)\nu_r \cdot_\beta \epsilon_0 \cdot_\beta \sigma, \sigma \rangle_\beta \\ &= (i_{\alpha_\sigma} k - (\text{tr}_b k)\alpha_\sigma)(\nu_r), \end{aligned}$$

where $\alpha_\sigma(X) = \langle X \cdot_\beta \epsilon_0 \cdot_\beta \sigma, \sigma \rangle_\beta$.

The second term we study is

$$\begin{aligned} \mathbf{i} \langle \mathcal{A}\nu_r \cdot_\gamma e_k \cdot_\gamma (\mathcal{A} - Id)(e_k) \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma \rangle_\gamma &= \mathbf{i} \langle \mathcal{A}\nu_r \cdot_\gamma \mathcal{A}\epsilon_k \cdot_\gamma (\mathcal{A} - Id)(\mathcal{A}\epsilon_k) \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma \rangle_\gamma \\ &= \mathbf{i} \langle \nu_r \cdot_\beta \epsilon_k \cdot_\beta \mathcal{A}^{-1} \circ (\mathcal{A} - Id) \circ \mathcal{A}(\epsilon_k) \cdot_\beta \sigma, \sigma \rangle_\beta \\ &\approx \mathbf{i} \langle \nu_r \cdot_\beta \epsilon_k \cdot_\beta B(\epsilon_k) \cdot_\beta \sigma, \sigma \rangle_\beta, \end{aligned}$$

but thanks to the same property as above

$$\nu_r \cdot \beta \sum_{k=2}^3 \epsilon_k \cdot \beta B(\epsilon_k) = B(\nu_r) - (\text{tr}_b B) \nu_r,$$

which induces

$$\begin{aligned} \mathbf{i} \langle \mathcal{A} \nu_r \cdot \gamma e_k \cdot \gamma (\mathcal{A} - Id)(e_k) \cdot \gamma (\mathcal{A} \sigma), \mathcal{A} \sigma \rangle_\gamma &\approx \mathbf{i} \langle B(\nu_r) - (\text{tr}_b B) \nu_r \cdot \beta \sigma, \sigma \rangle_\beta \\ &= (i_{\nabla V_\sigma} B - (\text{tr}_b B) dV_\sigma)(\nu_r), \end{aligned}$$

where $dV_\sigma(X) = \mathbf{i} \langle X \cdot \beta \sigma, \sigma \rangle_\beta$.

The last term we have to study is certainly the most difficult (summation convention $k \in \{2, 3\}, l \in \{1, 2, 3\}, m \in \{1, 2, 3\}$)

$$\begin{aligned} \left\langle \mathcal{A} \nu_r \cdot \gamma e_k \cdot \gamma (\bar{\nabla}_{e_k} - \tilde{\nabla}_{e_k})(\mathcal{A} \sigma), \mathcal{A} \sigma \right\rangle_\gamma &= \frac{1}{4} \langle (\bar{\omega}_{lm} - \tilde{\omega}_{lm})(e_k) \mathcal{A} \nu_r \cdot \gamma e_k \cdot \gamma e_l \cdot \gamma e_m \cdot \gamma (\mathcal{A} \sigma), \mathcal{A} \sigma \rangle_\gamma \\ &= \frac{1}{4} \langle (\bar{\omega}_{lm} - \tilde{\omega}_{lm}) \circ \mathcal{A}(\epsilon_k) \nu_r \cdot \beta \epsilon_k \cdot \beta \epsilon_l \cdot \beta \epsilon_m \cdot \beta \sigma, \sigma \rangle_\beta \\ &= \frac{1}{4} S \end{aligned}$$

$$\begin{aligned} S &= \sum_{k,l,m=2}^3 \langle (\bar{\omega}_{lm} - \tilde{\omega}_{lm})(e_k) \mathcal{A} \nu_r \cdot \gamma e_k \cdot \gamma e_l \cdot \gamma e_m \cdot \gamma (\mathcal{A} \sigma), \mathcal{A} \sigma \rangle_\gamma \\ &+ 2 \sum_{k,l=2}^3 \langle (\bar{\omega}_{1l} - \tilde{\omega}_{1l})(e_k) \mathcal{A} \nu_r \cdot \gamma e_k \cdot \gamma e_1 \cdot \gamma e_l \cdot \gamma (\mathcal{A} \sigma), \mathcal{A} \sigma \rangle_\gamma \\ &= S_1 + 2S_2. \end{aligned}$$

We will give estimates of each S_k , keeping in mind that they are real and that every term that is at least $O(|B|^2)$ can be neglected when $r \rightarrow +\infty$.

Estimate of S_1

$$S_1 = \sum_{k,l,m=2}^3 \langle (\bar{\omega}_{lm} - \tilde{\omega}_{lm})(e_k) \mathcal{A} \nu_r \cdot \gamma e_k \cdot \gamma e_l \cdot \gamma e_m \cdot \gamma (\mathcal{A} \sigma), \mathcal{A} \sigma \rangle_\gamma.$$

We can keep only the subscripts $l \neq m$ because of the skew-symmetry of $(\omega - \tilde{\omega})$. Besides if we suppose that $k = l$, we have terms like $\langle \mathcal{A} \nu_r \cdot \gamma e_m \cdot \gamma (\mathcal{A} \sigma), \mathcal{A} \sigma \rangle_\gamma$ which belong to $i\mathbb{R}$. So we can sum over k, l, m distinct subscripts without any loss of generality. On the other hand

$$(\bar{\omega}_{lm} - \tilde{\omega}_{lm})(e_k) = \frac{1}{2} \left(-g(\tilde{T}(e_k, e_l), e_m) + g(\tilde{T}(e_k, e_m), e_l) + g(\tilde{T}(e_l, e_m), e_k) \right)$$

where the two last terms of the right-hand side member are symmetric with respect to (l, k) , so they vanish when we sum over k and l distinct. Consequently

$$\begin{aligned} (\bar{\omega}_{lm} - \tilde{\omega}_{lm})(e_k) \epsilon_k \cdot \beta \epsilon_l \cdot \beta \epsilon_m &= \frac{1}{2} b (\mathcal{A}^{-1}(\bar{D}_{e_k} \mathcal{A}) \epsilon_l - \mathcal{A}^{-1}(\bar{D}_{e_l} \mathcal{A}) \epsilon_k, \epsilon_m) \epsilon_k \cdot \beta \epsilon_l \cdot \beta \epsilon_m \\ &= b (\mathcal{A}^{-1}(\bar{D}_{e_k} \mathcal{A}) \epsilon_l, \epsilon_m) \epsilon_k \cdot \beta \epsilon_l \cdot \beta \epsilon_m, \end{aligned}$$

but

$$\begin{aligned}
b(\mathcal{A}^{-1}(\overline{D}_{e_k}\mathcal{A})\epsilon_l, \epsilon_m) &= b(\mathcal{A}^{-1}(\overline{D}_{e_k}(\mathcal{A}\epsilon_l) - \mathcal{A}\overline{D}_{e_k}\epsilon_l), \epsilon_m) \\
&\approx b(\overline{D}_{e_k}(B\epsilon_l) - B(\overline{D}_{e_k}\epsilon_l), \epsilon_m) \\
&= b((\overline{D}_{e_k}B)\epsilon_l, \epsilon_m),
\end{aligned}$$

expression which is symmetric with respect to (l, m) , since $\overline{D}B$ is a symmetric endomorphism. Consequently

$$\sum_{k,l,m \text{ distinct}} (\overline{\omega}_{lm} - \tilde{\omega}_{lm})(e_k)\epsilon_k \cdot_\beta \epsilon_l \cdot_\beta \epsilon_m \approx 0,$$

when $r \rightarrow +\infty$.

Estimate of S_2

$$\begin{aligned}
S_2 &= \sum_{k,l=2}^3 \langle (\overline{\omega}_{1l} - \tilde{\omega}_{1l})(e_k)\mathcal{A}\nu_r \cdot_\gamma e_k \cdot_\gamma e_1 \cdot_\gamma e_l \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma \rangle_\gamma \\
&= - \sum_{k=2}^3 \langle (\overline{\omega}_{1k} - \tilde{\omega}_{1k})(e_k)\sigma, \sigma \rangle_\beta \\
&\quad + \sum_{k \neq l} \langle (\overline{\omega}_{1l} - \tilde{\omega}_{1l})(e_k)e_k \cdot_\gamma e_l \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma \rangle_\gamma,
\end{aligned}$$

but the second sum is in $\mathbf{i}\mathbb{R}$, so it remains

$$\Re e(S_2) = - \left(\sum_{k=1}^3 (\overline{\omega}_{1k} - \tilde{\omega}_{1k})(e_k) \right) V_\sigma.$$

We only have to compute

$$\begin{aligned}
- \sum_{k=1}^3 (\overline{\omega}_{1k} - \tilde{\omega}_{1k})(e_k) &= \sum_{k=1}^3 g((\overline{D}_{e_1}\mathcal{A})\mathcal{A}^{-1}e_k, e_k - (\overline{D}_{e_k}\mathcal{A})\mathcal{A}^{-1}e_1, e_k) \\
&= S'_2 - S''_2
\end{aligned}$$

We focus on

$$\begin{aligned}
S_2'' &= \sum_{k=1}^3 g((\overline{D}_{e_k} \mathcal{A}) \mathcal{A}^{-1} e_1, e_k) \\
&= \sum_{k=1}^3 b(\mathcal{A}^{-1} (\overline{D}_{e_k} \mathcal{A}) \epsilon_1, \epsilon_k) \\
&= \sum_{k=1}^3 b(\mathcal{A}^{-1} \overline{D}_{e_k} (\mathcal{A} \epsilon_1) - \overline{D}_{e_k} \epsilon_1, \epsilon_k) \\
&\approx \sum_{k=1}^3 b((\overline{D}_{e_k} B) \epsilon_1, \epsilon_k) \\
&\approx \sum_{k=1}^3 b(\epsilon_1, (\overline{D}_{e_k} B) \epsilon_k) \\
&= -\operatorname{div}_b B(\nu_r).
\end{aligned}$$

As regards the first term S_2' , we decompose the gauge endomorphism \mathcal{A} as follows: $\mathcal{A} \epsilon_i = \sum_{k=0}^3 \mathcal{A}_i^k \epsilon_k$. We remind that $\mathcal{A} \epsilon_0 = \epsilon_0$, $\mathcal{A}(TM) \subset TM$ and so we have $\mathcal{A}_0^k = \mathcal{A}_k^0 = 0$, $k \geq 1$.

$$\begin{aligned}
S_2' &= \sum_{k=1}^3 g((\overline{D}_{e_1} \mathcal{A}) \mathcal{A}^{-1} e_k, e_k) \\
&= \sum_{k=1}^3 b((\mathcal{A}^{-1} \overline{D}_{e_1} \mathcal{A}) \epsilon_k, \epsilon_k) \\
&= \sum_{k=1}^3 b(\mathcal{A}^{-1} \overline{D}_{e_1} (\mathcal{A} \epsilon_k) - \overline{D}_{e_1} \epsilon_k, \epsilon_k) \\
&\approx \sum_{k,l=1}^3 (e_1 \cdot \mathcal{A}_k^l) \{b(\epsilon_l, \epsilon_k) - b(B \epsilon_l, \epsilon_k)\} - \sum_{k=1}^3 b(\overline{D}_{e_1} \epsilon_k, \epsilon_k) \\
&\quad + \sum_{k,l=1}^3 \mathcal{A}_k^l \{b(\overline{D}_{e_1} \epsilon_l, \epsilon_k) - b(B \overline{D}_{e_1} \epsilon_l, \epsilon_k)\} \\
&\approx (\epsilon_1 \cdot \operatorname{tr}_b B) - \sum_{k=1}^3 b(\overline{D}_{e_1} \epsilon_k, \epsilon_k) + \sum_{k,l=1}^3 (\delta_l^k + B_k^l) \{b(\overline{D}_{e_1} \epsilon_l, \epsilon_k) - b(B \overline{D}_{e_1} \epsilon_l, \epsilon_k)\} \\
&\approx (\epsilon_1 \cdot \operatorname{tr}_b B) - \sum_{k=1}^3 \{b(B \overline{D}_{e_1} \epsilon_k, \epsilon_k) - b(\overline{D}_{e_1} \epsilon_k, B \epsilon_k)\} \\
&= d(\operatorname{tr}_b B)(\nu_r),
\end{aligned}$$

that entails

$$\Re e(S_2) \approx V_\sigma(d(\operatorname{tr}_b B) + \operatorname{div}_b B)(\nu_r).$$

We can conclude, taking $B = -\frac{1}{2}e$, that the real part of our boundary integrand is nothing but

$$\frac{1}{4} \left(-V_\sigma(\delta_b e + dtr_b e) - i_{\nabla^b V_\sigma} e + (tr_b e) dV_\sigma - 2i_{\alpha_\sigma^\sharp} k + 2(tr_b k) \alpha_\sigma \right) (\nu_r).$$

We can now make out the link between the energy-momentum and the imaginary Killing spinors with the

Proposition. *Let $\xi = f\mathcal{A}\sigma + \xi_0$, where $\sigma \in IKS(\Sigma)$ and ξ_0 is a compactly supported spinor field. Then we have*

$$\begin{aligned} \mathcal{H}(V_\sigma, \alpha_\sigma) &= 4 \lim_{r \rightarrow +\infty} \int_{S_r} \left\langle \widehat{\nabla}_{\mathcal{A}\nu_r} \xi + \mathcal{A}\nu_r \cdot \widehat{\mathfrak{D}}\xi, \xi \right\rangle_\gamma \\ &= 4 \int_M \left(|\widehat{\nabla}\xi|_\gamma^2 + \left\langle \widehat{\mathfrak{R}}\xi, \xi \right\rangle_\gamma \right) - 4 \int_M |\widehat{\mathfrak{D}}\xi|_\gamma^2. \end{aligned}$$

Remark. Although ξ is not an imaginary Killing spinor, we obtain $\mathcal{H}(V_\sigma, \alpha_\sigma)$ since the energy-momentum only takes into account the contribution of the integrand $\left\langle \widehat{\nabla}_{\mathcal{A}\nu_r} \xi + \mathcal{A}\nu_r \cdot \widehat{\mathfrak{D}}\xi, \xi \right\rangle_\gamma$ near the infinity of M . For r large enough this integrand is $\left\langle \widehat{\nabla}_{\mathcal{A}\nu_r} \mathcal{A}\sigma + \mathcal{A}\nu_r \cdot \widehat{\mathfrak{D}}\mathcal{A}\sigma, \mathcal{A}\sigma \right\rangle_\gamma$. \square
Now some analysis of the Dirac-Witten operator is necessary in order to obtain the non negativity of the energy-momentum functional \mathcal{H} when it is restricted to the couples $(V_\sigma, \alpha_\sigma)$ with $\sigma \in IKS(\Sigma)$.

3.6 A little analysis of $\widehat{\mathfrak{D}}$

3.6.1 M without inner boundary

We still follow the work of Andersson and Dahl [3] to study the analytical properties of $\widehat{\mathfrak{D}}$. Let us consider $C_0^\infty(\Sigma) = C_0^\infty$ the space of smooth and compactly supported spinors. We define a sesquilinear form on C_0^∞ by

$$a(\varphi, \psi) = \int_M \left\langle \widehat{\mathfrak{D}}\varphi, \widehat{\mathfrak{D}}\psi \right\rangle_\gamma d\mu_g,$$

where $d\mu_g$ denotes the standard volum form of the metric g . The form a is clearly bounded and non negative on C_0^∞ . We define the usual Sobolev space

$$H^1(\Sigma) = \left\{ \psi \in \Sigma / \int_M |\psi|_\gamma^2 + |\nabla\psi|_\gamma^2 < \infty \right\}.$$

Definition. We set $H(a) := \overline{C_0^\infty}^a$.

Lemma. There is a continuous embedding $H(a) \longrightarrow H^1$.

PROOF. For every $\psi \in C_0^\infty$ we have

$$\left(1 + \frac{n}{4}\right)^{-1} \|\psi\|_1^2 \leq \int_M |\widehat{\mathfrak{D}}\psi|_\gamma^2 = \int_M \left(|\nabla\psi|_\gamma^2 + \frac{n}{4} |\psi|^2 + \langle \widehat{\mathfrak{R}}\psi, \psi \rangle_\gamma \right).$$

□

We notice that for r great enough and for $\sigma \in IK_S(\Sigma)$

$$\begin{aligned} \widehat{\nabla}_X(\mathcal{A}\sigma) &= \nabla_X(\mathcal{A}\sigma) + \frac{\mathbf{i}}{2} X \cdot_\gamma (\mathcal{A}\sigma) \\ &= (\overline{\nabla}_X - \widetilde{\nabla}_X)(\mathcal{A}\sigma) - \frac{1}{2}(k(X) \cdot_\gamma e_0 + \mathbf{i}(\mathcal{A} - Id)X) \cdot_\gamma (\mathcal{A}\sigma). \end{aligned}$$

But the relations

$$\begin{cases} \widetilde{T}(X, Y) &= -((\overline{D}_X \mathcal{A})\mathcal{A}^{-1}Y - (\overline{D}_Y \mathcal{A})\mathcal{A}^{-1}X) \\ 2g(\widetilde{\nabla}_X Y - \overline{\nabla}_X Y, Z) &= g(\widetilde{T}(X, Y), Z) - g(\widetilde{T}(X, Z), Y) - g(\widetilde{T}(Y, Z), X) \end{cases}$$

tell us that $|(\widetilde{\omega}_{ij} - \overline{\omega}_{ij})(e_k)| \leq C|\mathcal{A}^{-1}|\overline{D}\mathcal{A}|$. We get an estimate

$$|\widehat{\mathfrak{D}}(\mathcal{A}\sigma)| \leq C|\mathcal{A}|(|\overline{D}\mathcal{A}| + |\mathcal{A} - Id| + |k|)|\sigma| \in L^2(M, d\mu_g),$$

which infers that $\widehat{\mathfrak{D}}(f\mathcal{A}\sigma) \in L^2(M, d\mu_g)$. We now consider the linear form l on $H(a)$ defined by

$$l(\psi) = \int_M \langle \widehat{\mathfrak{D}}(f\mathcal{A}\sigma), \widehat{\mathfrak{D}}\psi \rangle_\gamma d\mu_g.$$

Thanks to our estimate above we get $|l(\psi)|^2 \leq \left\| \widehat{\mathfrak{D}}(f\mathcal{A}\sigma) \right\|_{L^2}^2 a(\psi, \psi)$, that gives the continuity of l in $H(a)$. We can claim, thanks to Lax-Milgram theorem, that there exists a unique $\xi_0 \in H(a)$ such that $l = a(-\xi_0, \cdot)$. In other words

$$\int_M \langle (\widehat{\mathfrak{D}})^* \widehat{\mathfrak{D}}(f\mathcal{A}\sigma + \xi_0), \psi \rangle_\gamma = 0.$$

Since $\widehat{\mathfrak{D}}^* = \widehat{\mathfrak{D}} + \mathbf{i}n$, we have in the distributional sense $(\widehat{\mathfrak{D}} + \mathbf{i}n)\widehat{\mathfrak{D}}\xi = 0$, where we have set $\xi = f\mathcal{A}\sigma + \xi_0$. By an elliptic regularity argument, $\widehat{\mathfrak{D}}\xi$ is in fact smooth and $(\widehat{\mathfrak{D}})^k \xi$ are L^2 , for every $k \in \mathbb{N}$. It follows

$$\begin{aligned} \int_M \langle (\widehat{\mathfrak{D}})^2 \xi, (\widehat{\mathfrak{D}})^2 \xi \rangle_\gamma &= \int_M \langle (\widehat{\mathfrak{D}} + \mathbf{i}n)(\widehat{\mathfrak{D}})^2 \xi, \widehat{\mathfrak{D}}\xi \rangle_\gamma \\ &= \int_M \langle \widehat{\mathfrak{D}}(\widehat{\mathfrak{D}} + \mathbf{i}n)\widehat{\mathfrak{D}}\xi, \widehat{\mathfrak{D}}\xi \rangle_\gamma \\ &= 0, \end{aligned}$$

that implies $(\widehat{\mathfrak{D}})^2\xi = 0$, but we already know that $(\widehat{\mathfrak{D}} + \mathbf{i}n)\widehat{\mathfrak{D}}\xi = 0$, and thereby $\widehat{\mathfrak{D}}\xi = 0$. We now apply our integration formula to ξ

$$\begin{aligned}\mathcal{H}(V_\sigma, \alpha_\sigma) &= \lim_{r \rightarrow +\infty} \int_{S_r} \left\langle \widehat{\nabla}_{\mathcal{A}\nu_r} \xi + \mathcal{A}\nu_r \cdot \widehat{\mathfrak{D}}\xi, \xi \right\rangle_\gamma \\ &= 4 \int_M \left(\left| \widehat{\nabla}\xi \right|_\gamma^2 + \left\langle \widehat{\mathfrak{K}}\xi, \xi \right\rangle_\gamma \right) - 4 \int_M \left| \widehat{\mathfrak{D}}\xi \right|_\gamma^2 \\ &= 4 \int_M \left(\left| \widehat{\nabla}\xi \right|_\gamma^2 + \left\langle \widehat{\mathfrak{K}}\xi, \xi \right\rangle_\gamma \right) \geq 0,\end{aligned}$$

and we finally obtain the

Proposition. *For every $\sigma \in IK_S(\Sigma)$ there exists a unique $\xi_0 \in H(a)$ such that*

$$\mathcal{H}(V_\sigma, \alpha_\sigma) = \lim_{r \rightarrow +\infty} \int_{S_r} \left\langle \widehat{\nabla}_{\mathcal{A}\nu_r} \xi + \mathcal{A}\nu_r \cdot \widehat{\mathfrak{D}}\xi, \xi \right\rangle_\gamma = 4 \int_M \left(\left| \widehat{\nabla}\xi \right|_\gamma^2 + \left\langle \widehat{\mathfrak{K}}\xi, \xi \right\rangle_\gamma \right) \geq 0,$$

where $\xi = f\mathcal{A}\sigma + \xi_0 \in \text{Ker}\widehat{\mathfrak{D}}$.

3.6.2 M with inner boundary

We will consider, in this section, a Riemannian slice M that has a non empty inner boundary ∂M . $\check{g}, \check{\nabla}, \check{k}$ will denote respectively the induced metric, the connection and the second fundamental form which is defined by

$$\begin{aligned}\overline{\nabla}_X Y &= \check{\nabla}_X Y - \check{k}(X, Y)\nu \\ \overline{\nabla}_X \psi &= \check{\nabla}_X \psi - \frac{1}{2}\check{k}(X) \cdot \nu \cdot \psi,\end{aligned}$$

where ν is the normal to ∂M pointing toward infinity (that is to say pointing inside), and \cdot still denotes the Clifford action with respect to the metric γ . Consequently our integration formula has another boundary term

$$\int_{M_r} \left| \widehat{\mathfrak{D}}\psi \right|_\gamma^2 = \int_{M_r} \left(\left| \widehat{\nabla}\psi \right|_\gamma^2 + \left\langle \widehat{\mathfrak{K}}\psi, \psi \right\rangle_\gamma \right) - \int_{S_r} \left\langle \widehat{\nabla}_{\mathcal{A}\nu_r} \psi + \mathcal{A}\nu_r \cdot \widehat{\mathfrak{D}}\psi, \psi \right\rangle_\gamma + \int_{\partial M} \left\langle \widehat{\nabla}_\nu \psi + \nu \cdot \widehat{\mathfrak{D}}\psi, \psi \right\rangle_\gamma.$$

But if ψ is a compactly supported smooth spinor field then, making $r \rightarrow \infty$ one finds

$$\int_M \left| \widehat{\mathfrak{D}}\psi \right|_\gamma^2 = \int_M \left(\left| \widehat{\nabla}\psi \right|_\gamma^2 + \left\langle \widehat{\mathfrak{K}}\psi, \psi \right\rangle_\gamma \right) + \int_{\partial M} \left\langle \widehat{\nabla}_\nu \psi + \nu \cdot \widehat{\mathfrak{D}}\psi, \psi \right\rangle_\gamma.$$

We then have to estimate the boundary integrand $\left\langle \widehat{\nabla}_\nu \psi + \nu \cdot \widehat{\mathfrak{D}}\psi, \psi \right\rangle$.

Lemma. *If $(\nu = e_1, e_2, e_3)$ is a local orthonormal frame of $TM|_{\partial M}$ then*

$$\widehat{\nabla}_\nu \psi + \nu \cdot \widehat{\mathfrak{D}}\psi = \nu \cdot \sum_{k=2}^3 \widehat{\nabla}_{e_k} \psi \quad .$$

PROOF. Just remark that $\widehat{\nabla}_\nu \psi = -e_1 \cdot e_1 \cdot \widehat{\nabla}_{e_1} \psi$ □

Lemma. *Keeping our orthonormal frame $(\nu = e_1, e_2, e_3)$, we have*

$$\widehat{\nabla}_\nu \psi + \nu \cdot \widehat{\mathfrak{D}}\psi = \sum_{k=2}^3 \nu \cdot e_k \cdot \widetilde{\nabla}_{e_k} \psi + \frac{1}{2} \left\{ -\text{tr}\check{k} - (n-1)\mathbf{i}\nu + (\text{tr}k)\nu \cdot e_0 - k(\nu) \cdot e_0 \right\} \cdot \psi \quad .$$

PROOF. Using the formula above, we then express $\widehat{\nabla}$ in term of the 2-dimensional connection and second form, and the 3-dimensional second form. □

Let us define $F \in \text{End}(\Sigma|_{\partial M})$ by $F(\psi) = \mathbf{i}\nu \cdot \psi$. We sum up some basic properties of F in the following

Proposition. *The endomorphism F is symmetric, isometric with respect to $\langle \cdot, \cdot \rangle$, commutes to the action of $\nu \cdot$ and anticommutes to each $e_k \cdot$, ($k \neq 1$).*

Lemma. *If $F(\psi) = -\psi$ then*

$$\left\langle \widehat{\nabla}_\nu \psi + \nu \cdot \widehat{\mathfrak{D}}\psi, \psi \right\rangle_{|\partial M} = \frac{1}{2} \left\langle e_0 \cdot \left((-\text{tr}\check{k} + (n-1))e_0 + k(\nu) \right) \cdot \psi, \psi \right\rangle .$$

PROOF. Using the proposition above we know that $\nu \cdot e_k$ ($k \neq 1$) anticommutes with F and the formula follows since F respects $\langle \cdot, \cdot \rangle$. □

Assumption. *Let us suppose that the 4-vector $\vec{k} := (-\text{tr}\check{k} + (n-1))e_0 + k(\nu)$ is causal and positively oriented, that is to say $\gamma(\vec{k}, \vec{k}) \leq 0$ and $\text{tr}\check{k} \leq (n-1)$.*

This assumption (which is exactly the same as for $\widehat{\mathfrak{R}}$) guarantees the non negativity of the boundary integrand term $\left\langle \widehat{\nabla}_\nu \psi + \nu \cdot \widehat{\mathfrak{D}}\psi, \psi \right\rangle_{|\partial M} = \frac{1}{2} \left\langle e_0 \cdot \vec{k} \cdot \psi, \psi \right\rangle$, whenever the boundary condition $F(\psi) = -\psi$ is satisfied. Although this assumption is vectorial, it clearly extends the one given in [12].

Let us define $H_-(a) = \{\psi \in H(a) / F(\psi) = -\psi\}$ where $H(a)$ has been defined in section 3.6.1. Still taking ψ a compactly supported smooth spinor field in $H_-(a)$, we have

$$a(\psi, \psi) = \int_M \left(\left| \widehat{\nabla}\psi \right|_\gamma^2 + \left\langle \widehat{\mathfrak{R}}\psi, \psi \right\rangle_\gamma \right) + \frac{1}{2} \int_{\partial M} \left\langle e_0 \cdot \vec{k} \cdot \psi, \psi \right\rangle ,$$

whose each single term is non negative tanks to our assumption.

Remark. Weighted Poincaré inequality

$$\exists \omega \in L^1_{loc}(d\text{Vol}_g) \quad \text{ess}_M \inf \omega > 0 \quad \forall u \in C^1_0 \quad \int_M \omega |u|^2 d\text{Vol}_g \leq \int_M |\widehat{\nabla} u|^2 d\text{Vol}_g.$$

It is easy to see that Γ , the symmetric part of the connection $\widehat{\nabla}$ is given by $\Gamma_X = \frac{1}{2} \{k(X) \cdot e_0 - \mathbf{i}X\} \cdot$, and so satisfies the conditions (cf. [7]) in order to have the existence of a weighted Poincaré inequality that is to say $\Gamma \in L^n_{loc}(M)$ and $\limsup_{x \rightarrow 0} |x\Gamma_x| < \frac{n-1}{2}$. \square

Lemma. $H_-(a)$ continuously embeds in H^1_{loc} and furthermore

$$\left(\psi_k \xrightarrow{H_-(a)} \psi \right) \Rightarrow \left(\widehat{\nabla} \psi_k \xrightarrow{L^2(M)} \widehat{\nabla} \psi \quad \text{and} \quad \forall \Omega \subset M \quad |\Omega| < \infty \quad \psi_k \xrightarrow{H^1(\Omega)} \psi \right),$$

with a weighted Poincaré inequality.

PROOF. Let $(\psi_k)_{k \in \mathbb{N}} \in (C^\infty_0)^\mathbb{N}$ a Cauchy sequence with respect to the form a whose elements satisfy the boundary condition $F(\psi_k) = -\psi_k$. Then we have

$$\int_M \left| \widehat{\mathfrak{D}} \psi_k \right|_\gamma^2 = \int \left(\left| \widehat{\nabla} \psi_k \right|_\gamma^2 + \left\langle \widehat{\mathfrak{R}} \psi_k, \psi_k \right\rangle_\gamma \right) + \frac{1}{2} \int_{\partial M} \left\langle e_0 \cdot \vec{k} \cdot \psi_k, \psi_k \right\rangle,$$

and thus thanks to the weighted Poincaré inequality

$$\forall \Omega \subset M \quad |\Omega| < \infty \quad \psi_k \xrightarrow{L^2(\Omega)} \psi \quad \text{and} \quad \widehat{\nabla} \psi_k \xrightarrow{L^2(M)} \rho.$$

Now let us take a $\varphi \in C^1_0$ such that $\text{Supp} \varphi \subset K \subset (M \setminus \partial M)$ (K compact without boundary) and then

$$\begin{array}{ccc} \int_K \left\langle \widehat{\nabla}^* \varphi, \psi_k \right\rangle & \xlongequal{\quad} & \int_K \left\langle \varphi, \widehat{\nabla} \psi_k \right\rangle, \\ \downarrow k \rightarrow \infty & & \downarrow k \rightarrow \infty \\ \int_K \left\langle \widehat{\nabla}^* \varphi, \psi \right\rangle & \xlongequal{\quad} & \int_K \left\langle \varphi, \rho \right\rangle \end{array}$$

and therefore $\rho = \widehat{\nabla} \psi$ in the distributional sense. \square

We consider the linear form l on $H_-(a)$ defined by

$$l(\psi) = \int_M \left\langle \widehat{\mathfrak{D}}(f\mathcal{A}\sigma), \widehat{\mathfrak{D}}\psi \right\rangle_\gamma d\mu_g.$$

It still is a continuous linear form on the Hilbert space $H_-(a)$ (it is complete since the condition $F(\psi) = -\psi$ is closed) and applying again Lax-Milgram theorem we get the existence of a unique $\xi_0 \in H(a)$ such that $l = a(-\xi_0, \cdot)$. In other words

$$\forall \psi \in H_-(a) \quad \int_M \left\langle \chi, \widehat{\mathfrak{D}}\psi \right\rangle = 0,$$

where we have set $\xi = f\mathcal{A}\sigma + \xi_0$ and $\chi = \widehat{\mathfrak{D}}\xi$.

Remark. We have for any compactly supported smooth spinor fields φ_k , $k = 1, 2$, the integration by parts formula

$$\int_M \langle \varphi_1, \widehat{\mathfrak{D}}\varphi_2 \rangle = \int_M \langle \widehat{\mathfrak{D}}^*\varphi_1, \varphi_2 \rangle + \int_{\partial M} \langle \nu \cdot \varphi_1, \varphi_2 \rangle.$$

□

For any $\psi \in C_0^1$ we have

$$\int_M \langle \chi, \widehat{\mathfrak{D}}\psi \rangle = 0 = \int_M \langle \widehat{\mathfrak{D}}^*\chi, \psi \rangle + \int_{\partial M} \langle \nu \cdot \chi, \psi \rangle.$$

But remembering that $C_0^\infty(M \setminus \partial M)$ the space of smooth spinor fields compactly supported in $M \setminus \partial M$ is dense in $L^2(M)$ then we obtain that $\widehat{\mathfrak{D}}^*\chi = 0$ and $\chi \in H_+(a) = \{\psi \in H(a) / F(\psi) = +\psi\}$. By ellipticity χ is smooth and $\widehat{\mathfrak{D}}^k\chi \in L^2(M)$ for every $k \in \mathbb{N}$. Finally we notice that

$$\begin{aligned} \int_M |\widehat{\mathfrak{D}}\chi|^2 &= \int_M \langle \widehat{\mathfrak{D}}^*\widehat{\mathfrak{D}}\chi, \chi \rangle + \int_{\partial M} \langle \nu \cdot \widehat{\mathfrak{D}}\chi, \chi \rangle \\ &= 0 + \int_{\partial M} \langle -in\nu \cdot \chi, \chi \rangle \\ &= -n \int_{\partial M} |\chi|^2, \end{aligned}$$

and therefore $\widehat{\mathfrak{D}}\chi = 0$ which implies that $\chi = 0$. We can conclude with the

Proposition. For every $\sigma \in IK_S(\Sigma)$ there exists a unique $\xi_0 \in H_-(a)$ such that

$$\begin{aligned} \mathcal{H}(V_\sigma, \alpha_\sigma) &= \lim_{r \rightarrow +\infty} \int_{S_r} \langle \widehat{\nabla}_{\mathcal{A}\nu_r}\xi + \mathcal{A}\nu_r \cdot \widehat{\mathfrak{D}}\xi, \xi \rangle_\gamma \\ &= 4 \int_M \left(|\widehat{\nabla}\xi|_\gamma^2 + \langle \widehat{\mathfrak{R}}\xi, \xi \rangle_\gamma \right) + 2 \int_{\partial M} \langle e_0 \cdot \vec{k} \cdot \psi, \psi \rangle \geq 0, \end{aligned}$$

where $\xi = f\mathcal{A}\sigma + \xi_0 \in \text{Ker}\widehat{\mathfrak{D}} \cap H_-(a)$.

4 Proofs of the theorems

4.1 Positivity

On one hand we have found an application

$$\begin{aligned} \mathcal{K} : IK_S(\Sigma) \cong \mathbb{C}^2 \oplus \mathbb{C}^2 &\longrightarrow (\mathfrak{M} \oplus \mathfrak{sl}_2(\mathbb{C}))^{*\mathbb{R}} \cong \text{Ker } d\Phi_{(b,0)}^* , \\ w \oplus u &\longmapsto (V_{w \oplus u} \oplus \alpha_{w \oplus u}) \end{aligned}$$

which is $SL(2, \mathbb{C})$ -equivariant. On the other hand we know that the energy-momentum functional \mathcal{H} can be seen as a real linear form on $(\mathfrak{M} \oplus \mathfrak{sl}_2(\mathbb{C}))^{*\mathbb{R}}$ that is to say, as a vector $\mathcal{H} = M \oplus \Xi \in \mathfrak{M} \oplus \mathfrak{sl}_2(\mathbb{C})$. In the following, we will adopt the notations $\Xi = N \oplus \mathbf{i}R \in \mathfrak{G} \oplus \mathbf{i}\mathfrak{G}$, and $M = \Lambda(m_0, m)$, $N = \Lambda(0, n)$, $R = \Lambda(0, r)$, where Λ is the isomorphism defined in section 2.2. Thanks to the computations of section 2.4 we have

$$\begin{aligned} \mathcal{K}_{w \oplus u} \mathcal{H} &= V_{w \oplus u}(M) + \alpha_{w \oplus u}(\Xi) \\ &= 2(w^* \widehat{M} w + u^* M u) + 2(w^* \Xi u + u^* \Xi^* w), \end{aligned}$$

and consequently the application $w \oplus u \mapsto \mathcal{K}_{w \oplus u} \mathcal{H}$ is a sesquilinear form on $\mathbb{C}^2 \oplus \mathbb{C}^2$ whose matrix is

$$Q = 2 \begin{pmatrix} \widehat{M} & \Xi \\ \Xi^* & M \end{pmatrix} = 2 \begin{pmatrix} \Lambda(m_0, -m) & \Lambda(0, n) + \mathbf{i}\Lambda(0, r) \\ \Lambda(0, n) - \mathbf{i}\Lambda(0, r) & \Lambda(m_0, m) \end{pmatrix}.$$

Putting together the propositions of section 3.6 and the proposition and remark of section 3.5, we get the non negativity of Q and consequently the positive energy-momentum theorem stated in section 1.3. \square

Corollary. $m_0 = 0$ implies $\mathcal{H} = 0$.

PROOF. Making the sum of the order 2 principal minors of the matrix Q (cf. section 5) one gets $3m_0^2 - (|m|^2 + |n|^2 + |r|^2) \geq 0$. \square

Proposition. *If M is timelike, there exists a (non unique) representative element of the orbit of $\mathcal{H} = M \oplus \Xi$ under the natural action (cf. section 2.4) of $SL(2, \mathbb{C})$ on $\mathfrak{M} \oplus \mathfrak{sl}_2(\mathbb{C})$ which can be written*

$$m_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus n_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbf{i} \begin{pmatrix} r_1 & r_2 \\ r_2 & -r_1 \end{pmatrix},$$

with $m_0, n_1, r_1, r_2 \in \mathbb{R}$. The positive energy-momentum theorem then reduces to $m_0 \geq \sqrt{(|n_1| + |r_2|)^2 + r_1^2}$.

PROOF. Let us suppose that $M \in \mathfrak{M}$ is timelike. Thus considering the action of $SL(2, \mathbb{C})$ on $\mathfrak{M} \oplus \mathfrak{sl}_2(\mathbb{C})$ (cf. section 2.4), then there exists an element in the orbit of \mathcal{H} that can be written $m_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \Xi'$. Since the isotropy group of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is $SU(2)$ whose action on \mathfrak{G} is transitive, then there exists an element in the orbit of \mathcal{H} that can be written $m_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus n_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbf{i}R'$. But the isotropy group of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the one parameter group $\left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \theta \in \mathbb{R} \right\}$. Finally there exists an element (not unique since the isotropy

group of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is isomorphic to \mathbb{Z}_2) in the orbit of \mathcal{H} that can be written as announced in the proposition. The corresponding Hermitian matrix is

$$Q = 2 \begin{pmatrix} m_0 & 0 & n_1 + \mathbf{i}r_1 & \mathbf{i}r_2 \\ 0 & m_0 & \mathbf{i}r_2 & -n_1 - \mathbf{i}r_1 \\ n_1 - \mathbf{i}r_1 & -\mathbf{i}r_2 & m_0 & 0 \\ -\mathbf{i}r_2 & -n_1 + \mathbf{i}r_1 & 0 & m_0 \end{pmatrix}.$$

Since Q is non negative we have

$$\begin{aligned} m_0 &\geq 0 \\ m_0(m_0^2 - (n_1^2 + r_1^2 + r_2^2)) &\geq 0 \\ (m_0^2 - (n_1^2 + r_1^2 + r_2^2))^2 &\geq 4(n_1 r_2)^2, \end{aligned}$$

which implies $m_0 \geq \sqrt{(|n_1| + |r_2|)^2 + r_1^2}$. \square

Remark. The Kerr AdS metrics are asymptotically hyperbolic and parametrized by 2 real parameters: the mass and the angular momentum. The proposition above then shows that an asymptotically hyperbolic metric with non zero energy-momentum coefficients m_0, n_1, r_1, r_2 could not be obtained by the action of $\mathrm{SL}(2, \mathbb{C})$ on a Kerr AdS solution. As a consequence, an interesting question would be to find some (new!) asymptotically hyperbolic metrics which have an energy-momentum of the form given in the proposition above with non zero coefficients m_0, n_1, r_1, r_2 , and satisfy the dominant energy condition or the (stronger) cosmological vacuum constraints. \square

4.2 Rigidity

Let us suppose that Q is degenerate. Consequently, there exists some non zero $w \oplus u \in \mathbb{C}^2 \oplus \mathbb{C}^2$ such that $\mathcal{K}_{w \oplus u} \mathcal{H} = 0$. Let us denote by σ the corresponding β -imaginary Killing spinor $\sigma_{w \otimes \binom{1}{-i}}^{-1} + \sigma_{u \otimes \binom{1}{i}}^*$, and by $\xi = f\mathcal{A}\sigma + \xi_0$ (cf. the proposition of section 3.6) the unique corresponding spinor such that

$$\mathcal{K}_{w \oplus u} \mathcal{H} = \mathcal{H}(V_\sigma, \alpha_\sigma) = 4 \int_M \left(\left| \widehat{\nabla} \xi \right|_\gamma^2 + \langle \Re \xi, \xi \rangle_\gamma \right) = 0.$$

This implies that $\widehat{\nabla} \xi = 0$ on M which means ξ is a imaginary Killing spinor with respect to the metric γ , and $\langle \Re \xi, \xi \rangle = 0$. Besides it is important to notice that the function $\langle \xi, \xi \rangle > 0$ since the spinor ξ cannot vanish on M .

Proposition. *For every $X, Y \in \Gamma(TM)$ we have*

$$R_{X,Y}^\gamma = R_{X,Y}^g - \frac{1}{2} \left(d\bar{\nabla} k(X, Y) \cdot e_0 + \frac{1}{2} (k(X) \cdot k(Y) - k(Y) \cdot k(X)) \right) \cdot ,$$

where \cdot denotes the Clifford action with respect to the metric γ .

PROOF. It is a straightforward computation

$$\begin{aligned}
\nabla_X \nabla_Y &= \nabla_X \left(\bar{\nabla}_Y - \frac{1}{2} k(Y) \cdot e_0 \cdot \right) \\
&= \bar{\nabla}_X \bar{\nabla}_Y - \frac{1}{2} k(X) \cdot e_0 \cdot \bar{\nabla}_Y \\
&\quad - \frac{1}{2} (\nabla_X k(Y) \cdot e_0 \cdot + k(Y) \cdot (\nabla_X e_0) \cdot + k(Y) \cdot e_0 \cdot \nabla_X) \\
&= \bar{\nabla}_X \bar{\nabla}_Y - \frac{1}{2} (k(X) \cdot e_0 \cdot \bar{\nabla}_Y + k(Y) \cdot e_0 \cdot \bar{\nabla}_X) - \left(\frac{1}{2} \bar{\nabla}_X k(Y) \cdot e_0 + \frac{1}{4} k(X) \cdot k(Y) \right) \cdot ,
\end{aligned}$$

and the curvature formula above follows. \square

Let us denote by V the function $\langle \xi, \xi \rangle$, α the real 1-form defined by $\alpha(Y) = \langle Y \cdot e_0 \cdot \xi, \xi \rangle$ and finally the real symmetric 2-tensor E by

$$E := \text{Ric}^g + (n-1)g + (\text{tr}_g k)k - k \circ k.$$

A consequence of the curvature formula above is the usual Codazzi and Gauss equations that are obtained thanks to the natural isomorphism between $\text{Cl}_{3,1}$ and $\Lambda^2(\mathbb{R}^{3,1})$ (cf. [17] proposition 6.2):

$$\begin{aligned}
R^\gamma &= R^g - \frac{1}{2} k \otimes k \\
R^\gamma(X, Y, Z, e_0) &= d\bar{\nabla} k(X, Y, Z),
\end{aligned}$$

where \otimes denotes the Kulkarni-Nomizu product of two symmetric 2-tensors (cf. definition 1.110 of [11]). Using these equations, one finds

$$\begin{aligned}
\sum_{k=1}^3 e_k \cdot R_{X, e_k}^\gamma &= \frac{1}{2} \sum_{k=1}^3 \sum_{l, m=0}^3 R^\gamma(X, e_k, e_l, e_m) e_k \cdot e_l \cdot e_m \cdot \\
&= \sum_{k, m=1}^3 R^\gamma(X, e_k, e_0, e_m) e_k \cdot e_0 \cdot e_m \cdot + \frac{1}{2} \sum_{k, l, m=1}^3 R^\gamma(X, e_k, e_l, e_m) e_k \cdot e_l \cdot e_m \cdot \\
&= \sum_{k=1}^3 R^\gamma(X, e_k, e_0, e_k) e_0 \cdot + \frac{1}{2} \sum_{k, m=1}^3 R^\gamma(X, e_k, e_k, e_m) e_k \cdot e_k \cdot e_m \cdot \\
&= - \sum_{k=1}^3 \{ \bar{\nabla}_X k(e_k, e_k) - \bar{\nabla}_{e_k} k(e_k, X) \} \cdot e_0 \cdot \\
&\quad - \frac{1}{2} \sum_{k, m=1}^3 \{ R^g(X, e_k, e_k, e_m) - k(X, e_k)k(e_k, e_m) + k(X, e_m)k(e_k, e_k) \} \cdot \\
&= \frac{1}{2} (- \text{Ric}^g(X) - (\text{tr}_g k)k(X) + k \circ k(X) - 2(\text{dtr}_g k + \delta k)(X)e_0) \cdot ,
\end{aligned}$$

but since ξ is a γ -imaginary Killing spinor $R_{X,Y}^\gamma \xi = \frac{1}{4}(X \cdot Y - Y \cdot X) \cdot \xi$, and thereby $E(X) \cdot \xi = -2(\text{dtr}_g k + \delta k)(X)e_0 \cdot \xi$. Then we compute

$$\begin{aligned} E(X, Y)V &= \Re \langle E(X) \cdot \xi, Y \cdot \xi \rangle \\ &= -2(\text{dtr}_g k + \delta k)(X) \Re \langle e_0 \cdot \xi, Y \cdot \xi \rangle \\ &= 2(\text{dtr}_g k + \delta k) \odot \alpha(X, Y), \end{aligned}$$

where \odot denotes the symmetric product of 1-forms. Taking the trace of this expression with respect to g , one finds that

$$V (\text{Scal}^g + n(n-1) + (\text{tr}_g k)^2 - |k|_g^2) = 2 \langle (\text{dtr}_g k + \delta k), \alpha \rangle.$$

But the condition $\langle \widehat{\mathfrak{R}}\xi, \xi \rangle = 0$ implies

$$V (\text{Scal}^g + n(n-1) + (\text{tr}_g k)^2 - |k|_g^2) = -2 \langle (\text{dtr}_g k + \delta k), \alpha \rangle.$$

It follows, since the function V cannot vanish on M , that $\text{Scal}^g + n(n-1) + (\text{tr}_g k)^2 - |k|_g^2 = 0$ and $\text{dtr}_g k + \delta k = 0$ because of the dominant energy condition of section 3.3. But using $EV = 2(\text{dtr}_g k + \delta k) \odot \alpha = 0$, we can conclude that $E = 0$. But in dimension $n = 3$, R^g is totally determined by the Ricci curvature so that it is easy to see

$$R^g = \frac{1}{2}(g \otimes g + k \otimes k),$$

as regards the Riemannian curvature and

$$R_{X,Y}^g = \frac{1}{4}(X \cdot Y - Y \cdot X + k(X) \cdot k(Y) - k(Y) \cdot k(X)),$$

for the spinorial curvature. Finally using the formula of the proposition above against ξ one gets that $\text{d}^{\overline{\nabla}} k(X, Y) \cdot \xi = 0$ which implies that $\text{d}^{\overline{\nabla}} k(X, Y) = 0$. Consequently we obtain

$$\begin{aligned} R^g &= \frac{1}{2}(g \otimes g + k \otimes k) \\ \text{d}^{\overline{\nabla}} k &= 0. \end{aligned}$$

Furthermore, the couple $(V, Y) := (V, -\alpha^\sharp)$ is a Killing Initial Data (KID) [10]. If we consider $(\widetilde{M}, \widetilde{g}, \widetilde{k})$ the universal Riemannian covering of (M, g, k) , then we can make the Killing development of $(\widetilde{M}, \widetilde{g}, \widetilde{k})$ with respect to the KID $(\widetilde{V}, \widetilde{Y})$ which by definition is $\mathbb{R} \times \widetilde{M}$ endowed with the Lorentzian metric $\widetilde{\gamma} = \left(-\widetilde{N}^2 + |\widetilde{Y}|^2 \right) du^2 + 2\widetilde{Y}^b \odot du + \widetilde{g}$. By construction, \widetilde{M} is embedded in $(\mathbb{R} \times \widetilde{M}, \widetilde{\gamma})$ with induced metric \widetilde{g} and second fundamental form \widetilde{k} . Besides $\mathbb{R} \times \widetilde{M}$ is the universal covering of N , and $\widetilde{\gamma}$ which has sectional curvature -1, is a stationary solution of the vacuum Einstein equations with cosmological constant that is to say $G^{\widetilde{\gamma}} = \frac{n(n-1)}{2} \widetilde{\gamma}$. But $(\widetilde{M}, \widetilde{g})$ is complete since (M, g) is complete and therefore [1] $(\mathbb{R} \times \widetilde{M}, \widetilde{\gamma})$ is geodesically

complete. It follows that $(\mathbb{R} \times \widetilde{M}, \widetilde{\gamma})$ is AdS. It only remains to show that M is simply connected. We know that $\mathbb{R} \times \widetilde{M} \cong \mathbb{R}^4$ and thereby using the following compactly supported de Rham cohomology isomorphisms $\{0\} = H_{dR,c}^2(\mathbb{R} \times \widetilde{M}) = H_{dR,c}^2(\mathbb{R}^4) = H_{dR,c}^1(\widetilde{M})$, we obtain that \widetilde{M} has only one asymptotic end. This last fact compels the universal covering map $\widetilde{M} \rightarrow M$ to be trivial and as a consequence $(M, g, k) \equiv (\widetilde{M}, \widetilde{g}, \widetilde{k})$ is isometrically embedded in $\text{AdS} \equiv (\widetilde{N}, \widetilde{\gamma})$.

5 Appendix

Classical linear algebra results state that every principal minor of Q must be non negative which give rise to a set of inequalities on the coefficients of \mathcal{H} .

Order 1 principal minors

$$\begin{cases} m_0 + m_1 \geq 0 \\ m_0 - m_1 \geq 0 \end{cases} ,$$

Order 2 principal minors

$$\begin{cases} m_0^2 - |m|^2 \geq 0 \\ (m_0 + m_1)^2 - (n_2 + r_3)^2 - (r_2 - n_3)^2 \geq 0 \\ (m_0 - m_1)^2 - (n_2 - r_3)^2 - (r_2 + n_3)^2 \geq 0 \\ m_0^2 - m_1^2 - n_1^2 - r_1^2 \geq 0 \end{cases} .$$

Order 3 principal minors

$$\begin{cases} (m_0 + m_1)(m_0^2 - (|m|^2 + n_1^2 + r_1^2)) - (m_0 - m_1)((n_2 + r_3)^2 + (n_3 - r_2)^2) \\ \quad - 2((n_2 + r_3)(m_2 n_1 + m_3 r_1) + (-n_3 + r_2)(m_2 r_1 - m_3 n_1)) \geq 0 \\ (m_0 - m_1)(m_0^2 - (|m|^2 + n_1^2 + r_1^2)) - (m_0 + m_1)((n_2 - r_3)^2 + (n_3 + r_2)^2) \\ \quad + 2((n_2 - r_3)(m_2 n_1 - m_3 r_1) + (n_3 + r_2)(m_2 r_1 + m_3 n_1)) \geq 0 \end{cases} .$$

Determinant of Q

$$(m_0^2 - (|m|^2 + |n|^2 + |r|^2))^2 - 4(|m|^2|n|^2 + |m|^2|r|^2 + |n|^2|r|^2) + 4(\langle m, n \rangle^2 + \langle m, r \rangle^2 + \langle n, r \rangle^2) + 8m_0 \det_{\mathbb{R}^3}(m, n, r) \geq 0.$$

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Institut de Mathématiques et de Modélisation de Montpellier (I3M)
 Université Montpellier II
 UMR 5149 CNRS
 Place Eugène Bataillon
 34095 MONTPELLIER (FRANCE)
 email:maerten@math.univ-montp2.fr
 fax: 33 (0) 4 67 14 35 58
 tel: 33 (0) 4 67 14 48 47