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Pascal Auscher

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# On necessary and sufficient conditions for $L^p$ -estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates

Pascal Auscher \*

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## Abstract

This article focuses on  $L^p$  estimates for objects associated to elliptic operators in divergence form: its semigroup, the gradient of the semigroup, functional calculus, square functions and Riesz transforms. We introduce four critical numbers associated to the semigroup and its gradient that completely rule the ranges of exponents for the  $L^p$  estimates. It appears that the case  $p < 2$  already treated earlier is radically different from the case  $p > 2$  which is new. We thus recover in a unified and coherent way many  $L^p$  estimates and give further applications. The key tools from harmonic analysis are two criteria for  $L^p$  boundedness, one for  $p < 2$  and the other for  $p > 2$  but in ranges different from the usual intervals  $(1, 2)$  and  $(2, \infty)$ .

**Key words:** elliptic operators, divergence form, semigroup,  $L^p$  estimates, Calderón-Zygmund theory, good lambda inequalities, hypercontractivity, Riesz transforms, holomorphic functional calculus, Littlewood-Paley-Stein estimates.

**MSC 2000 numbers:** 42B20, 42B25, 47F05, 47B44, 35J15, 35J30, 35J45.

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\*Université Paris-Sud and CNRS UMR 8628, Department de Mathématiques, 91405 Orsay Cedex (France) Email: [pascal.auscher@math.u-psud.fr](mailto:pascal.auscher@math.u-psud.fr)

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## Introduction

Although the matter of this text applies *in extenso* to elliptic operators or systems in divergence form to any order in  $\mathbb{R}^n$ , we focus on second order operators in  $\mathbb{R}^n$ . A section will be devoted to these more general classes.

Let  $A = A(x)$  be an  $n \times n$  matrix of complex,  $L^\infty$  coefficients, defined on  $\mathbb{R}^n$ , and satisfying the ellipticity (or “accretivity”) condition

$$\lambda|\xi|^2 \leq \operatorname{Re} A\xi \cdot \bar{\xi} \text{ and } |A\xi \cdot \bar{\zeta}| \leq \Lambda|\xi||\zeta|,$$

for  $\xi, \zeta \in \mathbb{C}^n$  and for some  $\lambda, \Lambda$  such that  $0 < \lambda \leq \Lambda < \infty$ . We define a second order divergence form operator

$$Lf \equiv -\operatorname{div}(A\nabla f),$$

which we interpret in the sense of maximal accretive operators via a sesquilinear form. Here,  $\nabla$  denotes the array of first order partial derivatives.

The maximal accretivity condition implies the existence of an analytic contraction semigroup on  $L^2(\mathbb{R}^n)$  generated by  $-L$ . It also implies the existence of a holomorphic functional calculus that has the expected stability under commutation and convergence, allowing for example to define fractional powers. This in turn yields the possibility of defining various objects of interest both from functional and harmonic analysis points of view. Let us mention Littlewood-Paley-Stein type functionals such as

$$g_L(f)(x) = \left( \int_0^\infty |(L^{1/2}e^{-tL}f)(x)|^2 dt \right)^{1/2}$$

and

$$G_L(f)(x) = \left( \int_0^\infty |(\nabla e^{-tL}f)(x)|^2 dt \right)^{1/2}.$$

The “singular integral” pending to the latter square function is the so-called Riesz transform associated to  $L$  given for example by

$$\nabla L^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \sqrt{t} \nabla e^{-tL} \frac{dt}{t}.$$

Other objects of interest are 1) the operator of maximal regularity for the parabolic equation associated to  $L$ , 2) Riesz means and  $L^p$ -multipliers when  $L$

is, in addition, self-adjoint . . . They can be treated by the methods presented here but we have chosen not to go into such developments.

When the coefficients are constant, *e.g.* the Laplacian, one finds classical objects in harmonic analysis: multipliers, the Littlewood-Paley-Stein functionals and the original Riesz transforms. They belong to the well-understood class of Calderón-Zygmund operators. If the coefficients of  $L$  still have some smoothness, then the tools of pseudo-differential calculus or of Calderón-Zygmund theory can still be used. In absence of regularity of the coefficients, these operators fall beyond the above classes and this participate to Calderón's program <sup>1</sup> of defining algebras of differential operators with minimal smoothness.

The first step of that study is the action on  $L^2(\mathbb{R}^n)$ . First, there is a bounded holomorphic functional calculus on  $L^2$  basically as a consequence maximal accretivity and Von Neumann's inequality. One has

$$\|\varphi(L)f\|_2 \leq \|\varphi\|_\infty \|f\|_2$$

for  $\varphi$  bounded holomorphic in the open right half plane. Next,  $g_L$  and  $G_L$  are  $L^2$  bounded (see Section 7) and

$$\|g_L(f)\|_2 \sim \|f\|_2 \sim \|G_L(f)\|_2. \text{ }^2$$

The  $L^2$  boundedness of the Riesz transform has been proved recently and in fact, one has in all dimensions

$$\|L^{1/2}f\|_2 \sim \|\nabla f\|_2.$$

This implies that the domain of  $L^{1/2}$  is the Sobolev space  $W^{1,2}$ , which was known as Kato's conjecture. <sup>3</sup>

The second step is the action on  $L^p(\mathbb{R}^n)$  with  $1 < p < \infty$  but  $p \neq 2$ . The bounded holomorphic functional calculus on  $L^p$  consists in proving

<sup>1</sup> See [C].

<sup>2</sup> Here  $\sim$  is the equivalence in the sense of norms, with implicit constants  $C$  depending only on  $n$ ,  $\lambda$  and  $\Lambda$ .

<sup>3</sup> The one dimensional Kato's conjecture (that is the case  $n = 1$ ) is first proved by Coifman, M<sup>c</sup>Intosh & Meyer [CMcM, Théorème X] the two dimensional case by Hofmann & M<sup>c</sup>Intosh [HM, Theorem 1.4] and the general case in any dimension by Hofmann, Lacey, M<sup>c</sup>Intosh & Tchamitchian along with the author [AHLMcT, Theorem 1.4]. We refer to the latter reference for historical remarks and connections with other problems.

$L^p$  boundedness of  $\varphi(L)$  for an appropriate class of bounded holomorphic functions  $\varphi$ . It is completely understood <sup>4</sup> and one has

$$\|\varphi(L)(f)\|_p \lesssim \|\varphi\|_\infty \|f\|_p \quad \text{whenever } p_-(L) < p < p_+(L), \quad ^5$$

where  $p_-(L)$  and  $p_+(L)$  are the two critical exponents for the  $L^p$  uniform boundedness of the semigroup  $(e^{-tL})_{t>0}$ . It is clear that this interval is the largest open range of exponents for which such an inequality holds as  $\varphi$  can be an exponential. The  $L^p$  theory for square functions consists in comparing the  $L^p$  norms of  $g_L(f)$ ,  $f$  and  $G_L(f)$ . For  $g_L$ , what happens is completely understood in terms of functional calculus: <sup>6</sup> one has

$$\|g_L(f)\|_p \sim \|f\|_p \quad \text{whenever } p_-(L) < p < p_+(L).$$

It turns out that this interval is the largest open range of exponents for which this equivalence holds. The comparison between the  $L^p$  norms  $f$  and  $G_L(f)$  has not been done in general so far <sup>7</sup> and we shall see that

$$\|G_L(f)\|_p \sim \|f\|_p \quad \text{whenever } q_-(L) < p < q_+(L),$$

where  $q_-(L)$  and  $q_+(L)$  are the two critical exponents for the  $L^p$  uniform boundedness of the gradient of the semigroup  $(\sqrt{t} \nabla e^{-tL})_{t>0}$ , and that this open range is optimal. We also study the corresponding non-tangential Littlewood-Paley-Stein functionals (See Section 7).

The  $L^p$  theory for square roots consists in comparing  $L^{1/2}f$  and  $\nabla f$  in  $L^p$  norms. <sup>8</sup> There are two issues here, namely the Riesz transform  $L^p$  boundedness, that is an inequality  $\|\nabla f\|_p \lesssim \|L^{1/2}f\|_p$ , and its reverse  $\|L^{1/2}f\|_p \lesssim \|\nabla f\|_p$ . It turns out the ranges of  $p$  may be different. The state

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<sup>4</sup> This is essentially due to Blunck & Kunstmann [BK2, Proposition 2.3] although the authors did not introduce the numbers  $p_\pm(L)$ .

<sup>5</sup> Here  $\lesssim$  is the comparison in the sense of norms, with implicit constant  $C$  that may depend on  $L$  through ellipticity, dimension, its type and  $p_\pm(L)$ .

<sup>6</sup> This follows by combining works of Blunck & Kunstmann [BK2, Proposition 2.3] and Le Merdy [LeM, Theorem 3].

<sup>7</sup> After this paper was submitted, the author learned of a work by Yan [Yan] where the inequality  $\lesssim$  is obtained for  $2n/(n+2) < p \leq 2$ .

<sup>8</sup> This program was initialised by the author and P. Tchamitchian in [AT] for this class of complex operators. It arose from a different perspective towards applications to boundary value problems in the works of Dalbergh, Jerison, Kenig and their collaborators (see [Ke, problem 3.3.16]).

of the art for this class of operators  $L$  is as follows.<sup>9</sup> One has the Riesz transforms estimates<sup>10</sup>

$$\|\nabla f\|_p \lesssim \|L^{1/2}f\|_p \quad \text{if} \quad \begin{cases} n = 1 & \text{and} & 1 < p < \infty \\ n = 2 & \text{and} & 1 < p < 2 + \varepsilon \\ n \geq 3 & \text{and} & \frac{2n}{n+2} - \varepsilon < p < 2 + \varepsilon \end{cases},$$

and the reverse inequalities<sup>11</sup>

$$\|L^{1/2}f\|_p \lesssim \|\nabla f\|_p \quad \text{if} \quad \begin{cases} n = 1, 2 & \text{and} & 1 < p < \infty \\ n = 3, 4 & \text{and} & 1 < p < \frac{2n}{n-2} + \varepsilon \\ n \geq 5 & \text{and} & \frac{2n}{n+4} - \varepsilon < p < \frac{2n}{n-2} + \varepsilon \end{cases}.$$

Of course, if  $L$  possesses more properties then the ranges of exponents  $p$  improve. For example for constant coefficients operators these inequalities holds when  $1 < p < \infty$ , and for real operators in dimensions  $n \geq 3$ , the Riesz transform  $L^p$  boundedness is valid for  $1 < p < 2 + \varepsilon$  and the reverse inequality for  $1 < p < \infty$ .<sup>12</sup> Hence, it is interesting to have a theory that works for any single operator. In fact, the conclusion of the story for the

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<sup>9</sup> Some of the results were obtained prior to the Kato conjecture by making the  $L^2$  result an assumption.

<sup>10</sup> For  $n = 1$ , this is due to the author and Tchamitchian [AT1, Théorème A] for  $n = 2$  to the author and Tchamitchian [AT, Chapter IV, Theorem 1] combining the Gaussian estimates of the author, McIntosh and Tchamitchian [AMcT, Theorem 3.5] and the  $L^2$  result [AHLMcT, Theorem 1.4] and for  $n \geq 3$  and  $p_n = \frac{2n}{n+2} < p < 2$ , independently to Blunck & Kunstmann [BK2, Theorem 1.2], and to Hofmann & Martell [HM, Theorem 1.2]. The enlargement of the range below  $p_n$  is due to the author [A, Proposition 1] and above 2 is a consequence of the method of the author and Tchamitchian once the reverse inequality is established (see [AT, Chapter IV, Proposition 20] and [A, Corollary 4]).

<sup>11</sup> They are due, for  $n = 1$ , to the author and Tchamitchian ([AT1], Théorème A), for  $n = 2$ , to the author and Tchamitchian ([AT], Chapter IV, Theorem 1), again combining [AMcT], Theorem 3.5 with [AHLMcT], Theorem 1.4, for  $n = 3, 4$  to the author, Hofmann, Lacey, McIntosh & Tchamitchian ([AHLMcT], Proposition 6.2) and, for  $n \geq 5$  to the author ([A], Theorem 2).

<sup>12</sup> For constant coefficients, this goes back to Calderón-Zygmund original work [CZ] and for real operators, this is due to the author and Tchamitchian [AT, Chapter IV].



Riesz transform is <sup>13</sup>

$$\|\nabla f\|_p \lesssim \|L^{1/2}f\|_p \quad \text{if and only if } q_-(L) < p < q_+(L)$$

and we also show that

$$\|L^{1/2}f\|_p \lesssim \|\nabla f\|_p \quad \text{whenever } \tilde{p}_-(L) = \sup(1, (p_-(L))_*) < p < p_+(L).$$

This encapsulates all the above mentioned estimates (See Section 1 for the notation  $p_*$ ). Concerning the latter range, we show that  $p_+(L)$  is best possible in some sense, while we only know a lower bound on  $\tilde{p}_-(L)$ . <sup>14</sup> Staring at the formula given above for computing the Riesz transform this result seems to say that the integral yields a bounded operator on  $L^p$  if and only if the integrands are uniformly bounded operators on  $L^p$ . Said like this, the sufficiency looks astonishingly simple. But this is not quite the truth as there is a play on exponents in the proof. Note also the range of exponents for  $L^p$ -boundedness of the Riesz transform  $R(L) = \nabla L^{-1/2}$  is characterized. In particular, it is an open set. We also show that  $2 < p < q_+(L)$  if and only if the Hodge projector  $\nabla L^{-1} \operatorname{div}$  (or alternately, the second order Riesz transform  $R(L)R(L^*)^*$ ) is bounded on  $L^p$ . For  $p < 2$  the  $L^p$  boundedness of the Hodge projector is stronger than that of  $R(L)$ .

The objective of this paper is to present a complete, coherent and unified theory for all these topics. We present works of others and also original contributions. In particular, we have tried to be self-contained. Our main observation is the following: **four critical numbers** <sup>15</sup> **rule the  $L^p$  behavior**. These are  $p_{\pm}(L)$ , the limits of the interval of exponents  $p \in [1, \infty]$  for which the semigroup  $(e^{-tL})_{t>0}$  is  $L^p$  bounded, and  $q_{\pm}(L)$ , the limits of the interval of exponents  $p \in [1, \infty]$  for which the family  $(\sqrt{t} \nabla e^{-tL})_{t>0}$  is  $L^p$  bounded. We make a thorough study of these numbers, their inner relationships and

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<sup>13</sup> This problematic of finding the “smallest” exponent  $p$  is implicit in [BK2] and we present the counterpart for the “largest”, which turns out to require different arguments. After this paper was submitted, Shen [Sh] informed me of his independent and simultaneous work on the same problem for  $p > 2$  when  $L$  is real. He obtains a different characterization of  $q_+(L)$  in terms of reverse Hölder estimates for weak solutions of  $Lu = 0$ . We convince ourselves by e-mail discussions that this approach can be adapted to complex  $L$ .

<sup>14</sup> As said above, all applies to higher-order operators. For that extended class, this lower bound is optimal due to some existing appropriate counter-examples in the theory. Similar counter-examples for the second order case are not known.

<sup>15</sup> In fact they reduce to two:  $p_-(L)$  and  $q_+(L)$ .

their values in terms of dimension for the whole class of such  $L$  (Section 4). The key stone of this theory, which makes use of the divergence structure of our operators, is that the  $L^p$  boundedness of their semigroups (resp. of the gradient of their semigroups) is **equivalent** to some **off-diagonal estimates** and also to some **hypercontractivity**.<sup>16</sup> To be precise, we sometimes have to loosen the exponent  $p$ , but this is harmless for the kind of results we are after. Let us mention here that this equivalence is not powerful enough for treating  $L^\infty$  decay of the semigroup kernels whenever there is  $L^1$  or  $L^\infty$  boundedness of the semigroup. But, again, this is enough for our needs.

Next, we turn to studying the harmonic analysis objects (Sections 5, 6 & 7). On the one hand, finding necessary conditions on  $p$  for which one has  $L^p$  bounds for the functional calculus, the square functions, the Riesz transforms is intuitively easy and the critical numbers appear then. On the other hand, it is not clear at all why these conditions **alone** suffice. For this, appropriate criteria for  $L^p$  boundedness with minimal hypotheses are needed.

The  $L^p$  estimates obtained in Sections 5, 6 & 7 depend on the critical numbers of a given operator. Thus, they are individual results with sharp ranges of  $p$ 's, the operator norms depending on dimension, the ellipticity constants and some of the critical numbers. But when the critical numbers can be estimated for operators in some class, they become  $L^p$  estimates for the whole class. In this case, the optimality of the range of  $p$ 's is function of the sharpness of the bounds on the critical numbers. This is discussed in Section 4.5 for second order operators and in Section 8.2 for higher order.

As the reader may guess, the various critical numbers have no reason to be 1 or  $\infty$ . Hence, we have a class of operators that lies **beyond the class of Calderón-Zygmund operators**. We wish here to present an appropriate machinery to obtain  $L^p$  boundedness without caring about kernels of operators and for ranges of  $p$  different from the usual intervals  $(1, 2)$  or  $(2, \infty)$  (See Section 2 for more).

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<sup>16</sup> This equivalence for values of  $p$  different from 1, although not explicitly stated in the literature to our knowledge, is in the air of a number of works dealing with semigroups of elliptic operators. It appears first in Davies's work [Da]. See also [LSV] and the references therein.

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It is a pleasure to thank here many colleagues with whom I discussed these topics over the years and from whom I learnt a lot. Let me mention S. Blunck, T. Coulhon, G. David, X.T. Duong, S. Hofmann, A. McIntosh, E. Ouhabaz, P. Tchamitchian. Special thanks go to E. Ouhabaz who helped me with some historical comments and to S. Blunck for providing me with recent unpublished work. I also indebted to J.-M. Martell for a careful reading that helped correcting some misprints in the submitted version.

## 1 Notation

We constantly work on  $\mathbb{R}^n$ ,  $n \geq 1$ , equipped with its usual Lebesgue measure. If  $E$  is a measurable set in  $\mathbb{R}^n$ , we write

$$\|f\|_{L^p(E)} = \left( \int_E |f|^p \right)^{1/p}$$

for the norm in the Lebesgue space  $L^p(E)$ ,  $1 \leq p \leq \infty$ , with the usual modification if  $p = \infty$ . We do not indicate the integration variable and the measure unless this is necessary for comprehension. Also we drop  $E$  in the lower limit if  $E = \mathbb{R}^n$  and set  $L^p$  and  $\|f\|_p$  for  $L^p(\mathbb{R}^n)$  and  $\|f\|_{L^p(\mathbb{R}^n)}$  unless the value of  $n$  is of matter. For Hilbert-valued functions,  $|f|$  is replaced by the norm in the Hilbert space,  $|f|_H$ , but we do not introduce a specific notation for  $L^p$  as the situation will make it clear.

We use the notation  $p'$  for the dual exponent to  $p$ :  $p' = \frac{p}{p-1}$ .

The Sobolev space  $W^{m,p}(\mathbb{R}^n)$ ,  $m \in \mathbb{N}$ ,  $1 \leq p \leq \infty$ , is the space of those  $L^p$  functions  $f$  for which all derivatives up to and including order  $m$  are in  $L^p$ . The norm of  $f$  is the sum of the  $L^p$  norms of  $f$  and all its derivatives.

The homogeneous Sobolev space  $\dot{W}^{m,p}(\mathbb{R}^n)$ ,  $m \in \mathbb{N}$ ,  $1 \leq p < \infty$ , is the closure of  $C_0^\infty(\mathbb{R}^n)$  for the seminorm being the sum of the  $L^p$  norms of all derivatives of order  $m$ .

We are essentially interested in Sobolev spaces of order  $m = 1$ . The well-known Sobolev inequalities say that

$$\dot{W}^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$$

whenever,  $n > 1$ ,

$$1 \leq p < n, \quad \frac{n}{n-1} \leq q < \infty \quad \frac{n}{q} = \frac{n}{p} - 1$$

and

$$\|f\|_q \leq C(n, p, q) \|\nabla f\|_p.$$

We use the notation  $p^*$  ( $p$  upper star) for the Sobolev exponent of  $p$ , that is

$$p^* = \frac{np}{n-p}$$

with the convention that  $p^* = \infty$  if  $p \geq n$  and  $q_*$  ( $q$  lower star) for the reverse Sobolev exponent of  $q$ , that is

$$q_* = \frac{nq}{n+q}$$

Note that  $(p_*)' = (p')^*$  whenever  $p_* \geq 1$  and  $(p^*)' = (p')_*$  whenever  $p^* < \infty$ .

For  $1 \leq p, q \leq \infty$  we set

$$\gamma_{pq} = \left| \frac{n}{q} - \frac{n}{p} \right|$$

and in the special case where  $q = 2$

$$\gamma_p = \left| \frac{n}{2} - \frac{n}{p} \right|.$$

As usual, we use positive constants which do not depend on the parameters at stake and whose value change at each occurrence. Often, we do not mention about such constants as their meaning is self-explanatory.

## 2 Beyond Calderón-Zygmund operators

By definition, <sup>17</sup> a Calderón-Zygmund operator on  $\mathbb{R}^n$  <sup>18</sup> is a bounded operator on  $L^2$  which is associated with a kernel possessing some size and regularity properties, the latter being called Hörmander's condition. The fundamental

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<sup>17</sup> We take this terminology from Meyer [Me, Chapter VII].

<sup>18</sup> or more generally on a space of homogeneous type (see [CW]); we shall not be concerned with the development on non homogeneous spaces described in the excellent review by Verdera [V] although extensions to this setting of the results presented here would be interesting.

result <sup>19</sup> is that such operators are weak type  $(1, 1)$ , hence strong type  $(p, p)$  when  $1 < p < 2$  by the interpolation theorem of Marcinkiewicz and eventually strong type  $(p, p)$  for  $p > 2$  by a duality argument. Another route is to begin with their  $L^\infty - BMO$  boundedness, <sup>20</sup> interpolation between  $L^2$  and  $BMO$  <sup>21</sup> for  $2 < p < \infty$ , and duality for  $1 < p < 2$ . However, one should not forget that this interpolation is more involved than the Calderón-Zygmund décomposition used for weak type  $(1, 1)$ .

In applications, this is enough for numerous operators going from convolution operators such as the Hilbert transform and the classical Riesz transforms (the prototypes of Calderón-Zygmund operators) to the Cauchy integral on a Lipschitz curve and the double layer operator on a Lipschitz domain.

However, recently some interesting operators were found to be out of this class. That is they are strong type  $(2, 2)$  but the other properties fail. Some reasons are

1. their kernel does not possess regularity properties such as the Hörmander condition
2. they do not possess a kernel in any reasonable sense (but the distribution sense)
3. they are found to be strong type  $(p, p)$  for a range of values of  $p$  different from  $1 < p < 2$  or  $2 < p < \infty$  or their unions
4. duality does not apply

It is natural to ask the following question: **is there a general machinery to handle the  $L^p$  theory of such operators?**

The answer is yes. It turns out that the cases  $p < 2$  and  $p > 2$  are treated by different methods, which is useful when duality is not available.

Let us come now to statements. For simplicity, we work in the framework of  $\mathbb{R}^n$  equipped with the Lebesgue measure, although the original results are presented in spaces of homogeneous types.

We denote as above by  $B(x, r)$  the open (Euclidean) ball of radius  $r > 0$  and center  $x \in \mathbb{R}^n$ , and set  $|E|$  the measure of a set  $E$ .

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<sup>19</sup> This is due to Calderón & Zygmund [CZ] and Hörmander [Ho, Theorem 2.1] in the convolution case and is extended to non convolution operators in [CW].

<sup>20</sup> This is attributed to Peetre-Spanne-Stein (see [St2], p. 191)

<sup>21</sup> This result is due to Stampacchia [Sta].

Here is some further notation used throughout the paper. For a ball  $B$ , we let  $\lambda B$  be the ball with same center and radius  $\lambda$  times that of  $B$ . We set

$$C_1(B) = 4B \quad \text{and} \quad C_j(B) = 2^{j+1}B \setminus 2^j B, \text{ if } j \geq 2.$$

We adopt the similar notation,  $\lambda Q$  and  $C_j(Q)$ , for any cube  $Q$  with sides parallel to the axes.

Denote by  $M$  the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f|,$$

where  $B$  ranges over all open balls (or cubes) containing  $x$ .

**Theorem 2.1.** <sup>22</sup> *Let  $p_0 \in [1, 2)$ . Suppose that  $T$  is sublinear operator of strong type  $(2, 2)$ , and let  $A_r$ ,  $r > 0$ , be a family of linear operators acting on  $L^2$ . Assume for  $j \geq 2$*

$$(2.1) \quad \left( \frac{1}{|2^{j+1}B|} \int_{C_j(B)} |T(I - A_{r(B)})f|^2 \right)^{1/2} \leq g(j) \left( \frac{1}{|B|} \int_B |f|^{p_0} \right)^{1/p_0}$$

and for  $j \geq 1$

$$(2.2) \quad \left( \frac{1}{|2^{j+1}B|} \int_{C_j(B)} |A_{r(B)}f|^2 \right)^{1/2} \leq g(j) \left( \frac{1}{|B|} \int_B |f|^{p_0} \right)^{1/p_0}$$

for all ball  $B$  with  $r(B)$  the radius of  $B$  and all  $f$  supported in  $B$ . If  $\Sigma = \sum g(j)2^{nj} < \infty$ , then  $T$  is of weak type  $(p_0, p_0)$ , with a bound depending only on the strong type  $(2, 2)$  bound of  $T$ ,  $p_0$  and  $\Sigma$ , hence bounded on  $L^p$  for  $p_0 < p < 2$ .

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<sup>22</sup> This is due to Blunck & Kunstmann [BK1, Theorem 1.1] generalizing earlier work of Duong & McIntosh [DMc, Theorem 1] who obtained weak type  $(1, 1)$  under a weakened Hörmander condition, still assuming reasonable pointwise estimates for kernels but no regularity in the classical sense. The statement and proof here simplify that of [BK1]. Note that when  $p_0 = 1$  the assumptions are slightly different than those in [DMc]. One can find in Fefferman [Fef] weak type results for values of  $p$  not equal to one but no general statement is made.

**Theorem 2.2.** <sup>23</sup> Let  $p_0 \in (2, \infty]$ . Suppose that  $T$  is sublinear operator acting on  $L^2$ , and let  $A_r$ ,  $r > 0$ , be a family of linear operators acting on  $L^2$ . Assume

$$(2.3) \quad \left( \frac{1}{|B|} \int_B |T(I - A_{r(B)})f|^2 \right)^{1/2} \leq C(M(|f|^2))^{1/2}(y),$$

and

$$(2.4) \quad \left( \frac{1}{|B|} \int_B |TA_{r(B)}f|^{p_0} \right)^{1/p_0} \leq C(M(|Tf|^2))^{1/2}(y),$$

for all  $f \in L^2$ , all ball  $B$  and all  $y \in B$  where  $r(B)$  is the radius of  $B$ . If  $2 < p < p_0$  and  $Tf \in L^p$  when  $f \in L^p$  then  $T$  is strong type  $(p, p)$ . More precisely, for all  $f \in L^p \cap L^2$ ,

$$\|Tf\|_p \leq c\|f\|_p$$

where  $c$  depends only on  $n$ ,  $p$  and  $p_0$  and  $C$ .

*Remarks.* 1. The operators  $A_r$  play the role of approximate identities (as  $r \rightarrow 0$ ) eventhough  $A_r(1) = 1$  is not assumed. The boundedness of  $A_r$  on  $L^2$  is a consequence of linearity but the  $L^2$  bounds are not explicitly needed. In applications, the  $L^2$  bounds are uniform in  $r$  and used to check the hypotheses. The improvement in the exponents from  $p_0$  to 2 in (2.2) and from 2 to  $p_0$  in (2.4) expresses the regularizing effect of  $A_r$ . When  $p_0 = \infty$ , the left hand side of (2.4) is understood as the essential supremum on  $B$ .

2. Possible weakening of Theorem 2.1 is as follows: the exponent 2 in (2.1) can be changed to 1 and the exponent 2 in (2.2) can be changed to  $p > p_0$ .

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<sup>23</sup> This is due to the author, Coulhon, Duong & Hofmann [ACDH, Theorem 2.1] using ideas of Martell in [Ma] who developed a variant of the sharp function theory of Fefferman-Stein [FS] in the spirit of [DMc], again assuming there are reasonable pointwise estimates for kernels but no regularity in the classical sense. The proof here simplifies the exposition in [ACDH]. Shen independently proved a boundedness result similar in spirit by analogous methods [Sh, Theorem 3.1] which he attributes to Caffarelli and Peral [CP]. In fact, it is easy to recover Shen's theorem as a consequence of this one.

3. As we shall see, Theorem 2.2 has little to do with operators but rather with decomposition of functions in the spirit of Fefferman-Stein's argument for the sharp function and, in fact, it is an extension of it. This is why the regularised version  $TA_r$  of  $T$  is controlled by the maximal function of  $|Tf|^2$ , which may be surprising at first sight.
4. Define, for  $f \in L^2$ ,

$$\mathcal{M}_{T,A}^\# f(x) = \sup_{B \ni x} \left( \frac{1}{|B|} \int_B |T(I - A_{r(B)})f|^2 \right)^{1/2},$$

where the supremum is taken over all balls  $B$  in  $\mathbb{R}^n$  containing  $x$ , and  $r(B)$  is the radius of  $B$ . The assumption is that  $\mathcal{M}_{T,A}^\# f$  is controlled pointwise by  $(M(|f|^2))^{1/2}$ . In fact, rather than the exact form of the control what matters is that  $\mathcal{M}_{T,A}^\#$  is strong type  $(p, p)$  for the desired values of  $p$ .

5. The family  $(A_r)$  indexed by positive  $r$  could be replaced by a family  $(A_B)$  indexed by balls. Then  $A_{r(B)}$  is replaced by  $A_B$  in the statements. An example of such a family is given by mean values operators  $A_B f = \frac{1}{|B|} \int_B f$ .
6. Note that in Theorem 2.2,  $T$  acts on  $L^2$  but its boundedness is not needed in the proof. However, it is used in applications to check (2.4) and (2.3). Note also that  $T$  already acts on  $L^p$  and the purpose of the statement is to bound its norm. In concrete situations, this theorem is applied to suitable approximations of  $T$ , the uniformity of the bounds allowing a limiting argument to deduce  $L^p$  boundedness. So an argument to conclude for generic  $L^p$  functions is not needed here.
7. Both theorems are valid in the vector-valued context, that is when  $f$  is valued in a Banach space  $B_1$  and  $Tf$  is valued in another Banach space  $B_2$ . We leave to the reader the care of checking details. We apply this for square function estimates in Section 7.

**Proof of Theorem 2.1:** It begins with the classical Calderón-Zygmund decomposition which we recall. <sup>24</sup>

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<sup>24</sup> Many good references for this tool. One is [St1].



**Lemma 2.3.** *Let  $n \geq 1$ ,  $1 \leq p \leq \infty$  and  $\|f\|_p < \infty$ . Let  $\alpha > 0$ . Then, one can find a collection of cubes  $(Q_i)$ , functions  $g$  and  $b_i$  such that*

$$(2.5) \quad f = g + \sum_i b_i$$

and the following properties hold:

$$(2.6) \quad \|g\|_\infty \leq C\alpha,$$

$$(2.7) \quad \text{supp } b_i \subset Q_i \text{ and } \int_{Q_i} |b_i|^p \leq C\alpha^p |Q_i|,$$

$$(2.8) \quad \sum_i |Q_i| \leq C\alpha^{-p} \int_{\mathbb{R}^n} |f|^p,$$

$$(2.9) \quad \sum_i \mathbf{1}_{Q_i} \leq N,$$

where  $C$  and  $N$  depends only on dimension and  $p$ .

Let  $f \in L^{p_0} \cap L^2$ . We have to prove that for any  $\alpha > 0$ ,

$$\left| \left\{ x \in \mathbb{R}^n; |Tf(x)| > \alpha \right\} \right| \leq \frac{C}{\alpha^{p_0}} \int_{\mathbb{R}^n} |f|^{p_0}.$$

Write  $f = g + \sum_i b_i$  by the Calderón-Zygmund decomposition at height  $\alpha > 0$ . The construction of this decomposition implies that  $g \in L^2$  with  $\int |g|^2 \leq C\alpha^{2-p_0} \int |f|^{p_0}$ . Then  $Tg \in L^2$  with  $\|Tg\|_2 \leq C\|g\|_2$  by the assumption in Theorem 2.1. This with (2.6) yield

$$\left| \left\{ x \in \mathbb{R}^n; |Tg(x)| > \frac{\alpha}{3} \right\} \right| \leq \frac{C}{\alpha^2} \int_{\mathbb{R}^n} |g|^2 \leq \frac{C}{\alpha^{p_0}} \int_{\mathbb{R}^n} |f|^{p_0}.$$

To handle the remaining term, introduce for  $r \geq 0$  the operator  $B_r = I - A_r$  and let  $r_i$  be the radius of  $Q_i$ . Since

$$\left| T \left( \sum_i b_i \right) \right| \leq \sum_i |TB_{r_i} b_i| + \left| T \left( \sum_i A_{r_i} b_i \right) \right|$$

it is enough to estimate  $A = |\{x \in \mathbb{R}^n; \sum |TB_{r_i} b_i(x)| > \alpha/3\}|$  and  $B = |\{x \in \mathbb{R}^n; |T(\sum_i A_{r_i} b_i)(x)| > \alpha/3\}|$ . Let us bound the first term. First,

$$A \leq |\cup_i 4Q_i| + \left| \left\{ x \in \mathbb{R}^n \setminus \cup_i 4Q_i; \sum_i |TB_{r_i} b_i(x)| > \frac{\alpha}{3} \right\} \right|,$$

and by (2.8),  $|\cup_i 4Q_i| \leq \frac{C}{\alpha^{p_0}} \int |f|^{p_0}$ . To handle the other term, we observe that by Tchebychev inequality,

$$\left| \left\{ x \in \mathbb{R}^n \setminus \cup_i 4Q_i; \sum_i |TB_{r_i} b_i(x)| > \frac{\alpha}{3} \right\} \right| \leq \frac{C}{\alpha^2} \int \left| \sum_i h_i \right|^2$$

with  $h_i = \mathbf{1}_{(4Q_i)^c} |TB_{r_i} b_i|$ . To estimate the  $L^2$  norm, we dualize against  $u \in L^2$  with  $\|u\|_2 = 1$ . Write

$$\int |u| \sum_i h_i = \sum_i \sum_{j=2}^{\infty} A_{ij}$$

where

$$A_{ij} = \int_{C_j(Q_i)} |TB_{r_i} b_i| |u|.$$

By (2.1) and (2.7),

$$\begin{aligned} \|TB_{r_i} b_i\|_{L^2(C_j(Q_i))} &\leq |2^{j+1}Q_i|^{1/2} g(j) \left( \frac{1}{|Q_i|} \int_{Q_i} |b_i|^{p_0} \right)^{1/p_0} \\ &\leq C |2^{j+1}Q_i|^{1/2} g(j) \alpha \end{aligned}$$

for some appropriate constant  $C$ . Now remark that for any  $y \in Q_i$  and any  $j \geq 2$ ,

$$\left( \int_{C_j(Q_i)} |u|^2 \right)^{1/2} \leq \left( \int_{2^{j+1}Q_i} |u|^2 \right)^{1/2} \leq |2^{j+1}Q_i|^{1/2} (M(|u|^2)(y))^{1/2}.$$

Applying Hölder inequality, one obtains

$$A_{ij} \leq C \alpha 2^{nj} g(j) |Q_i| (M(|u|^2)(y))^{1/2}.$$

Averaging over  $Q_i$  yields

$$A_{ij} \leq C \alpha 2^{nj} g(j) \int_{Q_i} (M(|u|^2)(y))^{1/2} dy.$$

Summing over  $j \geq 2$  and  $i$ , we have

$$\begin{aligned} \int |u| \sum_i h_i &\leq C\alpha \int \sum_i \mathbf{1}_{Q_i}(y) (M(|u|^2)(y))^{1/2} dy \\ &\leq CN\alpha \int_{\cup_i Q_i} (M(|u|^2)(y))^{1/2} dy \\ &\leq C'N\alpha |\cup_i Q_i|^{1/2} \| |u|^2 \|_1^{1/2}. \end{aligned}$$

In the next to last inequality, we used (2.9), and in the last inequality, we used Kolmogorov's lemma and the weak type (1, 1) of the maximal function.<sup>25</sup> Hence from (2.8)

$$A \leq \frac{C''N}{\alpha^{p_0}} \int_{\mathbb{R}^n} |f|^{p_0}.$$

It remains to estimate the term  $B$ . To this end, we use that  $T$  is bounded on  $L^2$  to obtain

$$B \leq \frac{C}{\alpha^2} \int \left| T \left( \sum_i A_{r_i} b_i \right) \right|^2 \leq \frac{C}{\alpha^2} \int \left| \sum_i A_{r_i} b_i \right|^2.$$

To estimate the  $L^2$  norm, we dualize against  $u \in L^2$  with  $\|u\|_2 = 1$  and write

$$\int |u| \sum_i |A_{r_i} b_i| = \sum_i \sum_{j=1}^{\infty} B_{ij}$$

where

$$B_{ij} = \int_{C_j(Q_i)} |A_{r_i} b_i| |u|.$$

Using (2.2) and (2.7),

$$\begin{aligned} \|A_{r_i} b_i\|_{L^2(F_{ij})} &\leq |2^{j+1}Q_i|^{1/2} g(j) \left( \frac{1}{|Q_i|} \int_{Q_i} |b_i|^{p_0} \right)^{1/p_0} \\ &\leq C |2^{j+1}Q_i|^{1/2} g(j) \alpha \end{aligned}$$

for  $j \geq 1$ . From here, we may argue as before and conclude that  $B$  is bounded by  $\frac{C}{\alpha^{p_0}} \int |f|^{p_0}$  as desired.  $\square$

*Remark.* As the reader can see, there is absolutely no use of mean value properties of the  $b_i$ 's.

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<sup>25</sup> This idea is borrowed from Hofmann & Martell [HM].

**Proof of Theorem 2.2:** We begin with a useful localisation lemma.

**Lemma 2.4.** *There is  $K_0$  depending only on dimension such that the following holds. For every  $f \in L^1_{loc}$  and every cube  $Q$  and every  $\lambda > 0$  for which there exists  $\bar{x} \in 4Q$  for which  $Mf(\bar{x}) \leq \lambda$ , then for every  $K \geq K_0$ ,*

$$\{\chi_Q Mf > K\lambda\} \subset \{M(f\chi_{8Q}) > \frac{K}{K_0}\lambda\}$$

**Proof:** We use that  $M$  is comparable to the centered maximal function  $M_c$ : there is  $K_0$  depending only on the doubling constant such that  $M \leq K_0 M_c$ .

Let  $x \in Q$  with  $Mf(x) > K\lambda$ . Then  $M_c f(x) > \frac{K}{K_0}\lambda$ . Hence, there is a cube centered at  $x$  with radius  $r$  such that

$$\frac{1}{|Q(x, r)|} \int_{Q(x, r)} |f| > \frac{K}{K_0}\lambda.$$

If  $\frac{K}{K_0} > 1$ ,  $\bar{x} \notin Q(x, r)$  since  $Mf(\bar{x}) \leq \lambda$ . The conditions  $x \in Q$ ,  $\bar{x} \in 4Q$  and  $\bar{x} \notin Q(x, r)$  imply  $Q(x, r) \subset 8Q$ . Hence,

$$\frac{K}{K_0}\lambda < \frac{1}{|Q(x, r)|} \int_{Q(x, r)} |f\chi_{8Q}| \leq M(f\chi_{8Q})(x).$$

□

We continue with a two parameters family of good lambda inequalities.

**Proposition 2.5.** *Fix  $1 < q \leq \infty$  and  $a > 1$ . Then, there exist  $C = C(q, n, a)$  and  $K'_0 = K'_0(n, a)$  with the following property: If  $F, G$  are non-negative measurable functions such that for every cube  $Q$  there exist non-negative functions  $G_Q, H_Q$  with*

$$\begin{aligned} F &\leq G_Q + H_Q \quad \text{a.e. on } Q, \\ \left( \frac{1}{|Q|} \int_Q H_Q^q \right)^{1/q} &\leq aMF(x) \quad \text{for all } x \in Q \\ \frac{1}{|Q|} \int_Q G_Q &\leq G(x) \quad \text{for all } x \in Q. \end{aligned}$$

Then for all  $\lambda > 0$ , for all  $K > K'_0$  and  $\gamma < 1$ ,

$$|\{MF > K\lambda, G \leq \gamma\lambda\}| \leq C \left( \frac{1}{K^q} + \frac{\gamma}{K} \right) |\{MF > \lambda\}|.$$

If  $q = \infty$ , we understand the average in  $L^q$  as an essential supremum. In this case,  $\frac{1}{K^q} = 0$ .

**Proof:** Let  $E_\lambda = \{MF > \lambda\}$ . We assume this is a proper subset in  $\mathbb{R}^n$  otherwise there is nothing to prove. Since  $E_\lambda$  is open, the Whitney decomposition yields a family of non overlapping dyadic cubes  $Q_i$  such that  $E_\lambda = \cup_i Q_i$  and  $4Q_i$  contains at least one point  $\bar{x}_i$  outside  $E_\lambda$ , that is

$$MF(\bar{x}_i) \leq \lambda.$$

Let  $B_\lambda = \{MF > K\lambda, G \leq \gamma\lambda\}$ . If  $K \geq 1$  then  $B_\lambda \subset E_\lambda$ , hence

$$|B_\lambda| \leq \sum_i |B_\lambda \cap Q_i|.$$

Fix  $i$ . If  $B_\lambda \cap Q_i = \emptyset$ , we have nothing to do. If not, there is a point  $\bar{y}_i \in Q_i$  such that

$$G(\bar{y}_i) \leq \gamma\lambda.$$

By the localisation lemma applied to  $F$  on  $Q_i$ , if  $K \geq K_0$ , then

$$|B_\lambda \cap Q_i| \leq |\{MF > K\lambda\} \cap Q_i| \leq |\{M(F\chi_{8Q_i}) > \frac{K}{K_0}\lambda\}|.$$

Now use  $F \leq G_i + H_i$  with  $G_i = G_{8Q_i}$  and  $H_i = H_{8Q_i}$  to deduce

$$|\{M(F\chi_{8Q_i}) > \frac{K}{K_0}\lambda\}| \leq |\{M(G_i\chi_{8Q_i}) > \frac{K}{2K_0}\lambda\}| + |\{M(H_i\chi_{8Q_i}) > \frac{K}{2K_0}\lambda\}|.$$

Now by using the weak type  $(1, 1)$  and  $(q, q)$  of the maximal operator with respective constant  $c_1$  and  $c_q$ , we have

$$\begin{aligned} |\{M(G_i\chi_{8Q_i}) > \frac{K}{2K_0}\lambda\}| &\leq \frac{2K_0c_1}{K\lambda} \int_{8Q_i} G_i \\ &\leq \frac{2K_0c_1}{K\lambda} |8Q_i| G(\bar{y}_i) \\ &\leq \frac{2K_0c_1}{K} |8Q_i| \gamma, \end{aligned}$$

and, if  $q < \infty$

$$\begin{aligned} |\{M(H_i\chi_{8Q_i}) > \frac{K}{2K_0}\lambda\}| &\leq \left(\frac{2K_0c_q}{K\lambda}\right)^q \int_{8Q_i} H_i^q \\ &\leq \left(\frac{2K_0c_q}{K\lambda}\right)^q |8Q_i| (aMF(\bar{x}_i))^q \\ &\leq \left(\frac{2K_0c_q}{K}\right)^q |8Q_i| a^q. \end{aligned}$$

Hence, summing over  $i$  yields

$$|B_\lambda| \leq \sum_i C \left( \frac{a^q}{K^q} + \frac{\gamma}{K} \right) |8Q_i| \leq C8^n \left( \frac{a^q}{K^q} + \frac{\gamma}{K} \right) |E_\lambda|.$$

If  $q = \infty$ , then

$$\|M(H_i \chi_{8Q_i})\|_\infty \leq \|H_i \chi_{8Q_i}\|_\infty \leq aMF(\bar{x}_i) \leq a\lambda,$$

so that, choosing  $K \geq 2K_0a$  leads us to  $\{M(H_i \chi_{8Q_i}) > \frac{K}{2K_0}\lambda\} = \emptyset$ .  $\square$

Let us now assume that  $F, G$  are so that the conclusion of the proposition holds. Let  $0 < p < q$ . Then, we have

$$\|MF\|_p^p \leq CK^p \left( \frac{a^q}{K^q} + \frac{\gamma}{K} \right) \|MF\|_p^p + \frac{K^p}{\gamma^p} \|G\|_p^p.$$

Hence, if furthermore  $\|MF\|_p < \infty$ , since  $p < q$ , one can choose  $K$  large enough and  $\gamma$  small enough so that

$$CK^p \left( \frac{a^q}{K^q} + \frac{\gamma}{K} \right) \leq 1 - 2^{-p}.$$

This choice depends therefore on  $p, q, n, a$  and we have

$$\|MF\|_p^p \leq \frac{(2K)^p}{\gamma^p} \|G\|_p^p.$$

Let us now prove Theorem 2.2. Let  $f \in L^2 \cap L^p$ . We let  $q = \frac{p_0}{2}$  and set  $F = |Tf|^2 \in L^{p/2}$ . By sublinearity of  $T$ , we have for any cube  $Q$  that  $F \leq G_Q + H_Q$  with  $G_Q = 2|T(I - A_{r(Q)})f|^2$  and  $H_Q = 2|TA_{r(Q)}f|^2$ . Hence the hypotheses of the proposition apply with  $a = 2C^2$  and  $G = 2C^2M(f^2)$ . Let  $2 < p < p_0$ . Since we know that  $MF \in L^{p/2}$  from the hypothesis, we obtain

$$\|Tf\|_p^2 \leq \|MF\|_{p/2} \leq C\|G\|_{p/2} \leq C'\|f\|_p^2.$$

In the last inequality, we have used the assumption on  $T(I - A_r)$ .  $\square$

*Remark.* With  $q = \infty$  and given  $F$ , we let  $G_Q = |F - m_Q F|$  and  $H_Q = |m_Q F|$ ,  $a = 1$  and  $G = M^\# F$ , the sharp function of Fefferman-Stein. We obtain if  $0 < p < \infty$  and  $\|MF\|_p < \infty$  that

$$\|MF\|_p \leq C_p \|M^\# F\|_p,$$

hence this argument contains in particular Fefferman-Stein's. To recover the sharp function introduced by Martell, we take  $G_Q = |F - A_{r(Q)}F|$  and  $H_Q = |A_{r(Q)}F|$  and, as proved Martell,  $\sup_{Q \ni x} H_Q \leq aMF(x)$  under kernel upper bounds on  $A_r$ .

### 3 Basic $L^2$ theory for elliptic operators

#### 3.1 Definition

Let  $A = A(x)$  be an  $n \times n$  matrix of complex,  $L^\infty$  coefficients, defined on  $\mathbb{R}^n$ , and satisfying the ellipticity (or ‘‘accretivity’’) condition

$$(3.1) \quad \lambda|\xi|^2 \leq \operatorname{Re} A\xi \cdot \bar{\xi} \text{ and } |A\xi \cdot \bar{\zeta}| \leq \Lambda|\xi||\zeta|,$$

for  $\xi, \zeta \in \mathbb{C}^n$  and for some  $\lambda, \Lambda$  such that  $0 < \lambda \leq \Lambda < \infty$ . We define a second order divergence form operator

$$(3.2) \quad Lf \equiv -\operatorname{div}(A\nabla f),$$

which we first interpret in the sense of maximal accretive operators via a sesquilinear form. That is,  $\mathcal{D}(L)$  is the largest subspace contained in  $W^{1,2}$  for which

$$\left| \int_{\mathbb{R}^n} A\nabla f \cdot \nabla g \right| \leq C\|g\|_2$$

for all  $g \in W^{1,2}$  and we set  $Lf$  by

$$\langle Lf, g \rangle = \int_{\mathbb{R}^n} A\nabla f \cdot \bar{\nabla} g$$

for  $f \in \mathcal{D}(L)$  and  $g \in W^{1,2}$ . Thus defined,  $L$  is maximal-accretive operator on  $L^2$  and  $\mathcal{D}(L)$  is dense in  $W^{1,2}$ .<sup>26</sup>

Since  $W^{1,2}$  is dense in its homogeneous version  $\dot{W}^{1,2}$  (for the semi-norm  $\|\nabla f\|_2$ ),  $L$  extends to a bounded operator invertible operator from  $\dot{W}^{1,2}$  into its dual space  $\dot{W}^{-1,2}$ , which justifies the divergence notation in (3.2). In particular, one has

$$(3.3) \quad \|\nabla L^{-1} \operatorname{div} f\|_2 \leq c\|f\|_2.$$

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<sup>26</sup> For precise definitions, details and proofs, see Kato's book [K2, Chapter VI].

## 3.2 Holomorphic functional calculus on $L^2$

Let  $L$  be as above. There exists  $\omega \in [0, \frac{\pi}{2})$  depending only on the ellipticity constants such that for all  $f \in \mathcal{D}(L)$ ,

$$(3.4) \quad |\arg\langle Lf, f \rangle| \leq \omega.$$

We fix the smallest such  $\omega$ . and the following  $L^2$  estimate is easily proved: for all  $\mu \in (\omega, \pi)$  and all complex numbers  $\lambda \in \Sigma_{\pi-\mu}$ ,

$$\|(L + \lambda)^{-1}f\|_2 \leq \frac{c\|f\|_2}{|\lambda|},$$

where we have set  $\Sigma_\mu = \{z \in \mathbb{C}^*; |\arg z| < \mu\}$ .<sup>27</sup> Hence,  $L$  is of type  $\omega$  on  $L^2$ . Several consequences follow.

In particular,  $-L$  generates a semigroup  $(e^{-tL})_{t>0}$  which has an analytic extension to a complex semigroup  $(e^{-zL})_{z \in \Sigma_{\frac{\pi}{2}-\omega}}$  of contractions on  $L^2$ .

Since  $L$  is also maximal-accretive operator, it has a bounded holomorphic functional calculus on  $L^2$ . In particular, for any  $\mu \in (\omega, \pi)$  and any  $\varphi$  holomorphic and bounded in  $\Sigma_\mu$ , the operator  $\varphi(L)$  is bounded on  $L^2$  with the estimate

$$\|\varphi(L)f\|_2 \leq c\|\varphi\|_\infty\|f\|_2,$$

the constant  $c$  depending only on  $\omega$  and  $\mu$ . If  $\varphi$  satisfies the technical condition

$$(3.5) \quad |\varphi(\zeta)| \leq c|\zeta|^s(1 + |\zeta|)^{-2s}$$

for all  $\zeta \in \Sigma_\mu$  for some positive constants  $c, s$ , then  $\varphi(L)$  can be computed using the semigroup. Let  $\omega < \theta < \nu < \mu < \frac{\pi}{2}$ . One has

$$(3.6) \quad \varphi(L) = \int_{\Gamma_+} e^{-zL}\eta_+(z) dz + \int_{\Gamma_-} e^{-zL}\eta_-(z) dz$$

where  $\Gamma_\pm$  is the half-ray  $\mathbb{R}^+e^{\pm i(\frac{\pi}{2}-\theta)}$ ,

$$(3.7) \quad \eta_\pm(z) = \frac{1}{2\pi i} \int_{\gamma_\pm} e^{\zeta z} \varphi(\zeta) d\zeta, \quad z \in \Gamma_\pm,$$

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<sup>27</sup> See, for example, [AT, Preliminary Chapter] for a proof.



with  $\gamma_{\pm}$  being the half-ray  $\mathbb{R}^+ e^{\pm i\nu}$  (the orientation on paths is irrelevant in the arguments where this representation is used so we do not insist on that). For general bounded holomorphic functions,  $\varphi(L)$  is defined by a limiting procedure which we do not need in this work.

Finally, one can define unbounded operators  $\varphi(L)$  for  $\varphi$  holomorphic in  $\Sigma_{\mu}$ ,  $\omega < \mu < \pi$ , satisfying

$$(3.8) \quad |\varphi(\zeta)| \leq c \sup(|\zeta|^s, |\zeta|^{-s'})$$

for some  $c, s, s' \geq 0$ . This includes fractional powers of  $L$ . We call  $\mathcal{F}(\Sigma_{\mu})$  the class of such holomorphic functions.<sup>28</sup>

### 3.3 $L^2$ off-diagonal estimates

A very important ingredient in this paper is the off-diagonal estimates of Gaffney type. They are crucial to our analysis because when dealing with complex operators, we do not have at our disposal contractivity of the semi-group on all  $L^p$  spaces, and in fact this is false in general.

*Definition.* Let  $\mathcal{T} = (T_t)_{t>0}$  be a family of operators. We say that  $\mathcal{T}$  satisfies  $L^2$  off-diagonal estimates if for some constants  $C \geq 0$  and  $\alpha > 0$  for all closed sets  $E$  and  $F$ , all  $h \in L^2$  with support in  $E$  and all  $t > 0$  we have

$$(3.9) \quad \|T_t h\|_{L^2(F)} \leq C e^{-\frac{cd(E,F)^2}{t}} \|h\|_2.$$

Here, and subsequently,  $d(E, F)$  denotes the semi-distance induced on sets by the Euclidean distance.

In case  $\mathcal{T} = (T_t)_{t>0}$  is replaced by a family  $\mathcal{T} = (T_z)_{z \in \Sigma_{\mu}}$  defined on a complex sector  $\Sigma_{\mu}$  with  $0 \leq \mu < \frac{\pi}{2}$ , then we adopt the same definition and replacing  $t$  by  $|z|$  in the right hand side of the inequality. In this case, the constants  $C$  and  $\alpha$  may depend on the angle  $\mu$ .

**Proposition 3.1.** *For all  $\mu \in (0, \frac{\pi}{2} - \omega)$ , the families  $(e^{-zL})_{z \in \Sigma_{\mu}}$ ,  $(zLe^{-zL})_{z \in \Sigma_{\mu}}$  and  $(\sqrt{z} \nabla e^{-zL})_{z \in \Sigma_{\mu}}$  satisfies  $L^2$  off-diagonal estimates.*

*Proof.* We begin with the case of real times  $t > 0$ . Let  $\varphi$  be a Lipschitz function on  $\mathbb{R}^n$  with Lipschitz norm 1 and  $\rho > 0$ . We may define  $L_{\rho} = e^{\rho\varphi} L e^{-\rho\varphi}$  by the form method. This operator is of second order type with

<sup>28</sup> For definitions, we refer to [Mc] and [CDMcY].

same principal term as  $L$  and lower order terms with bounded coefficients. More precisely, let  $Q_\rho$  be the associated form. Then it is bounded on  $W^{1,2}$  and one can find  $c$  depending only on dimension and the ellipticity constants of  $L$  (not on  $\varphi$ ) such that

$$\Re Q_\rho(f) \geq \frac{\lambda}{2} \|\nabla f\|_2^2 - c\rho^2 \|f\|_2^2, \quad f \in W^{1,2}.$$

The construction guarantees that  $L_\rho + c\rho^2$  is maximal-accretive on  $L^2$ . Hence, the semigroup  $(e^{-tL_\rho})_{t>0}$  exists on  $L^2$  and its analyticity gives us:

$$\|e^{-tL_\rho} f\|_2 + \left\| t \frac{d}{dt} e^{-tL_\rho} f \right\|_2 + \|\sqrt{t} \nabla e^{-tL_\rho} f\|_2 \leq C e^{c\rho^2 t} \|f\|_2$$

for all  $t > 0$  where  $C$  depends on the ellipticity constants of  $L$  and dimension only. Let  $E$  and  $F$  be two closed sets and  $f \in L^2$ , with compact support contained in  $E$ . Choose  $\varphi(x) = d(x, E)$ . Then,

$$(3.10) \quad e^{-tL} f = e^{-\rho\varphi} e^{-tL_\rho} f.$$

Hence, for all  $t > 0$ , all  $\rho > 0$

$$\|e^{-tL} f\|_{L^2(F)} \leq C e^{-\rho d(E,F)} e^{c\rho^2 t} \|f\|_2.$$

Optimizing with respect to  $\rho > 0$  yields

$$\|e^{-tL} f\|_{L^2(F)} \leq C e^{-\frac{d(E,F)^2}{4ct}} \|f\|_2.$$

Next, differentiating (3.10), one has

$$\frac{d}{dt} e^{-tL} f = e^{-\rho\varphi} \frac{d}{dt} e^{-tL_\rho} f$$

and the same argument applies. Eventually, applying the gradient operator to (3.10) yields

$$\nabla e^{-tL} f = -\rho(e^{-\rho\varphi} e^{-tL_\rho} f)(\nabla\varphi) + e^{-\rho\varphi} \nabla e^{-tL_\rho} f.$$

Hence,

$$\|\nabla e^{-tL} f\|_{L^2(F)} \leq C\rho e^{-\frac{d(E,F)^2}{4ct}} \|f\|_2 + Ct^{-1/2} e^{-\rho d(E,F)} e^{c\rho^2 t} \|f\|_2$$

and choosing  $\rho = \frac{d(E,F)}{2ct}$  yields

$$\|\nabla e^{-tL} f\|_{L^2(F)} \leq \frac{C}{\sqrt{t}} \left(1 + \frac{d(E,F)}{\sqrt{t}}\right) e^{-\frac{d(E,F)^2}{4ct}} \|f\|_2.$$

A density argument (since  $f$  was supposed with compact support in  $E$ ) concludes the proof for real times. To go to complex times, we notice that this applies to  $e^{i\alpha}L$ , which is an operator in the same class as  $L$  (with coefficients  $e^{i\alpha}A(x)$ ) as long as  $|\alpha| < \frac{\pi}{2} - \omega$ . Hence, the above estimates apply and the constants remain uniform when  $\alpha$  is contained in a compact subset of  $(-\frac{\pi}{2} + \omega, \frac{\pi}{2} - \omega)$ . Finally, the desired estimates follow easily from the observation that  $e^{-zL} = e^{-t(e^{i\alpha}L)}$  when  $z = te^{i\alpha}$ .  $\square$

### 3.4 Square root

As  $L$  is a maximal-accretive operator, it has a square root, which we denote by  $L^{1/2}$ , defined as the unique maximal-accretive operator such that

$$L^{1/2}L^{1/2} = L$$

as unbounded operators.<sup>29</sup> Many formulas can be used to compute  $L^{1/2}$ . The one we are going to use is

$$L^{1/2}f = \pi^{-1/2} \int_0^\infty e^{-tL} Lf \frac{dt}{\sqrt{t}}.$$

This equality is valid as a Bochner integral when  $f \in \mathcal{D}(L)$  and as the limit in  $L^2$  of the truncated Bochner integrals  $\int_\varepsilon^R \dots$  as  $\varepsilon \downarrow 0$  and  $R \uparrow \infty$  when  $f \in \mathcal{D}(L^{1/2})$ . Also, this construction implies that  $\mathcal{D}(L)$  is dense in  $\mathcal{D}(L^{1/2})$ .

The determination of the domain of the square root of  $L$  has become known as the Kato square root problem and it is now a theorem in all dimensions, as recalled in the Introduction, that  $\mathcal{D}(L^{1/2}) = W^{1,2}$  with

$$(3.11) \quad \|L^{1/2}f\|_2 \sim \|\nabla f\|_2$$

for all  $f \in \mathcal{D}(L)$ , hence by density in  $\dot{W}^{1,2}$ . In particular,  $L^{1/2}$  extends to an isomorphism from  $\dot{W}^{1,2}$  to  $L^2$  and the formula

$$g = L^{1/2}L^{-1/2}g$$

---

<sup>29</sup> For an explicit construction, we refer the reader to Kato's book [K2, p. 281] or to Meyer and Coifman's book [MeCo, Chapter XIV].

extends to all  $L^2$  functions  $g$ .

This implies the following representation formula

**Lemma 3.2.** *If  $f, h \in \dot{W}^{1,2}$  then*

$$\langle (L^*)^{1/2} f, L^{1/2} h \rangle = \int_{\mathbb{R}^n} \nabla f \cdot \overline{A \nabla h}.$$

*Proof.* Since  $f, h \in \dot{W}^{1,2}$ ,  $(L^*)^{1/2} f, L^{1/2} h \in L^2$ . Hence both sides of the equality are well-defined. It suffices to obtain the equality if, in addition,  $h \in \mathcal{D}(L)$ , as  $\mathcal{D}(L)$  is dense in  $\dot{W}^{1,2}$ .

In this case,  $L^{1/2} h$  belongs to the domain of  $L^{1/2}$ , so  $\langle (L^*)^{1/2} f, L^{1/2} h \rangle = \langle f, Lh \rangle$  and the latter is equal to  $\int_{\mathbb{R}^n} \nabla f \cdot \overline{A \nabla h}$  by construction of  $L$ .  $\square$

### 3.5 The conservation property

For real operators, the semigroup is contracting on  $L^\infty$  and the conservation property  $e^{-tL}1 = 1$  is a classical consequence of the probabilistic interpretation of the semigroup or of the maximum principle for parabolic equations. But for complex operators, the semigroup may not act from  $L^\infty$  into  $L^\infty$  (see the section on  $L^p$  theory for the semigroup). Yet, thanks to  $L^2$  off-diagonal estimates, the action of the semigroup on  $L^\infty$  can be defined in the  $L^2_{loc}$  sense and the conservation property still holds in this sense:<sup>30</sup>

$$(3.12) \quad e^{-tL}1 = 1, \quad t > 0.$$

The first step is to show that  $e^{-tL}$  maps  $L^2$  functions with compact supports in  $L^1$ . Fix  $t > 0$  and  $\phi \in L^2$  supported in a cube  $Q$ . Cover  $\mathbb{R}^n$  with a family of nonoverlapping cubes  $(Q_k)$  of constant size with  $Q_0 \supset 2Q$ . Using the  $L^2$  off-diagonal estimates,

$$(3.13) \quad \int_{Q_k} |e^{-tL}\phi| \leq |Q_k|^{1/2} e^{-\frac{cd(Q_k, Q)^2}{t}} \|\phi\|_2$$

so that summing in  $k$  gives us the result.

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<sup>30</sup> This is proved under  $L^1 - L^\infty$  off diagonal estimates of the semigroup in [AT] (Chapter I, Proposition 25) and is a consequence of Corollary 4.6 in [ABBO] under the weaker  $L^\infty$ -boundedness. It is mentioned in all generality but without proof in [AHLMcT], p. 638.

Applying the first step to  $L^*$  means that one can define  $e^{-tL}1$  in  $L^2_{loc}$  by

$$\int_{\mathbb{R}^n} e^{-tL}1 \bar{\phi} = \int_{\mathbb{R}^n} \overline{e^{-tL^*}\phi}$$

for all  $\phi$  in  $L^2$  with compact support.

Next, let  $\mathcal{X}$  be a smooth function with  $\mathcal{X}(x) = 1$  if  $|x| \leq 1$  and  $\mathcal{X}(x) = 0$  if  $|x| \geq 2$ . Let  $\mathcal{X}_R(x) = \mathcal{X}(x/R)$  for  $R > 0$ . If  $\phi$  is an  $L^2$  compactly supported function, then for  $R > 0$  and  $t > 0$

$$(3.14) \quad \int_{\mathbb{R}^n} e^{-tL}1 \bar{\phi} = \int_{\mathbb{R}^n} \mathcal{X}_R \overline{e^{-tL^*}\phi} + \int_{\mathbb{R}^n} (1 - \mathcal{X}_R) \overline{e^{-tL^*}\phi}.$$

We use this representation twice, first to show that the left hand side does not depend on  $t > 0$  and, second, to find  $\int_{\mathbb{R}^n} \bar{\phi}$  as its value. This, indeed, shows that  $e^{-tL}1 = 1$  in the sense of  $L^2_{loc}$ .

Let us begin with differentiating (3.14) with respect to  $t$ . Indeed, the first step applies also to  $\frac{d}{dt}e^{-tL^*}$  by the  $L^2$  off-diagonal estimates and this allows us to use the Lebesgue differentiation theorem to see that

$$\frac{d}{dt} \int_{\mathbb{R}^n} e^{-tL}1 \bar{\phi} = \int_{\mathbb{R}^n} \mathcal{X}_R \overline{\frac{d}{dt}e^{-tL^*}\phi} + \int_{\mathbb{R}^n} (1 - \mathcal{X}_R) \overline{\frac{d}{dt}e^{-tL^*}\phi}.$$

Fix  $t > 0$  and let  $R \rightarrow \infty$ . By Lebesgue dominated convergence, the latter integral tends to 0. Now, since  $\nabla \mathcal{X}_R \in L^2$  and  $\frac{d}{dt}e^{-tL^*}\phi \in \mathcal{D}(L^*)$ , we have that

$$\int_{\mathbb{R}^n} \mathcal{X}_R \overline{\frac{d}{dt}e^{-tL^*}\phi} = \int_{\mathbb{R}^n} A \nabla \mathcal{X}_R \cdot \overline{\nabla e^{-tL^*}\phi}.$$

Again, using the  $L^2$  off-diagonal estimates for  $\nabla e^{-tL^*}$  and arguing as for (3.13), this integral is bounded by  $CR^{n/2-1}e^{-cR^2/t}\|\phi\|_2$  for  $R$  large enough so that it tends to 0. This shows that the left hand side of (3.14) is independent of  $t > 0$ . In the right hand side, choose and fix  $R$  large enough so that the supports of  $\phi$  and  $(1 - \mathcal{X}_R)$  are far apart. It follows from (3.13) that  $\int_{\mathbb{R}^n} (1 - \mathcal{X}_R) \overline{e^{-tL^*}\phi}$  tends to 0 with  $t$  by dominated convergence. Eventually, since  $e^{-tL^*}$  is a strongly continuous in  $L^2$  at  $t = 0$ , we obtain that  $\int_{\mathbb{R}^n} \mathcal{X}_R \overline{e^{-tL^*}\phi}$  tends to  $\int_{\mathbb{R}^n} \mathcal{X}_R \bar{\phi} = \int_{\mathbb{R}^n} \bar{\phi}$  as  $t$  tends to 0. This proves (3.12).

## 4 $L^p$ theory for the semigroup

This section is devoted to establishing the basic properties concerning uniform boundedness of the semigroup  $(e^{-tL})_{t>0}$  and of the family  $(\sqrt{t}\nabla e^{-tL})_{t>0}$  on  $L^p$  spaces.

### 4.1 Hypercontractivity and uniform boundedness

The point of this section is to present a general statement allowing to pass from hypercontractivity properties for the semigroup to uniform boundedness properties. The bridge between both are off-diagonal estimates.<sup>31</sup>

We introduce a few definitions.

*Definition.* Let  $\mathcal{T} = (T_t)_{t>0}$  be a family of uniformly bounded operators on  $L^2$ . We say that  $\mathcal{T}$  is  $L^p - L^q$  bounded for some  $p, q \in [1, \infty]$  with  $p \leq q$  if for some constant  $C$ , for all  $t > 0$  and all  $h \in L^p \cap L^2$

$$(4.1) \quad \|T_t h\|_q \leq C t^{-\gamma_{pq}/2} \|h\|_p.$$

We say that  $\mathcal{T}$  satisfies  $L^p - L^q$  off-diagonal estimates for some  $p, q \in [1, \infty]$  with  $p \leq q$  if for some constants  $C, c > 0$ , for all closed sets  $E$  and  $F$ , all  $h \in L^p \cap L^2$  with support in  $E$  and all  $t > 0$  we have

$$(4.2) \quad \|T_t h\|_{L^q(F)} \leq C t^{-\gamma_{pq}/2} e^{-\frac{cd(E,F)^2}{t}} \|h\|_p.$$

Recall that the numbers  $\gamma_{pq}$  are defined in the notation section. Such estimates depend on dimension and on the “parabolic” character of the family through this number. If  $p = q$  we speak of  $L^p$  boundedness and  $L^p$  off-diagonal estimates. Remark that these notions dualize. A family  $(T_t)_{t>0}$  is  $L^p - L^q$  bounded (resp. satisfies  $L^p - L^q$  off-diagonal estimates) if and only if the dual family  $((T_t)^*)_{t>0}$  is  $L^{q'} - L^{p'}$  bounded (resp. satisfies  $L^{q'} - L^{p'}$  off-diagonal estimates).

Let us also state a useful result whose easy proof is skipped.

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<sup>31</sup> This tool was introduced for that purpose by Davies [Da]. See [LSV] for further historical comments. This notion turns out to be equivalent to some  $L^p - L^q$  boundedness for perturbed semigroups by exponential weights (See [LSV]). We do not need to go into this here.

**Proposition 4.1.** *If  $(T_t)_{t>0}$  satisfies  $L^p - L^q$  boundedness (resp. off-diagonal estimates) and  $(S_t)_{t>0}$  satisfies  $L^q - L^r$  boundedness (resp. off-diagonal estimates) then  $(S_t T_t)_{t>0}$  satisfies  $L^p - L^r$  boundedness (resp. off-diagonal estimates).*

For a semigroup, the terminology **hypercontractivity** is often used for  $L^p - L^q$  boundedness for some  $p < q$ . The relation between hypercontractivity, boundedness and off-diagonal estimates is the following result.

**Proposition 4.2.** *Let  $p \in [1, 2)$  and  $n \geq 1$ . Let  $\mathcal{S} = (e^{-tL})_{t>0}$ .*

1. *If  $\mathcal{S}$  is  $L^p$  bounded then it is  $L^p - L^2$  bounded.*
2. *If  $\mathcal{S}$  is  $L^p - L^2$  bounded, then for all  $q \in (p, 2)$  it satisfies  $L^q - L^2$  off-diagonal estimates.*
3. *If  $\mathcal{S}$  satisfies  $L^p - L^2$  off-diagonal estimates then it is  $L^p$  bounded.*

*Remark.* The result applies when  $2 < p \leq \infty$  by duality: replace  $L^p - L^2$  by  $L^2 - L^p$  everywhere. We have privileged the central role of  $L^2$  for reasons of simplicity and usefulness.  $L^2$  could be replaced by  $L^q$  for  $q$  larger than 2 if necessary. It occurs in Section 5.5.

*Proof.* The proof of item 1 is obtained from Nash type inequalities.<sup>32</sup> We start from the Gagliardo-Nirenberg inequality

$$\|f\|_2^2 \leq C \|\nabla f\|_2^{2\alpha} \|f\|_p^{2\beta}$$

with

$$\alpha + \beta = 1 \quad \text{and} \quad (1 + \gamma_p)\alpha = \gamma_p.$$

This yields the Nash inequality

$$\|e^{-tL} f\|_2^2 \leq C \|\nabla e^{-tL} f\|_2^{2\alpha} \|e^{-tL} f\|_p^{2\beta}$$

for all  $t > 0$  and  $f \in L^2 \cap L^p$ . By ellipticity, one has

$$\|\nabla e^{-tL} f\|_2^2 \leq \lambda^{-1} \Re \int_{\mathbb{R}^n} A \nabla e^{-tL} f \cdot \overline{\nabla e^{-tL} f} = -(2\lambda)^{-1} \frac{d}{dt} \|e^{-tL} f\|_2^2.$$

---

<sup>32</sup> The proof follows that of [VSC, Theorem II.3.2] done for  $p = 1$ .

Assume  $f \in L^2 \cap L^p$  with  $\|f\|_p = 1$ . Using  $L^p$  boundedness of the semigroup in the Nash inequality, one obtains the differential inequality

$$\varphi(t)^{1/\alpha} \leq -C\varphi'(t),$$

where  $\varphi(t) = \|e^{-tL}f\|_2^2$ . Integrating between  $t$  and  $2t$  and using that  $\varphi$  is nonincreasing, one finds easily that

$$\varphi(t) \leq Ct^{-\frac{\alpha}{\alpha-1}} = Ct^{-\gamma_p},$$

which is the desired estimate.

The proof of item 2 consists in interpolating by the Riesz-Thorin theorem the  $L^p - L^2$  boundedness assumption with the  $L^2$  off-diagonal estimates, once we fix the sets  $E$  and  $F$  in the definition of the off-diagonal estimates.

The proof of item 3 can be seen by invoking the following simple lemma which has nothing to do with semigroups.

**Lemma 4.3.** <sup>33</sup> *If  $1 \leq p \leq q \leq \infty$  and  $T$  is a linear operator which satisfies  $L^p - L^q$  off-diagonal estimates in the form  $\|Tf\|_{L^q(F)} \leq g(d(E, F))\|f\|_{L^p(E)}$  whenever  $E, F$  are closed cubes and  $f$  is supported in  $E$  and  $g$  is some function. Then  $T$  is bounded on  $L^p$  with norm bounded by  $s^{\gamma_{pq}} \sum_{k \in \mathbb{Z}^n} g(\sup(|k| - 1, 0)s)$  for any  $s > 0$  provided this sum is finite.*

*Proof.* We may assume  $p \leq q < \infty$ . Let  $(Q_k)_{k \in \mathbb{Z}^n}$  be a partition of  $\mathbb{R}^n$  by cubes having sidelength  $s$  and  $\mathcal{X}_\ell$  be the indicator function of  $Q_\ell$ . Then

$$\begin{aligned} \|Tf\|_q^q &= \sum_k \left\| \sum_\ell T(f\mathcal{X}_\ell) \right\|_{L^q(Q_k)}^q \\ &\leq \sum_k \left( \sum_\ell g(\sup(|k - \ell| - 1, 0)s) \|f\|_{L^p(Q_\ell)} \right)^q \\ &\leq \sum_k \left( \sum_\ell g(\sup(|k - \ell| - 1, 0)s) s^{\gamma_{pq}} \|f\|_{L^q(Q_\ell)} \right)^q \\ &\leq \left( \sum_k g(\sup(|k - \ell| - 1, 0)s) s^{\gamma_{pq}} \right)^q \sum_\ell \|f\|_{L^q(Q_\ell)}^q \\ &= \left( \sum_k g(\sup(|k| - 1, 0)s) s^{\gamma_{pq}} \right)^q \|f\|_q^q \end{aligned}$$

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<sup>33</sup> This result is implicit in the proof of [Da, Theorem 25].



where we used that the discrete convolution with an  $\ell^1$  sequence is bounded on  $\ell^q$ .  $\square$

We come back to the proof of the proposition. It suffices to apply this lemma to  $T = e^{-tL}$  from  $L^p$  to  $L^2$  with  $g(u) = ct^{-\gamma p/2}e^{-cu^2/t}$  and choose  $s = t^{1/2}$ . This yields  $L^p$  boundedness of  $\mathcal{S}$ .  $\square$

*Remark.* For the **same**  $p$ ,  $L^p - L^2$  off-diagonal estimates  $\implies L^p$  boundedness  $\implies L^p - L^2$  boundedness for the semigroup  $\mathcal{S}$ . We do not know the status of the converses for this class of semigroups. It would be of great interest when  $p = 1$ .

## 4.2 $W^{1,p}$ elliptic estimates and hypercontractivity

In this section, we show how to obtain hypercontractivity properties using  $W^{1,p}$  elliptic estimates. We proceed independently of dimension although specific arguments for dimensions 1 and 2 yield much better results.

**Lemma 4.4.** <sup>34</sup> *There is an  $r \in [1, 2)$  depending on dimension and the ellipticity constants only, such that  $I + L$  extends to a bounded and invertible operator from  $W^{1,p}$  onto  $W^{-1,p}$  for  $|\frac{1}{2} - \frac{1}{p}| < |\frac{1}{2} - \frac{1}{r}|$ .* <sup>35</sup>

*Proof.* That  $I + L$  is bounded from  $W^{1,p}$  into  $W^{-1,p}$  for all  $p$  with  $1 \leq p \leq \infty$  is obvious.

Let  $A$  denote the operator of pointwise multiplication with  $A(x)$  and  $\|A\|$  its norm acting on  $L^p$  spaces of  $\mathbb{C}^n$ -valued functions (endowing matrices with the norm inherited from the Hermitian structure on  $\mathbb{C}^n$ ). This is the same number for all  $1 \leq p \leq \infty$ . By ellipticity, there exists a large constant  $c$  such that  $\|A - cI\| < c$ . Thus,  $A = c(I + M)$  where  $\|M\| < 1$ . This means that

$$L = (I - c\Delta - c \operatorname{div} M \nabla) = J(I - J^{-1} \operatorname{div} M \nabla J^{-1})J,$$

where  $\Delta$  denotes the ordinary Laplacian and  $J = c^{-1/2}(I - c\Delta)^{1/2}$ .

Remark that by standard multiplier theorems (or kernel estimates and Calderón-Zygmund theory) the array of operators  $\nabla J^{-1}$  is bounded from  $L^p$

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<sup>34</sup> In this generality, this is [AMcT, Proposition 3.1]. See also [BLP] and [IS]. Note that when  $n = 1$ , invertibility holds for  $1 \leq p \leq \infty$  [AMcT, Theorem 2.2]. If  $n \geq 2$ ,  $r$  may be arbitrary close to 2.

<sup>35</sup> There is also the corresponding homogeneous statement for  $L$  from  $\dot{W}^{1,p}$  onto  $\dot{W}^{-1,p}$ . See Section 5.5.

(of  $\mathbb{C}$ -valued functions) to  $L^p$  (of  $\mathbb{C}^n$ -valued functions) for  $1 < p < \infty$  ( $p = 1$  and  $p = \infty$  included if  $n = 1$ ). Moreover, a Fourier transform argument shows that  $c_2=1$  if  $c_p$  denotes the norm on  $L^p$ .

Thus  $R = J^{-1} \operatorname{div} M \nabla J^{-1}$  is bounded on  $L^p$  for  $1 < p < \infty$  with norm bounded above by  $\|M\|c_p^2$ . Since  $c_p$  is controlled by a convex function, the operator norm of  $R$  on  $L^p$  remains less than 1 provided  $p$  is close to 2. Therefore, one can invert  $I - R$  by a converging Neumann series in the space bounded operators on  $L^p$  for  $p$  close to 2. Since  $J$  is bounded and invertible from  $W^{s,p}$  onto  $W^{s-1,p}$  for  $s = 0, 1$  and  $1 < p < \infty$ , this proves the invertibility of  $I + L$  from  $W^{1,p}$  onto  $W^{-1,p}$  for  $p$  close to 2.  $\square$

**Corollary 4.5.** *Let  $r_* = \frac{nr}{n+r}$  with  $r$  as in Lemma 4.4. Let  $p \in [1, 2]$  be such that*

$$\begin{cases} p > r_*, & \text{if } r_* \geq 1 \\ p = 1 & \text{if } r_* < 1. \end{cases}$$

*The semigroup  $(e^{-tL})_{t>0}$  and its dual  $(e^{-tL^*})_{t>0}$  are  $L^p - L^2$  bounded and the best constant  $C$  in*

$$(4.3) \quad \|e^{-tL} f\|_2 + \|e^{-tL^*} f\|_2 \leq Ct^{-\gamma_p/2} \|f\|_p, \quad f \in L^2 \cap L^p$$

*depends only on dimension, ellipticity and  $p$ .*

*Proof.* Assume first that  $t = 1$ . By Lemma 4.4 and the Sobolev embedding theorem, in a finite number of steps  $(I+L)^{-k}$  extends to a bounded map from  $L^p$  into  $L^2$ . Note that  $k$  depends only on  $r$ , hence ellipticity and dimension. Let  $f \in L^2 \cap L^p$ . Since  $f$  is in  $L^2$ , the equality

$$e^{-L} f = e^{-L}(I+L)^k(I+L)^{-k} f$$

is justified. As  $e^{-L}(I+L)^k$  extends to bounded operator on  $L^2$  by analyticity of the semigroup on  $L^2$ , we have obtained that  $\|e^{-L} f\|_2 \leq C\|f\|_p$ , with a constant  $C$  that depends only on ellipticity, dimension and  $p$ .

If  $t \neq 1$ , then the affine change of variable in  $\mathbb{R}^n$  defined by  $g(x) = f(t^{1/2}x)$  gives us  $e^{-tL} f(x) = (e^{-L_t} g)(t^{-1/2}x)$  with  $L_t$  the second order operator associated with the matrix of coefficients  $A(t^{1/2}x)$ . Since  $L_t$  has same ellipticity constants as  $L$ , the previous bound applies and yields the desired estimate in function of  $t$  by change of variables.

The same argument applies to  $L^*$ .  $\square$

**Corollary 4.6.** <sup>36</sup> *If  $n = 1$  or  $2$ , then the semigroup  $(e^{-tL})_{t>0}$  is  $L^p - L^2$  bounded for  $1 \leq p < 2$  and  $L^2 - L^p$  bounded for  $2 < p \leq \infty$ . Hence, it is  $L^p$ -bounded for all  $p \in (1, \infty)$ . If  $n \geq 3$ , there exists  $\varepsilon > 0$  depending only on dimension and the ellipticity constants such that the semigroup  $(e^{-tL})_{t>0}$  is  $L^p$ -bounded for all  $p$  contained in the interval  $(p_n - \varepsilon, (p_n - \varepsilon)')$  for  $p_n = \frac{2n}{n+2}$ .*

*Proof.* In dimensions  $n = 1, 2$ , we have  $r_* < 1$ . In dimension  $n \geq 3$ , the value of  $r_*$  may or may not be less than 1, but it is less than  $2_* = \frac{2n}{n+2}$ . It suffices to combine Corollary 4.5 and Proposition 4.2 to finish the proof.  $\square$

We introduce here two critical exponents for  $L$ . Let  $\mathcal{J}(L)$  denote the maximal interval of exponents  $p$  in  $[1, \infty]$  for which the semigroup  $(e^{-tL})_{t>0}$  is  $L^p$  bounded. <sup>37</sup> We write  $\text{int}\mathcal{J}(L) = (p_-(L), p_+(L))$ . Note that  $(p_+(L))' = p_-(L^*)$  and vice versa. One has shown  $p_+(L) = \infty$  and  $p_-(L) = 1$  if  $n = 1, 2$ , and  $p_+(L) > \frac{2n}{n-2}$  and  $p_-(L) < \frac{2n}{n+2}$  if  $n \geq 3$ . In specific situations, much more can be said. For example, we have from the well-known formula for the heat kernel  $p_-( -\Delta) = 1$ ,  $p_+( -\Delta) = \infty$ . From the maximum principle for real parabolic equations, one has  $p_-(L) = 1$ ,  $p_+(L) = \infty$  if  $L$  has real coefficients.

### 4.3 Gradient estimates

Let us consider the possible estimates for  $\nabla e^{-tL}$ . Let  $\mathcal{N}(L)$  denote the maximal interval (if nonempty) of exponents  $p$  in  $[1, \infty]$  for which the family  $(\sqrt{t}\nabla e^{-tL})_{t>0}$  is  $L^p$  bounded. We write  $\text{int}\mathcal{N}(L) = (q_-(L), q_+(L))$ . A dichotomy between the cases  $p > 2$  and  $p < 2$  appears immediatly.

**Proposition 4.7.** *Let  $1 \leq p < 2$ . If  $(\sqrt{t}\nabla e^{-tL})_{t>0}$  is  $L^p$ -bounded, then  $(e^{-tL})_{t>0}$  is  $L^q$ -bounded for  $p < q < 2$ . Conversely, if  $(e^{-tL})_{t>0}$  is  $L^p$ -bounded then  $(\sqrt{t}\nabla e^{-tL})_{t>0}$  is  $L^q$ -bounded for  $p < q < 2$ . Hence  $q_-(L) = p_-(L)$ .*

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<sup>36</sup> For  $n = 1$ , this is in [AMcT], Theorem 2.21, for  $n = 2$  in [AMcT], Theorem 3.5. There,  $L^1$  and  $L^\infty$  boundedness and even  $L^1 - L^\infty$  off-diagonal estimates are proved, but further arguments such as the so-called Davies' trick are needed. This method, sufficient for our needs here, is not powerful enough. For  $n \geq 3$ , this follows from combining the works of Davies ([Da]) for  $p_n \leq p \leq (p_n)'$  and the perturbation method in [AT], Chapter I.

<sup>37</sup> In [LSV], a systematic study of this interval is made for a class of real elliptic operators. The methods heavily rely on real functions.

*Proof.* Let us see the direct part first. We have nothing to prove if  $n \leq 2$  by Corollary 4.6 as the conclusion holds true. If  $n \geq 3$ , by interpolation and Sobolev embeddings we have that  $(e^{-tL})_{t>0}$  is  $L^q - L^{q^*}$  bounded for  $p \leq q \leq 2$ . Let  $p_k$  be defined by  $p_0 = p$ ,  $p_{k+1} = (p_k)^*$  and stop whenever  $2_* \leq p_k < 2$ . By composition and the semigroup property, we have that  $(e^{-tL})_{t>0}$  is  $L^p - L^{p_k}$  bounded. By Corollary 4.5, since  $p_k \geq 2_*$ ,  $(e^{-tL})_{t>0}$  is  $L^{p_k} - L^2$  bounded. By composition again,  $(e^{-tL})_{t>0}$  is  $L^p - L^2$  bounded. The conclusion follows from Proposition 4.2.

For the converse, by Proposition 4.2,  $(e^{-tL})_{t>0}$  is  $L^p - L^2$  bounded. Since  $(\sqrt{t} \nabla e^{-tL})_{t>0}$  is  $L^2$  bounded, this self-improves by composition to  $L^p - L^2$  boundedness. Lemma 4.3 applied to  $T = \nabla e^{-tL}$  yields the conclusion.

The relation  $q_-(L) = p_-(L)$  follows immediately.  $\square$

*Remark.* 1. We see that the semigroup acting on  $L^p$  self-improves into  $W^{1,p}$  if  $p < 2$  up to allowing the  $p$ 's to vary. This is false when  $p > 2$ .

2. Although we do not need such a refinement here, it would be interesting to know whether the conclusions of  $L^p$  boundedness hold at the endpoint  $p$ . The arguments show, nevertheless, that, for the **same**  $p$ ,  $L^p - L^2$  boundedness for  $(e^{-tL})_{t>0}$  is equivalent to  $L^p - L^2$  boundedness for  $(\sqrt{t} \nabla e^{-tL})_{t>0}$ , and  $L^p - L^2$  off-diagonal estimates for  $(e^{-tL})_{t>0}$  implies  $L^p - L^2$  off-diagonal estimates for  $(\sqrt{t} \nabla e^{-tL})_{t>0}$ . The converse of the latter is not clear.

Let us record here the following consequences of the above argument for later use.

**Corollary 4.8.** *Let  $p \in [1, 2)$  and  $n \geq 1$ . Let  $\mathcal{N} = (\sqrt{t} \nabla e^{-tL})_{t>0}$ .*

1. *If  $\mathcal{N}$  is  $L^p$  bounded then it is  $L^p - L^2$  bounded.*
2. *If  $\mathcal{N}$  is  $L^p - L^2$  bounded, then it satisfies  $L^q - L^2$  off-diagonal estimates for  $p < q < 2$ .*
3. *If  $\mathcal{N}$  satisfies  $L^p - L^2$  off-diagonal estimates then it is  $L^p$  bounded.*

Next, let us consider the case  $p > 2$ . We have the same statement as the corollary but with a different argument.

**Proposition 4.9.** *Let  $p \in (2, \infty]$  and  $n \geq 1$ . Let  $\mathcal{N} = (\sqrt{t} \nabla e^{-tL})_{t>0}$ .*

1. *If  $\mathcal{N}$  is  $L^p$  bounded then it is  $L^2 - L^p$  bounded.*

2. If  $\mathcal{N}$  is  $L^2 - L^p$  bounded, then it satisfies  $L^2 - L^q$  off-diagonal estimates for  $2 < q < p$ .

3. If  $\mathcal{N}$  satisfies  $L^2 - L^p$  off-diagonal estimates then it is  $L^p$  bounded.

*Proof.* Let us prove the first item. Assume that  $n \geq 3$  and also that  $p < \infty$ . If  $\nabla e^{-tL}$  is bounded on  $L^p$  with bound  $Ct^{-1/2}$ , then the same is true for all  $q \in [2, p]$ . By Sobolev embeddings, for all  $q \in [2, p]$  with  $q < n$ ,

$$e^{-tL} : L^q \rightarrow L^{q^*}$$

with bound  $Ct^{-1/2}$ . Since we know that  $e^{-tL} : L^2 \rightarrow L^q$  for any  $q \in \mathbb{R}$  with  $2 \leq q \leq 2^*$  from Corollary 4.5, it follows as in the proof of Proposition 4.7 that  $e^{-tL}$  maps  $L^2$  to  $L^q$  for all  $q \in \mathbb{R}$  with  $2 \leq q \leq p^*$ . Writing  $\nabla e^{-tL}$  as  $\nabla e^{-(t/2)L} e^{-(t/2)L}$  shows that this operator is bounded from  $L^2$  into  $L^p$  with the appropriate norm growth.

Assume next  $n \geq 3$  and  $p = \infty$ . We just need to prove that the semigroup is  $L^2 - L^\infty$  bounded and the rest of the argument applies. By Morrey's embedding, if  $n < q < \infty$  and  $\alpha = 1 - \frac{n}{q}$ ,  $t > 0$  and  $x, y \in \mathbb{R}^n$ ,

$$|e^{-tL}f(x) - e^{-tL}f(y)| \lesssim \|\nabla e^{-tL}f\|_q |x - y|^\alpha \lesssim t^{-\frac{1}{2} - \frac{\gamma q}{2}} \|f\|_2 |x - y|^\alpha.$$

In the last inequality, we used that  $\mathcal{N}$  is  $L^2 - L^q$  bounded as we have just proved it. Fix  $x$  and average the square of this inequality on a ball  $B$  with center  $x$  and radius  $r = \sqrt{t}$  to find

$$|e^{tL}f(x)| \lesssim |B|^{-1/2} \|e^{-tL}f\|_{L^2(B)} + r^{\frac{\alpha}{2}} t^{-\frac{1}{2} - \frac{\gamma q}{2}} \|f\|_2 \lesssim t^{-n/4} \|f\|_2$$

by using the  $L^2$  boundedness of  $e^{-tL}$ . This prove the  $L^2 - L^\infty$  boundedness of the semigroup.

If  $n \leq 2$ , we already know that  $e^{-tL} : L^2 \rightarrow L^q$  for any  $q \in [2, \infty]$  from Corollary 4.5 and the rest of the above argument applies.

The second item is a consequence of interpolation between the hypothesis and the  $L^2$  off-diagonal estimates for  $(\sqrt{t} \nabla e^{-tL})_{t>0}$ .

For the third item, it is enough to apply Lemma 4.3 to  $T = e^{-tL^*}$  div with  $s = t^{1/2}$  and duality.  $\square$

**Corollary 4.10.**  $q_+(L) > 2$ .

*Proof.* Let  $p > 2$  such that  $|\frac{1}{2} - \frac{1}{p}| < |\frac{1}{2} - \frac{1}{r}|$  where  $r$  is given in Lemma 4.4. Let  $f \in L^2$ . After a finite number of steps (depending only on  $r$ )  $(I + L)^{-k}$  maps  $L^2$  into  $W^{1,p}$ . In particular, since  $(I + L)^k e^{-L} f \in L^2$  by analyticity,

$$\nabla e^{-L} f = \nabla (I + L)^{-k} (I + L)^k e^{-L} f \in L^p.$$

This proves the boundedness of  $\nabla e^{-L}$  from  $L^2$  into  $L^p$ . The  $L^2 - L^p$  boundedness of the family  $(\sqrt{t} \nabla e^{-tL})_{t>0}$  follows by the rescaling argument of the proof of Corollary 4.5. Hence,  $q_+(L) \geq p$ .  $\square$

**Corollary 4.11.**  $\mathcal{N}(L)$  is not empty and contains a neighborhood of 2.

*Proof.* We have  $q_-(L) = p_-(L) < 2$  and  $q_+(L) > 2$ .  $\square$

More is true in dimension 1.

**Proposition 4.12.** If  $n = 1$ , we have  $q_+(L) = \infty$ .

*Proof.* In dimension 1, the operator  $L$  takes the form  $-\frac{d}{dx}(a\frac{d}{dx})$ . By Proposition 4.9, it suffices to establish that  $(\sqrt{t} \frac{d}{dx} e^{-tL})_{t>0}$  is  $L^2 - L^\infty$  bounded.<sup>38</sup> Let  $f \in L^2$  and  $t > 0$  and set  $g = \sqrt{t} a \frac{d}{dx} e^{-tL} f$ . We know that  $g \in L^2$  with norm  $O(1)$ , while  $g' = -\sqrt{t} L e^{-tL} f \in L^\infty$  with norm  $O(t^{-1/2})$ . The last fact can be seen from writing  $L e^{-tL} = e^{-(t/2)L} L e^{-(t/2)L}$  and using that  $(tL e^{-tL})_{t>0}$  is  $L^2$  bounded and  $(e^{-tL})_{t>0}$  is  $L^2 - L^\infty$  bounded by Corollary 4.6. This implies that  $g \in L^\infty$  with norm  $O(t^{-1/4})$ . Indeed, let  $I$  be an interval of size  $t^{1/4}$  and  $x, y \in I$ . We have  $g(x) - g(y) = \int_y^x g'(s) ds$ . Hence,  $|g(x)| \leq |g(y)| + Ct^{-1/4}$ . Averaging squares over  $I$  with respect to  $y$  yields  $|g(x)|^2 \leq Ct^{-1/2} \|g\|_2^2 + Ct^{-1/2} = O(t^{-1/2})$ . Since  $I$  and  $x$  are arbitrary, this gives us the desired  $L^\infty$  bound on  $\mathbb{R}$ .  $\square$

**Proposition 4.13.** Assume  $2 < p \leq \infty$  and  $(\sqrt{t} \nabla e^{-tL})_{t>0}$  is  $L^p$ -bounded.

1.  $(e^{-tL})_{t>0}$  is  $L^q$ -bounded for  $2 \leq q \leq p^*$  except when  $p = n$  for which we conclude only for  $2 \leq q < \infty$ . In particular, one has  $p_+(L) \geq (q_+(L))^*$ .
2. If  $p < \infty$ ,  $(t \nabla e^{-tL} \operatorname{div})_{t>0}$  is  $L^p$ -bounded.

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<sup>38</sup> In fact, this family satisfy the  $L^1 - L^\infty$  off-diagonal estimates, which implies all the other boundedness properties. See [AT1, p. 728] where it is mentioned, and [AMcT, Theorem 2.21] for details.

*Proof.* The first part has been seen in the proof of Proposition 4.9. For the second part, the  $L^p$  boundedness of  $(\sqrt{t}\nabla e^{-tL})_{t>0}$  implies in particular that  $(e^{-tL})_{t>0}$  is  $L^q$  bounded for  $q$  in a neighborhood of  $p$ , thus, that  $(e^{-tL^*})_{t>0}$  is  $L^q$  bounded for  $q$  in a neighborhood of  $p'$ . Applying Proposition 4.7 shows, in particular, that  $(\sqrt{t}\nabla e^{-tL^*})_{t>0}$  is  $L^{p'}$ -bounded, hence, that  $(\sqrt{t}e^{-tL}\operatorname{div})_{t>0}$  is  $L^p$ -bounded by duality. We conclude by composition.  $\square$

*Remark.* If  $n = 1$ , combining Corollary 4.6, Proposition 4.12 and Proposition 4.13, one has that the semigroup is bounded from  $W^{-1,p}$  into  $W^{1,p}$  for  $1 < p < \infty$  and  $t > 0$ .<sup>39</sup>

## 4.4 Summary

This section is nothing but a summary of results obtained so far and gathered here for the reader's convenience.

Recall that  $\mathcal{J}(L)$  denotes the maximal interval of exponents  $p$  in  $[1, \infty]$  for which the semigroup  $(e^{-tL})_{t>0}$  is  $L^p$  bounded. We write  $\operatorname{int}\mathcal{J}(L) = (p_-(L), p_+(L))$ . Note that  $(p_+(L))' = p_-(L^*)$  and vice versa. One has  $p_+(L) = \infty$  and  $p_-(L) = 1$  if  $n = 1, 2$  and  $p_+(L) > \frac{2n}{n-2}$  and  $p_-(L) < \frac{2n}{n+2}$  if  $n \geq 3$ .

Next,  $\mathcal{N}(L)$  denotes the maximal interval of exponents  $p$  in  $[1, \infty]$  for which the family  $(\sqrt{t}\nabla e^{-tL})_{t>0}$  is  $L^p$  bounded. We write  $\operatorname{int}\mathcal{N}(L) = (q_-(L), q_+(L))$ . We have  $q_-(L) = 1$  and  $q_+(L) = \infty$  if  $n = 1$ ,  $q_-(L) = 1$  if  $n = 2$ ,  $q_-(L) < \frac{2n}{n+2}$  if  $n \geq 3$  and  $q_+(L) > 2$  if  $n \geq 2$ . In particular,  $\operatorname{int}\mathcal{N}(L)$  is a neighborhood of 2.

The relation between  $p_{\pm}(L)$  and  $q_{\pm}(L)$  are as follows:

$$\begin{aligned} q_-(L) &= p_-(L), \\ p_+(L) &\geq (q_+(L))^*. \end{aligned}$$

## 4.5 Sharpness issues

Denote by  $\mathcal{E}$  the class of all complex elliptic operators under study.

It is known that if  $n \geq 5$ , there exists  $L \in \mathcal{E}$  for which  $p_-(L) > 1$ .<sup>40</sup>

<sup>39</sup> More is true, in fact, this holds for the endpoints  $p = 1$  and  $p = \infty$  and one even has  $L^1 - L^\infty$  off-diagonal estimates for the family  $(t\frac{d}{dx}e^{-tL}\frac{d}{dx})_{t>0}$ . See [AMcT].

<sup>40</sup> This is proved in [ACT], based on counterexamples built in [MNP]. See [AT] for a shorter argument due to Davies.

It is not known if the inequality  $p_-(L) < \frac{2n}{n+2}$  ( $n \geq 3$ ), or equivalently by taking the adjoint  $p_+(L) > \frac{2n}{n-2}$  is sharp: that is, if given  $p < \frac{2n}{n+2}$ , there exists  $L \in \mathcal{E}$  for which  $p_-(L) > p$ .

In view of this a natural conjecture is

**Conjecture 4.14.** *If  $n \geq 3$ , the inequality  $p_-(L) < \frac{2n}{n+2}$  is sharp for the class  $\mathcal{E}$ .*

It is known that the inequality  $q_+(L) > 2$  is sharp in dimensions  $n \geq 2$ : for any  $\varepsilon > 0$ , there exists  $L \in \mathcal{E}$  (in fact,  $L$  is real and symmetric) such that  $q_+(L) \leq 2 + \varepsilon$ .<sup>41</sup> As a consequence, there is no general upper bound of  $p_+(L)$  in terms of  $q_+(L)$  since  $p_+(L) = \infty$  if  $n = 2$ .

It is not known whether the inequality  $p_+(L) \geq q_+(L)^*$  is best possible ( $n \geq 3$ ): that is, if given  $\varepsilon > 0$ , one can find  $L \in \mathcal{E}$  with  $p_+(L) < q_+(L)^* + \varepsilon$ .

## 4.6 Analytic extension

For technical reasons, we often need to apply the above results to the analytic extension of the semigroup associated to  $L$ . We come to this now.

The definition 4.1 of  $L^p - L^q$  boundedness and  $L^p - L^q$  off-diagonal estimates applies to a family  $\mathcal{T} = (T_z)_{z \in \Sigma_\beta}$  defined on a complex sector  $\Sigma_\beta$  with  $0 \leq \beta < \frac{\pi}{2}$ , replacing  $t$  by  $|z|$  in the right hand side of the inequalities (4.1) and (4.2).

Proposition 4.1 extends right away to such complex families. The statement of Proposition 4.2 is also true for the analytic extension of the semigroup. However, more is true. It suffices to know boundedness of the semigroup to obtain all properties for its analytic extension with optimal angles of the sectors, that is, the sectors are  **$p$ -independent**.<sup>42</sup>

<sup>41</sup> This follows from Meyers' example. See [AT].

<sup>42</sup> The history of  $p$ -independence of sector of holomorphy for  $1 \leq p < \infty$  begins with that of  $p$ -independence of spectra. Hempel and Voigt [HV] proved that the spectrum is  $p$ -independent for a large class of Schrödinger operators acting on  $L^p(\mathbb{R}^n)$ . Arendt [Are] extended this to elliptic operators on domains of  $\mathbb{R}^n$  under the assumption that the semigroup of the given operator satisfies  $L^1 - L^\infty$  off-diagonal estimates. Ouhabaz realized in his PhD thesis (published later in [Ou]) that the same assumption yields holomorphy of the semigroup up to  $L^1$  with optimal angles in the case of a self-adjoint operator on a subset of  $\mathbb{R}^n$ . Independently, Arendt & ter Elst [AE, Theorem 5.4], and Hieber [Hi] removed the self-adjointness assumption. Finally, Davies [Da2] extended Ouhabaz argument to the setting of doubling spaces (for non-negative self-adjoint operators) to obtain in a simpler manner the result of  $p$ -independence of spectrum of Arendt.



Recall that  $\mathcal{J}(L)$  (resp.  $\mathcal{N}(L)$ ) is the maximal interval of exponents  $p$  in  $[1, \infty]$  for which the semigroup  $(e^{-tL})_{t>0}$  (resp. the family  $(\sqrt{t} \nabla e^{-tL})_{t>0}$ ) is  $L^p$  bounded.

**Proposition 4.15.** *Let  $\omega$  be the type of  $L$ . Then the semigroup  $(e^{-tL})_{t>0}$  has an analytic extension to  $\Sigma_{\frac{\pi}{2}-\omega}$  on  $L^p$  for  $p \in \text{int}\mathcal{J}(L)$ . Moreover, for  $p \in \text{int}\mathcal{J}(L)$  and all  $\beta \in (0, \frac{\pi}{2}-\omega)$ , the families  $(e^{-zL})_{z \in \Sigma_\beta}$  and  $(zLe^{-zL})_{z \in \Sigma_\beta}$  are  $L^p$  bounded, satisfy  $L^p - L^2$  (resp.  $L^2 - L^p$ ) off-diagonal estimates and are  $L^p - L^2$  (resp.  $L^2 - L^p$ ) bounded if  $p < 2$  (resp. if  $p > 2$ ).*

*Proof.* The analyticity is a consequence of Stein's complex interpolation theorem together with holomorphy of the semigroup on  $L^2$ . However, this does not yield the best angle for the sector of holomorphy. Here is a better argument.

For  $\alpha \in (-\frac{\pi}{2} + \omega, \frac{\pi}{2} - \omega)$ , set  $L_\alpha = e^{i\alpha}L$ . It is an operator in the same class as  $L$ , associated to the matrix of coefficients  $e^{i\alpha}A(x)$ . Hence, Proposition 4.2 applies to the semigroup associated to  $L_\alpha$ . Furthermore, a careful check of the proof shows that the various constants are independent of  $\alpha$  as long as  $\alpha$  is restricted to a compact subset of  $(-\frac{\pi}{2} + \omega, \frac{\pi}{2} - \omega)$ . Let  $\beta \in (0, \frac{\pi}{2} - \omega)$ . If  $z = te^{i\alpha} \in \Sigma_\beta$ , then  $e^{-zL} = e^{-tL_\alpha}$  and the reasoning above shows that the statement of Proposition 4.2 extends to the complex family  $(e^{-zL})_{z \in \Sigma_\beta}$ . Hence, for  $p \in \text{int}\mathcal{J}(L)$ , it remains to showing that this family is  $L^p - L^2$  (resp.  $L^2 - L^p$ ) bounded if  $p < 2$  (resp. if  $p > 2$ ).

The case  $p > 2$  can be handled by duality, we restrict attention to  $p < 2$ . Let  $z \in \Sigma_\beta$ . Choose  $\beta < \beta' < \frac{\pi}{2} - \omega$ . Elementary geometry shows that one can decompose  $z = \zeta + t$  with  $\zeta \in \Sigma_{\beta'}$ ,  $t > 0$  and  $|z| \sim |\zeta| \sim t$  where the implicit constants only depend on  $\beta$  and  $\beta'$ . By assumption and Proposition 4.2,  $(e^{-tL})_{t>0}$  is  $L^p - L^2$  bounded. Since  $(e^{-\zeta L})_{\zeta \in \Sigma_{\beta'}}$  is  $L^2$  bounded, writing  $e^{-zL} = e^{-\zeta L}e^{-tL}$ , this shows that  $(e^{-zL})_{z \in \Sigma_\beta}$  is  $L^p - L^2$  bounded.  $\square$

**Proposition 4.16.** *Let  $\omega$  be the type of  $L$ . Then, for  $p \in \text{int}\mathcal{N}(L)$  and all  $\beta \in (0, \frac{\pi}{2} - \omega)$ , its analytic extension  $(\sqrt{z} \nabla e^{-zL})_{z \in \Sigma_\beta}$  is  $L^p$  bounded, satisfies  $L^p - L^2$  (resp.  $L^2 - L^p$ ) off-diagonal estimates and is  $L^p - L^2$  (resp.  $L^2 - L^p$ ) bounded if  $p < 2$  (resp. if  $p > 2$ ).*

It suffices to adapt the above proof. Further details are left to the reader.

## 5 $L^p$ theory for square roots

Let  $L$  be as the introduction. We study here the following sets:

1. the maximal interval of exponents  $p$  in  $(1, \infty)$  for which one has the  $L^p$  boundedness of the Riesz transform  $\nabla L^{-1/2}$ , which we call  $\mathcal{I}(L)$ .
2. the maximal interval of exponents  $p$  in  $(1, \infty)$  for which one has the *a priori* inequality  $\|L^{1/2}f\|_p \lesssim \|\nabla f\|_p$  for  $f \in C_0^\infty$ , which we call  $\mathcal{R}(L)$ .

We characterize the limits of  $\mathcal{I}(L)$  and obtain bounds on the limits of  $\mathcal{R}(L)$ .

## 5.1 Riesz transforms on $L^p$

We prove here the following theorem.

**Theorem 5.1.** *The interior of  $\mathcal{I}(L)$  equals  $(p_-(L), q_+(L))$ .*

Recall that  $p_-(L)$  is the lower limit of both  $\mathcal{J}(L)$  and  $\mathcal{N}(L)$ , and that  $q_+(L)$  is the upper limit of  $\mathcal{N}(L)$ . The cases  $p < 2$  and  $p > 2$  are treated by different methods.

### 5.1.1 The case $p < 2$

We introduce the following sets.

$$\begin{aligned} \mathcal{I}_-(L) &= \mathcal{I}(L) \cap (1, 2) \\ \mathcal{J}_-(L) &= \mathcal{J}(L) \cap [1, 2) \\ \mathcal{K}_-(L) &= \{1 \leq p < 2; (e^{-tL})_{t>0} \text{ is } L^p - L^2 \text{ bounded}\} \\ \mathcal{M}_-(L) &= \{1 \leq p < 2; (e^{-tL})_{t>0} \text{ satisfies } L^p - L^2 \text{ off-diagonal estimates}\} \end{aligned}$$

It is quite clear from interpolation that these sets are intervals with 2 as upper limit, provided they are nonempty. It follows directly from Proposition 4.2 that  $\mathcal{J}_-(L)$ ,  $\mathcal{K}_-(L)$  and  $\mathcal{M}_-(L)$  have the same interiors (that is, the same lower limit). Also, by Corollary 4.5 the interior of  $\mathcal{J}_-(L)$  is not empty.

**Theorem 5.2.** <sup>43</sup> *All the above sets are intervals with common interiors.*

Let us derive a corollary for results in the range  $1 < p < 2$ .

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<sup>43</sup> The fact that off-diagonal estimates implies boundedness of the Riesz transforms is proved in [BK2] and this idea is also the main tool in [HM]. The converse was not noticed in these works.

**Proposition 5.3.**  $\nabla L^{-1/2}$  is bounded on  $L^p$  for  $1 < p < 2$  if, and only if,  $(e^{-tL})_{t>0}$  is  $L^p - L^2$  bounded for  $1 < p < 2$ . In particular, if  $(e^{-tL})_{t>0}$  is  $L^1 - L^2$  bounded, then  $\nabla L^{-1/2}$  is bounded on  $L^p$  for  $1 < p < 2$ .

*Proof.* The equivalence is contained in the theorem above since  $\text{int}\mathcal{I}_-(L) = \text{int}\mathcal{K}_-(L)$ . Next, if  $(e^{-tL})_{t>0}$  is  $L^1 - L^2$  bounded, then by interpolation it is also  $L^p - L^2$  bounded for  $1 < p < 2$   $\square$

*Remark.* Usually, the  $L^p$  boundedness of the Riesz transform for  $1 < p < 2$  is obtained under the stronger assumption that the semigroup kernel has good pointwise upper bounds.<sup>44</sup>

Let us prove Theorem 5.2. According to the above remark, the following steps suffice to prove this result:

**Step 1.**  $p \in \mathcal{I}_-(L)$  implies  $p \in \mathcal{K}_-(L)$ .

**Step 2.**  $p_0 \in \mathcal{M}_-(L)$  and  $p_0 < p < 2$  imply  $p \in \mathcal{I}_-(L)$ .

**Step 1.**  $p \in \mathcal{I}_-(L)$  implies  $p \in \mathcal{K}_-(L)$ . First the conclusion is true if  $n \leq 2$  from Corollary 4.6. We assume, therefore, that  $n \geq 3$ .

Let  $p \in \mathcal{I}_-(L)$ . If  $p \geq 2_* = \frac{2n}{n+2}$ , we already know that  $(e^{-tL})_{t>0}$  is  $L^p - L^2$ . Assume now that  $p < 2_*$ . Remark that any  $q \in [p, 2]$  belongs to  $\mathcal{I}_-(L)$ . By Sobolev embedding's, we see that  $L^{-1/2}: L^q \rightarrow L^{q^*}$ . Let  $p_0 = p$  and  $p_k = (p_{k-1})^*$  and stop when  $k$  is the largest integer that satisfies  $p_k < 2$ . We have that  $L^{-k/2}$  is bounded from  $L^p$  into  $L^{p_k}$ . Write for  $f \in L^2 \cap L^p$ ,

$$e^{-tL} f = (e^{-(t/2)L} L^{k/2}) e^{-(t/2)L} L^{-k/2} f,$$

the equality being justified by the fact that  $f \in L^2$ . We successively have that  $g = L^{-k/2} f \in L^{p_k}$ , then  $h = e^{-(t/2)L} g \in L^2$  by Corollary 4.5 since  $p_k \geq \frac{2n}{n+2}$ , and  $e^{-(t/2)L} L^{k/2} h \in L^2$  by the bounded holomorphic calculus of  $L$  on  $L^2$ . Hence,  $e^{-tL}$  is bounded from  $L^p$  into  $L^2$  by density and we obtain the right bound for its operator norm by keeping track of the bounds from each step.

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<sup>44</sup> In [AT, Chapter IV], this is obtained via an  $\mathcal{H}^1$  estimate under  $L^1 - L^\infty$  off diagonal estimates plus Hölder regularity on the heat kernel. The regularity was removed in [DMc1] to obtain weak type (1,1). In [CD], the use of weighted  $L^2$  estimates similar to the  $L^1 - L^2$  boundedness was stressed in the context of the Riesz transform for the Laplace-Beltrami on Riemannian manifolds.

**Step 2.**  $p_0 \in \mathcal{M}_-(L)$  and  $p_0 < p < 2$  imply  $p \in \mathcal{I}_-(L)$ . We apply Theorem 2.1 to the operator  $T = \nabla L^{-1/2}$  to obtain weak type  $(p_0, p_0)$ . We first introduce the operators  $A_r = I - (I - e^{-r^2 L})^m$  where  $m$  is some integer to be specified later.

Observe that  $A_r = \sum_{k=1}^m c_k e^{-kr^2 L}$  for some numbers  $c_k$ . A direct consequence of the assumption that  $L$  satisfies  $L^{p_0} - L^2$  off-diagonal estimates is that if  $B$  is any ball, with radius  $r$ , and  $f \in L^2 \cap L^{p_0}$  with support in  $B$ , and  $j \geq 1$ , then

$$\left( \int_{C_j(B)} |A_r f|^2 \right)^{1/2} \leq C r^{-\gamma} e^{-\alpha 4^j} \left( \int_B |f|^{p_0} \right)^{1/p_0}$$

where  $\gamma = \gamma_{p_0} = |\frac{n}{2} - \frac{n}{p_0}|$ . Recall that  $C_j(B)$  denotes the ring  $2^{j+1}B \setminus 2^j B$  if  $j \geq 2$  and  $C_1(B) = 4B$ . Hence the assumption (2.2) in Theorem 2.1 holds with  $g(j) = C(m)2^{-nj/2}e^{-\alpha 4^j}$  for any  $m \geq 1$ . It remains to check the assumption (2.1) and this is where we use the role of  $m$ .

**Lemma 5.4.** *Let  $p_0 \in \mathcal{M}_-(L)$ . There exists  $C \geq 0$ , such that for all balls  $B$  with radius  $r > 0$  and  $f \in L^2 \cap L^{p_0}$  with support in  $B$ , and  $j \geq 2$*

$$(5.1) \quad \|\nabla L^{-1/2}(I - e^{-r^2 L})^m f\|_{L^2(C_j(B))} \leq C r^{-\gamma} 2^{-j(2m+\gamma)} \|f\|_{L^{p_0}(B)},$$

where  $\gamma = \gamma_{p_0} = |\frac{n}{2} - \frac{n}{p_0}|$ .

*Proof.* By expanding  $(I - e^{-r^2 L})^m$  in the representation of the square root, we obtain

$$\begin{aligned} \nabla L^{-1/2}(I - e^{-r^2 L})^m f &= \pi^{-1/2} \int_0^\infty \nabla e^{-tL} (I - e^{-r^2 L})^m f \frac{dt}{\sqrt{t}} \\ &= \pi^{-1/2} \int_0^\infty g_{r^2}(t) \nabla e^{-tL} f dt \end{aligned}$$

where using the usual notation for the binomial coefficient,

$$g_s(t) = \sum_{k=0}^m \binom{m}{k} (-1)^k \frac{\chi_{\{t > ks\}}}{\sqrt{t - ks}},$$

and  $\chi$  is the indicator function of  $(0, \infty)$ .

Observe that  $(\sqrt{t}\nabla e^{-tL})_{t>0}$  satisfies  $L^{p_0} - L^2$  off-diagonal estimates by applying the composition to the families  $(\sqrt{t}\nabla e^{-(t/2)L})_{t>0}$  and  $(e^{-(t/2)L})_{t>0}$  which satisfy respectively  $L^2$  and  $L^{p_0} - L^2$  off-diagonal estimates. By Minkowski integral inequality, we have that

$$\|\nabla L^{-1/2}(I - e^{-r^2L})^m f\|_{L^2(C_j(B))} \leq C \int_0^\infty |g_{r^2}(t)| e^{-\frac{\alpha 4^j r^2}{t}} t^{-\gamma/2} \frac{dt}{\sqrt{t}} \|f\|_{L^{p_0}(B)}.$$

The latter integral can be estimated as follows. Elementary analysis yields the following estimates for  $g_{r^2}$ :

$$|g_{r^2}(t)| \leq \frac{C}{\sqrt{t - kr^2}} \quad \text{if } kr^2 < t \leq (k+1)r^2 \leq (m+1)r^2$$

and

$$|g_{r^2}(t)| \leq Cr^{2m} t^{-m-\frac{1}{2}} \quad \text{if } t > (m+1)r^2.$$

The latter estimate comes from the inequality

$$\left| \sum_{k=0}^m \binom{m}{k} (-1)^k v(t - ks) \right| \leq Cs^m \sup_{u \geq t/(m+1)} |v^{(m)}(u)|$$

for  $t > (m+1)s \geq 0$  after expanding  $v(t - ks)$  using Taylor's formula about  $t$  and using the classical relations  $\sum_{k=0}^m \binom{m}{k} (-1)^k k^\ell = 0$  for  $\ell \in \mathbb{N}$ ,  $\ell < m$ . This readily yields the estimates

$$\int_0^\infty |g_{r^2}(t)| e^{-\frac{\alpha 4^j r^2}{t}} t^{-\gamma/2} \frac{dt}{\sqrt{t}} \leq Cr^{-\gamma} 2^{-j(2m+\gamma)}.$$

**Alternate proof of the lemma:** By the representation of the square root, one obtains

$$(5.2) \quad \nabla L^{-1/2}(I - e^{-r^2L})^m f = \pi^{-1/2} \int_0^\infty \sqrt{t} \nabla e^{-tL} (I - e^{-r^2L})^m f \frac{dt}{t}.$$

The function  $\varphi(z) = e^{-tz}(1 - e^{r^2z})^m$  is holomorphic and satisfies the technical condition (3.5) in any sector  $\Sigma_\mu$ ,  $\mu < \frac{\pi}{2}$ . Hence, one may use the representation (3.6) to compute  $\varphi(L)$ . With the same choices of the parameters as in Section 3.2, for some positive constant  $c$ , the functions  $\eta_\pm$  given by (3.7) satisfy

$$|\eta_\pm(z)| \leq \int_{\gamma_\pm} e^{-c|\zeta|(|z|+t)} |1 - e^{-r^2\zeta}|^m |d\zeta|,$$

and one finds

$$(5.3) \quad |\eta_{\pm}(z)| \leq \frac{C}{|z|+t} \inf \left( 1, \frac{r^{2m}}{(|z|+t)^m} \right), \quad z \in \Gamma_{\pm}.$$

Observe that for any  $0 < \beta < \beta' < \frac{\pi}{2} - \omega$ ,  $(\sqrt{z} \nabla e^{-zL})_{z \in \Sigma_{\beta}}$  satisfies  $L^{p_0} - L^2$  off-diagonal estimates by applying the composition lemma to the families  $(\sqrt{z} \nabla e^{-zL})_{z \in \Sigma_{\beta'}}$  and  $(e^{-tL})_{t>0}$  which satisfy respectively  $L^2$  and  $L^{p_0} - L^2$  off-diagonal estimates. It suffices to decompose  $z \in \Sigma_{\beta}$  into  $z' + t$  with  $z' \in \Sigma_{\beta'}$  and  $t > 0$  with  $|z| \sim |z'| \sim t$ . Using this in (3.6) and the estimate for  $\eta_{\pm}$ ,  $\|\sqrt{t} \nabla e^{-tL} (I - e^{-r^2 L})^m f\|_{L^2(C_j(B))}$  is bounded by

$$C \int_{\Gamma_+} e^{-\frac{c4^j r^2}{|z|} t^{1/2}} \frac{1}{|z|^{1/2}} \frac{1}{|z|^{\gamma/2}} \frac{1}{(|z|+t)} \frac{r^{2m}}{(|z|+t)^m} |dz| \|f\|_{L^{p_0}(B)}$$

plus the similar term corresponding to integration on  $\Gamma_-$ . Here  $\gamma = \gamma_{p_0} = |\frac{n}{2} - \frac{n}{p_0}|$ . Using the inequality (5.4) below for the integral, this gives us the bound

$$\frac{C}{4^{jm}(2^j r)^{\gamma}} \inf \left( \left( \frac{t}{4^j r^2} \right)^{1/2}, \left( \frac{4^j r^2}{t} \right)^{m-1/2} \right) \|f\|_{L^{p_0}(B)}.$$

Integrating with respect to  $t$  in (5.2), one finds (5.1) by Minkowski inequality since  $m \geq 1 > 1/2$ .  $\square$

Thus, (2.1) holds with  $g(j) = 2^{-nj/2} 2^{-j(2m+\gamma)}$ . Hence, the summability condition  $\sum_{j \geq 1} 2^{nj} g(j) < \infty$  is granted if  $2m + \gamma > n/2$ . This finishes the proof modulo the proof of (5.4) which we do in the following lemma.

**Lemma 5.5.** *Let  $\gamma \geq 0$ ,  $\alpha \geq 0$ ,  $m > 0$  be fixed parameters, and  $c$  a positive constant. For some  $C$  independent of  $j \in \mathbb{N}$ ,  $r > 0$  and  $t > 0$ , the integral*

$$I = \int_0^{\infty} e^{-\frac{c4^j r^2}{s}} \frac{1}{s^{\gamma/2}} \frac{t^{\alpha}}{(s+t)^{1+\alpha}} \frac{r^{2m}}{(s+t)^m} ds$$

satisfies the estimate

$$(5.4) \quad I \leq \frac{C}{4^{jm}(2^j r)^{\gamma}} \inf \left( \left( \frac{t}{4^j r^2} \right)^{\alpha}, \left( \frac{4^j r^2}{t} \right)^m \right).$$

*Proof.* Set  $\beta = \frac{4^j r^2}{t}$  for the argument. Split  $I = I_1 + I_2$  where  $I_1 = \int_t^\infty \dots$  and  $I_2 = \int_0^t \dots$

If  $t \leq s$ , write  $\frac{t^\alpha}{(s+t)^{1+\alpha}} \leq \frac{t^\alpha}{s^{1+\alpha}}$ ,  $\frac{r^{2m}}{(s+t)^m} \leq \frac{r^{2m}}{s^m}$  and change variable by setting  $\frac{4^j r^2}{s} = u$ . Then,

$$I_1 \leq \int_0^{\frac{4^j r^2}{t}} e^{-cu} \left( \frac{u}{4^j r^2} \right)^{\gamma/2} \left( \frac{tu}{4^j r^2} \right)^\alpha \frac{u^m}{4^{jm}} \frac{du}{u} = \frac{\beta^{-\alpha}}{4^{jm}(2^j r)^\gamma} \int_0^\beta e^{-cu} u^{\gamma/2+\alpha+m} \frac{du}{u}.$$

As  $\gamma/2 + \alpha + m > 0$ , we obtain

$$I_1 \leq \frac{C}{4^{jm}(2^j r)^\gamma} \inf(\beta^{\gamma/2+m}, \beta^{-\alpha}).$$

If  $t \geq s$  and  $\gamma > 0$ , write  $\frac{t^\alpha}{(s+t)^{1+\alpha}} \leq \frac{1}{s}$ ,  $\frac{r^{2m}}{(s+t)^m} \leq \frac{r^{2m}}{t^m}$  and change variable by setting  $\frac{4^j r^2}{s} = u$ . Then,

$$I_2 \leq \int_{\frac{4^j r^2}{t}}^\infty e^{-cu} \left( \frac{u}{4^j r^2} \right)^{\gamma/2} \frac{r^{2m}}{t^m} \frac{du}{u} = \frac{\beta^m}{4^{jm}(2^j r)^\gamma} \int_\beta^\infty e^{-cu} u^{\gamma/2} \frac{du}{u}.$$

Since  $\gamma > 0$ , we obtain

$$I_2 \leq \frac{C}{4^{jm}(2^j r)^\gamma} \inf(\beta^m, e^{-c\beta})$$

where the value of  $c$  has changed. The conclusion follows readily.

To treat the case  $\gamma = 0$  when  $\beta \leq 1$ , then use instead  $\frac{t^\alpha}{(s+t)^{1+\alpha}} \leq \frac{1}{t}$  and that  $\int_\varepsilon^\infty e^{-cu} \frac{du}{u^2}$  is asymptotic to  $c\varepsilon^{-1}$  as  $\varepsilon$  tends to 0. Further details are left to the reader.  $\square$

### 5.1.2 The case $p > 2$

Introduce the following sets:

$$\mathcal{I}_+(L) = \mathcal{I}(L) \cap (2, \infty)$$

$$\mathcal{N}_+(L) = \mathcal{N}(L) \cap (2, \infty]$$

$$\mathcal{K}_+(L) = \{2 < p \leq \infty; (\sqrt{t} \nabla e^{-tL})_{t>0} \text{ is } L^2 - L^p \text{ bounded}\}$$

$$\mathcal{M}_+(L) = \{2 < p \leq \infty; (\sqrt{t} \nabla e^{-tL})_{t>0} \text{ satisfies } L^2 - L^p \text{ off-diagonal estimates}\}$$

Observe that these sets, if nonempty, are intervals with 2 as lower limit. We know that  $\mathcal{N}_+(L)$  is not empty and that it has the same interior as  $\mathcal{K}_+(L)$  and  $\mathcal{M}_+(L)$ .

**Theorem 5.6.** *The above sets are intervals with common interiors.*

Let us state a result for the range  $2 < p < \infty$ .

**Proposition 5.7.**  $\nabla L^{-1/2}$  is bounded on  $L^p$  for  $2 < p < \infty$  if, and only if,  $(\sqrt{t} \nabla e^{-tL})_{t>0}$  is  $L^2 - L^p$  bounded for  $2 < p < \infty$ . In particular, if  $(\sqrt{t} \nabla e^{-tL})_{t>0}$  is  $L^2 - L^\infty$  bounded, then  $\nabla L^{-1/2}$  is bounded on  $L^p$  for  $2 < p < \infty$ .

*Proof.* The equivalence is contained in the theorem above. If  $(\sqrt{t} \nabla e^{-tL})_{t>0}$  is  $L^2 - L^\infty$  bounded then by interpolation with the  $L^2$ -boundedness, it is also  $L^2 - L^p$  bounded for  $2 < p < \infty$ .  $\square$

Let us prove Theorem 5.6. In view of the above remarks, it is enough to show

**Step 1.**  $p \in \mathcal{I}_+(L)$  implies  $p \in \mathcal{K}_+(L)$ .

**Step 2.**  $p_0 \in \mathcal{M}_+(L)$  and  $2 < p < p_0$  imply  $p \in \mathcal{I}_+(L)$ .

**Step 1.**  $p \in \mathcal{I}_+(L)$  implies  $p \in \mathcal{K}_+(L)$ .

Assume first that  $n \geq 3$ . By Sobolev embedding's, if  $q \in [2, p]$  with  $q < n$  then

$$L^{-1/2}: L^q \rightarrow L^{q^*}.$$

Following the method of Section 5.1.1 applied to the dual semigroup, we obtain that if  $q \in \mathbb{R}$  with  $2 \leq q \leq p^*$ ,

$$e^{-tL}: L^2 \rightarrow L^q$$

with bound  $Ct^{-\gamma_q}/2$ . Since,  $L^{1/2}e^{-(t/2)L}$  is bounded on  $L^2$  with bound  $Ct^{-1/2}$ , we obtain  $\nabla e^{-tL} = \nabla L^{-1/2}e^{-(t/2)L}L^{1/2}e^{-(t/2)L}$  is bounded from  $L^2$  to  $L^p$  with the desired bound on the operator norm.

If  $n \leq 2$ , we already know that

$$e^{-tL}: L^2 \rightarrow L^q$$

for all  $q \in [2, \infty]$  (See Section 4). The argument is then similar.



**Step 2.**  $p_0 \in \mathcal{M}_+(L)$  and  $2 < p < p_0$  imply  $p \in \mathcal{I}_+(L)$ : We apply Theorem 2.2 to  $T = \nabla L^{-1/2}$ . We set again  $A_r = I - (I - e^{-r^2 L})^m$  for some integral number  $m$  to be chosen. We have to check (2.3) and (2.4).

We begin with the following lemma.

**Lemma 5.8.** *For every ball  $B$  with radius  $r > 0$ ,*  
(5.5)

$$\|\nabla L^{-1/2}(I - e^{-r^2 L})^m f\|_{L^2(B)} \leq |B|^{1/2} \sum_{j \geq 1} g(j) \left( \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f|^2 \right)^{1/2}$$

with  $g(j) = C2^{(n/2)j}4^{-mj}$ .

Hence, (2.3) follows provided  $m > n/4$ , since  $\left( \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f|^2 \right)^{1/2}$  is controlled by  $(M(|f|^2)(y))^{1/2}$  for any  $y \in B$ .

*Proof.* Fix a ball  $B$  and  $r = r(B)$  its radius. Decompose  $f$  as  $f_1 + f_2 + f_3 + \dots$  with  $f_j = f\chi_{C_j}$  where  $\chi_{C_j}$  is the indicator function of  $C_j = C_j(B)$ . By Minkowski inequality we have that

$$\|\nabla L^{-1/2}(I - e^{-r^2 L})^m f\|_{L^2(B)} \leq \sum_{j \geq 1} \|\nabla L^{-1/2}(I - e^{-r^2 L})^m f_j\|_{L^2(B)}.$$

For  $j = 1$  we use  $L^2$  boundedness of  $\nabla L^{-1/2}(I - e^{-r^2 L})^m$ :

$$\|\nabla L^{-1/2}(I - e^{-r^2 L})^m f_1\|_{L^2(B)} \lesssim \|f\|_{L^2(4B)} \leq |4B|^{1/2} \left( \frac{1}{|4B|} \int_{4B} |f|^2 \right)^{1/2}.$$

**First proof in the case  $j \geq 2$ :** Expanding  $(I - e^{-r^2 L})^m$  in the representation of the square root, we find

$$\begin{aligned} \nabla L^{-1/2}(I - e^{-r^2 L})^m f_j &= \pi^{-1/2} \int_0^\infty \nabla e^{-tL} (I - e^{-r^2 L})^m f_j \frac{dt}{\sqrt{t}} \\ &= \pi^{-1/2} \int_0^\infty g_{r^2}(t) \nabla e^{-tL} f_j dt \end{aligned}$$

where, as in Section 5.1.1,

$$g_s(t) = \sum_{k=0}^m \binom{m}{k} (-1)^k \frac{\chi_{\{t > ks\}}}{\sqrt{t - ks}}.$$

By Minkowski integral inequality and the  $L^2$  off-diagonal estimates for  $(\sqrt{t}\nabla e^{-tL})_{t>0}$  using the support of  $f_j$ , we have that

$$\|\nabla L^{-1/2}(I - e^{-r^2L})^m f_j\|_{L^2(B)} \leq C \int_0^\infty |g_{r^2}(t)| e^{-\frac{c4^j r^2}{t}} \frac{dt}{\sqrt{t}} \|f\|_{L^2(C_j)}.$$

As in Section 5.1.1, the latter integral is bounded above by  $C4^{-jm}$  uniformly over  $r > 0$ . Next,

$$\|f\|_{L^2(C_j)} \leq |2^{j+1}B|^{1/2} \left( \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f|^2 \right)^{1/2}$$

and we obtain (5.5).

**Second proof in the case  $j \geq 2$ :** Write again (5.2):

$$\nabla L^{-1/2}(I - e^{-r^2L})^m f_j = \pi^{-1/2} \int_0^\infty \sqrt{t} \nabla e^{-tL} (I - e^{-r^2L})^m f_j \frac{dt}{t}.$$

As in Section 5.1.1, one may use the representation (3.6) with the function  $\varphi(z) = e^{-tz}(1 - e^{r^2z})^m$ . The functions  $\eta_\pm$  in (3.7) satisfy the estimates (5.3). Since for any  $0 < \beta < \frac{\pi}{2} - \omega$ ,  $(\sqrt{z}\nabla e^{-zL})_{z \in \Sigma_\beta}$  satisfies  $L^2$  off-diagonal estimates, using (3.6) and the estimate for  $\eta_\pm$ ,  $\|\sqrt{t}\nabla e^{-tL}(I - e^{-r^2L})^m f_j\|_{L^2(B)}$  is bounded by

$$C \int_{\Gamma_+} e^{-\frac{c4^j r^2}{|z|}} \frac{t^{1/2}}{|z|^{1/2} (|z| + t)} \frac{1}{(|z| + t)^m} r^{2m} |dz| \|f\|_{L^2(C_j)}$$

plus the similar term corresponding to integration on  $\Gamma_-$ . Using (5.4), this gives us the bound

$$\frac{C}{4^{jm}} \inf \left( \left( \frac{t}{4^j r^2} \right)^{1/2}, \left( \frac{4^j r^2}{t} \right)^{m-1/2} \right) \|f\|_{L^2(C_j)}.$$

Integrating with respect to  $t$  in (5.2), one finds (5.5) by Minkowski inequality since  $m \geq 1 > 1/2$ .  $\square$

We now show that (2.4) holds. By expanding  $A_r = I - (I - e^{-r^2L})^m$  it suffices to show

$$(5.6) \quad \left( \frac{1}{|B|} \int_B |\nabla e^{-kr^2L} f|^{p_0} \right)^{1/p_0} \leq \sum_{j \geq 1} g(j) \left( \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |\nabla f|^2 \right)^{1/2}$$

for every ball  $B$  with  $r = r(B)$ , and  $k = 1, 2, \dots, m$  with  $\sum g(j) < \infty$ . Recall that  $m$  is chosen larger than  $n/4$ . This, applied to  $f = L^{-1/2}g$  for appropriate  $g$ , gives us (2.4).

Using the conservation property (3.12) of the semigroup, we have

$$\nabla e^{-kr^2L} f = \nabla e^{-kr^2L}(f - f_{4B})$$

where  $f_E$  is the mean of  $f$  over  $E$ . Write  $f - f_{4B} = f_1 + f_2 + f_3 + \dots$  where  $f_j = (f - f_{4B})\chi_{C_j(B)}$ . For  $j = 1$ , we use the fact that  $p_0 \in \mathcal{K}_+$ , that is that  $(\sqrt{t}\nabla e^{-tL})_{t>0}$  satisfies  $L^2 - L^{p_0}$  off-diagonal estimates, and Poincaré inequalities to obtain

$$\begin{aligned} \left( \frac{1}{|B|} \int_B |\nabla e^{-kr^2L} f_1|^{p_0} \right)^{1/p_0} &\leq C \left( \frac{1}{r^2|4B|} \int_{4B} |f - f_{4B}|^2 \right)^{1/2} \\ &\leq C \left( \frac{1}{|4B|} \int_{4B} |\nabla f|^2 \right)^{1/2} \end{aligned}$$

For  $j \geq 2$ , we have similarly by the assumption on  $p_0$  and the various support assumptions,

$$\left( \frac{1}{|B|} \int_B |\nabla e^{-kr^2L} f_j|^{p_0} \right)^{1/p_0} \leq \frac{C e^{-\alpha 4^j} 2^{jn/2}}{r} \left( \frac{1}{|2^{j+1}B|} \int_{C_j(B)} |f_j|^2 \right)^{1/2}.$$

But

$$\begin{aligned} \int_{C_j(B)} |f_j|^2 &\leq \int_{2^{j+1}B} |f - f_{4B}|^2, \\ |f - f_{4B}| &\leq |f - f_{2^{j+1}B}| + \sum_{\ell=2}^j |f_{2^\ell B} - f_{2^{\ell+1}B}| \end{aligned}$$

and observe that by Poincaré inequality,

$$|f_{2^\ell B} - f_{2^{\ell+1}B}|^2 \leq \frac{2^n}{|2^{\ell+1}B|} \int_{2^{\ell+1}B} |f - f_{2^{\ell+1}B}|^2 \leq \frac{C(2^\ell r)^2}{|2^{\ell+1}B|} \int_{2^{\ell+1}B} |\nabla f|^2.$$

Hence, by Minkowski inequality, we easily obtain

$$\left( \frac{1}{|B|} \int_B |\nabla e^{-kr^2L} f_j|^{p_0} \right)^{1/p_0} \leq C e^{-\alpha 4^j} 2^{jn/2} \sum_{\ell=1}^j 2^\ell \left( \frac{1}{|2^{\ell+1}B|} \int_{2^{\ell+1}B} |\nabla f|^2 \right)^{1/2}$$

and summing over  $j \geq 2$  gives us (5.6).

This concludes the proof of Theorem 5.6.

## 5.2 Reverse inequalities

In this section, we study the reverse inequality to the Riesz transform  $L^p$  boundedness. Recall the maximal interval,  $\mathcal{R}(L)$ , of exponents  $p$  in  $(1, \infty)$  for which one has the *a priori* inequality  $\|L^{1/2}f\|_p \lesssim \|\nabla f\|_p$  for  $f \in C_0^\infty$  and write  $\text{int}\mathcal{R}(L) = (r_-(L), r_+(L))$ . We know so far that  $\text{int}\mathcal{I}(L) = (p_-(L), q_+(L))$  and that  $p_+(L) \geq (q_+(L))^*$ . We show the following bounds.

**Theorem 5.9.** *We have*

$$\begin{aligned} r_-(L) &\leq \sup(1, (p_-(L))^*), \\ r_+(L) &\geq p_+(L). \end{aligned}$$

Hence,  $\mathcal{R}(L)$  contains a neighborhood of the closure (in  $\mathbb{R}$ ) of  $\mathcal{I}(L)$ .

We discuss the optimality of these bounds in Section 6.4.

This result is a consequence of Lemma 5.10 and Proposition 5.11 below. We begin with a duality principle which applies for all  $p$  in  $(1, \infty)$  but which gives us the bound in Theorem 5.9 only for  $r_+(L)$ .

**Lemma 5.10.** *If  $1 < p < \infty$  and  $\nabla L^{-1/2}$  is bounded on  $L^p$  then  $\|(L^*)^{1/2}f\|_{p'} \lesssim \|\nabla f\|_{p'}$  holds for  $f \in C_0^\infty$ . Hence,  $r_+(L) \geq p_+(L)$ .*

*Proof.* Let  $f \in C_0^\infty$ . Then  $(L^*)^{1/2}f$  is defined and belongs to  $L^2$ . We estimate  $\|(L^*)^{1/2}f\|_{p'}$  by testing against  $g \in L^2 \cap L^p$ . Since  $g \in L^2$ , we have  $g = L^{1/2}L^{-1/2}g$  and  $h = L^{-1/2}g \in \dot{W}^{1,2}$ . Hence, by Lemma 3.2,

$$\langle (L^*)^{1/2}f, g \rangle = \int_{\mathbb{R}^n} \nabla f \cdot \overline{A\nabla h}.$$

Since  $\|\nabla h\|_p \lesssim \|g\|_p$ , it follows that  $\|(L^*)^{1/2}f\|_{p'} \lesssim \|\nabla f\|_{p'}$  as desired.

Let  $2 < p < p_+(L)$ . By duality,  $p_-(L^*) < p' < 2$ , hence,  $\nabla(L^*)^{-1/2}$  is bounded on  $L^{p'}$  by Theorem 5.2 and  $p \in \mathcal{R}(L)$ . Thus,  $r_+(L) > p$  and the conclusion follows.  $\square$

The above lemma does not give an interesting information for  $r_-(L)$ . The next proposition yields a much better bound. Define  $\tilde{p}_-(L) = \sup(1, (p_-(L))^*)$ .

**Proposition 5.11.** *Let  $1 < p < 2$ . If  $\nabla L^{-1/2}$  is bounded on  $L^p$  and  $p \geq \frac{n}{n-1}$  then  $\|L^{1/2}f\|_q \lesssim \|\nabla f\|_q$  for  $p_* < q < 2$ . If  $\nabla L^{-1/2}$  is bounded on  $L^p$  and  $p < \frac{n}{n-1}$  then  $\|L^{1/2}f\|_q \lesssim \|\nabla f\|_q$  for  $1 < q < 2$ . Hence,  $r_-(L) \leq \tilde{p}_-(L)$ .*

The inequality on  $r_-(L)$  follows from the definition of  $r_-(L)$  and the identification of  $p_-(L)$  as the lower limit of  $\mathcal{I}(L)$ .

Let us remark that for  $n \geq 2$  and  $p < 2$ , there exists an  $L$  such that  $p \in \mathcal{I}_-(L)$  but  $p' \notin \mathcal{I}_+(L^*)$ .<sup>45</sup> Hence, no duality argument can help us here. In fact, the argument will rely on a Calderón-Zygmund decomposition for Sobolev functions which is proved in the Appendix.

**Lemma 5.12.** *Let  $n \geq 1$ ,  $1 \leq p \leq \infty$  and  $f \in \mathcal{D}'(\mathbb{R}^n)$  be such that  $\|\nabla f\|_p < \infty$ . Let  $\alpha > 0$ . Then, one can find a collection of cubes  $(Q_i)$ , functions  $g$  and  $b_i$  such that*

$$(5.7) \quad f = g + \sum_i b_i$$

and the following properties hold:

$$(5.8) \quad \|\nabla g\|_\infty \leq C\alpha,$$

$$(5.9) \quad b_i \in W_0^{1,p}(Q_i) \text{ and } \int_{Q_i} |\nabla b_i|^p \leq C\alpha^p |Q_i|,$$

$$(5.10) \quad \sum_i |Q_i| \leq C\alpha^{-p} \int_{\mathbb{R}^n} |\nabla f|^p,$$

$$(5.11) \quad \sum_i \mathbf{1}_{Q_i} \leq N,$$

where  $C$  and  $N$  depend only on dimension and  $p$ .

The space  $W_0^{1,p}(\Omega)$  denotes the closure of  $C_0^\infty(\Omega)$  in  $W^{1,p}(\Omega)$ . The point is in the fact that the functions  $b_i$  are supported in cubes as the original Calderón-Zygmund decomposition applied to  $\nabla f$  would not give this.

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<sup>45</sup> This is due to Kenig. See [AT, Chapter IV, Theorem 7].

**Proof of Proposition 5.11:** By Theorem 5.2 we may transform the hypothesis on the Riesz transform, *i.e.*  $p \in \mathcal{I}_-(L)$ , into an hypothesis on the semigroup. Proposition 5.11 is therefore a consequence of the next result combined with Marcinkiewicz interpolation.

**Lemma 5.13.** *Let  $\rho \in \mathcal{M}_-(L)$ . Then we have*

$$(5.12) \quad \|L^{1/2}f\|_{p,\infty} \lesssim \|\nabla f\|_p, \quad \text{if } 1 \leq \rho_* < p < 2,$$

$$(5.13) \quad \|L^{1/2}f\|_{1,\infty} \lesssim \|\nabla f\|_1, \quad \text{if } \rho_* < 1.$$

*Proof.* Recall that  $\rho_* = \frac{n\rho}{n+\rho}$ . Of course, it is enough to pick  $\rho$  and  $p$  as small as possible. If  $n = 1, 2$ , then  $\rho$  may be chosen with  $\rho_* < 1$  (as a consequence of Corollary 4.6) and we set  $p = 1$  in the proof. If  $n \geq 3$ , then one can always assume that  $\rho < p_n = \frac{2n}{n+2}$  and we pick  $p$  so that  $\rho_* < p < p_n$ . Let  $f \in C_0^\infty$ . We have to establish the weak type estimate

$$(5.14) \quad |\{x \in \mathbb{R}^n; |L^{1/2}f(x)| > \alpha\}| \leq \frac{C}{\alpha^p} \int |\nabla f|^p,$$

for all  $\alpha > 0$ . We use the following resolution of  $L^{1/2}$ :

$$L^{1/2}f = c \int_0^\infty e^{-t^2L} Lf dt$$

where  $c = 2\pi^{-1/2}$  is forgotten from now on. It suffices to obtain the result for the truncated integrals  $\int_\varepsilon^R \dots$  with bounds independent of  $\varepsilon, R$ , and then to let  $\varepsilon \downarrow 0$  and  $R \uparrow \infty$ . For the truncated integrals, all the calculations are justified. We ignore this step and argue directly on  $L^{1/2}$ . Apply the Calderón-Zygmund decomposition of Lemma 5.12 to  $f$  at height  $\alpha^p$  and write  $f = g + \sum_i b_i$ . By construction,  $\|\nabla g\|_p \leq c\|\nabla f\|_p$ . Interpolating with (5.8) yields  $\int |\nabla g|^2 \leq c\alpha^{2-p} \int |\nabla f|^p$ . Hence

$$\left| \left\{ x \in \mathbb{R}^n; |L^{1/2}g(x)| > \frac{\alpha}{3} \right\} \right| \leq \frac{C}{\alpha^2} \int |L^{1/2}g|^2 \leq \frac{C}{\alpha^2} \int |\nabla g|^2 \leq \frac{C}{\alpha^p} \int |\nabla f|^p$$

where we used the  $L^2$ -estimate (3.11) for square roots. To compute  $L^{1/2}b_i$ , let  $r_i = 2^k$  if  $2^k \leq \ell_i = \ell(Q_i) < 2^{k+1}$  and set  $T_i = \int_0^{r_i} e^{-t^2L} L dt$  and  $U_i = \int_{r_i}^\infty e^{-t^2L} L dt$ . It is enough to estimate  $A = |\{x \in \mathbb{R}^n; |\sum_i T_i b_i(x)| > \alpha/3\}|$  and  $B = |\{x \in \mathbb{R}^n; |\sum_i U_i b_i(x)| > \alpha/3\}|$ . Let us bound the first term.

First,

$$A \leq |\cup_i 4Q_i| + \left| \left\{ x \in \mathbb{R}^n \setminus \cup_i 4Q_i; \left| \sum_i T_i b_i(x) \right| > \frac{\alpha}{3} \right\} \right|,$$

and by (5.10),  $|\cup_i 4Q_i| \leq \frac{C}{\alpha^p} \int |\nabla f|^p$ .

For the other term, we have

$$\left| \left\{ x \in \mathbb{R}^n \setminus \cup_i 4Q_i; \left| \sum_i T_i b_i(x) \right| > \frac{\alpha}{3} \right\} \right| \leq \frac{C}{\alpha^2} \int \left| \sum_i h_i \right|^2$$

with  $h_i = \mathbf{1}_{(4Q_i)^c} |T_i b_i|$ . To estimate the  $L^2$  norm, we dualize against  $u \in L^2$  with  $\|u\|_2 = 1$ :

$$\int |u| \sum_i h_i = \sum_i \sum_{j=2}^{\infty} A_{ij}$$

where

$$A_{ij} = \int_{C_j(Q_i)} |T_i b_i| |u|.$$

Let  $q = 2$  if  $n \leq 2$  and  $q = p^* = \frac{np}{n-p}$  if  $n \geq 3$ . Observe that  $\rho < q \leq 2$  and that  $p \leq q \leq p^*$ .

Since  $\rho < q \leq 2$ , the family  $(tLe^{-tL})_{t>0}$  satisfies  $L^q - L^2$  off-diagonal estimates on combining Proposition 4.2 and Proposition 4.15. Hence, using also  $r_i \sim \ell_i$ ,

$$\|e^{-t^2 L} L b_i\|_{L^2(C_j(Q_i))} \leq \frac{C}{t^{\gamma+2}} e^{-\frac{c^4 j r_i^2}{t^2}} \|b_i\|_q$$

where  $\gamma = \gamma_q = |\frac{n}{2} - \frac{n}{q}|$ . By Poincaré-Sobolev inequality (since  $p \leq q \leq p^*$ ) and (5.9),

$$\|b_i\|_q \leq c \ell_i^{1 - (\frac{n}{p} - \frac{n}{q})} \|\nabla b_i\|_p \leq c \alpha \ell_i^{1 + \frac{n}{q}}.$$

Hence, by Minkowski integral inequality, for some appropriate positive constants  $C, c$ ,

$$\begin{aligned} \|T_i b_i\|_{L^2(C_j(Q_i))} &\leq \int_0^{r_i} \|e^{-t^2 L} L b_i\|_{L^2(C_j(Q_i))} dt \\ &\leq C \alpha e^{-c^4 j} \ell_i^{\frac{n}{2}}, \end{aligned}$$

Now remark that for any  $y \in Q_i$  and any  $j \geq 2$ ,

$$\left( \int_{C_j(Q_i)} |u|^2 \right)^{1/2} \leq \left( \int_{2^{j+1}Q_i} |u|^2 \right)^{1/2} \leq (2^{n(j+1)}|Q_i|)^{1/2} (M(|u|^2)(y))^{1/2}.$$

Applying Hölder inequality, one obtains

$$A_{ij} \leq C\alpha 2^{nj/2} e^{-c4^j} \ell_i^n (M(|u|^2)(y))^{1/2}.$$

Averaging over  $Q_i$  yields

$$A_{ij} \leq C\alpha 2^{nj/2} e^{-c4^j} \int_{Q_i} (M(|u|^2)(y))^{1/2} dy.$$

Summing over  $j \geq 2$  and  $i$ , we have

$$\int |u| \sum_i h_i \leq C\alpha \int \sum_i \mathbf{1}_{Q_i}(y) (M(|u|^2)(y))^{1/2} dy.$$

As in the proof of Theorem 2.1 using finite overlap (5.11) of the cubes  $Q_i$  and Kolmogorov's inequality, one obtains

$$\int |u| \sum_i h_i \leq C' N\alpha |\cup_i Q_i|^{1/2} \| |u|^2 \|_1^{1/2}.$$

Hence

$$\left| \left\{ x \in \mathbb{R}^n \setminus \cup_i 4Q_i; \left| \sum_i T_i b_i(x) \right| > \frac{\alpha}{3} \right\} \right| \leq C |\cup_i Q_i| \leq \frac{C}{\alpha^p} \int |\nabla f|^p$$

by (5.11) and (5.10).

It remains to handling the term  $B$ . Using functional calculus for  $L$  one can compute  $U_i$  as  $r_i^{-1}\psi(r_i^2 L)$  with  $\psi$  the holomorphic function on the sector  $|\arg z| < \frac{\pi}{2}$  given by

$$(5.15) \quad \psi(z) = \int_1^\infty e^{-t^2 z} z dt.$$

It is easy to show that  $|\psi(z)| \leq C|z|^{1/2} e^{-c|z|^2}$ , uniformly on subsectors  $|\arg z| \leq \mu < \frac{\pi}{2}$ .

We invoke the following lemma proved by duality in Section 7. <sup>46</sup>

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<sup>46</sup> It can also be proved directly using the vector-valued extension of Theorem 2.1.



**Lemma 5.14.** *If  $\rho \in \mathcal{M}_-$  then for  $\rho < q \leq 2$*

$$(5.16) \quad \left\| \sum_{k \in \mathbb{Z}} \psi(4^k L) \beta_k \right\|_q \lesssim \left\| \left( \sum_{k \in \mathbb{Z}} |\beta_k|^2 \right)^{1/2} \right\|_q,$$

whenever the right hand side is finite.

To apply this lemma, observe that the definitions of  $r_i$  and  $U_i$  yield

$$\sum_i U_i b_i = \sum_{k \in \mathbb{Z}} \psi(4^k L) \beta_k$$

with

$$\beta_k = \sum_{i, r_i = 2^k} \frac{b_i}{r_i}.$$

Using the bounded overlap property (5.11), one has that

$$\left\| \left( \sum_{k \in \mathbb{Z}} |\beta_k|^2 \right)^{1/2} \right\|_q^q \leq C \int \sum_i \frac{|b_i|^q}{r_i^q}.$$

By (9.2), and  $p \leq q \leq p^*$ , together with  $\ell_i \sim r_i$ ,

$$\int \sum_i \frac{|b_i|^q}{r_i^q} \leq C \alpha^q \sum_i |Q_i|.$$

Hence, by (5.10)

$$\left| \left\{ x \in \mathbb{R}^n; \left| \sum_i U_i b_i(x) \right| > \frac{\alpha}{3} \right\} \right| \leq C \sum_i |Q_i| \leq \frac{C}{\alpha^p} \int |\nabla f|^p.$$

□

### 5.3 Invertibility

We finish the study of square roots with their invertibility properties on  $L^p$  spaces. This is summed up in the following theorem. Recall that  $\mathcal{I}(L)$  is the maximal interval of exponents  $p \in (1, \infty)$  for which  $\nabla L^{-1/2}$  is bounded on  $L^p$ .

**Theorem 5.15.** *Let  $1 < p < \infty$ . Then  $p \in \mathcal{I}(L)$  if and only if  $L^{1/2}$ , a priori defined from  $C_0^\infty$  into  $L^2$ , extends to an isomorphism from  $\dot{W}^{1,p}$  onto  $L^p$  with  $\|L^{1/2}f\|_p \sim \|\nabla f\|_p$ . Furthermore,  $\mathcal{I}(L)$  is an open interval.*

*Proof.* For  $p = 2$ , the equivalence is a consequence of the solution of the Kato's conjecture (3.11) as recalled in Section 3.4.

For other values of  $p$ , one implication is straightforward. Conversely, we know from Theorem 5.9 that  $\mathcal{R}(L)$ , the maximal interval of exponents  $p \in (1, \infty)$  for which one has the *a priori* inequality  $\|L^{1/2}f\|_p \lesssim \|\nabla f\|_p$ , contains a neighborhood of the closure (in  $\mathbb{R}$ ) of  $\mathcal{I}(L)$ .

In particular, if  $\nabla L^{-1/2}$  is bounded on  $L^p$  then for the same  $p$ ,  $L^{1/2}$  can be extended boundedly from  $\dot{W}^{1,p}$  into  $L^p$ . The isomorphism property is now easy. Indeed, from  $\|L^{1/2}f\|_p \sim \|\nabla f\|_p$ , we deduce that this extension is one-one and has closed range in  $L^p$ . It remains to establish the density of this range. If  $g \in L^p \cap L^2$ , we have  $g = L^{1/2}L^{-1/2}g$  and by  $\|\nabla L^{-1/2}g\|_p \leq c\|g\|_p < \infty$ , we conclude that  $g$  is in  $L^{1/2}(\dot{W}^{1,p})$ . Thus  $L^{1/2}(\dot{W}^{1,p})$  contains a dense subspace of  $L^p$ .

The openness of  $\mathcal{I}(L)$  is a consequence of the following result.

**Lemma 5.16.** <sup>47</sup> *Let  $X^s, Y^s$ , be two scales of complex interpolation Banach spaces,  $s$  describing an open interval  $I$ . If  $T: X^s \rightarrow Y^s$  is bounded for each  $s \in I$ , then the set of  $s \in I$  for which  $T$  is an isomorphism from  $X^s$  onto  $Y^s$  is open.*

Apply this result (changing  $s$  to  $p$ ) with  $I = \text{int}\mathcal{R}(L)$ ,  $X^p = \dot{W}^{1,p}$ ,  $Y^p = L^p$  and  $T = L^{1/2}$ : we just proved that the set of  $p \in \text{int}\mathcal{R}(L)$  for which  $L^{1/2}$  is an isomorphism from  $\dot{W}^{1,p}$  onto  $L^p$  is  $\mathcal{I}(L)$ . Thus,  $\mathcal{I}(L)$  is open.  $\square$

## 5.4 Applications

We have developed the necessary theory to reobtain all the results mentioned in the Introduction. Let us start with results holding in all generality.

**Proposition 5.17.** <sup>48</sup> *If  $n = 1$ , we have  $\|L^{1/2}f\|_p \sim \|\frac{d}{dx}f\|_p$  and  $L^{1/2}$  extends to an isomorphism from  $\dot{W}^{1,p}$  onto  $L^p$  when  $1 < p < \infty$ .*

<sup>47</sup> This is due to I. Sneiberg ([Sn]).

<sup>48</sup> This is first proved in [AT1], Theorem A. More is proved there: in that  $a\frac{d}{dx}L^{-1/2}$  is a Calderón-Zygmund operator and the boundedness properties at  $p = 1$  and  $p = \infty$  are studied.

*Proof.* We know from Corollary 4.6 that  $p_-(L) = 1$  and from Proposition 4.12 that  $q_+(L) = \infty$ . Hence  $\mathcal{I}(L) = (1, \infty)$  and the conclusion follows from Theorem 5.15.  $\square$

**Proposition 5.18.** <sup>49</sup> *If  $n = 2$ , we have  $\|\nabla L^{-1/2} f\|_p \lesssim \|f\|_p$  for  $1 < p < 2 + \varepsilon$  and  $\|L^{1/2} f\|_p \lesssim \|\nabla f\|_p$  for  $1 < p < \infty$ . Furthermore,  $L^{1/2}$  extends to an isomorphism from  $\dot{W}^{1,p}$  onto  $L^p$  when  $1 < p < 2 + \varepsilon$ .*

*Proof.* We know from Corollary 4.6 that  $p_-(L) = 1$  and from Corollary 4.10 that  $q_+(L) > 2$ . Hence,  $\mathcal{I}(L) = (1, 2 + \varepsilon)$  and we have the isomorphism property from Theorem 5.15.

Now,  $r_-(L) = 1$  and  $r_+(L) \geq p_+(L) = (p_-(L^*))' = \infty$ . Hence  $\|L^{1/2} f\|_p \lesssim \|\nabla f\|_p$  for  $1 < p < \infty$  by Theorem 5.9.  $\square$

**Proposition 5.19.** <sup>50</sup> *If  $n \geq 3$ , then  $\|\nabla L^{-1/2} f\|_p \lesssim \|f\|_p$  for  $\frac{2n}{n+2} - \varepsilon < p < 2 + \varepsilon'$  and  $\|L^{1/2} f\|_p \lesssim \|\nabla f\|_p$  for  $\sup(1, \frac{2n}{n+4} - \varepsilon_1) < p < \frac{2n}{n-2} + \varepsilon'_1$  with  $\varepsilon, \varepsilon', \varepsilon_1, \varepsilon'_1 > 0$  depending on dimension and the ellipticity constants of  $L$ . Furthermore,  $L^{1/2}$  extends to an isomorphism from  $\dot{W}^{1,p}$  onto  $L^p$  when  $\frac{2n}{n+2} - \varepsilon < p < 2 + \varepsilon'$ .*

*Proof.* We know from Corollary 4.6 that  $p_-(L) < \frac{2n}{n+2}$  and from Corollary 4.10 that  $q_+(L) > 2$ . Hence,  $\mathcal{I}(L) = (\frac{2n}{n+2} - \varepsilon, 2 + \varepsilon')$  and the isomorphism property follows from Theorem 5.15. By Theorem 5.9,  $r_-(L) \leq \sup(1, (p_-(L))^*)$  and  $r_+(L) \geq p_+(L) = (p_-(L^*))'$ . Thus  $\|L^{1/2} f\|_p \lesssim \|\nabla f\|_p$  for  $\sup(1, \frac{2n}{n+4} - \varepsilon_1) < p < \frac{2n}{n-2} + \varepsilon'_1$ .  $\square$

*Remark.* If  $n \geq 2$ , the bound  $p < 2 + \varepsilon'$  for the Riesz transform  $L^p$  boundedness is sharp as  $q_+(L) > 2$  is optimal (see Section 4.5).

A new fact is the following negative result.

**Corollary 5.20.** *If  $n \geq 5$ , there exists  $p \in (1, 2)$  and an operator  $L$  for which  $\nabla L^{-1/2}$  is not bounded on  $L^p$ .*

*Proof.* If  $n \geq 5$ , there exists an operator  $L$  for which  $p_-(L) > 1$  (see Section 4.5). It remains to invoke Theorem 5.2.  $\square$

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<sup>49</sup> This follows from [AT], Chapter IV, Theorem 1, using [AMcT], Theorem 3.5 and [AHLMcT], Theorem 1.4.

<sup>50</sup> The first inequality is proved in [BK2], [HM] for the range  $\frac{2n}{n+2} < p < 2$  and the extension to this larger range is in [A], the second inequality is in [HM] for the range  $2 < p < \frac{2n}{n-2}$ , and in [A] for the rest of the range.

Let us come to results where further hypotheses may be taken on  $L$ .

**Corollary 5.21.** <sup>51</sup> *If  $n \geq 3$  and  $L$  has **real** coefficients, then  $\|\nabla L^{-1/2} f\|_p \lesssim \|f\|_p$  for  $1 < p < 2 + \varepsilon$  and  $\|L^{1/2} f\|_p \lesssim \|\nabla f\|_p$  for  $1 < p < \infty$ . Furthermore,  $L^{1/2}$  extends to an isomorphism from  $W^{1,p}$  onto  $L^p$  when  $1 < p < 2 + \varepsilon$ .*

*Proof.* The semigroup  $(e^{-tL})_{t>0}$  is contracting on  $L^1$ . By Proposition 4.2, part 1, this implies  $L^1 - L^2$  boundedness. By Proposition 5.3, we obtain  $\|\nabla L^{-1/2} f\|_p \lesssim \|f\|_p$  for  $1 < p < 2$ . The rest of the proof is as the one of Proposition 5.18.  $\square$

*Remark.* If  $n \geq 2$  and  $L$  has continuous and periodic coefficients with common period then a careful spectral analysis of the semigroup yields that the Riesz transform is bounded on  $L^p$  for all  $p \in (1, \infty)$ . <sup>52</sup> As a consequence, one obtains uniform gradient bounds on the heat operator from  $L^p$  to  $L^p$ , for any  $1 < p < \infty$ .

*Remark.* If  $n \geq 2$  and  $L$  has real, Hölder continuous and quasiperiodic coefficients satisfying Koslov condition then by homogeneization techniques, the Riesz transform is proved to be bounded on  $L^p$  for all  $p \in (1, \infty)$ . <sup>53</sup> As a consequence, one obtains uniform gradient bounds on the heat operator from  $L^p$  to  $L^p$ , for any  $1 < p < \infty$ .

*Remark.* If  $n \geq 2$  and  $L$  has almost-periodic coefficients then the semigroup is  $L^1 - L^2$  bounded (in fact, much more is true), which is enough to conclude that the Riesz transform is bounded on  $L^p$  for all  $p \in (1, 2)$ . The situation for  $p > 2$  is unclear. <sup>54</sup>

## 5.5 Riesz transforms and Hodge decomposition

An  $L^p$  Hodge decomposition adapted to the operator  $L$  consists in writing a field  $f \in L^p$  into the sum of fields  $g + \nabla h$  where  $g, \nabla h \in L^p$  with  $\|g\|_p + \|\nabla h\|_p \leq c\|f\|_p$  and  $\operatorname{div}(Ag) = 0$ . This amounts to the boundedness of the Hodge projection  $\nabla L^{-1} \operatorname{div}$  on  $L^p$ , that is an inequality of the type

$$(5.17) \quad \|\nabla L^{-1} \operatorname{div} f\|_p \lesssim \|f\|_p.$$

<sup>51</sup> This is first proved in [AT], Chapter 4, using [AHLMcT], Theorem 1.4 and Aronson's estimates in [Ar].

<sup>52</sup> This is in [ERS], Theorem 1.1.

<sup>53</sup> This is in [Al], Theorem 1.2.

<sup>54</sup> This is in [DER], Theorem 1.1.

Indeed, formally, one has

$$\nabla L^{-1} \operatorname{div}(Af) = \nabla h.$$

Alternately, it can be seen as the boundedness for the second order Riesz transform,  $\nabla L^{-1/2}(\nabla(L^*)^{-1/2})^*$ .

For  $p = 2$ , the inequality 5.17 is automatic by construction of  $L$ . For  $p \neq 2$ , we see here the connection to Riesz transforms estimates. Before, we put aside the one dimensional case.

**Proposition 5.22.** *If  $n = 1$ ,  $\frac{d}{dx}L^{-1}\frac{d}{dx}$  extends to a bounded operator on  $L^p$ ,  $1 < p < \infty$ , equal to the negative of the operator of pointwise multiplication with  $\frac{1}{a(x)}$ .*

*Proof.* Let  $D$  be the space of compactly supported and  $C^1$  functions  $f$  with  $\int_{-\infty}^{\infty} \frac{f(t)}{a(t)} dt = 0$ . It is easy to see that  $D$  is dense in  $L^p$  when  $1 < p < \infty$ . For  $f$  in  $D$ , the unique solution in  $\dot{W}^{1,2}$  of  $Lu = \frac{df}{dx}$  is given by

$$u(x) = \int_x^{\infty} \frac{f(t)}{a(t)} dt, \quad x \in \mathbb{R}.$$

We have,  $u'(x) = -\frac{f(x)}{a(x)}$  and  $u' \in L^p$  with  $\|u'\|_p \leq \|\frac{1}{a}\|_{\infty} \|f\|_p$  since  $\frac{1}{a}$  is a bounded function. Thus the boundedness property holds from  $D$  into  $L^p$  and the conclusion of the proposition follows readily.  $\square$

Recall that Riesz transforms inequalities hold in the full range  $1 < p < \infty$  when  $n = 1$ .

We now restrict our attention to dimensions  $n \geq 2$ . In that case, (5.17) is always valid for  $|\frac{1}{2} - \frac{1}{p}| < \varepsilon$ , the value of  $\varepsilon$  depending on the ellipticity constant of  $L$  and dimension.

**Theorem 5.23.** *1. Let  $2 < p < \infty$  and  $n \geq 2$ . Then  $\nabla L^{-1} \operatorname{div}$  is bounded on  $L^p$  if and only if  $\nabla L^{-1/2}$  is bounded on  $L^p$ .*

*2. Let  $1 < p < 2$  and  $n \geq 3$ . If  $\nabla L^{-1} \operatorname{div}$  is bounded on  $L^p$  then  $\nabla L^{-1/2}$  is bounded on  $L^q$  for  $\sup(1, p_*) < q < 2$ .*

*Proof.* Let us begin the argument with the second statement. Let  $1 < p < 2$  and  $n \geq 3$ . Assume  $\|\nabla L^{-1} \operatorname{div} f\|_p \lesssim \|f\|_p$  for all  $f \in L^p$ . Using interpolation with the corresponding  $L^2$ -inequality, there is no loss of generality to assume

$p_* > 1$ . By Sobolev embeddings we deduce that  $L^{-1}: L^{p_*} \rightarrow L^{p^*}$  for any such  $p$ . Take  $p_0$  with  $p_* < p_0 < p$ . Define  $p_k$  by  $p_k = ((p_{k-1})^*)^*$  if  $p_0 \leq 2_*$  and stop when  $2_* < p_k \leq 2^*$ .

If  $k = 0$ , then we already know that  $(e^{-tL})_{t>0}$  is  $L^{p_0} - L^2$  bounded, hence  $p_0 \in \mathcal{J}_-(L)$ . By Theorem 5.2, we conclude that  $(p_0, 2) \subset \mathcal{I}_-(L)$ . Since  $p_0$  was arbitrary, we have shown that  $(p_*, 2) \subset \mathcal{I}_-(L)$  in this case.

Assume now, that  $k \geq 1$ . By construction,  $L^{-k}$  is bounded from  $L^{p_0}$  to  $L^{p_k}$ . Since  $2_* \leq p_k \leq 2^*$ ,  $(e^{-tL})_{t>0}$  is an analytic semigroup on  $L^{p_k}$  by Proposition 4.15. As in Step 2 of the proof of Theorem 5.2, we obtain that  $e^{-tL}$  is bounded from  $L^{p_0}$  to  $L^{p_k}$ . We also have that  $(e^{-tL})_{t>0}$  satisfies  $L^{p_k}$  off-diagonal estimates by interpolation between  $L^2$  off-diagonal estimates and  $L^r$  boundedness for  $r$  chosen so that  $p_k$  is between 2 and  $r$ . By interpolation again, we deduce that if  $p_0 < q < p_k$ ,  $(e^{-tL})_{t>0}$  satisfies  $L^q - L^{p_k}$  off-diagonal estimates. Using Lemma 4.3, this implies that  $(e^{-tL})_{t>0}$  is  $L^q$  bounded. Hence,  $q \in \mathcal{J}_-(L)$ . We conclude as above that  $(p_*, 2) \subset \mathcal{I}_-(L)$ .

Let us now consider the case  $p > 2$ . Assume  $\|\nabla L^{-1} \operatorname{div} f\|_p \lesssim \|f\|_p$ . We first claim that  $\|(L^*)^{1/2} f\|_{p'} \lesssim \|\nabla f\|_{p'}$ . Indeed, this always holds if  $n = 2$  by Proposition 5.18. If  $n \geq 3$ , by duality we have  $\|\nabla(L^*)^{-1} \operatorname{div} f\|_{p'} \lesssim \|f\|_{p'}$  and the preceding case tells us that  $\nabla(L^*)^{-1/2}$  is  $L^{p'}$  bounded. Applying Theorem 5.9 proves the claim.

Secondly, let  $h \in L^{p'} \cap \dot{W}^{1,2}$ . Since  $\operatorname{div} h \in L^2$ , we have that

$$(L^*)^{-1/2} \operatorname{div} h = (L^*)^{1/2} (L^*)^{-1} \operatorname{div} h,$$

hence

$$\|(L^*)^{-1/2} \operatorname{div} h\|_{p'} = \|(L^*)^{1/2} L^{*-1} \operatorname{div} h\|_{p'} \lesssim \|\nabla(L^*)^{-1} \operatorname{div} h\|_{p'} \lesssim \|h\|_{p'}.$$

Since  $L^{p'} \cap \dot{W}^{1,2}$  is dense in  $L^{p'}$ , we obtain the  $L^{p'}$ -boundedness of  $(L^*)^{-1/2} \operatorname{div}$  which, by duality, means the  $L^p$ -boundedness of the Riesz transform  $\nabla L^{-1/2}$ .

For the converse, we deduce from the characterization of  $\mathcal{I}(L)$  and the relation between  $p_{\pm}(L)$  and  $q_{\pm}(L)$  that the  $L^p$ -boundedness of the Riesz transform  $\nabla L^{-1/2}$  and  $p > 2$  imply the  $L^{p'}$ -boundedness of the Riesz transform  $\nabla(L^*)^{-1/2}$ , which by duality means that  $L^{-1/2} \operatorname{div}$  is bounded on  $L^p$ . Hence,  $\nabla L^{-1} \operatorname{div} = \nabla L^{-1/2} L^{-1/2} \operatorname{div}$  is bounded on  $L^p$ .  $\square$

*Remark.* By Proposition 5.18, the second statement is meaningless if  $n = 2$ , hence the assumption  $n \geq 3$ . Its converse is false for any  $p < 2$ . For example, for any  $\varepsilon > 0$ , there exist real symmetric operators  $L$  such that

$\|\nabla L^{-1} \operatorname{div} f\|_p \lesssim \|f\|_p$  only when  $|\frac{1}{2} - \frac{1}{p}| < \varepsilon$ , yet its associated Riesz transform is  $L^p$  bounded when  $1 < p < 2$ .

*Remark.* Let us describe a geometric interpretation of the reverse inequalities for square roots. Let  $\mathcal{H}_p(L) = \{g \in L^p; \operatorname{div}(Ag) = 0\}$  and  $\mathcal{G}_p = \nabla(\dot{W}^{1,p})$ . These are closed subspaces of  $L^p$ . The Hodge decomposition in  $L^p$  is equivalent to having  $\mathcal{H}_p(L) + \mathcal{G}_p = L^p$  as a topological direct sum. By duality, it is also equivalent to  $\mathcal{H}_{p'}(L^*) + \mathcal{G}_{p'} = L^{p'}$  is a topological direct sum.

Assume that  $\|L^{1/2}f\|_p \lesssim \|\nabla f\|_p$  holds. Then, by duality  $(L^*)^{1/2}$  is bounded from  $L^{p'}$  into  $\dot{W}^{-1,p'}$ . If  $\nabla h \in \mathcal{G}_{p'}$  then  $(L^*)^{-1/2} \operatorname{div}(A^* \nabla h) = (L^*)^{1/2} h$  makes sense. Hence, the restriction of  $(L^*)^{-1/2} \operatorname{div}(A^* \cdot)$  to  $\mathcal{G}_{p'}$  is bounded into  $L^{p'}$ . If  $g \in \mathcal{H}_{p'}(L^*)$  then  $(L^*)^{-1/2} \operatorname{div}(A^* g) = 0$ , hence the restriction of  $(L^*)^{-1/2} \operatorname{div}(A^* \cdot)$  to  $\mathcal{H}_{p'}(L^*)$  is bounded into  $L^{p'}$  (without any hypothesis). Conversely, these two facts imply  $\|L^{1/2}f\|_p \lesssim \|\nabla f\|_p$ . Thus, this inequality means that  $(L^*)^{-1/2} \operatorname{div}(A^* \cdot)$  is bounded on closed subspaces of  $L^{p'}$  into  $L^{p'}$  even without knowing whether they are in direct sum in  $L^{p'}$ .

If the topological direct sum holds then  $(L^*)^{-1/2} \operatorname{div}(A^* \cdot)$  is bounded on  $L^{p'}$ , hence  $\nabla L^{-1/2}$  is bounded on  $L^p$ . This is what we proved above.

This also illustrates why  $\|L^{1/2}f\|_p \lesssim \|\nabla f\|_p$  is possible even when  $\nabla L^{-1/2}$  is not bounded on  $L^p$ .

Let us finish with another identification of  $q_+(L)$ .

**Corollary 5.24.**  $q_+(L)$  is the supremum of exponents  $p$  for which one has the Hodge decomposition in  $L^p$ , or alternatively for which  $L$  extends to an isomorphism from  $\dot{W}^{1,p}$  onto  $\dot{W}^{-1,p}$ .<sup>55</sup> The interval of values of  $p$  for which this holds is, therefore,  $((q_+(L^*))', q_+(L))$ .

*Proof.* If  $n = 1$  we have  $q_+(L) = \infty$  and the Hodge projections are bounded on all  $L^p$  spaces as recalled above. If  $n \geq 2$ , this follows right away from the previous theorem and the fact that  $q_+(L)$  is the supremum of  $\mathcal{I}(L)$ .  $\square$

## 6 Riesz transforms and functional calculi

In this section, we present the theorem of Blunck & Kunstmann concerning  $H^\infty$  functional calculus on  $L^p$  spaces. We also discuss Hardy-Littlewood-Sobolev inequalities. Combining this with Riesz transform estimates, we obtain a family of inequalities which we summed up in what we call the Hardy-Littlewood-Sobolev-Kato diagram.

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<sup>55</sup> See Lemma 4.4.

## 6.1 Blunck & Kunstmann's theorem

Let  $L$  be as usual and  $\omega$  be the type of  $L$  defined in (3.4). We know that  $L$  admits a bounded holomorphic functional calculus on  $L^2$ .

Let  $p \in (1, \infty)$ . We say that  $L$  has a bounded holomorphic functional calculus on  $L^p$  if one has the following property: for any  $\mu \in (\omega, \pi)$  and any  $\varphi$  holomorphic and bounded in  $\Sigma_\mu$ , and all  $f \in L^2 \cap L^p$

$$\|\varphi(L)f\|_p \leq c\|\varphi\|_\infty\|f\|_p$$

the constant  $c$  depending only on  $p$ ,  $\omega$  and  $\mu$ . The operator  $\varphi(L)$  is defined on  $L^p$  by density.<sup>56</sup>

We define  $\mathcal{H}(L)$  as the sets of those exponents  $p$  with the above property. By interpolation, these sets are intervals (if nonempty).

Recall that  $\mathcal{J}(L)$  is the set of exponents  $p$  for which  $(e^{-tL})_{t>0}$  is  $L^p$  bounded and  $\mathcal{J}(L) \cap [1, 2) = \mathcal{J}_-(L)$ .

**Theorem 6.1.** *The sets  $\mathcal{J}(L)$  and  $\mathcal{H}(L)$  have same interiors.*

By Theorem 5.2, this gives us

**Corollary 6.2.** *The sets  $\mathcal{I}_-(L)$  and  $\mathcal{H}_-(L)$  have same interiors.*

We turn to the proof of the theorem.

*Proof.* It is enough to show that  $\mathcal{J}_-(L)$  and  $\mathcal{H}_-(L)$  have same interiors as one can use duality in this context.

Let  $p < 2$ . If  $L$  has a bounded holomorphic functional calculus on  $L^p$  then the semigroup  $(e^{-tL})_{t>0}$  is  $L^p$  bounded. Hence  $\mathcal{H}_-(L) \subset \mathcal{J}_-(L)$ .

Conversely, it is enough to show that  $p_0 \in \text{int}\mathcal{K}_-(L)$ , that is  $(e^{-zL})_{z \in \Sigma_\beta}$  satisfies  $L^{p_0} - L^2$  off-diagonal estimates for any  $\beta \in (0, \frac{\pi}{2} - \omega)$  (see Proposition 4.15), implies that  $\varphi(L)$  is weak-type  $(p_0, p_0)$  whenever  $\varphi$  is holomorphic and bounded in  $\Sigma_\mu$  for  $\mu > \omega$  and we may choose  $\mu < \frac{\pi}{2}$  by convenience. To this end, we apply Theorem 2.1 to  $T = \varphi(L)$ . It is enough to assume further on  $\varphi$  the technical condition (3.5).<sup>57</sup>

We set  $A_r = I - (I - e^{-r^2L})^m$ ,  $r > 0$ , for some large integral number  $m$ .

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<sup>56</sup> Strictly speaking, one should incorporate a statement about convergence of operators to allow limiting procedures. In fact, this follows from the  $L^2$ -functional calculus and density.

<sup>57</sup> This follows from the Convergence Lemma in [Mc].



Assume  $p_0 < 2$  is such that  $(e^{-zL})_{z \in \Sigma_\beta}$  satisfies  $L^{p_0} - L^2$  off-diagonal estimates for any  $\beta \in (0, \frac{\pi}{2} - \omega)$ . It is enough to check (2.1) as (2.2) is granted from the assumption on the semigroup. Our goal is to establish the inequality

$$(6.1) \quad \left( \frac{1}{|2^{j+1}B|} \int_{C_j(B)} |\varphi(L)(I - e^{r^2L})^m f|^2 \right)^{1/2} \leq g(j) \left( \frac{1}{|B|} \int_B |f|^{p_0} \right)^{1/p_0}$$

for all ball  $B$  with  $r$  the radius of  $B$  and all  $f$  supported in  $B$  and all  $j \geq 2$ , with  $\sum 2^{nj}g(j) < \infty$ . As before,  $C_j(B)$  denotes the ring  $2^{j+1}B \setminus 2^jB$ .

To do this, let  $\psi(z) = \varphi(z)(1 - e^{-r^2z})^m$  so that  $\psi(L) = \varphi(L)(I - e^{r^2L})^m$ . Represent  $\psi(L)$  using the representation formula (3.6). Using the exact form of  $\psi$  and the definition of  $\eta_\pm$ , it is easy to obtain

$$|\eta_\pm(z)| \leq C \|\varphi\|_\infty |z|^{-1} \inf(1, r^{2m}|z|^{-m}), \quad z \in \Gamma_\pm.$$

Hence, by Minkowski integral inequality in (3.6), the hypothesis on  $p_0$  and on the support of  $f$ ,  $\|\varphi(L)(I - e^{r^2L})^m f\|_{L^2(C_j(B))}$  is bounded above by

$$C \int_{\Gamma_+} e^{-\frac{e^{4j}r^2}{|z|}} \frac{1}{|z|^{\gamma_{p_0}/2}} \inf(1, r^{2m}|z|^{-m}) \frac{d|z|}{|z|} \|f\|_{p_0}$$

plus the term corresponding to integration on  $\Gamma_-$ . A calculation gives us a bound

$$Cr^{-\gamma_{p_0}} 2^{-j(\gamma_{p_0}+2m)} \|f\|_{p_0}.$$

Using the value of  $\gamma_{p_0}$  we obtain (6.1) with  $g(j) = C2^{-nj/2}2^{-j(\gamma_{p_0}+2m)}$ . Choosing  $m$  with  $\gamma_{p_0} + 2m > n/2$  concludes the argument.  $\square$

## 6.2 Hardy-Littlewood-Sobolev estimates

We prove the following result.

**Proposition 6.3.** *Let  $p_-(L) < p < q < p_+(L)$ . Then  $L^{-\alpha}$  is bounded from  $L^p$  into  $L^q$  provided*

$$\alpha = \frac{1}{2} \left( \frac{n}{p} - \frac{n}{q} \right).$$

*Proof.* We first observe that with the following choice of  $p$  and  $q$  then  $(e^{-tL})_{t>0}$  is  $L^p - L^q$  bounded. Indeed, we know it if  $p = 2$  or if  $q = 2$  by Proposition 4.2. If  $p < 2 < q$ , it suffices to use composition. If  $p < q < 2$ , interpolate between  $L^p$  boundedness and  $L^p - L^2$  boundedness. And if  $2 < p < q$ , interpolate between  $L^q$  boundedness and  $L^2 - L^q$  boundedness.

Next, by  $L^2$  functional calculus, we have

$$f = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} L^\alpha e^{-tL} f dt$$

for all  $f \in L^2$ , where the integrals  $\int_\varepsilon^R \dots$  converge in  $L^2$  as  $\varepsilon \downarrow 0$  and  $R \uparrow \infty$ . Set  $T_{\varepsilon,R} = \Gamma(\alpha)^{-1} \int_\varepsilon^R t^{\alpha-1} e^{-tL} dt$ . Let  $f \in L^2 \cap L^p$  with  $\|f\|_p = 1$ . Fix  $\alpha = \frac{1}{2}(\frac{n}{p} - \frac{n}{q})$ . For  $a > 0$  and  $p < q_0 < q < q_1 < q_+(L)$ , we easily obtain with uniform constant  $C$ ,

$$\left\| \int_\varepsilon^a t^{\alpha-1} e^{-tL} f dt \right\|_{q_1} \leq C a^{\frac{1}{2}(\frac{n}{q} - \frac{n}{q_1})}$$

and

$$\left\| \int_a^R t^{\alpha-1} e^{-tL} f dt \right\|_{q_0} \leq C a^{\frac{1}{2}(\frac{n}{q_0} - \frac{n}{q})}.$$

Hence, by the argument of Marcinkiewicz interpolation theorem, we have if  $\lambda > 0$

$$|\{x \in \mathbb{R}^n; |T_{\varepsilon,R}f(x)| > \lambda\}| \leq C\lambda^{-q_1} a^{\frac{q_1}{2}(\frac{n}{q} - \frac{n}{q_1})} + C\lambda^{-q_0} a^{\frac{q_0}{2}(\frac{n}{q_0} - \frac{n}{q})}.$$

Choosing  $a^{\frac{n}{2q}} = \lambda$  yields,

$$|\{x \in \mathbb{R}^n; |T_{\varepsilon,R}f(x)| > \lambda\}| \leq 2C\lambda^{-q}.$$

Hence,  $T_{\varepsilon,R}f$  belongs to the Lorentz space  $L^{q,\infty}$ . Since this holds for all  $q$  as above, by interpolation again, we conclude that  $T_{\varepsilon,R}f \in L^q$  and

$$\|T_{\varepsilon,R}f\|_q \leq C\|f\|_p$$

whenever  $f \in L^2 \cap L^p$  with uniform constant with respect to  $\varepsilon, R$ .

It remains to pass to the limit. If  $0 \leq \alpha \leq 1/2$ , we claim that  $L^\alpha$  defines an isomorphism between the fractional Sobolev space  $\dot{H}^{2\alpha}$  (defined as the closure of  $C_0^\infty$  for the seminorm  $\|f\|_{\dot{H}^{2\alpha}} \equiv \|(-\Delta)^\alpha f\|_2$ ) and  $L^2$  with

$$\|L^\alpha f\|_2 \sim \|f\|_{\dot{H}^{2\alpha}}.$$

Indeed, if  $\Re\alpha = 0$ ,  $z \mapsto z^\alpha$  is bounded holomorphic in  $\Sigma_\pi$ , hence

$$\|L^\alpha f\|_2 \sim \|f\|_2$$

with implicit constants independent of  $\Im\alpha$ . For  $\Re\alpha = 1/2$ , combining the square root problem (3.11) with the  $L^2$  functional calculus, we have

$$\|L^\alpha f\|_2 \sim \|\nabla f\|_2$$

with implicit constants independent of  $\Im\alpha$ . The claim follows by complex interpolation. By construction, if  $f \in L^2$  then

$$\lim_{\varepsilon \downarrow 0, R \uparrow \infty} \|T_{\varepsilon, R} f - L^{-\alpha} f\|_{\dot{H}^{2\alpha}},$$

hence, for  $\alpha$  sufficiently small so that Sobolev embeddings  $\dot{H}^{2\alpha} \subset L^r$  applies with  $r < \infty$ , we also have the convergence in  $L^r$ .

Now, if  $f \in L^2 \cap L^p$ , we combine the uniform bound of  $T_{\varepsilon, R} f$  in  $L^q$  and its convergence in  $L^r$  to conclude for the  $L^p - L^q$  bounded extension of  $L^{-\alpha}$  for all small positive  $\alpha$ .

We obtain all possible values of  $\alpha$  by writing  $L^{-\alpha} = (L^{-\alpha/k})^k$  for  $k$  large enough.  $\square$

*Remark.* The isomorphism property used in the proof holds  $0 \leq \alpha < 1/2$  without knowing the solution of the Kato square root problem using that the domain of  $L^\alpha$  is an interpolation space between the form domain (*i.e.*  $W^{1,2}$ ) and  $L^2$ .<sup>58</sup>

### 6.3 The Hardy-Littlewood-Sobolev-Kato diagram

In the plane  $\{(\frac{1}{p}, s)\}$ , where  $p$  is a Lebesgue exponent and  $s$  a regularity index, we introduce a convex set on which we have a rule for boundedness from  $\dot{W}^{s,p}$  to  $\dot{W}^{\sigma,q}$  of functions of  $L$ . We call it the Hardy-Littlewood-Sobolev-Kato diagram of  $L$  because it includes all previously seen estimates from functional calculus and Kato type inequalities (see below).

Pick exponents  $p_0, q_0, p_1, q_1$  as follows:

$$q_-(L) = p_-(L) < p_0 < 2 < q_0 < q_+(L)$$

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<sup>58</sup> This complex interpolation result follows from Kato [K1] and Lions [Li].

and

$$q_-(L^*) = p_-(L^*) < p_1 < 2 < q_1 < q_+(L^*).$$

Hence,  $p_0, q_0$  (resp.  $p_1, q_1$ ) are in the range of  $L^p$ -boundedness of the Riesz transform  $\nabla L^{-1/2}$  (resp.  $\nabla(L^*)^{-1/2}$ ). Consider the closed convex polygon  $\mathcal{P} = ABDFEC$  in the plane  $\{(\frac{1}{p}, s)\}$  with  $A = (\frac{1}{q_0}, 1)$ ,  $B = (\frac{1}{p_0}, 1)$ ,  $D = (\frac{1}{p_0}, 0)$ ,  $F = (\frac{1}{(q_1)', -1})$ ,  $E = (\frac{1}{(p_1)', -1})$  and  $C = (\frac{1}{(p_1)', 0})$ . See Figure 1. Let  $M = (\frac{1}{p}, s)$ ,  $N = (\frac{1}{q}, \sigma)$  in  $\mathcal{P}$ . We call  $\overrightarrow{MN}$  an authorized arrow if  $p \leq q$ . Set in this case,

$$\alpha(M, N) = \frac{\sigma - s}{2} + \frac{1}{2} \left( \frac{n}{p} - \frac{n}{q} \right).$$

In other words, we are allowed to move in  $\mathcal{P}$  horizontally from the right to the left, vertically up and down and all possible combinations. In fact, an accurate correspondance would be the convex three dimensional set of authorized arrows  $\overrightarrow{MN}$  between two parrallel copies of  $\mathcal{P}$  in  $\mathbb{R}^3$  whenever  $M$  in the first copy and  $N$  in the second copy.

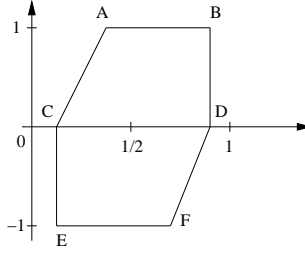


Figure 1: **The Hardy-Littlewood-Sobolev-Kato diagram  $\mathcal{P}$ .**

**Theorem 6.4.** *Let  $\mu \in (\omega, \pi)$ . For any  $M = (\frac{1}{p}, s)$ ,  $N = (\frac{1}{q}, \sigma) \in \mathcal{P}$  with  $p \leq q$  and any  $\varphi \in \mathcal{F}(\Sigma_\mu)$ ,*

$$\|\varphi(L)f\|_{\dot{W}^{\sigma,q}} \lesssim \|z^{\alpha(M,N)}\varphi\|_{L^\infty(\Sigma_\mu)} \|f\|_{\dot{W}^{s,p}},$$

*provided the quantity  $\|z^{\alpha(M,N)}\varphi\|_{L^\infty(\Sigma_\mu)}$  is finite. The implicit constant depends on  $\mu, \omega, p, q, s, \sigma$ .*

A few explanations are necessary. First, for  $0 < s$ ,  $\dot{W}^{s,p}$  is defined as the closure of  $C_0^\infty$  for  $\|(-\Delta)^{s/2}f\|_p$  and  $\dot{W}^{-s,p'}$  is its dual space. If  $0 < \alpha < 1$ ,

$z \mapsto z^\alpha$  is the analytic continuation in  $\Sigma_\pi$  of  $t \mapsto t^\alpha$  defined on  $(0, \infty)$ . If  $k < \alpha < k + 1$  with  $k$  integral number, then  $z \mapsto z^\alpha = z^k z^{\alpha-k}$  is analytic in  $\Sigma_\pi$ . Hence, if  $\varphi \in \mathcal{F}(\Sigma_\mu)$  then  $z^\alpha \varphi \in \mathcal{F}(\Sigma_\mu)$  and  $L^\alpha \varphi(L)$  is well-defined (See Section 3.2). Also, we shall write *a priori* inequalities for suitable functions  $f$  and we shall leave to the reader the care of providing the density arguments.

When  $s = \sigma = 0$ , this includes  $L^p - L^q$  estimates for negative fractional powers of  $L$  (Hardy-Littlewood-Sobolev estimates). For  $p = q$ , we have a comparison between fractional powers of  $L$  with the ones of the Laplacian  $-\Delta$  (Kato type estimates): the operators  $(-\Delta)^{\sigma/2} L^{(s-\sigma)/2} (-\Delta)^{-s/2}$  are bounded on  $L^p$  provided  $(\frac{1}{p}, s)$  and  $(\frac{1}{p}, \sigma)$  belong to  $\mathcal{P}$ . More precisely

**Corollary 6.5.** *If  $(\frac{1}{p}, s)$  and  $(\frac{1}{p}, \sigma)$  belong to  $\mathcal{P}$ , then  $L^{(s-\sigma)/2}$  extends to an isomorphism from  $\dot{W}^{s,p}$  onto  $\dot{W}^{\sigma,p}$*

**Proof:** The previous remark means that

$$\|L^{(s-\sigma)/2} f\|_{\dot{W}^{\sigma,p}} \sim \|f\|_{\dot{W}^{s,p}}$$

which implies that the extension from  $C_0^\infty$  to  $\dot{W}^{s,p}$  is bounded and one-to-one. Since the same thing is true for  $L^*$  in the dual range, we conclude that this extension is onto by classical arguments.  $\square$

In particular we recover the simultaneous  $L^p$  boundedness of the Riesz transform and the Hodge projection when  $p > 2$ . We pursue the discussion on Kato type estimates in the next section.

**Proof of Theorem 6.4:** By convexity and complex interpolation, it suffices to prove the result for the following extremal authorized arrows:

1. the minimal ones:  $\overrightarrow{MM}$  where  $M$  is one of the six vertices of  $\mathcal{P}$ .
2. the maximal horizontal ones:  $\overrightarrow{BA}$ ,  $\overrightarrow{DC}$ ,  $\overrightarrow{FE}$ .
3. the maximal vertical ones:  $\overrightarrow{DB}$ ,  $\overrightarrow{FH}$ ,  $\overrightarrow{GA}$ ,  $\overrightarrow{EC}$  and their respective opposite where  $G = (\frac{1}{p_0}, -1)$  and  $H = (\frac{1}{(q_1)', 1})$ .

### Step 1. Minimal arrows

- $\overrightarrow{AA}$ : Write

$$\|\nabla \varphi(L) f\|_{q_0} \lesssim \|L^{1/2} \varphi(L) f\|_{q_0} \sim \|\varphi(L) L^{1/2} f\|_{q_0} \lesssim \|\varphi\|_\infty \|L^{1/2} f\|_{q_0} \lesssim \|\nabla f\|_{q_0}.$$

The first inequality holds because the Riesz transform is  $L^{q_0}$  bounded, the second by the commutative property of the functional calculus, the third by Theorem 6.1 and the last by the reverse inequalities at  $q_0$ .

- $\overrightarrow{BB}$ : Same as  $\overrightarrow{AA}$  by changing  $q_0$  to  $p_0$ .
- $\overrightarrow{CC}$ : We have  $(p_1)' \in \text{int}\mathcal{J}(L)$ , hence Theorem 6.1 applies and yields  $\|\varphi(L)f\|_{(p_1)'} \lesssim \|\nabla f\|_{(p_1)'}$ .
- $\overrightarrow{DD}$ : Same as  $\overrightarrow{CC}$  by changing  $(p_1)'$  to  $p_0$ .
- $\overrightarrow{EE}$ : By duality, we have to show  $\|\nabla\varphi(L^*)f\|_{p_1} \lesssim \|\nabla f\|_{p_1}$ . This is the same as  $\overrightarrow{BB}$  by changing  $L$  to  $L^*$  and  $p_0$  to  $p_1$ .
- $\overrightarrow{FF}$ : By duality, this is the same as  $\overrightarrow{AA}$  by changing  $L$  to  $L^*$  and  $q_0$  to  $q_1$ .

## Step 2. Maximal horizontal arrows

- $\overrightarrow{BA}$ : As for  $\overrightarrow{AA}$ , we begin with

$$\|\nabla\varphi(L)f\|_{q_0} \lesssim \|L^{1/2}\varphi(L)f\|_{q_0} \sim \|\varphi(L)L^{1/2}f\|_{q_0}.$$

Next, we continue with

$$\|\varphi(L)L^{1/2}f\|_{q_0} \lesssim \|z^{\alpha(B,A)}\varphi\|_{\infty} \|L^{-\alpha(B,A)}L^{1/2}f\|_{q_0}$$

by Theorem 6.1, then

$$\|L^{-\alpha(B,A)}L^{1/2}f\|_{q_0} \lesssim \|L^{1/2}f\|_{p_0} \lesssim \|\nabla f\|_{p_0},$$

by the Hardy-Littlewood-Sobolev inequality for  $L$  and the definition of  $\alpha(B, A)$ , and the reverse inequality at  $p_0$ .

- $\overrightarrow{DC}$ : Since  $p_-(L) < p_0 < (p_1)' < p_+(L)$ , by Proposition 6.3 and Theorem 6.1,

$$\|\varphi(L)f\|_{(p_1)'} \lesssim \|z^{\alpha(D,C)}\varphi\|_{\infty} \|L^{-\alpha(D,C)}f\|_{(p_1)'} \lesssim \|f\|_{p_0}.$$

- $\overrightarrow{FE}$ : By duality, this is the same as  $\overrightarrow{BA}$  by changing  $L$  to  $L^*$ ,  $p_0$  to  $p_1$  and  $q_0$  to  $q_1$ .

## Step 3. Maximal vertical arrows

Recall that  $G$  and  $H$  are points in  $\mathcal{P}$  given by  $G = (\frac{1}{p_0}, -1)$  and  $H = (\frac{1}{(q_1)'}, 1)$ .

• $\overrightarrow{DB}$ : Using that the Riesz transform is  $L^{p_0}$  bounded and Theorem 6.1, one has

$$\|\nabla\varphi(L)f\|_{p_0} \lesssim \|L^{1/2}\varphi(L)f\|_{p_0} \lesssim \|z^{1/2}\varphi\|_\infty\|f\|_{p_0}.$$

• $\overrightarrow{BD}$ : We have

$$\|\varphi(L)f\|_{p_0} \lesssim \|L^{-1/2}\varphi(L)L^{1/2}f\|_{p_0} \lesssim \|z^{-1/2}\varphi\|_\infty\|L^{1/2}f\|_{p_0} \lesssim \|\nabla f\|_{p_0}.$$

• $\overrightarrow{FH}$ : Since  $(q_1)' > p_-(L)$ , one has the  $L^{(q_1)'}$  boundedness of the Riesz transform associated to  $L$ :

$$\|\nabla\varphi(L)\operatorname{div}f\|_{(q_1)'} \lesssim \|L^{1/2}\varphi(L)\operatorname{div}f\|_{(q_1)'}$$

Then writing  $L^{1/2}\varphi(L) = L\varphi(L)L^{-1/2}$  and using Theorem 6.1 yield

$$\|L^{1/2}\varphi(L)\operatorname{div}f\|_{(q_1)'} \lesssim \|z\varphi\|_\infty\|L^{-1/2}\operatorname{div}f\|_{(q_1)'}$$

Next, using that the Riesz transform associated to  $L^*$  is bounded on  $L^{q_1}$ , one concludes by

$$\|L^{-1/2}\operatorname{div}f\|_{(q_1)'} \lesssim \|f\|_{(q_1)'}$$

• $\overrightarrow{HF}$ : Since  $(q_1)' > p_-(L)$ ,  $L^{1/2}$  is bounded from  $L^{(q_1)'}$  to  $\dot{W}^{-1,(q_1)'}$ , hence

$$\|\varphi(L)f\|_{\dot{W}^{-1,(q_1)'}} = \|L^{1/2}\varphi(L)L^{-1/2}f\|_{\dot{W}^{-1,(q_1)'}} \lesssim \|\varphi(L)L^{-1/2}f\|_{(q_1)'}$$

Next, by Theorem 6.1,

$$\|\varphi(L)L^{-1/2}f\|_{(q_1)'} \lesssim \|z^{-1}\varphi\|_\infty\|L^{1/2}f\|_{(q_1)'}$$

We finish with  $L^{1/2}$  bounded from  $\dot{W}^{(q_1)'}$  to  $L^{(q_1)'}$  since  $(q_1)' > p_-(L)$ .

• $\overrightarrow{GA}$ : Same as  $\overrightarrow{FH}$  by changing  $(q_1)'$  to  $q_0$ .

• $\overrightarrow{AG}$ : Same as  $\overrightarrow{HF}$  by changing  $(q_1)'$  to  $q_0$ .

• $\overrightarrow{EC}$ : By duality, this is the same as  $\overrightarrow{DB}$  by changing  $L$  to  $L^*$  and  $p_0$  to  $p_1$ .

• $\overrightarrow{CE}$ : By duality, this is the same as  $\overrightarrow{BD}$  by changing  $L$  to  $L^*$  and  $p_0$  to  $p_1$ .

We conclude this section with the following corollary.

**Corollary 6.6.** *If  $q_-(L) < p < q_+(L)$ , then  $L$  has a bounded holomorphic functional calculus on  $\dot{W}^{1,p}$ .*<sup>59</sup>

*Proof.*  $(q_-(L), q_+(L)) = \mathcal{I}(L)$ , so this functional calculus corresponds to minimal arrows  $\overrightarrow{MM}$  for  $M = (\frac{1}{p}, 1)$ .  $\square$

<sup>59</sup> We do not know if the converse holds.

## 6.4 More on the Kato diagram

As we have seen, we can move vertically up and down in  $\mathcal{P}$ . However, one can authorize more downward arrows. Doing this and modifying slightly the definition of the numbers  $r_{\pm}(L)$  in Theorem 5.9, we shall show that  $r_+(L) = p_+(L)$  and obtain a lower bound on  $r_-(L)$ .

First we claim that the proof given for the reverse inequalities  $\|L^{1/2}f\|_p \lesssim \|\nabla f\|_p$  extends as follows.

**Proposition 6.7.** *For all  $\mu \in (\omega, \pi)$  and all  $\varphi \in H^\infty(\Sigma_\mu)$ ,*

$$\|L^{1/2}\varphi(L)f\|_p \lesssim \|\varphi\|_\infty \|\nabla f\|_p$$

whenever  $\tilde{p}_-(L) = r_-(L) < p < p_+(L)$ .

*Proof.* The case  $p > 2$  is a simple consequence of this inequality for  $L^{1/2}$  together with the bounded holomorphic functional calculus on  $L^p$  for  $p < p_+(L)$ .

We now turn to  $p < 2$ . We begin with the representation formula (3.6) for  $\psi(z) = z^{1/2}\varphi(z)$ . One has

$$\psi(L) = \int_{\Gamma_+} e^{-zL}\eta_+(z) dz + \int_{\Gamma_-} e^{-zL}\eta_-(z) dz$$

and

$$\eta_{\pm}(z) = \frac{1}{2\pi i} \int_{\gamma_{\pm}} e^{\zeta z} \psi(\zeta) d\zeta, \quad z \in \Gamma_{\pm},$$

valid provided the technical assumption  $|\varphi(\zeta)| \leq C(1 + |\zeta|)^{-1/2-s}$  for some  $s > 0$ . If one defines the primitive  $N_{\pm}(z)$  of respectively  $\eta_{\pm}(z)$  which vanishes at infinity then, under the technical assumption, one may integrate by parts and, since the terms at 0 cancel each other, one finds

$$\psi(L) = \int_{\Gamma_+} L e^{-zL} N_+(z) dz + \int_{\Gamma_-} L e^{-zL} N_-(z) dz.$$

Furthermore, one has

$$|N_{\pm}(z)| \leq C|z|^{-1/2} \|\varphi\|_\infty, \quad z \in \Gamma_{\pm}.$$

Hence, by repeating the argument for  $L^{1/2}$  where integration on the positive axis is replaced by integration on half-rays  $\Gamma_{\pm}$ , we obtain

$$\|\psi(L)f\|_p \lesssim \|\varphi\|_\infty \|\nabla f\|_p$$



whenever  $\tilde{p}_-(L) < p < 2$  and  $\varphi$  satisfies the technical condition, which is removed by a limiting argument.  $\square$

Let us discuss the sharpness of the bounds obtained above.

**Proposition 6.8.** *Assume that for all  $\mu \in (\omega, \pi)$  and all  $\varphi \in H^\infty(\Sigma_\mu)$ ,*

$$\|L^{1/2}\varphi(L)f\|_p \lesssim \|\varphi\|_\infty \|\nabla f\|_p.$$

*Then  $\sup(\bar{p}_-(L), 1) \leq p \leq p_+(L)$  where  $\bar{p}_-(L)$  is defined as follows: the point  $\bar{B} = (\frac{1}{\bar{p}_-(L)}, 1)$  is symmetric to  $F = (\frac{1}{(q_+(L^*))'}, -1)$  with respect to  $D = (\frac{1}{(p_+(L^*))'}, 0) = (\frac{1}{p_-(L)}, 0)$ .*

One can see that  $\bar{p}_-(L) \leq \tilde{p}_-(L)$ . This lower bound  $\bar{p}_-(L)$  is the best one can obtain by a convexity method. If  $n \leq 4$ ,  $\tilde{p}_-(L) = 1$  so there is nothing more to say. In dimensions  $n \geq 5$ , the lower bound is optimal for any operator  $L$  for which this inequality cannot be improved, which is the same as saying that  $p_+(L^*) \geq (q_+(L^*))^*$  is optimal (See Section 4.5 for this).

*Proof.* Fix  $\varphi \in H^\infty(\Sigma_\mu)$  with  $\|\varphi\|_\infty = 1$ . We have for all  $t \in \mathbb{R}$

$$\|L^{1/2+it}\varphi(L)f\|_p \lesssim \|\nabla f\|_p = \|f\|_{\dot{W}^{1,p}}.$$

We also have from the HLSK diagram that for all  $(q_+(L^*))' < q < p_+(L)$  and  $t \in \mathbb{R}$ ,

$$\|L^{-1/2+it}\varphi(L)f\|_q \lesssim \|f\|_{\dot{W}^{-1,q}}.$$

By complex interpolation, we obtain

$$\|\varphi(L)f\|_r \lesssim \|f\|_r$$

for  $1/r$  the middle of  $1/p$  and  $1/q$ . By Theorem 6.1, it is necessary that  $p_-(L) \leq r \leq p_+(L)$ .

Choosing  $q$  arbitrarily close to  $p_+(L)$  forces  $p \leq p_+(L)$ . Choosing  $q$  arbitrarily close to  $(q_+(L^*))'$  forces  $p \geq \bar{p}_-(L)$  given the definition of this number.  $\square$

Geometrically, this provides us with a family of authorized downarrows that may not be contained in  $\mathcal{P}$ . We assume that  $\tilde{p}_0 = (p_0)_* > 1$  otherwise we exclude from this discussion points  $\{(\frac{1}{p}, s)\}$  with  $\frac{1}{p} \geq 1$ . Consider the closed convex polygon  $\mathcal{P}_{in} = A\tilde{B}DFEC$  in the plane  $\{(\frac{1}{p}, s)\}$  with  $A, D, F, E, C$  as before and  $\tilde{B} = (\frac{1}{\tilde{p}_0}, 1)$  and also the closed convex polygon  $\mathcal{P}_{ext} = \tilde{A}\tilde{B}\tilde{D}FE$ , where  $\tilde{A} = (\frac{1}{(p_1)}, 1)$  and  $\tilde{D} = (\frac{1}{\tilde{p}_0}, 0)$ . See Figure 2.

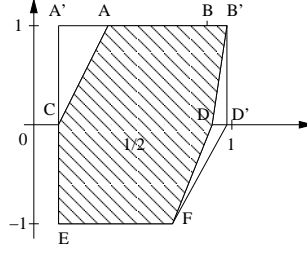


Figure 2: **The convex polygons  $\mathcal{P}_{in}$  (shaded) and  $\mathcal{P}_{ext}$ . The points  $A', B', D'$  correspond to  $\tilde{A}, \tilde{B}, \tilde{D}$  in the text. Downarrows  $MN$  with  $M \in \mathcal{P}_{in}$  and  $N \in \mathcal{P}_{ext}$  or  $M \in \mathcal{P}_{ext}$  and  $N \in \mathcal{P}_{in}$  are authorized.**

**Proposition 6.9.** *For any downarrow  $\overrightarrow{MN}$ ,  $M = (\frac{1}{p}, s)$   $N = (\frac{1}{p}, \sigma)$  with one extremity  $\mathcal{P}_{in}$  and the other in  $\mathcal{P}_{ext}$  we have for all  $\mu \in (\omega, \pi)$  and all  $\varphi \in H^\infty(\Sigma_\mu)$ ,*

$$\|\varphi(L)f\|_{\dot{W}^{\sigma,p}} \lesssim \|z^{\alpha(M,N)}\varphi\|_{L^\infty(\Sigma_\mu)} \|f\|_{\dot{W}^{s,p}},$$

*provided the quantity  $\|z^{\alpha(M,N)}\varphi\|_{L^\infty(\Sigma_\mu)}$  is finite. The implicit constant depends on  $\mu, \omega, p, q, s, \sigma$ .*

*Proof.* The previous proposition shows that for the maximal downarrows  $\overrightarrow{\tilde{A}C}$  and  $\overrightarrow{\tilde{B}D}$  the corresponding inequality of the statement are valid. Combining this with all possible vertical authorized arrows in  $\mathcal{P}$ , the convexity property of complex interpolation gives us the desired result.  $\square$

Of course, the downarrows can be reversed exactly when  $M$  and  $N$  belong to the HLSK diagram  $\mathcal{P}$ .

The expected maximal convex set  $\mathcal{P}_{in}$  would be with  $\tilde{B}$  replaced by  $\overline{B}$  where  $\overline{B}$  is the symmetric point of  $F$  with respect to  $D$ . But our arguments do not suffice to prove the inequality corresponding to the vertical downarrow  $\overrightarrow{\overline{B}D}$  with  $\overline{D}$  the vertical projection of  $\overline{B}$  on the  $\frac{1}{p}$  axis.

## 7 Square function estimates

In this section, we study vertical square functions of two different types which are representative of larger classes of square functions. Then, we prove some weak type and strong type inequalities for non-tangential square functions.

## 7.1 Necessary and sufficient conditions for boundedness of vertical square functions

Define the quadratic functionals for  $f \in L^2$

$$g_L(f)(x) = \left( \int_0^\infty |(L^{1/2}e^{-tL}f)(x)|^2 dt \right)^{1/2}$$

and

$$G_L(f)(x) = \left( \int_0^\infty |(\nabla e^{-tL}f)(x)|^2 dt \right)^{1/2}.$$

The  $L^2$  theory of quadratic estimates for operators having a bounded holomorphic functional calculus on  $L^2$  implies

$$(7.1) \quad \|g_L(f)\|_2 \sim \|f\|_2.$$

In fact, such an inequality is equivalent to the bounded holomorphic functional calculus on  $L^2$ . Moreover, the family  $((tL)^{1/2}e^{-tL})_{t>0}$  in  $g_L$  can be replaced by more general functions of  $L$ .<sup>60</sup>

As for  $G_L$ ,

$$(7.2) \quad \|G_L(f)\|_2 \sim \|f\|_2$$

is a consequence of ellipticity and

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = 2\Re \iint_{\mathbb{R}^n \times (0, \infty)} A(x)(\nabla e^{-tL}f)(x) \cdot \overline{(\nabla e^{-tL}f)(x)} dx dt.$$

This equality is obtained as follows. We have

$$\begin{aligned} \|f\|_2^2 &= - \int_0^\infty \frac{d}{dt} \|e^{-tL}f\|_2^2 dt \\ &= 2\Re \iint_{\mathbb{R}^n \times (0, \infty)} (Le^{-tL}f)(x) \overline{(e^{-tL}f)(x)} dx dt \end{aligned}$$

and it remains to integrate by parts in the  $x$  variable using the definition of  $L$ .

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<sup>60</sup> All this is due to McIntosh and Yagi, [McY] and [Y].

We are interested in the  $L^p$  counterparts of these results. Define

$$\mathcal{S}(L) = \{1 < p < \infty; \forall f \in L^2 \cap L^p \quad \|g_L(f)\|_p \sim \|f\|_p\}$$

and  $\mathcal{S}_-(L) = \mathcal{S}(L) \cap (1, 2)$  and

$$\mathcal{G}(L) = \{1 < p < \infty; \forall f \in L^2 \cap L^p \quad \|G_L(f)\|_p \sim \|f\|_p\},$$

$\mathcal{G}_-(L) = \mathcal{G}(L) \cap (1, 2)$  and  $\mathcal{G}_+(L) = \mathcal{G}(L) \cap (2, \infty)$ . Recall  $\text{int}\mathcal{J}(L) = (p_-(L), p_+(L))$  and  $\text{int}\mathcal{N}(L) = (q_-(L), q_+(L))$ .

**Theorem 7.1.** 1.  $\text{int}\mathcal{S}(L) = (p_-(L), p_+(L))$ .<sup>61</sup>

2.  $\text{int}\mathcal{G}(L) = (q_-(L), q_+(L))$ .<sup>62</sup>

Roughly, this theorem says that, up to endpoints, the range of  $p \in (1, \infty)$  for which  $g_L$  defines a new norm on  $L^p$  is the same as the one of boundedness of the semigroup and that the range of  $p \in (1, \infty)$  for which  $G_L$  defines a new norm on  $L^p$  is the same as the one of boundedness of  $(\sqrt{t} \nabla e^{-tL})_{t>0}$ .

Due to Theorems 5.2 and 5.6, the connections to intervals of exponents  $p$  for which one has  $L^p$  boundedness for the Riesz transform  $\nabla L^{-1/2}$  is as follows.

**Corollary 7.2.** 1. *The intervals  $\mathcal{I}_-(L)$ ,  $\mathcal{S}_-(L)$  and  $\mathcal{G}_-(L)$  have same interiors.*

2. *The intervals  $\mathcal{I}_+(L)$  and  $\mathcal{G}_+(L)$  have same interiors.*

Hence, there is again a dichotomy  $p < 2$  vs  $p > 2$  in the description. We turn to the proof of the theorem.

*Proof.* The argument has several steps. By Proposition 4.2, we may freely replace  $\mathcal{J}_\pm(L)$  by one of the intervals  $\mathcal{K}_\pm(L)$ ,  $\mathcal{M}_\pm(L)$ .

<sup>61</sup> We develop an argument based on the extension of Calderón-Zygmund theory in Section 2. However, there is an other argument as follows: we have shown that the interior of  $\mathcal{J}(L)$  is the maximal open interval of exponents  $p$  for which  $L$  has a bounded holomorphic functional calculus of  $L^p$ . Take such a  $p$ . Then, apply Le Merdy's Theorem 3 in [LeM] to conclude that  $g_L$  defines a new norm on  $L^p$ . More is proved in [LeM]: the operators  $(tL)^{1/2}e^{-tL}$  in the definition of  $g_L$  can be replaced by more general functions of  $L$ . We choose not to develop this here.

<sup>62</sup> The statement is valid for square functions where the semigroup in the definition of  $G_L$  is replaced by more general functions of  $L$ . Again, we do not go into such considerations.

**Step 1.**  $p_0 > 2$  and  $(\sqrt{t}\nabla e^{-tL})_{t>0}$  satisfies  $L^2 - L^{p_0}$  off-diagonal estimates imply  $\|G_L(f)\|_p \lesssim \|f\|_p$  for  $2 < p < p_0$ .

**Step 2.**  $p_0 > 2$  and  $(e^{-tL})_{t>0}$  satisfies  $L^2 - L^{p_0}$  off-diagonal estimates imply  $\|g_L(f)\|_p \lesssim \|f\|_p$  for  $2 < p < p_0$ .

**Step 3.**  $p_0 < 2$  and  $(e^{-tL})_{t>0}$  satisfies  $L^{p_0} - L^2$  off-diagonal estimates imply  $\|g_L(f)\|_p \lesssim \|f\|_p$  for  $p_0 < p < 2$ .

**Step 4.**  $p_0 < 2$  and  $(\sqrt{t}\nabla e^{-tL})_{t>0}$  satisfies  $L^{p_0} - L^2$  off-diagonal estimates imply  $\|G_L(f)\|_p \lesssim \|f\|_p$  for  $p_0 < p < 2$ .

**Step 5.** Reverse  $L^p$  inequality for  $g_L$  when  $p_-(L) < p < p_+(L)$ .

**Step 6.**  $\|g_L(f)\|_p \sim \|f\|_p$  implies  $(e^{-tL})_{t>0}$   $L^p$  bounded.

**Step 7.**  $\|G_L(f)\|_p \lesssim \|f\|_p$  implies  $(\sqrt{t}\nabla e^{-tL})_{t>0}$   $L^q$ -bounded for  $q$  in the interval between 2 and  $p$ .

**Step 8.** Reverse  $L^p$  inequality for  $G_L$  when  $1 < p < \infty$ .

The combination of steps 2, 3 and 5 shows that  $\text{int}\mathcal{J}(L)$  is contained in  $\mathcal{S}(L)$ . Step 6 implies that  $\mathcal{S}(L)$  is contained in  $\mathcal{J}(L)$ .

The combination of steps 1 and 4 shows that  $\text{int}\mathcal{N}(L)$  is contained in  $\text{int}\mathcal{G}(L)$ . Step 7 and Step 8 show the converse.

**Step 1.**  $p_0 > 2$  and  $(\sqrt{t}\nabla e^{-tL})_{t>0}$  satisfies  $L^2 - L^{p_0}$  off-diagonal estimates implies  $\|G_L(f)\|_p \lesssim \|f\|_p$  for  $2 < p < p_0$ . We apply Theorem 2.2 to the sublinear operator  $T = G_L$ . We choose as usual  $A_r = I - (I - e^{-r^2L})^m$  for  $m$  a large enough integral number.

We have to check (2.4). Let  $B$  be a ball and  $r = r(B)$  be its radius and  $k = 1, \dots, m$ . By Minkowski integral inequality and  $p_0 > 2$ , we have

$$\left( \frac{1}{|B|} \int_B |G_L(e^{-kr^2L}f)|^{p_0} \right)^{2/p_0} \leq \int_0^\infty \left( \frac{1}{|B|} \int_B |\nabla e^{-tL}(e^{-kr^2L}f)|^{p_0} \right)^{2/p_0} dt$$

Using the hypothesis of  $p_0$ , the commutativity property of the semigroup and applying the scalar inequality (5.6) to  $e^{-tL}f$  for each  $t > 0$ , we have

$$\left( \frac{1}{|B|} \int_B |\nabla e^{-tL}(e^{-kr^2L}f)|^{p_0} \right)^{1/p_0} \leq \sum_{j \geq 1} g(j) \left( \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |\nabla e^{-tL}f|^2 \right)^{1/2}$$

Squaring this and using that  $\sum g(j) < \infty$ , we obtain

$$\int_0^\infty \left( \frac{1}{|B|} \int_B |\nabla e^{-tL}(e^{-kr^2L}f)|^{p_0} \right)^{2/p_0} dt \leq C \int_0^\infty \sum_{j \geq 1} \frac{g(j)}{|2^{j+1}B|} \int_{2^{j+1}B} |\nabla e^{-tL}f|^2 dt$$

Exchanging the sum and the integral, the latter is equal to

$$C \sum_{j \geq 1} g(j) \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |G_L(f)|^2$$

which is controlled by  $CM(G_L(f)^2)(y)$  for any  $y \in B$ .

Next, we establish (2.3). Let  $B$  be a ball and  $r = r(B)$  be its radius. Let  $f \in L^2$ . Decompose  $f = f_1 + f_2 + f_3 \dots$  where  $f_j = f\chi_{C_j}$  and  $C_j$  are defined as usual. We start from

$$\left( \frac{1}{|B|} \int_B |G_L((I - e^{-r^2L})^m f)|^2 \right)^{1/2} \leq \sum_{j \geq 1} \left( \frac{1}{|B|} \int_B |G_L((I - e^{-r^2L})^m f_j)|^2 \right)^{1/2}.$$

For  $j = 1$ , we merely use the  $L^2$  boundedness of  $G_L$  in (7.2) and that of  $(I - e^{-r^2L})^m$  to obtain

$$\left( \frac{1}{|B|} \int_B |G_L((I - e^{-r^2L})^m f_1)|^2 \right)^{1/2} \leq C \left( \frac{1}{|4B|} \int_{4B} |f|^2 \right)^{1/2}.$$

For  $j \geq 2$ , we write

$$\frac{1}{|B|} \int_B |G_L((I - e^{-r^2L})^m f_j)|^2 = \int_0^\infty \frac{1}{|B|} \int_B |\sqrt{t} \nabla (I - e^{-kr^2L})^m e^{-tL} f_j|^2 dx \frac{dt}{t}.$$

and we use the estimates in the second argument of the proof of Theorem 5.6 to obtain a bound

$$\frac{C2^{nj}}{|2^{j+1}B|} \int_{2^{j+1}B} |f|^2 \left( \int_0^\infty \frac{1}{4^{jm}} \inf \left( \left( \frac{t}{4^j r^2} \right)^{1/2}, \left( \frac{4^j r^2}{t} \right)^{m-1/2} \right) \frac{dt}{t} \right)^2,$$

hence as  $m \geq 1$ ,

$$C2^{nj}2^{-4mj} \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f|^2.$$

Choosing further  $m > n/4$  allows to sum in  $j \geq 2$  and to conclude for (2.3).

**Step 2.**  $p_0 > 2$  and  $(e^{-tL})_{t>0}$  satisfies  $L^2 - L^{p_0}$  off-diagonal estimates imply  $\|g_L(f)\|_p \lesssim \|f\|_p$  for  $2 < p < p_0$ . We apply Theorem 2.2 to the sublinear operator  $T = g_L$ . We choose as usual  $A_r = I - (I - e^{-r^2L})^m$  for  $m$  a large enough integral number.

We have to check (2.4). Let  $B$  be a ball and  $r = r(B)$  be its radius and  $k = 1, \dots, m$ . By Minkowski integral inequality and  $p_0 > 2$ , we have

$$\left( \frac{1}{|B|} \int_B |g_L(e^{-kr^2L}f)|^{p_0} \right)^{2/p_0} \leq \int_0^\infty \left( \frac{1}{|B|} \int_B |(tL)^{1/2}e^{-tL}(e^{-kr^2L}f)|^{p_0} \right)^{2/p_0} \frac{dt}{t}.$$

Using the hypothesis of  $p_0$ , and following already used arguments, we have that

$$\left( \frac{1}{|B|} \int_B |(e^{-kr^2L}g)|^{p_0} \right)^{2/p_0} \leq \sum_{j \geq 1} \frac{c_j}{|2^{j+1}B|} \int_{2^{j+1}B} |g|^2$$

with  $c_j = Ce^{-c4^j}$  for some positive constants  $c, C$ . Applying the commutativity property of the semigroup and this inequality to  $g = (tL)^{1/2}e^{-tL}f$  for each  $t > 0$ , we have

$$\left( \frac{1}{|B|} \int_B |g_L(e^{-kr^2L}f)|^{p_0} \right)^{2/p_0} \leq \int_0^\infty \left\{ \sum_{j \geq 1} \frac{c_j}{|2^{j+1}B|} \int_{2^{j+1}B} |(tL)^{1/2}e^{-tL}f|^2 \right\} \frac{dt}{t}.$$

As the latter expression equals

$$\sum_{j \geq 1} \frac{c_j}{|2^{j+1}B|} \int_{2^{j+1}B} |g_L(f)|^2$$

we obtain a bound in  $CM(g_L(f)^2)(y)$  for any  $y \in B$  as desired.

Next, we establish (5.5). Let  $B$  be a ball and  $r = r(B)$  be its radius. Let  $f \in L^2$ . Decompose  $f = f_1 + f_2 + f_3 \dots$  where  $f_j = f\chi_{C_j}$  and  $C_j = C_j(B)$  are defined as usual. We start from

$$\left( \frac{1}{|B|} \int_B |g_L((I - e^{-kr^2L})^m f)|^2 \right)^{1/2} \leq \sum_{j \geq 1} \left( \frac{1}{|B|} \int_B |g_L((I - e^{-kr^2L})^m f_j)|^2 \right)^{1/2}.$$

For  $j = 1$ , we merely use the  $L^2$  boundedness of  $g_L$  in (7.1) and that of  $(I - e^{r^2L})^m$  to obtain

$$\left( \frac{1}{|B|} \int_B |g_L((I - e^{-kr^2L})^m f_1)|^2 \right)^{1/2} \leq C \left( \frac{1}{|4B|} \int_{4B} |f|^2 \right)^{1/2}.$$

For  $j \geq 2$ , we write

$$\frac{1}{|B|} \int_B |g_L((I - e^{-kr^2L})^m f_j)|^2 = \int_0^\infty \frac{1}{|B|} \int_B |(tL)^{1/2} e^{-tL} (I - e^{-kr^2L})^m f_j|^2 dx \frac{dt}{t}.$$

As in Section 5.1.1, one may use the representation (3.6) with the function  $\varphi(z) = (tz)^{1/2} e^{-tz} (1 - e^{r^2z})^m$ . The corresponding functions  $\eta_\pm$  satisfy the estimates

$$(7.3) \quad |\eta_\pm(z)| \leq \frac{Ct^{1/2}}{(|z| + t)^{3/2}} \inf \left( 1, \frac{r^{2m}}{(|z| + t)^m} \right), \quad z \in \Gamma_\pm.$$

Since for any  $0 < \beta < \frac{\pi}{2} - \omega$ ,  $(e^{-zL})_{z \in \Sigma_\beta}$  satisfies  $L^2$  off-diagonal estimates, using (3.6) and the above estimate for  $\eta_\pm$ ,  $\|(tL)^{1/2} e^{-tL} (I - e^{-kr^2L})^m f_j\|_{L^2(B)}$  is bounded by

$$C \int_{\Gamma_+} e^{-\frac{c4^j r^2}{|z|}} \frac{t^{1/2}}{(|z| + t)^{3/2}} \frac{r^{2m}}{(|z| + t)^m} |dz| \|f\|_{L^2(C_j)}$$

plus the similar term corresponding to integration on  $\Gamma_-$ . Using (5.4), this gives us the bound

$$\frac{C}{4^j m} \inf \left( \left( \frac{t}{4^j r^2} \right)^{1/2}, \left( \frac{4^j r^2}{t} \right)^m \right) \|f\|_{L^2(C_j)}.$$

Squaring and integrating with respect to  $t$ , we find

$$\frac{1}{|B|} \int_B |g_L((I - e^{-kr^2L})^m f_j)|^2 \leq C 2^{jn} 4^{-mj} \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} |f|^2$$

and this readily implies (5.5).

**Step 3.**  $p_0 < 2$  and  $(e^{-tL})_{t>0}$  satisfies  $L^{p_0} - L^2$  off-diagonal estimates imply  $\|g_L(f)\|_p \lesssim \|f\|_p$  for  $p_0 < p < 2$ . We apply Theorem 2.1 to  $T = g_L$ . We choose as usual  $A_r = I - (I - e^{-r^2L})^m$  for  $m$  a large enough integral number.

Equation (2.2) is a direct consequence of the assumption on the semi-group. We then turn to the verification of (2.1). Let  $B$  be a ball and  $r = r(B)$  its radius and  $j \geq 2$ . Let  $f$  be a function supported in  $B$ . We have

$$\frac{1}{|2^{j+1}B|} \int_{C_j} |g_L((I - e^{-kr^2L})^m f)|^2 = \int_0^\infty \frac{1}{|2^{j+1}B|} \int_{C_j} |(tL)^{1/2} e^{-tL} (I - e^{-kr^2L})^m f|^2 dx \frac{dt}{t}.$$



Using similar arguments as in the previous step and using the hypothesis on the semigroup,  $\|(tL)^{1/2}e^{-tL}(I - e^{-r^2L})^m f\|_{L^2(C_j)}$  is bounded by

$$C \int_{\Gamma_+} e^{-\frac{c4^j r^2}{|z|}} \frac{1}{|z|^{\gamma/2}} \frac{t^{1/2}}{(|z| + t)^{3/2}} \frac{r^{2m}}{(|z| + t)^m} |dz| \|f\|_{L^{p_0}(B)}$$

plus the similar term corresponding to integration on  $\Gamma_-$ . Here  $\gamma = \gamma_{p_0} = |\frac{n}{2} - \frac{n}{p_0}|$ . Using (5.4), this gives us the bound

$$\frac{C}{4^{jm}(2^j r)^\gamma} \inf \left( \left( \frac{t}{4^j r^2} \right)^{1/2}, \left( \frac{4^j r^2}{t} \right)^m \right) \|f\|_{L^{p_0}(B)}.$$

Squaring and integrating with respect to  $t$ , we find

$$\left( \frac{1}{|2^{j+1}B|} \int_{C_j} |g_L((I - e^{-r^2L})^m f)|^2 \right)^{1/2} \leq C 2^{-jn/2} 4^{-mj} 2^{-j\gamma} \left( \frac{1}{|B|} \int_B |f|^{p_0} \right)^{1/p_0}$$

and this readily implies (2.1) and  $m$  is chosen with  $2m + \gamma > n/2$ .

**Step 4.**  $p_0 < 2$  and  $(\sqrt{t} \nabla e^{-tL})_{t>0}$  satisfies  $L^{p_0} - L^2$  off-diagonal estimates imply  $\|G_L(f)\|_p \lesssim \|f\|_p$  for  $p_0 < p < 2$ . We apply Theorem 2.1 to  $T = G_L$ . We choose as usual  $A_r = I - (I - e^{-r^2L})^m$  for  $m$  a large enough integral number.

Equation (2.2) is a direct consequence of the assumption on the semigroup. We then turn to the verification of (2.1). Let  $B$  be a ball and  $r = r(B)$  its radius and  $j \geq 2$ . Let  $f$  be a function supported in  $B$ . We have

$$\frac{1}{|2^{j+1}B|} \int_{C_j} |G_L((I - e^{-r^2L})^m f)|^2 = \int_0^\infty \frac{1}{|2^{j+1}B|} \int_{C_j} |\sqrt{t} \nabla e^{-tL} (I - e^{-kr^2L})^m f|^2 dx \frac{dt}{t}.$$

Proposition 4.16 shows that we may replace the hypothesis by  $(\sqrt{z} \nabla e^{-zL})_{z \in \Sigma_\beta}$  satisfies  $L^{p_0} - L^2$  off-diagonal estimates for any  $\beta \in (0, \frac{\pi}{2} - \omega)$  (up to changing  $p_0$  to an arbitrary larger value). Using then similar arguments as in the step 1,  $\|\sqrt{t} \nabla e^{-tL} (I - e^{-kr^2L})^m f\|_{L^2(C_j)}$  is bounded by

$$C \int_{\Gamma_+} e^{-\frac{c4^j r^2}{|z|}} \frac{1}{|z|^{\gamma/2+1/2}} \frac{t^{1/2}}{(|z| + t)} \frac{r^{2m}}{(|z| + t)^m} |dz| \|f\|_{L^{p_0}(B)}$$

plus the similar term corresponding to integration on  $\Gamma_-$ . Here  $\gamma = \gamma_{p_0} = |\frac{n}{2} - \frac{n}{p_0}|$ . Using (5.4), this gives us the bound

$$\frac{C}{4^{jm}(2^j r)^\gamma} \inf \left( \left( \frac{t}{4^j r^2} \right)^{1/2}, \left( \frac{4^j r^2}{t} \right)^{m-1/2} \right) \|f\|_{L^{p_0}(B)}.$$

Squaring and integrating with respect to  $t$ , we find

$$\left( \frac{1}{|2^{j+1}B|} \int_{C_j} |G_L((I - e^{-kr^2L})^m f)|^2 \right)^{1/2} \leq C 2^{-jn/2} 4^{-mj} 2^{-j\gamma} \left( \frac{1}{|B|} \int_B |f|^{p_0} \right)^{1/p_0}$$

and this readily implies (2.1) if  $m$  is chosen with  $2m + \gamma > n/2$ .

**Step 5. Reverse  $L^p$  inequality for  $g_L$  when  $p_-(L) < p < p_+(L)$ .** By functional calculus for  $L$ , we have the representation formula for the identity on  $L^2$

$$(7.4) \quad f = 2 \int_0^\infty t L e^{-2tL} f \frac{dt}{t}.$$

Hence, dualizing against  $g$  in  $L^2$ , writing

$$\langle t L e^{-2tL} f, g \rangle = \langle (tL)^{1/2} e^{-tL} f, (tL^*)^{1/2} e^{-tL^*} g \rangle$$

and using Cauchy-Schwarz inequality yield

$$\left| \int_{\mathbb{R}^n} f \bar{g} \right| \leq \int_{\mathbb{R}^n} g_L(f) g_{L^*}(g).$$

Recall that  $\text{int}\mathcal{J}(L) = (p_-(L), p_+(L))$  is the largest open interval of exponents  $p$  in  $(1, \infty)$  for which the semigroup  $(e^{-tL})_{t>0}$  is  $L^p$  bounded. Then, by steps 2 and 3,  $g_L$  is  $L^p$  bounded for  $p \in \text{int}\mathcal{J}(L)$ . Let  $p \in \text{int}\mathcal{J}(L)$ . Applying this to  $L^*$  since  $p' \in \text{int}\mathcal{J}(L^*)$ , we have

$$\|g_{L^*}(g)\|_{p'} \leq C \|g\|_{p'},$$

hence,

$$\left| \int_{\mathbb{R}^n} f \bar{g} \right| \leq C \|g_L(f)\|_p \|g\|_{p'}.$$

In conclusion,

$$\|f\|_p \leq C \|g_L(f)\|_p$$

for  $p \in \text{int}\mathcal{J}(L)$ .

**Step 6.**  $\|g_L(f)\|_p \sim \|f\|_p$  **implies**  $(e^{-tL})_{t>0}$   $L^p$  **bounded.** It follows easily from the definition of  $g_L$  and the commutativity properties of the semigroup that for all  $s > 0$ ,

$$g_L(e^{-sL}f) \leq g_L(f)$$

in the pointwise sense, hence in  $L^p$  norm. In particular, this and the hypothesis yield  $\|e^{-sL}f\|_p \leq C\|f\|_p$  uniformly in  $s > 0$ .

**Step 7.**  $\|G_L(f)\|_p \lesssim \|f\|_p$  **implies**  $(\sqrt{t} \nabla e^{-tL})_{t>0}$   $L^q$ -**bounded for  $q$  between 2 and  $p$ .**<sup>63</sup>

The argument of step 5 does not apply for the semigroup and the gradient operator do not commute. We rely instead on some trick using complex interpolation. Define

$$\tilde{G}_L(f)(x) = \left( \int_0^\infty |(\nabla(tL)e^{-tL}f)(x)|^2 dt \right)^{1/2}.$$

We assume  $p > 2$ . The argument for  $p < 2$  is entirely similar. We show that  $\|G_L(f)\|_p \lesssim \|f\|_p$  implies  $\|\tilde{G}_L(f)\|_q \lesssim \|f\|_q$  for  $2 < q < p$ . Assume this is done. We prove the  $L^q$  boundedness of  $\sqrt{s} \nabla e^{-sL}$ . Without loss of generality, we may assume  $s = 1$ . Write

$$\begin{aligned} |\nabla e^{-L}f| &\leq \int_1^2 |\nabla e^{-sL}f| ds + \int_1^2 |\nabla e^{-L}f - \nabla e^{-sL}f| ds \\ &\leq \int_1^2 |\nabla e^{-sL}f| ds + \int_1^2 \left| \int_1^s \nabla L e^{-tL}f dt \right| ds \\ &\leq \int_1^2 |\nabla e^{-sL}f| ds + \int_1^2 |\nabla(tL)e^{-tL}f| dt. \end{aligned}$$

Hence,

$$|\nabla e^{-L}f| \leq G_L(f) + \tilde{G}_L(f).$$

and the  $L^q$  boundedness of  $\nabla e^{-L}$  follows from that of  $G_L$  and  $\tilde{G}_L$ .

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<sup>63</sup> In practice, quadratic functionals are built from operators whose individual boundedness properties are known. The converse, thus, is never considered, so that this result appears new. Note that the argument could be written for arbitrary quadratic functionals made after an analytic family of operators. We leave to the reader the care of stating this general result.

It remains to prove the  $L^q$  boundedness of the latter. To this end, we follow the proof of Stein's complex interpolation theorem after dualizing. Fix  $q$  with  $2 < q < p$ .

Let  $f$  be a  $\mathbb{C}$ -valued simple function with  $\|f\|_q = 1$  and  $g = (g_t)_{t>0}$  be a  $H$ -valued simple function where  $H = L^2((0, \infty), \mathbb{C}^n; dt)$  with  $\|g\|_{q'} = 1$ . Write  $f(x) = \sum_k a_k \chi_{E_k}(x)$  and  $g(t, x) = \sum_k b_k(t) \chi_{E_k}(x)$  where  $E_k$  are pairwise disjoint measurable sets,  $a_k$  are complex numbers and  $b_k(t)$  are  $\mathbb{C}^n$ -valued, and set  $B_k = (\int_0^\infty |b_k(t)|^2 dt)^{1/2}$ .

Let

$$f_z(x) = \sum_k |a_k|^{\alpha(z)} \frac{a_k}{|a_k|} \chi_{E_k}(x)$$

and

$$g_z(t, x) = \sum_k B_k^{\beta(z)} \frac{b_k(t)}{B_k} \chi_{E_k}(x)$$

with

$$\alpha(z) = \frac{q}{p}(1-z) + \frac{q}{2}z \quad \text{and} \quad \beta(z) = \frac{q'}{p'}(1-z) + \frac{q'}{2}z.$$

Pick  $\beta \in (0, \frac{\pi}{2} - \omega)$  and consider the function

$$F(z) = \int_{\mathbb{R}^n} \int_0^\infty \nabla e^{-te^{i\beta}z} L f_z(x) \cdot g_z(t, x) dx dt.$$

defined for  $z$  in the strip  $0 \leq \Re z \leq 1$ . This function is clearly continuous, and it is analytic in the interior of this strip. Moreover, for  $\Re z = 0$ , one finds easily from Cauchy-Schwarz and Hölder inequalities

$$|F(z)| \leq \|G_L(f_z)\|_p \|g_z\|_{p'} \leq C_0 \|f_z\|_p \|g_z\|_{p'} = C_0$$

and for  $\Re(z) = 1$ , by the square function estimate (7.2) for  $e^{i\beta}L$ ,

$$|F(z)| \leq \|G_{e^{i\beta}L} f_z\|_2 \|g_z\|_2 \leq C_1 \|f_z\|_2 \|g_z\|_2 = C_1.$$

Hence, by the three lines theorem, we have for  $0 < \Re z < 1$

$$|F(z)| \leq \max(C_0, C_1).$$

This implies that  $L^q$  boundedness of  $G_{e^{i\beta}c_q L}$  when  $\Re c_q = \frac{\frac{1}{q} - \frac{1}{p}}{\frac{1}{2} - \frac{1}{p}}$ . Now, this is true for all  $\beta$  in  $(0, \frac{\pi}{2} - \omega)$  and also in  $(-\frac{\pi}{2} + \omega, 0)$  by changing  $\beta$  to  $-\beta$ .

Keep  $q$  as before and let  $f, g$  be simple functions as above. If  $\beta$  is small enough, the function

$$G(z) = \int_{\mathbb{R}^n} \int_0^\infty \nabla e^{-te^{i\beta z} L} f(x) \cdot g(t, x) dx dt$$

is continuous in the strip  $-1 \leq \Re z \leq 1$  and, by the previous argument, analytic in the open strip minus the real axis. Thus it is analytic in the open strip by Morera's theorem. Applying the three lines theorem again (this times,  $f$  and  $g$  are fixed and the exponent remains  $q$ ), we deduce a bound for  $G'(0)$ , which is equivalent to the  $L^q$  boundedness of  $\tilde{G}_L$ .

**Step 8. Reverse  $L^p$  inequality for  $G_L$  when  $1 < p < +\infty$ .** What we have proved so far applies to any operator  $L$  in our class, and in particular, to  $L = -\Delta$ . The explicit formula for the heat kernel implies that  $p_-(-\Delta) = 1$  and  $q_+(-\Delta) = \infty$ . Hence, we recover from our method the well-known estimate <sup>64</sup>

$$\|G_{-\Delta}(f)\|_p \lesssim \|f\|_p$$

for all  $1 < p < \infty$ .

Now, let  $f, g \in L^2$  and observe that <sup>65</sup>

$$\begin{aligned} \int_{\mathbb{R}^n} f \bar{g} &= \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} e^{-\varepsilon L} f \overline{e^{\varepsilon \Delta} g} - \lim_{R \uparrow \infty} \int_{\mathbb{R}^n} e^{-RL} f \overline{e^{R\Delta} g} \\ &= - \int_0^\infty \frac{d}{dt} \int_{\mathbb{R}^n} (e^{-tL} f)(x) \overline{(e^{t\Delta} g)(x)} dx dt \\ &= \iint_{\mathbb{R}^n \times (0, \infty)} (A(x) + I)(\nabla e^{-tL} f)(x) \cdot \overline{(\nabla e^{t\Delta} g)(x)} dx dt. \end{aligned}$$

The last equality is obtained by integration by parts in the  $x$  variable after computing the time derivative. Hence, we obtain

$$\left| \int_{\mathbb{R}^n} f \bar{g} \right| \leq (\|A\|_\infty + 1) \int_{\mathbb{R}^n} G_L(f) G_{-\Delta}(g).$$

Thus, if  $1 < p < \infty$ , the  $L^{p'}$  boundedness of  $G_{-\Delta}$  yields

$$\left| \int_{\mathbb{R}^n} f \bar{g} \right| \lesssim \|G_L(f)\|_p \|g\|_{p'}$$

<sup>64</sup> due to Stein. See [St1].

<sup>65</sup> we learned this trick from J. Pipher in an unpublished manuscript. See [AMcN1] where it is used.

and it follows

$$\|f\|_p \lesssim \|G_L(f)\|_p.$$

Of course this inequality is meaningful whenever the right hand side is finite.  $\square$

Let us draw some consequences of our results.

**Corollary 7.3.** *If  $n = 1$ , we have for  $1 < p < \infty$ ,*

$$\|G_L(f)\|_p \sim \|f\|_p \sim \|g_L(f)\|_p.$$

**Corollary 7.4.** *If  $n = 2$ , we have  $\|g_L(f)\|_p \sim \|f\|_p$  for  $1 < p < \infty$  and  $\|G_L(f)\|_p \sim \|f\|_p$  for  $1 < p < 2 + \varepsilon'$ .*

**Corollary 7.5.** *If  $n \geq 3$ , we have  $\|g_L(f)\|_p \sim \|f\|_p$  for  $\frac{2n}{n+2} - \varepsilon < p < \frac{2n}{n-2} + \varepsilon'$  and  $\|G_L(f)\|_p \sim \|f\|_p$  for  $\frac{2n}{n+2} - \varepsilon < p < 2 + \varepsilon'_1$ .*

**Corollary 7.6.** *If  $n \geq 3$  and  $L$  has **real** coefficients, then we have  $\|g_L(f)\|_p \sim \|f\|_p$  for  $1 < p < \infty$  and  $\|G_L(f)\|_p \sim \|f\|_p$  for  $1 < p < 2 + \varepsilon$ .*

**Corollary 7.7.** *If  $n \geq 5$ , there exists  $L$  such that  $g_L$  and  $G_L$  are not bounded for some  $p$  close to 1 (hence  $g_{L^*}$  is not bounded on  $L^p$  for some  $p$  close to  $\infty$ ).*

We finish this section by proving Lemma 5.14.

**Proof of Lemma 5.14:** The assumption of the lemma is that  $(e^{-tL})_{t>0}$  satisfies  $L^\rho - L^2$  off-diagonal estimates. By duality,  $(e^{-tL^*})_{t>0}$  satisfies  $L^2 - L^{\rho'}$  off-diagonal estimates. Hence, it follows from the method of step 2 above (applied to discrete times  $4^k$ ,  $k \in \mathbb{Z}$ , and changing  $(tL)^{1/2}e^{-tL}$  to  $\psi(tL)$  with  $\psi$  given by (5.15)) that for  $2 < q' < \rho'$ ,

$$\left\| \left( \sum_{k \in \mathbb{Z}} |\psi(4^k L^*) f|^2 \right)^{1/2} \right\|_{q'} \leq C \|f\|_{q'}.$$

By duality, we obtain (5.2).

## 7.2 On inequalities of Stein and Fefferman for non-tangential square functions

Consider the functional

$$g_\lambda^*(f)(x) = \left( \iint_{\mathbb{R}_+^{n+1}} \left( \frac{t}{|x-y|+t} \right)^{n\lambda} |t \nabla u(y, t)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}$$

where  $u(x, t)$  is the harmonic extension of  $f$  and  $1 < \lambda$ . It is bounded on  $L^p$  ( $1 < p < \infty$ ) if and only if  $\lambda > \frac{2}{p}$ .<sup>66</sup> At the critical case  $\lambda = \frac{2}{p}$ , it is weak-type  $(p, p)$ .<sup>67</sup> Of course, the main tool for this is Calderón-Zygmund decomposition for  $L^p$  functions.

We show how to use our technology in this situation by replacing functions of the Laplacian by functions of  $L$ . Again, the main point is that  $L^p$  boundedness of the semigroup suffices. We also separate the cases  $p < 2$  and  $p > 2$ .

Define

$$g_\lambda^*(f)(x)^2 = \iint_{\mathbb{R}_+^{n+1}} \left( \frac{\sqrt{t}}{|x-y|+\sqrt{t}} \right)^{n\lambda} |\mathcal{T}(e^{-tL}f)(y)|^2 \frac{dy dt}{\sqrt{t}^n},$$

where for fixed  $t > 0$ ,

$$|\mathcal{T}g(y, t)|^2 = |\nabla_y g(y, t)|^2 + |L^{1/2}g(y, t)|^2.$$

In particular, this square function controls non tangential square functions where integration is performed on parabolic cones  $|x-y| \leq c\sqrt{t}$ :

$$g_{NT}(f)(x)^2 = \iint_{|x-y| < c\sqrt{t}} |\mathcal{T}e^{-tL}f(y)|^2 \frac{dy dt}{\sqrt{t}^n}.$$

**Proposition 7.8.** *If  $q_-(L) < p < 2$  and  $\lambda = \frac{2}{p}$  then  $g_\lambda^*$  has weak type  $(p, p)$ .*

**Proposition 7.9.** *If  $2 < p < q_+(L)$  and  $\lambda > 1$  then  $g_\lambda^*$  has strong type  $(p, p)$ .*

**Corollary 7.10.** *If  $q_-(L) < p < q_+(L)$  and  $\lambda > \frac{2}{p}$ , then  $g_{NT}, g_\lambda^*$  are bounded on  $L^p$  and one has<sup>68</sup>*

$$\|g_\lambda^*f\|_p \sim \|g_{NT}(f)\|_p \sim \|f\|_p.$$

<sup>66</sup> This is due to Stein [St1].

<sup>67</sup> This is in C. Fefferman's thesis [Fef].

<sup>68</sup> While Le Merdy's theorem mentioned above works well for the vertical square function of abstract operators on  $L^p$  spaces, it is not clear it applies to non-tangential square functions which are more geometrical objects in an abstract setting.

Note that the result for  $g_{NT}$  holds for  $q_-(L) < p < q_+(L)$  since it is independent of  $\lambda$ . The proof of the corollary is simple: we have a pointwise control of  $g_{NT}$  by  $g_\lambda^*$  for any  $\lambda > \frac{2}{p}$  and we obtain boundedness. The reverse inequalities are obtained as for  $g_L$  and  $G_L$  following the heuristic idea below. Suppose we can write  $\int f(y)g(y) dy = \iint f_t(y)g_t(y)dydt$  and that  $\int h(x, y, t) dx = 1$  for all  $y, t$ . Then

$$\int f(y)g(y) dy = \int \left( \iint f_t(y)g_t(y)h(x, y, t) dydt \right) dx.$$

It suffices to apply Cauchy-Schwarz in the variables  $y, t$  to obtain non-tangential square function by choosing  $h$  and then Hölder in the  $x$  variable to deduce reverse inequalities from direct ones. We skip further details.

*Remark.* The limitations on  $p$  are only due to the presence of the spatial gradient  $\nabla_y$  in the definition of  $g_\lambda^*$ . If one drops this gradient to keep only the  $L^{1/2}$  part then the range of  $p$  becomes  $p_-(L) < p < p_+(L)$ .

Let us turn to the proof of Proposition 7.8. Due to the fact that  $x$  and  $y$  may be far apart, Theorem 2.1 does not apply directly and one has to do some transformations. The key identity of this proof is that for any closed set  $F$ ,

$$\int_F g_\lambda^*(f)(x)^2 dx = \iint_{\mathbb{R}_+^{n+1}} |\mathcal{T}(e^{-tL}f)(y)|^2 J_{\lambda, F}(y, t) dydt$$

with

$$J_{\lambda, F}(y, t) = \frac{1}{\sqrt{t}^n} \int_F \left( \frac{\sqrt{t}}{|x-y| + \sqrt{t}} \right)^{n\lambda} dx.$$

First  $J_{\lambda, F} \leq C$  so that

$$\|g_\lambda^*(f)\|_2^2 \leq C\|g_L(f)\|_2^2 + C\|G_L(f)\|_2^2$$

where  $g_L$  and  $G_L$  are the square functions defined earlier. The  $L^2$  boundedness of  $g_L$  and  $G_L$  implies the  $L^2$  boundedness of  $g_\lambda^*$ .

Second, we have also if  $y$  lies in some cube  $Q$  and  $F = \mathbb{R}^n \setminus (2Q)$  that

$$J_{\lambda, F}(y, t) \leq C\sqrt{t}^{n(\lambda-1)}|Q|^{-(\lambda-1)/2}.$$

We begin as in Theorem 2.1, by looking at  $\{g_\lambda^*(f) > \alpha\}$  and decomposing  $f = g + \sum b_i$  according to the threshold  $\alpha^p$  for  $|f|^p$ . For  $g$  use the  $L^2$  boundedness of  $g_\lambda^*$ .



Next write again

$$b_i = A_{r_i} b_i + (1 - A_{r_i}) b_i$$

where  $A_r$  is the operator that works for  $g_L$  and  $G_L$  in the previous section. The term  $\sum A_{r_i} b_i$  is again in  $L^2$  with the right bound so that the  $L^2$  boundedness of  $g_\lambda^*$  suffices again. It remains to estimate the size of the set

$$\{x \in \mathbb{R}^n; g_\lambda^*(\sum (1 - A_{r_i}) b_i)(x) > \alpha/3\}.$$

Again we take away the union of the dilated Whitney cubes  $4Q_i$ , whose mass is under control. It remains to estimate what is left on its complement  $F$ . By Tchebychev's inequality, it is enough to estimate  $\int_F g_\lambda^*(\sum (1 - A_{r_i}) b_i)(x)^2 dx$  which we rewrite as

$$\iint_{\mathbb{R}_+^{n+1}} \left| \sum_i \mathcal{T}(e^{-tL}(1 - A_{r_i}) b_i)(y) \right|^2 J_{\lambda, F}(y, t) dy dt.$$

The non local part of the  $i$ th summand is when  $y \notin 2Q_i$ . We bound  $J_{\lambda, F}$  by a constant and we are back to the estimates performed to obtain the weak type  $(p, p)$  for  $g_L$  and  $G_L$ . We refer the reader to steps 3 and 4 in the previous section.

It remains to localise each  $i$ th summand on  $2Q_i$ . By the bounded overlaps of  $2Q_i$ 's<sup>69</sup> and the second upper bound on  $J_{\lambda, F}$ , we have an upper bound

$$N \sum_i \iint_{2Q_i \times \mathbb{R}_+} \frac{\sqrt{t}^{n(\lambda-1)}}{|Q_i|^{(\lambda-1)/2}} |\mathcal{T}(e^{-tL}(1 - A_{r_i}) b_i)(y)|^2 dy dt.$$

Now for each  $i$ , we integrate on the full upper half space: the integral in  $y$  (with  $t$  fixed) and the solution of the Kato problem<sup>70</sup> allows us to bound the term with  $\nabla_y$  by the one with  $L^{1/2}$ . Next if  $a = \frac{n}{2}(\lambda - 1) = \frac{n}{p} - \frac{n}{2}$ , the  $L^p$  bounds of the semigroup implies by Proposition 6.3 the Hardy-Littlewood-Sobolev inequality

$$\|L^{-a} f\|_2 \leq C \|f\|_p.$$

Using the square function estimate of McIntosh-Yagi based on  $(\sqrt{t}L)^a e^{-tL}$  we obtain that the  $i$ th term is bounded by

$$\frac{C |Q_i|^{1-2/p} \|L^{-a}(1 - A_{r_i}) b_i\|_2^2}{\sqrt{\phantom{x}}} \leq C |Q_i|^{1-2/p} \|(1 - A_{r_i}) b_i\|_p^2 \leq C |Q_i|^{1-2/p} \|b_i\|_p^2.$$

<sup>69</sup> We have to make sure in the construction of the Whitney cubes that this actually holds and may be twice the cubes is not appropriate but certainly  $cQ_i$  with some  $c > 1$  is.

<sup>70</sup> One can proceed also using Remark 6.2.

It remains to sum other  $i$  and we are done.  $\square$

Let us turn to the proof of the Proposition 7.9. Here also, we cannot apply directly Theorem 2.2 but rather its spirit and its proof. We let  $f \in L^2(\mathbb{R}^n)$  and  $B$  be a ball with radius  $r$ . We also let  $2 < p_0 < q_+(L)$  and  $m > n/4$  an integer. Assume that we have proved that for  $k = 1, 2, \dots, m$

$$(7.5) \quad \left( \frac{1}{|B|} \int_B |g_\lambda^*(e^{-kr^2L}f)|^{p_0} \right)^{1/p_0} \leq C \inf_{x \in B} M(g_\lambda^*(f)^2)^{1/2}(x)$$

then we can argue as follows. For  $A_r = I - (I - e^{r^2L})^m$ , we have

$$g_\lambda^*(f)^2(x) \leq 2g_\lambda^*(A_r f)^2(x) + 2g_\lambda^*((I - A_r)f)^2(x).$$

Write

$$\begin{aligned} g_\lambda^*((I - A_r)f)^2(x) &= \iint_{y \in 2B} h(x - y, t) |\mathcal{T}(e^{-tL}(I - A_r)f)(y)|^2 dy dt \\ &\quad + \iint_{y \notin 2B} h(x - y, t) |\mathcal{T}(e^{-tL}(I - A_r)f)(y)|^2 dy dt \end{aligned}$$

with  $h(x, t) = \sqrt{t}^n \left( \frac{\sqrt{t}}{|x| + \sqrt{t}} \right)^{n\lambda}$ . As  $h(x - y, t) \sim h(z - y, t)$  for  $x, z \in B$  and  $y \notin 2B$ , the second integral is bounded by

$$2g_\lambda^*(A_r f)^2(x) + 2 \inf_{z \in B} g_\lambda^*(f)^2(z).$$

Hence, we can apply Proposition 2.5 as in the proof of Theorem 2.2 with

$$G_B(x) = 2 \iint_{y \in 2B} h(x - y, t) |\mathcal{T}(e^{-tL}(I - A_r)f)(y)|^2 dy dt$$

and

$$H_B(x) = 4g_\lambda^*(A_r f)^2(x) + 2 \inf_{z \in B} g_\lambda^*(f)^2(z)$$

provided we show that

$$\frac{1}{|B|} \int_B G_B \leq C \inf_{z \in B} M(|f|^2)(z).$$

But,  $\int_B h(x - y, t) dx \leq 1$  since  $\lambda > 1$ , hence

$$\int_B G_B(x) dx \leq 2 \iint_{2B \times \mathbb{R}^+} |\mathcal{T}(e^{-tL}(I - A_r)f)(y)|^2 dy dt$$

and we are back to the calculations made in steps 1 and 2 of the previous section that give us

$$\iint_{2B \times \mathbb{R}^+} |\mathcal{T}(e^{-tL}(I - A_r)f)(y)|^2 dy dt \leq C \inf_{z \in B} M(|f|^2)(z).$$

Hence, it remains to establish (7.5). To do that, assume that  $B$  is the unit ball and  $r = 1$  (One can treat the general case by rescaling and translation, which changes  $L$  to another operator with the same properties and the same critical numbers). Assume also for simplicity that  $k = 1$ . We write for  $x \in B$ ,

$$g_\lambda^*(e^{-r^2L}f)^2(x) = I + II$$

where

$$I = \iint_{(y,t) \notin E} h(x-y, t) |\mathcal{T}(e^{-tL}e^{-r^2L}f)(y)|^2 dy dt$$

and

$$II = \iint_{(y,t) \in E} h(x-y, t) |\mathcal{T}(e^{-tL}e^{-r^2L}f)(y)|^2 dy dt$$

and  $(y, t) \notin E$  means  $y \in 2B$  and  $t \leq r(2B)^2$ . Let us treat the first term.

Using Minkowski integral inequality (with respect to  $t$ ) and  $\int_{2B} h(x-y, t) dx \leq 1$  whenever  $y \in 2B$ , we obtain

$$\left( \int_B I^{p_0/2} \right)^{2/p_0} \leq \int_0^4 \left( \int_{2B} |\mathcal{T}(e^{-r^2L}e^{-tL}f)(y)|^{p_0} dy \right)^{2/p_0} dt.$$

Following again the calculations in steps 1 and 2 of the previous section

$$\left( \int_{2B} |\mathcal{T}(e^{-tL}(I - A_r)f)(y)|^{p_0} dy \right)^{2/p_0} \leq \sum_{j \geq 2} \frac{c_j}{|2^{j+1}B|} \int_{2^{j+1}B} |\mathcal{T}(e^{-tL}f)(y)|^2 dy$$

with  $c_j = Ce^{-c_4^j}$  and with the limitation  $p_0 < q_+(L)$  from  $\nabla_y$  and  $p_0 < p_+(L)$  from  $L^{1/2}$  in the definition of  $\mathcal{T}$ . Now since  $t \leq 4$  and  $j \geq 2$ ,

$$\int_{2^{j+1}B} |\mathcal{T}(e^{-tL}f)(y)|^2 dy dt \leq \frac{A}{\sqrt{t}^n} \int_{2^{j+1}B} \int_{|x-y| \leq \sqrt{t}} |\mathcal{T}(e^{-tL}f)(y)|^2 dy dx$$

for some  $A > 0$ . Hence, we obtain readily

$$\left( \int_B I^{p_0/2} \right)^{2/p_0} \leq AC \inf_{x \in B} M((g_{NT}f)^2)^{1/2}(x).$$

For the second term  $II$ , we first observe that for  $x \in B$  and  $(y, t) \in E$ , then  $h(x - y, t) \sim h(y, t)$  (recall that  $B$  is the unit ball so that 0 is its center), thus  $(\int_B II^{p_0/2})^{2/p_0} \leq \sup_B II$ . Next, decompose  $E$  as the union of  $E_k$ ,  $k \geq -1$ , as follows:  $E_{-1} = E \cap \{|y| < \sqrt{t}\}$  and  $E_k = E \cap \{2^k \sqrt{t} < |y| \leq 2^{k+1} \sqrt{t}\}$ . Then for  $(y, t) \in E_k$ ,  $h(y, t) \leq 2^{-kn\lambda} \sqrt{t}^{-n}$ . A crucial geometrical observation is that if  $(y, t) \in E_k$  then  $2^k \sqrt{t} \geq 1$ . Using the method of the second argument in the proof of Theorem 5.6 (recall that  $r = 1$ ),

$$\begin{aligned} \int_{|y| \leq 2^k \sqrt{t}} |\mathcal{T} e^{-r^2 L} (e^{-tL} f)(y)|^2 dy &\leq C \int_{|z| \leq 2^{k+1} \sqrt{t}} |\mathcal{T} (e^{-tL} f)(z)|^2 dz \\ &+ \sum_{j \geq k+2} \int_{|z| \leq 2^j \sqrt{t}} C e^{-c4^j t} |\mathcal{T} (e^{-tL} f)(z)|^2 dz. \end{aligned}$$

Hence

$$II \leq \iint m(z, t) |\mathcal{T} (e^{-tL} f)(z)|^2 dz dt$$

where

$$m(z, t) = C \sum_{k \geq -1} 2^{-kn\lambda} \sqrt{t}^{-n} 1_{2^k \sqrt{t} \geq 1} (1_{|z| \leq 2^{k+1} \sqrt{t}} + \sum_{j \geq k+2} e^{-c4^j t} 1_{|z| \leq 2^j \sqrt{t}}).$$

Tedious but elementary verifications show that  $m(z, t) \leq Ch(x - z, t)$  for all  $x \in B$  and  $(z, t) \in \mathbb{R}_+^{n+1}$  using only  $\lambda > 0$ . Hence,  $II \leq C \inf_{x \in B} g_\lambda^*(f)^2(x)$ .  $\square$

## 8 Miscellani

### 8.1 Local theory

Let  $L$  be as in the Introduction. We have developed a global (or homogeneous)  $L^p$  theory by making global in time assumptions on the semigroup. Reasons for this theory not to apply to a particular  $L$  at  $p$  are that the semigroup is not  $L^p$  bounded for some (or all)  $t > 0$  (in which case this is the end of the story) or that the semigroup operators are bounded on  $L^p$  but not uniformly, usually with an exponential blow up. In the second case, adding a large  $s$  to  $L$  gives us back the uniformity. The local  $L^p$  theory consists in working with  $L + s$  instead of  $L$ . Hence, the above results may be adapted with minor modifications in the proofs by changing systematically  $L$  to  $L + s$

for  $s > 0$ . One may define the four critical exponents  $p_{\pm}(L+s)$  and  $q_{\pm}(L+s)$  which may depend on  $s$  or not. Indeed, the  $L^2$  theory developed in Section 3 works with  $s = 0$  and the numbers  $p_{-}(L+s)$ ,  $q_{-}(L+s)$  are non increasing, and  $p_{+}(L+s)$ ,  $q_{+}(L+s)$  non decreasing as  $s$  grows.

We have the following assertions for  $s > 0$ .

1.  $p_{-}(L+s) = q_{-}(L+s)$  and  $(q_{+}(L+s))^* \leq p_{+}(L+s)$ .
2.  $q_{+}(L+s)$  is the supremum of exponents  $p$  for which one has the invertibility of  $L+s$  from  $W^{1,p}$  onto  $W^{-1,p}$ .
3. For the Riesz transform  $\nabla(L+s)^{-1/2}$  the range for  $L^p$  boundedness is the open interval  $(p_{-}(L+s), q_{+}(L+s))$ . For  $p$  in this range  $L+s$  is an isomorphism from  $W^{1,p}$  onto  $L^p$ .
4. There is bounded holomorphic functional calculus for  $L+s$  on  $L^p$  essentially when  $p_{-}(L+s) < p < p_{+}(L+s)$ .
5. There is a Hardy-Littlewood-Sobolev-Kato diagram.
6. There is an equivalent  $L^p$  norm defined by  $g_{L+s}$  essentially for  $p_{-}(L+s) < p < p_{+}(L+s)$ .
7. There is an equivalent  $L^p$  norm defined by  $G_{L+s}$  essentially for  $q_{-}(L+s)' < p < q_{+}(L+s)$ .

This applies to operators whose coefficients have some smoothness. If the coefficients are, in addition, BUC (bounded uniformly continuous) or in the closure of  $BUC$  for the  $bmo$  norm, then it is known that  $p_{-}(L+s) = q_{-}(L+s) = 1$  and  $p_{+}(L+s) = q_{+}(L+s) = \infty$  for  $s$  large enough.<sup>71</sup> This gives  $L^p$  estimates for Riesz transforms, functional calculi, square functions in the range  $1 < p < \infty$ .

One can also add to  $L$  perturbation by lower order terms and develop the similar theory.

One can probably develop this theory for operators on domains with Lipschitz boundaries at least with Dirichlet or Neumann boundary conditions. This is left to the interested reader.

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<sup>71</sup> This is a consequence of [AMT] (see also [IS])

Another interesting direction is to test this theory for other classes of elliptic operators such as Schrödinger operators for which criteria for the determination of  $p_-(L)$  and  $p_+(L)$  have been given.<sup>72</sup> This theory already applies in the range  $p < 2$ .<sup>73</sup> It remains to study the range  $p > 2$ .

## 8.2 Higher order operators and systems

Consider an homogeneous elliptic operator  $L$  of order  $m$ ,  $m \in \mathbb{N}$ ,  $m \geq 2$ , defined by

$$(8.1) \quad Lf = (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha (a_{\alpha\beta} \partial^\beta f),$$

where the coefficients  $a_{\alpha\beta}$  are complex-valued  $L^\infty$  functions on  $\mathbb{R}^n$ , and we assume

$$(8.2) \quad \left| \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} a_{\alpha\beta}(x) \partial^\beta f(x) \partial^\alpha \bar{g}(x) dx \right| \leq \Lambda \|\nabla^m f\|_2 \|\nabla^m g\|_2$$

and the strong Gårding inequality

$$(8.3) \quad \operatorname{Re} \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} a_{\alpha\beta}(x) \partial^\beta f(x) \partial^\alpha \bar{f}(x) dx \geq \lambda \|\nabla^m f\|_2^2$$

for some  $\lambda > 0$  and  $\Lambda < +\infty$  independent of  $f, g \in W^{m,2}$ . Here,  $\nabla^k$  is the array of all  $k$ th order derivatives.

One can also generalize second order or higher order operators to elliptic systems of any even order verifying the strong Gårding inequality. For simplicity of exposition we stick to the scalar case but all works similarly for systems.

The  $L^2$  theory for the semigroup is analogous. There are

1. bounded holomorphic functional calculus on  $L^2$ ,

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<sup>72</sup> See [LSV]

<sup>73</sup> [BK1] and [BK2] for results on the functional calculus and Riesz transforms

2.  $L^2$  off-diagonal estimates for the families  $(t^{k/2m} \nabla^k e^{-tL})_{t>0}$  and their analytic extensions for  $0 \leq k \leq m$  where the gaussian decay  $e^{-cu^2}$  is changed to  $e^{-cu \frac{2m}{2m-1}}$  and the homogeneity changes from  $\sqrt{t}$  to  $t^{1/2m}$ .

These estimates yield the generalized conservation property

$$e^{-tL}P = P$$

for all  $t > 0$  in the  $L^2_{loc}$  sense for  $P$  polynomial of degree less than  $m$ .<sup>74</sup> Since  $L$  is constructed as before as a maximal-accretive operator, it has a square root and one has in all dimensions<sup>75</sup>

$$(8.4) \quad \|L^{1/2}f\|_2 \sim \|\nabla^m f\|_2.$$

Moreover, the square functions  $g_L$  and  $G_L$  define equivalent norms on  $L^2$  (in  $G_L$  replace  $\nabla$  by  $\nabla^m$ ).

Then one can develop the  $L^p$  theory of the semigroup, introducing the limits  $p_{\pm}(L)$  for the  $L^p$  boundedness of  $(e^{-tL})_{t>0}$  and the limits  $q_{\pm}(L)$  for the  $L^p$  boundedness of  $(\sqrt{t} \nabla^m e^{-tL})_{t>0}$ . The results are similar with more technical burden in the arguments as one often has to control intermediate families  $(t^{k/2m} \nabla^k e^{-tL})_{t>0}$  for  $1 \leq k \leq m-1$ .<sup>76</sup> One has that

$$\begin{aligned} p_-(L) &= q_-(L) \\ p_+(L) &\geq (q_+(L))^{*m} \end{aligned}$$

where  $p^{*m}$  means  $m$  times the operation  $p \mapsto p^*$ .

By Sobolev embeddings plus perturbation results (such as Lemma 4.4) we have

$$p_-(L) \begin{cases} = 1, & \text{if } n \leq 2m \\ < \frac{2n}{n+2m}, & \text{if } n > 2m. \end{cases}$$

Furthermore, this upper bound is sharp for the class of **all** higher order operators with  $n > 2m \geq 4$ : for any  $n$  and  $m \geq 2$  with  $2m < n$  and  $p < \frac{2n}{n+2m}$ , there exists an operator (scalar or system)  $L$  of order  $2m$  such

<sup>74</sup> See [AHMcT] for a proof under additional hypotheses. The argument has the same structure in the general case.

<sup>75</sup> This is [AHMcT], Theorem 1.1.

<sup>76</sup> see [AT], Chapter I, [AQ] and [Da]

that  $p_-(L) > p$ .<sup>77</sup> One has also

$$q_+(L) \begin{cases} = +\infty, & \text{if } n = 1 \\ > 2, & \text{if } n \geq 2. \end{cases}$$

If  $L^*$  is an example with  $p_-(L^*) \sim \frac{2n}{n+2m}$  ( $n > 2m$ ), then  $p_+(L) \sim \frac{2n}{n-2m}$  and one has  $p_+(L) \sim (q_+(L))^{*m}$  (Here,  $\sim$  means within some arbitrary small  $\varepsilon$ ). Thus the inequality  $p_+(L) \geq (q_+(L))^{*m}$  is best possible.

Next, the Riesz transform becomes  $\nabla^m L^{-1/2}$  and

$$\|\nabla^m L^{-1/2} f\|_p \lesssim \|f\|_p \quad \text{if and only if} \quad q_-(L) < p < q_+(L)$$

and one can show the reverse inequalities (with  $t \in \mathbb{R}$ )

$$\|L^{1/2+it} f\|_p \lesssim \|\nabla^m f\|_p \quad \text{whenever} \quad \sup(1, (p_-(L))_{*m}) < p < p_+(L).$$

The bounds for  $p > 2$  are merely obtained by duality from the Riesz transform bounds and  $p_+(L)$  is best possible. A tool to obtain the estimates for  $p < 2$  is the extension of the Calderón-Zygmund decomposition to Sobolev  $W^{m,p}$  functions.<sup>78</sup> The lower limit  $(p_-(L))_{*m}$  (if not  $\leq 1$ , which implies large dimensions), is best possible if  $L$  is an operator for which  $p_+(L^*) \geq q_+(L^*)^{*m}$  is best possible. We have seen there exist such operators. The connexion with Hodge theory is analogous to the second order case and the dichotomy  $p > 2$  vs  $p < 2$  appears again.

The critical numbers  $\frac{2n}{n\pm 2}$  and  $\frac{2n}{n+4}$  which appear in the  $L^p$  theory of square roots for second order operators (namely Propositions 5.18 and 5.19) become  $\frac{2n}{n\pm 2m}$  and  $\frac{2n}{n+4m}$ .<sup>79</sup> Hence, the range of exponents  $p$  for the Riesz transform  $L^p$  estimate is

$$1 < p < \infty, \text{ when } n = 1, m \geq 1,$$

$$1 < p < 2 + \varepsilon, \text{ if } 1 < n \leq 2m,$$

$$\frac{2n}{n+2m} - \varepsilon < p < 2 + \varepsilon', \text{ when } n > 2m.$$

<sup>77</sup> This is in Davies [Da1] based on examples of Maz'ya and de Giorgi.

<sup>78</sup> See [A] for the proof.

<sup>79</sup> See [A] where all this is explicated. The case  $\frac{2n}{n+2m} < p < 2$  when  $n > 2m$  is due to Blunck & Kunstmann [BK1]. The other cases follow from the methods in [AT] although this is not explicated.



The discussion above show that these open ranges are best possible. The range of exponents  $p$  for the reverse inequality <sup>80</sup> is

$$\begin{aligned} 1 < p < \infty & \text{ if } n \leq 2m, \\ 1 < p < \frac{2n}{n-2m} + \varepsilon & \text{ if } 2m < n \leq 4m, \\ \frac{2n}{n+4m} - \varepsilon_1 < p < \frac{2n}{n-2m} + \varepsilon & \text{ if } n > 4m. \end{aligned}$$

Again, these open ranges are best possible. <sup>81</sup>

The bounded holomorphic functional calculus extends on  $L^p$  for  $p_-(L) < p < p_+(L)$ . <sup>82</sup>

The theory of square functions also generalize similarly. <sup>83</sup> For  $g_L$  the range of  $p$ 's is  $p_-(L) < p < p_+(L)$  and for  $G_L$ ,  $q_-(L) < p < q_+(L)$ . One could treat also variants of the non-tangential square functions.

If the strong Gårding inequality is weakened by a term  $+\kappa\|f\|_2^2$  in the right hand side (in both the operator and system cases), then one has to replace  $L + \lambda$  for  $\lambda \geq \kappa$  and the local theory applies.

One can also add pertubation by lower order terms with bounded measurable coefficients without any harm to the theory.

## 9 Calderón-Zygmund decomposition for Sobolev functions

Here we prove the Calderón-Zygmund decomposition for Sobolev functions in Lemma 5.12. The notation are those of the statement.

*Proof.* If  $p = \infty$ , set  $g = f$ . Assume next that  $p < \infty$ . Let  $\Omega = \{x \in \mathbb{R}^n; M(|\nabla f|^p)(x) > \alpha^p\}$  where  $M$  is the uncentered maximal operator over

<sup>80</sup> This is proved in [A] for  $p < 2$  and  $n > 4m$ . The other cases were done earlier as a consequence of the methods in [AT] for  $n \leq 2m$  and in [AHLMcT] for  $2m \leq n < 4m$ .

<sup>81</sup> In particular, this means that the methods used here cannot improve the ranges of  $p$  for second order operators (as improved methods would apply for higher order as well) unless they use specific features of second order operators.

<sup>82</sup> This is due to Blunck & Kunstmann [BK2].

<sup>83</sup> For  $g_L$ , as for second order operators, one can also combine results of Blunck & Kunstmann and of Le Merdy; for  $G_L$  this is new.

cubes of  $\mathbb{R}^n$ . If  $\Omega$  is empty, then set  $g = f$ . Otherwise, the maximal theorem gives us

$$|\Omega| \leq C\alpha^{-p} \int_{\mathbb{R}^n} |\nabla f|^p.$$

Let  $F$  be the complement of  $\Omega$ . By the Lebesgue differentiation theorem,  $|\nabla f| \leq \alpha$  almost everywhere on  $F$ . We also have,

**Lemma 9.1.** *One can redefine  $f$  on a null set of  $F$  so that for all  $x \in F$ , for all cube  $Q$  centered at  $x$ ,*

$$(9.1) \quad |f(x) - m_Q f| \leq C\alpha\ell(Q)$$

where  $\ell(Q)$  is the sidelength of  $Q$  and for all  $x, y \in F$ ,

$$(9.2) \quad |f(x) - f(y)| \leq C\alpha|x - y|.$$

The constant  $C$  depends only on dimension and  $p$ .

Here  $m_E f$  denotes the mean of  $f$  over  $E$ . It is well-defined if  $E$  is a cube as  $f$  is locally integrable. Let us postpone the proof of this lemma and continue the argument.

Let  $(Q_i)$  be a Whitney decomposition of  $\Omega$  by dyadic cubes. Hence,  $\Omega$  is the disjoint union of the  $Q_i$ 's, the cubes  $2Q_i$  are contained in  $\Omega$  and have the bounded overlap property, but the cubes  $4Q_i$  intersect  $F$ . As usual,  $\lambda Q$  is the cube co-centered with  $Q$  with sidelength  $\lambda$  times that of  $Q$ . Hence (5.10) and (5.11) are satisfied by the cubes  $2Q_i$ . Let us now define the functions  $b_i$ . Let  $(\mathcal{X}_i)$  be a partition of unity on  $\Omega$  associated to the covering  $(Q_i)$  so that for each  $i$ ,  $\mathcal{X}_i$  is a  $C^1$  function supported in  $2Q_i$  with  $\|\mathcal{X}_i\|_\infty + \ell_i \|\nabla \mathcal{X}_i\|_\infty \leq c(n)$ ,  $\ell_i$  being the sidelength of  $Q_i$ . Pick a point  $x_i \in 4Q_i \cap F$ . Set

$$b_i = (f - f(x_i))\mathcal{X}_i.$$

It is clear that  $b_i$  is supported in  $2Q_i$ . Let us estimate  $\int_{2Q_i} |\nabla b_i|^p$ . Introduce  $\tilde{Q}_i$  the cube centered at  $x_i$  with sidelength  $8\ell_i$ . Then  $2Q_i \subset \tilde{Q}_i$ . Set  $c_i = m_{2Q_i} f$  and  $\tilde{c}_i = m_{\tilde{Q}_i} f$  and write

$$b_i = (f - c_i)\mathcal{X}_i + (c_i - \tilde{c}_i)\mathcal{X}_i + (\tilde{c}_i - f(x_i))\mathcal{X}_i.$$

By (9.1) and (5.11) for the cubes  $2Q_i$ ,  $|\tilde{c}_i - f(x_i)| \leq C\alpha\ell_i$ , hence  $\int_{2Q_i} |\tilde{c}_i - f(x_i)|^p |\nabla \mathcal{X}_i|^p \leq C\alpha^p |2Q_i|$ . Next, using the  $L^p$ -Poincaré inequality and the

fact that  $\tilde{Q}_i \cap F$  is not empty,

$$|c_i - \tilde{c}_i| \leq \frac{1}{|2Q_i|} \int_{\tilde{Q}_i} |f - \tilde{c}_i| \leq C\ell_i \left( \frac{1}{|\tilde{Q}_i|} \int_{\tilde{Q}_i} |\nabla f|^p \right)^{1/p} \leq C\alpha\ell_i.$$

Hence,  $\int_{2Q_i} |c_i - \tilde{c}_i|^p |\nabla \mathcal{X}_i|^p \leq C\alpha^p |2Q_i|$ . Lastly, since  $\nabla((f - c_i)\mathcal{X}_i) = \mathcal{X}_i \nabla f + (f - c_i) \nabla \mathcal{X}_i$ , we have again by the  $L^p$ -Poincaré inequality and the fact that the average of  $|\nabla f|^p$  on  $2Q_i$  is controlled by  $C\alpha^p$  that

$$\int_{2Q_i} |\nabla((f - c_i)\mathcal{X}_i)|^p \leq C\alpha^p |2Q_i|.$$

Thus (5.9) is proved.

Set  $h(x) = \sum_i f(x_i) \nabla \mathcal{X}_i(x)$ . Note that this sum is locally finite and  $h(x) = 0$  for  $x \in F$ . Note also that  $\sum_i \mathcal{X}_i(x)$  is 1 on  $\Omega$  and 0 on  $F$ . Since it is also locally finite we have  $\sum_i \nabla \mathcal{X}_i(x) = 0$  for  $x \in \Omega$ . We claim that  $h(x) \leq C\alpha$ . Indeed, fix  $x \in \Omega$ . Let  $Q_j$  be the Whitney cube containing  $x$  and let  $I_x$  be the set of indices  $i$  such that  $x \in 2Q_i$ . We know that  $\#I_x \leq N$ . Also for  $i \in I_x$  we have that  $C^{-1}\ell_i \leq \ell_j \leq C\ell_i$  and  $|x_i - x_j| \leq C\ell_j$  where the constant  $C$  depends only on dimension (see [St1]). We have

$$|h(x)| = \left| \sum_{i \in I_x} (f(x_i) - f(x_j)) \nabla \mathcal{X}_i(x) \right| \leq C \sum_{i \in I_x} |f(x_i) - f(x_j)| \ell_i^{-1} \leq CN\alpha,$$

by the previous observations.

It remains to obtain (5.7) and (5.8). We easily have using  $\sum_i \nabla \mathcal{X}_i(x) = 0$  for  $x \in \Omega$ , that

$$\nabla f = (\nabla f) \mathbf{1}_F + h + \sum_i \nabla b_i, \quad \text{a.e.}$$

Now  $\sum_i b_i$  is a well-defined distribution on  $\mathbb{R}^n$ . Indeed, for a test function  $u$ , using the properties of the Whitney cubes,

$$\sum_i \int |b_i u| \leq C \int \left( \sum_i |b_i(x)| \ell_i^{-1} \right) |u(x)| d(x, F) dx$$

and the last sum converges in  $L^p$  as a consequence of (5.10) and

**Lemma 9.2.** Set  $p^* = \frac{np}{n-p}$  if  $p < n$  and  $p^* = \infty$  otherwise, then for all real numbers  $r$  with  $p \leq r \leq p^*$ ,

$$(9.3) \quad \left\| \sum_i |b_i| \ell_i^{-1} \right\|_r^r \leq C \alpha^r \sum_i |Q_i|.$$

Admit this lemma and set  $g = f - \sum_i b_i$ . Then  $\nabla g = (\nabla f) \mathbf{1}_F + h$  in the sense of distributions and, hence,  $\nabla g$  is a bounded function with  $\|\nabla g\|_\infty \leq C\alpha$ .  $\square$

**Proof of Lemma 9.2:** By (5.11) and the Poincaré-Sobolev inequality:

$$\left\| \sum_i |b_i| \ell_i^{-1} \right\|_r^r \leq N \sum_i \left\| |b_i| \ell_i^{-1} \right\|_r^r \leq NC \sum_i \ell_i^{r\theta} \|\nabla b_i\|_p^r$$

where  $\theta = \frac{n}{r} - \frac{n}{p}$ . By (5.9),  $\ell_i^{r\theta} \|\nabla b_i\|_p^r \leq \alpha^r \ell_i^{nr/p}$ , hence

$$\left\| \sum_i |b_i| \ell_i^{-1} \right\|_r^r \leq CN \alpha^r \sum_i \ell_i^n.$$

**Proof of Lemma 9.1:** Let  $x$  be a point in  $F$ . Fix such cube  $Q$  with center  $x$  and let  $Q_k$  be co-centered cubes with  $\ell(Q_k) = 2^k \ell(Q)$  for  $k$  a negative integer. Then, by Poincaré's inequality

$$\begin{aligned} |m_{Q_{k+1}} f - m_{Q_k} f| &\leq 2^n |m_{Q_{k+1}}(f - m_{Q_{k+1}} f)| \\ &\leq C 2^n \ell(Q_k) (m_{Q_{k+1}} |\nabla f|^p)^{1/p} \\ &\leq C 2^k \ell(Q) \alpha \end{aligned}$$

since  $Q_{k+1}$  contains  $x \in F$ . It easily follows that  $m_Q f$  has a limit as  $|Q|$  tends to 0. If, moreover,  $x$  is in the Lebesgue set of  $f$ , then this limit is equal to  $f(x)$ . Redefine  $f$  on the complement of the Lebesgue set in  $F$  so that  $m_Q f$  tends to  $f(x)$  with  $Q$  centered at  $x$  with  $|Q| \rightarrow 0$ . Moreover, summing over  $k$  the previous inequality gives us (9.1). To see (9.2), let  $Q_x$  be the cube centered at  $x$  with sidelength  $2|x-y|$  and  $Q_y$  be the cube centered at  $y$  with sidelength  $8|x-y|$ . It is easy to see that  $Q_x \subset Q_y$ . As before, one can see that  $|m_{Q_x} f - m_{Q_y} f| \leq C\alpha|x-y|$ . Hence by the triangle inequality and (9.1), one obtains (9.2) readily.

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