A characterization of Markov processes enjoying the time-inversion property

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Abstract

We give a necessary and sufficient condition for a homogeneous Markov process taking values in $\mathbb{R}^n$ to enjoy the time-inversion property of degree α. The condition sets the shape for the semigroup densities of the process and allows to further extend the class of known processes satisfying the time-inversion property. As an application we recover the result of Watanabe in [24] for continuous and conservative Markov processes on $\mathbb{R}_+$. As new examples we generalize Dunkl processes and construct a matrix-valued process with jumps related to the Wishart process by a skew-product representation.

Keywords: homogeneous Markov processes; time-inversion property; Bessel processes; Dunkl processes; Wishart processes; semi-stable processes.

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1 Introduction

Let $\{(X_t, t \geq 0);\ (\mathrm{P}_x)_{x\in\mathbb{R}^n}\}$ be a homogeneous Markov process with semigroup densities (assumed to exist):

$$P_t(x, dy) = p_t(x, y)dy. \tag{1.1}$$

For all $x \in \mathbb{R}^n$ and some $\alpha > 0$, the process $\{(t^\alpha X_{\frac{1}{t}}, t > 0);\ \mathrm{P}_x\}$ is Markov and in general inhomogeneous.

Definition 1.1. The process $\{(X_t, t \geq 0);\ \mathrm{P}_x\}$ is said to enjoy the *time-inversion property of degree* α if the Markov process $\{(t^\alpha X_{\frac{1}{t}}, t > 0);\ \mathrm{P}_x\}$ is homogeneous.

Celebrated examples of Markov processes, known to enjoy this property for $\alpha = 1$, are Brownian motions with drift in $\mathbb{R}^n$ and Bessel processes with drift (see [21, 24]). Gallardo and Yor [12] recently worked out a sufficient condition on the semigroup densities for a Markov process to enjoy the time-inversion property. Their argument extended the class of processes to processes with jumps such as the Dunkl process [23] and matrix-valued processes such as the Wishart process [3]. The aim of the paper is to show that the former condition is actually necessary and sufficient and to provide some new examples.

Section 2 contains the main theorem of the paper, which is proved in section 3 using straightforward analytical arguments. Section 4 considers an application of the theorem to Markov processes on $\mathbb{R}_+$. The result is shown to be strong enough to entirely characterize the class of diffusion processes on $\mathbb{R}_+$ that enjoy the time-inversion property and thus provide a different proof than that

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of Watanabe in [24]. Section 5 gives new examples of processes that enjoy the time-inversion property. We review how the generalized Dunkl process fits the requirements of the theorem and then introduce a matrix-valued process with jumps. The relation of the latter process to the Wishart process mimics the relation between the one-dimensional Dunkl and Bessel processes.

2 Markov processes which enjoy time-inversion

Fix $x \in \mathbb{R}^n$. Recall that under P_x, the process $(t^\alpha X_{\frac{1}{t}}, t > 0)$ is Markov, inhomogeneous, with transitional probability densities $q^{(x)}_{s,t}(z,y)$, $(s < t; z, y \in \mathbb{R}^n)$, which satisfy the following relation:

$$E_x\left[f(t^\alpha X_{\frac{1}{t}}) \mid s^\alpha X_{\frac{1}{s}} = z\right] = \int dy\ f(y)\ q^{(x)}_{s,t}(z,y) \tag{2.1}$$

where

$$q^{(x)}_{s,t}(a,b) = t^{-n\alpha}\ \frac{p_{\frac{1}{t}}\left(x, \frac{b}{t^\alpha}\right)\ p_{\frac{1}{s}-\frac{1}{t}}\left(\frac{b}{t^\alpha}, \frac{a}{s^\alpha}\right)}{p_{\frac{1}{s}}\left(x, \frac{a}{s^\alpha}\right)}. \tag{2.2}$$

Throughout the paper, we assume the semigroup densities $p_t(x,y)$ to be positive over the domain and regular enough to be at least twice differentiable in the space and time variables.

The process $(t^\alpha X_{\frac{1}{t}}, t > 0)$ is not uniquely defined from the knowledge of the semigroup densities $p_t(x,y)$. Actually, there exists at least one transformation that leaves the semigroup densities $q^{(x)}_{s,t}(a,b)$ unchanged: Doob's h-transform.

Definition 2.1. Doob's h-transform is the transformation

$$T_h : \mathrm{P}_x|_{\mathcal{F}_t} \mapsto \frac{h(X_t)}{h(x)} e^{-\nu t}\ \mathrm{P}_x|_{\mathcal{F}_t}. \tag{2.3}$$

for some function h and some constant $\nu > 0$.

This remark leads to our first two assertions.

Proposition 2.2. *Two processes related by h-transforms yield the same process by time-inversion.*

Proposition 2.3. *T_h is an equivalence relation over the class of homogeneous Markov processes.*

It is hence legitimate to research for a criterium to classify all processes that enjoy the time-inversion property up to h-transforms. The following theorem gives a concise statement of our main result:

Theorem 2.4. *The Markov process $(t^\alpha X_{\frac{1}{t}}, t > 0)$ is homogeneous if and only if the semigroup densities of $(X_t, t \geq 0)$ are of the form:*

$$p_t(x,y) = t^{-\frac{n\alpha}{2}}\ \Phi\left(\frac{x}{t^{\frac{\alpha}{2}}}, \frac{y}{t^{\frac{\alpha}{2}}}\right)\ \theta\left(\frac{y}{t^{\frac{\alpha}{2}}}\right)\ \exp\left\{\rho\left(\frac{x}{t^{\frac{\alpha}{2}}}\right) + \rho\left(\frac{y}{t^{\frac{\alpha}{2}}}\right)\right\} \tag{2.4}$$

or if they are in h-transform relationship with it. The functions Φ, θ, ρ have the following properties for $\lambda > 0$:

1. $\Phi(\lambda x, y) = \Phi(x, \lambda y)$;
2. $\theta(\lambda y) = \lambda^\beta\ \theta(y)$ *for some* $\beta \in \mathbb{R}$;
3. $\rho(\lambda x) = \lambda^{\frac{2}{\alpha}}\ \rho(x)$;

Moreover, if the symmetry condition $\Phi(x,y) = \Phi(y,x)$ is satisfied, then the semigroup densities are related as follows:

$$q^{(x)}_t(a,b) = \frac{\Phi(x,b)}{\Phi(x,a)}\ \exp\left\{t\rho(x)\right\}\ p_t(a,b). \tag{2.5}$$

Definition 2.5. A Markov process $(X_t, t \geq 0)$ is called semi-stable with index γ in the sense of Lamperti [18] if:

$$\Big\{(X_{ct}, t \geq 0); \mathrm{P}_x\Big\} \overset{(d)}{=} \Big\{(c^\gamma X_t, t \geq 0); \mathrm{P}_{x/c^\gamma}\Big\}. \tag{2.6}$$

As a consequence, the semigroup densities of a semi-stable process with index γ have the following property:

$$p_t(x,y) = t^{-n\gamma}\ p_1\left(\frac{x}{t^\gamma}, \frac{y}{t^\gamma}\right). \tag{2.7}$$

Remarking that the expression for the semigroup densities in Theorem 2.4 satisfies this property for $\gamma = \alpha/2$ yields the following corollary:

Corollary 2.6. *A Markov process that enjoys the time-inversion property of degree α is a semi-stable process with index $\alpha/2$, or is in h-transform relationship with it. The converse is not true.*

3 Proof of the Theorem

3.1 Sufficiency

If $p_t(x,y)$ satisfies the condition (2.4), then from formula (2.2) the semigroup densities $q^{(x)}_{s,t}(a,b)$ can be written as

$$q^{(x)}_{s,t}(a,b) = (t-s)^{-\frac{n\alpha}{2}} \prod_{i=1}^{3} R^{(i)}_{s,t}$$

where

$$\begin{aligned}
R^{(1)}_{s,t} &= \Phi\left(x\ t^{\frac{\alpha}{2}}, \frac{b}{t^\alpha}\ t^{\frac{\alpha}{2}}\right)\ \Phi\left(\frac{b}{t^\alpha}\left(\frac{st}{t-s}\right)^{\frac{\alpha}{2}}, \frac{a}{s^\alpha}\left(\frac{st}{t-s}\right)^{\frac{\alpha}{2}}\right)\ \Phi\left(x\ s^{\frac{\alpha}{2}}, \frac{a}{s^\alpha}\ s^{\frac{\alpha}{2}}\right)^{-1} \\
R^{(2)}_{s,t} &= \theta\left(\frac{b}{t^\alpha}\ t^{\frac{\alpha}{2}}\right)\ \theta\left(\frac{a}{s^\alpha}\left(\frac{st}{t-s}\right)^{\frac{\alpha}{2}}\right)\ \theta\left(\frac{a}{s^\alpha}\ s^{\frac{\alpha}{2}}\right)^{-1} \\
R^{(3)}_{s,t} &= \exp\left\{\rho\left(x\ t^{\frac{\alpha}{2}}\right) + \rho\left(\frac{b}{t^\alpha}\ t^{\frac{\alpha}{2}}\right) + \rho\left(\frac{b}{t^\alpha}\left(\frac{st}{t-s}\right)^{\frac{\alpha}{2}}\right) + \rho\left(\frac{a}{s^\alpha}\left(\frac{st}{t-s}\right)^{\frac{\alpha}{2}}\right)\right. \\
&\qquad \left. -\rho\left(x\ s^{\frac{\alpha}{2}}\right) - \rho\left(\frac{a}{s^\alpha}\ s^{\frac{\alpha}{2}}\right)\right\}
\end{aligned}$$

Using the properties of Φ, θ, ρ described in Theorem 2.4, we obtain:

$$\begin{aligned}
R^{(1)}_{s,t} &= \frac{\Phi(x,b)}{\Phi(x,a)}\ \Phi\left(\frac{b}{(t-s)^{\frac{\alpha}{2}}}, \frac{a}{(t-s)^{\frac{\alpha}{2}}}\right) \\
R^{(2)}_{s,t} &= \theta\left(\frac{b}{(t-s)^{\frac{\alpha}{2}}}\right) \\
R^{(3)}_{s,t} &= \exp\left\{(t-s)\rho(x) + \rho\left(\frac{a}{(t-s)^{\frac{\alpha}{2}}}\right) + \rho\left(\frac{b}{(t-s)^{\frac{\alpha}{2}}}\right)\right\}.
\end{aligned}$$

Hence there is no separate dependence on s and t, but only on the difference $t-s$, which allows to conclude that

$$q^{(x)}_{s,t}(a,b) = q^{(x)}_{t-s}(a,b)$$

and proves homogeneity for the process $(t^\alpha X_{\frac{1}{t}}, t > 0)$. If in addition $\Phi(x,y) = \Phi(y,x)$, we obtain (2.5).

3.2 Necessity

For simplicity of notation, we prove the necessity of condition (2.4) in the case $\alpha = 1$. The extension to $\alpha > 0$ is immediate by change of variables $X_t \mapsto X_t^\alpha$. Recall first the following definition of homogeneous functions:

Definition 3.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies for $x = (x_1, \ldots, x_n)$, $\forall \lambda \in \mathbb{R}_+$,

$$f(\lambda x) = \lambda^\beta f(x)$$

is called *homogeneous of degree* $\beta \in \mathbb{R}$. If $f \in C^1(\mathbb{R}^n)$, Euler's Homogeneous Function Theorem gives a necessary and sufficient condition for the function $f(x)$ to be homogeneous:

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} f(x) = \beta f(x).$$

Consider the function of $2n+1$ variables: $l(x, y, t) = \ln(p_t(x, y)),\ x, y \in \mathbb{R}^n$. From (2.2), l must satisfy for $s = t - h$:

$$\frac{\partial}{\partial t}\left[n \ln \frac{1}{t} + l\left(x, \frac{b}{t}, \frac{1}{t}\right) + l\left(\frac{b}{t}, \frac{a}{t-h}, \frac{1}{t-h} - \frac{1}{t}\right) - l\left(x, \frac{a}{t-h}, \frac{1}{t-h}\right)\right] = 0. \tag{3.1}$$

3.2.1 The kernel $\Phi(x, y)$

Taking derivatives with respect to b_i and a_j for some $0 \le i, j \le n$ yields:

$$\frac{\partial}{\partial b_i}\frac{\partial}{\partial a_j}\frac{\partial}{\partial t}\, l\left(\frac{b}{t}, \frac{a}{t-h}, \frac{1}{t-h} - \frac{1}{t}\right) = 0.$$

If we set $\phi\,(x, y, t) = \frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j} l\,(x, y, t)$, the latter becomes

$$-\frac{1}{t^2(t-h)}\left(\phi + \frac{b}{t} \cdot \nabla_1 \phi\right) - \frac{1}{t(t-h)^2}\left(\phi + \frac{a}{t-h} \cdot \nabla_2 \phi\right) + \frac{1}{t(t-h)}\left(\frac{1}{t^2} - \frac{1}{(t-h)^2}\right)\partial\phi = 0,$$

with the notation $\nabla_1 = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})^T,\ \nabla_2 = (\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n})^T$ and $\phi = \phi\left(\frac{b}{t}, \frac{a}{t-h}, \frac{1}{t-h} - \frac{1}{t}\right)$. For clarity, we change variables to

$$z_1 = \frac{b}{t} \quad , \quad z_2 = \frac{a}{t-h} \quad , \quad t_1 = -\frac{1}{t} \quad , \quad t_2 = \frac{1}{t-h}.$$

Then $\phi = \phi(z_1, z_2, t_1 + t_2)$ and

$$t_1\Big(\phi + z_1 \cdot \nabla_1 \phi \ + t_1\ \partial\phi\Big) = t_2\Big(\phi + z_2 \cdot \nabla_2 \phi \ + t_2\ \partial\phi\Big),$$

or equivalently

$$t_1\Big(\phi + z_1 \cdot \nabla_1 \phi \ + (t_1 + t_2)\ \partial\phi\Big) = t_2\Big(\phi + z_2 \cdot \nabla_2 \phi \ + (t_1 + t_2)\ \partial\phi\Big). \tag{3.2}$$

We change variables once more: $u = \frac{t_1}{t_2}, v = t_1 + t_2$ to get:

$$\phi + z_1 \cdot \nabla_1 \phi \ + v\ \partial\phi = u\big(\phi + z_2 \cdot \nabla_2 \phi \ + v\ \partial\phi\big). \tag{3.3}$$

This gives the following proposition.

Proposition 3.2. *Equation (3.3) is satisfied if and only if*

$$\phi(\lambda x, \mu y, \lambda\mu t) = \frac{1}{\lambda\mu}\ \phi(x, y, t). \tag{3.4}$$

Proof. As the LHS of equation (3.3) is independent of u, one can readily take the equivalent condition:

$$\begin{aligned} \phi + z_1 \cdot \nabla_1\phi\ + v\ \partial\phi &= 0, & (3.5)\\ \phi + z_2 \cdot \nabla_2\phi\ + v\ \partial\phi &= 0. & (3.6) \end{aligned}$$

Let $g(\lambda) = \phi(\lambda z_1, z_2, \lambda v)$ and $h(\mu) = \phi(z_1, \mu z_2, \mu v)$. Equation (3.5) implies $g(\lambda)+\lambda g'(\lambda) = 0$, which solves to $g(\lambda) = \lambda^{-1}g(1)$ and hence

$$\phi(\lambda z_1, z_2, \lambda v) = \frac{1}{\lambda}\phi(z_1, z_2, v).$$

Equation (3.6) implies $h(\mu) + \mu h'(\mu) = 0$, which solves to $h(\mu) = \mu^{-1}h(1)$ and hence

$$\phi(z_1, \mu z_2, \mu v) = \frac{1}{\mu}\phi(z_1, z_2, v).$$

Combining the latter two equations yields (3.4).

Conversely, if g is homogeneous of degree -1 and h is homogeneous of degree -1, we get:

$$g(\lambda) + \lambda g'(\lambda) = h(\mu) + \mu h'(\mu) = 0,$$

which is equivalent to (3.5) and (3.6) and concludes the proof. □

Note that the previous result remains valid for all second derivatives of $l(x, y, t)$ with respect to x_i and y_j, $\forall i, j \in \{1, \ldots, n\}$. By integration over x_i and y_j, one can thus already make an assumption on the general shape of the function l:

$$l(x, y, t) = k(x, y, t) + \varphi_1(x, t) + \varphi_2(y, t) + h_1(x) + h_2(y) + \tau(t), \tag{3.7}$$

where $k : \mathbb{R}^{2n+1} \to \mathbb{R}$, $\varphi_1, \varphi_2 : \mathbb{R}^{n+1} \to \mathbb{R}$, $\tau : \mathbb{R} \to \mathbb{R}$ and $h_1, h_2 : \mathbb{R}^n \to \mathbb{R}$. Moreover, the kernel $k(x, y, t)$ must satisfy the following property:

$$k(\lambda x, \mu y, \lambda\mu t) = k(x, y, t).$$

This is immediate from (3.4) and

$$\begin{aligned} \phi(\lambda x, \mu y, \lambda\mu t) &= \frac{\partial}{\partial\lambda x_i}\frac{\partial}{\partial\mu y_j} l(\lambda x, \mu y, \lambda\mu t) = \frac{\partial}{\partial\lambda x_i}\frac{\partial}{\partial\mu y_j} k(\lambda x, \mu y, \lambda\mu t) = \frac{1}{\lambda\mu}\frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j} k(\lambda x, \mu y, \lambda\mu t),\\ \frac{1}{\lambda\mu}\phi(x, y, t) &= \frac{1}{\lambda\mu}\frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j} l(x, y, t) = \frac{1}{\lambda\mu}\frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j} k(x, y, t). \end{aligned}$$

This property is actually equivalent to

$$k\left(\lambda\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}, 1\right) = k\left(\frac{x}{\sqrt{t}}, \lambda\frac{y}{\sqrt{t}}, 1\right)$$

Setting $\Phi\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) = \exp k(x, y, t)$ completes the proof of the first condition of the theorem, i.e.

$$\Phi(\lambda x, y) = \Phi(x, \lambda y).$$

3.2.2 The function $\rho(x)$

We now focus on the functions $\varphi_1(x,t), \varphi_2(y,t)$. From (3.1), they must satisfy

$$\frac{\partial}{\partial t} \quad \left[\varphi_1\left(x, \frac{1}{t}\right) + \varphi_2\left(\frac{b}{t}, \frac{1}{t}\right) + \varphi_1\left(\frac{b}{t}, \frac{1}{t-h} - \frac{1}{t}\right) + \varphi_2\left(\frac{a}{t-h}, \frac{1}{t-h} - \frac{1}{t}\right) \right.$$
$$\left. - \varphi_1\left(x, \frac{1}{t-h}\right) - \varphi_2\left(\frac{a}{t-h}, \frac{1}{t-h}\right)\right] = 0. \tag{3.8}$$

Recall that the variables a, b, x are independent from each other and that φ_1, φ_2 are defined to be explicitly dependent on their variables. So taking the derivative of (3.8) with respect to x_i gives

$$\frac{1}{t^2}\ \partial_t \psi_1\left(x, \frac{1}{t}\right) = \frac{1}{(t-h)^2}\ \partial_t \psi_1\left(x, \frac{1}{t-h}\right)$$

where $\psi_1(x,t) = \frac{\partial}{\partial x_i}\varphi_1(x,t)$. Hence $\psi_1(x,t)$ must be of the form $\psi_1(x,t) = \frac{1}{t}\psi_{11}(x) + \psi_{12}(x)$, which implies

$$\varphi_1(x,t) = \frac{1}{t}\varphi_{11}(x) + \varphi_{12}(x).$$

We omit the second term as it can always be recast as part of $h(x)$ in (3.7). There are so far no further conditions to add on the function $\varphi_{11}(x)$.

Taking the derivative of (3.8) with respect to b_i leads to

$$\frac{\partial}{\partial b_i}\frac{\partial}{\partial t}\left[\varphi_2\left(\frac{b}{t}, \frac{1}{t}\right) + \frac{t(t-h)}{h}\ \varphi_{11}\left(\frac{b}{t}\right)\right] = 0,$$

which develops to

$$-\frac{1}{t^2}\partial_i\varphi_2 - \frac{b}{t^3}\cdot\nabla\partial_i\varphi_2 - \frac{1}{t^3}\partial_t\partial_i\varphi_2 + \frac{2t-h}{ht}\partial_i\varphi_{11} - \frac{t(t-h)}{h}\frac{1}{t^2}\partial_i\varphi_{11} - \frac{t(t-h)}{h}\frac{b}{t^3}\cdot\nabla\partial_i\varphi_{11} = 0.$$

Recall that $\psi_{11}(x) = \frac{\partial}{\partial x_i}\varphi_{11}(x)$ and set $\psi_2(x,t) = \frac{\partial}{\partial x_i}\varphi_2(x,t)$. In this notation, we get

$$-\frac{1}{t}\psi_2 - \frac{b}{t^2}\cdot\nabla\psi_2 - \frac{1}{t^2}\partial_t\psi_2 + \frac{2t-h}{h}\psi_{11} - \frac{t(t-h)}{h}\frac{1}{t}\psi_{11} - \frac{t(t-h)}{h}\frac{b}{t^2}\cdot\nabla\psi_{11} = 0.$$

Let $z = \frac{b}{t},\ t_1 = \frac{1}{t},\ t_2 = \frac{1}{t-h}$, then we have

$$\psi_2(z,t_1) + z\cdot\nabla\psi_2(z,t_1) + t_1\partial_t\psi_2(z,t_1) = \frac{t_2}{t_1(t_2-t_1)}\psi_{11}(z) - \frac{1}{t_2-t_1}z\cdot\nabla\psi_{11}(z).$$

As the LHS is independent of t_2, so must be the RHS, which implies

$$z\cdot\nabla\psi_{11}(z) = \psi_{11}(z).$$

Hence ψ_{11} is homogeneous of degree 1. In that case,

$$\psi_2(z,t_1) + z\cdot\nabla\psi_2(z,t_1) + t_1\partial_t\psi_2(z,t_1) = \frac{1}{t_1}\psi_{11}(z).$$

We solve the equation for $g(\lambda) = \psi_2(\lambda z, \lambda t_1)$ to get $g(\lambda) = \frac{1}{t_1}\psi_{11}(z) + \frac{1}{\lambda}c(z,t_1)$, where $c(z,t_1)$ turns out to be a homogeneous function of degree -1. Integration over z_i yields some conditions for $\varphi_2(z,t)$. Once again, since the index $i \in \{1,\dots,n\}$ was chosen arbitrarily, we obtain the following characteristics:

$$\varphi_1(\lambda z, \lambda t) = \frac{\varphi_{11}(\lambda z)}{\lambda t} \quad \text{with} \quad \varphi_{11}(\lambda z) = \lambda^2\varphi_{11}(z)$$

and

$$\varphi_2(\lambda z, \lambda t) = \frac{\varphi_{21}(\lambda z)}{\lambda t} + \hat{c}(\lambda z, \lambda t),$$

where $\varphi_{21}(\lambda z) = \lambda^2\varphi_{21}(z)$ and even $\varphi_{21}(z) = \varphi_{11}(z)$ without loss of generality. Let $\rho(z) = \varphi_{11}(z) = \varphi_{12}(z)$, then we recover the third condition of the theorem, that is

$$\rho(\lambda z) = \lambda^2\rho(z).$$

3.2.3 The function $\theta(y)$

In order to further investigate the properties of $\hat{c}(z,t)$, we derive (3.8) by a_i:

$$\frac{\partial}{\partial a_i}\frac{\partial}{\partial t}\left[\frac{1}{h}\varphi(a)+\hat{c}\left(\frac{a}{t-h},\frac{1}{t-h}-\frac{1}{t}\right)-\hat{c}\left(\frac{a}{t-h},\frac{1}{t-h}\right)\right]=0.$$

This leads to

$$c(z,t_2-t_1)+z\cdot\nabla c(z,t_2-t_1)+\frac{t_1^2-t_2^2}{t_2}\,\partial_t c(z,t_2-t_1)=c(z,t_2)+z\cdot\nabla c(z,t_2)+t_2\partial_t c(z,t_2).$$

Since $c(z,t)$ is homogeneous of degree -1, the RHS is zero. This only proves that $\frac{\partial}{\partial t}c(z,t)=\frac{\partial}{\partial t}\frac{\partial}{\partial z_i}\hat{c}(z,t)=0,\forall i$. Hence, $\hat{c}(z,t)=\hat{c}_1(z)+\hat{c}_2(t)$. Going back to equation (3.8) derived by b_i gives this time for $\hat{c}(z,t)$,

$$\frac{\partial}{\partial b_i}\frac{\partial}{\partial t}\left[\hat{c}\left(\frac{b}{t},\frac{1}{t}\right)\right]=\frac{\partial}{\partial b_i}\frac{\partial}{\partial t}\left[\hat{c}_1\left(\frac{b}{t}\right)\right]=0,$$

which clearly implies $\hat{c}_1(\frac{b}{t})=\hat{c}_{11}(b)+\hat{c}_{12}(\frac{1}{t})$. Using equation (3.8) once more gives

$$\frac{\partial}{\partial t}\left[\hat{c}_{12}\left(\frac{1}{t}\right)+\hat{c}_2\left(\frac{1}{t}\right)+\hat{c}_2\left(\frac{1}{t-h}-\frac{1}{t}\right)-\hat{c}_2\left(\frac{1}{t-h}\right)\right]=0, \tag{3.9}$$

which is equivalent to

$$t_1^2\;\hat{c}'_{12}(t_1)+t_1^2\;\hat{c}'_2(t_1)+(t_1+t_2)(t_2-t_1)\;\hat{c}'_2(t_2-t_1)-t_2^2\;\hat{c}'_2(t_2)=0.$$

We set $g(t)=t\hat{c}'_2(t)$ so that

$$t_1^2\;\hat{c}'_{12}(t_1)+t_1\;g(t_1)=t_2\big(g(t_2)-g(t_2-t_1)\big)-t_1 g(t_2-t_1).$$

Assuming $\lim\limits_{t_1\to 0}\dfrac{LHS}{t_1}=\dfrac{\beta}{2}\in\mathbb{R}$ with $|\beta|<\infty$, we divide the equation by t_1 and take the limit as $t_1\to 0$. This yields $\frac{1}{2}\beta=t_2 g'(t_2)-g(t_2)$, which solves for $g(t)=-\frac{1}{2}\beta+\gamma t,\ \gamma\in\mathbb{R}$. Hence,

$$\hat{c}_2(t)=-\frac{1}{2}\beta\ln t+\gamma t\quad\text{and}\quad\hat{c}_{12}(t)=\beta\ln t,$$

where any additional constant term has been set to 0. It will be convenient to set $\gamma=0$. A term $\gamma\neq 0$ corresponds to the independent case of an affine function of t in the sum (3.7) and is included in the discussion for $\tau(t)$. To summarize, the function $\hat{c}(z,t)$ must be expressed as

$$\hat{c}(z,t)=\hat{c}_1(z)-\frac{\beta}{2}\ln t,$$

where $c_1(z)$ has the following property: $\hat{c}_1(\lambda z)=\hat{c}_1(z)+\beta\ln\lambda$. Let $\theta(z)=\exp\hat{c}_1(z)$. We then recover condition 2 of the theorem, i.e.

$$\theta(\lambda z)=\lambda^\beta\;\theta(z).$$

3.2.4 The functions $h_1(x), h_2(y)$

The obvious solution to (3.1) in this case is $h(z)=h_1(z)=-h_2(z)$. $\exp h$ hence defines an h-transform, which is neglected since we classify all processes up to h-transforms.

3.2.5 The function $\tau(t)$

It remains to explicitly formulate the form of the function $\tau : \mathbb{R}_+ \to \mathbb{R}$ that is defined to satisfy

$$\frac{\partial}{\partial t}\left[-n\ln t + \tau\left(\frac{1}{t}\right) + \tau\left(\frac{1}{t-h} - \frac{1}{t}\right) - \tau\left(\frac{1}{t-h}\right)\right] = 0.$$

But this equation is similar to (3.9), so we immediately have the solution ($\gamma \in \mathbb{R}$):

$$\tau(t) = -\frac{n}{2}\ln t + \gamma t.$$

Once again $e^{\gamma t}$ can be included in an h-transform, so γ is set to 0 and $e^{\tau(t)} = t^{-\frac{n}{2}}$ gives the first factor in the semigroup densities (2.4).

3.2.6 The general case $\alpha > 0$

For $\alpha > 0$, we use the change of variables $X_t \mapsto X_t^\alpha$. The semigroup densities for X_t^α become

$$p_t^\alpha(x,y) = p_t\left(x^{\frac{1}{\alpha}}, y^{\frac{1}{\alpha}}\right) J(y)$$

where $J(y)$ is the Jacobian of the inverse transformation, that is $J(y) = \alpha^{-n}\ y_1^{\frac{1}{\alpha}-1}\cdots y_n^{\frac{1}{\alpha}-1}$, and we use the slight abuse of notation $x^{\frac{1}{\alpha}} = \left(x_1^{\frac{1}{\alpha}}, \ldots, x_n^{\frac{1}{\alpha}}\right)$. The semigroup densities can be recast into

$$p_t^\alpha(x,y) = t^{-\frac{n\alpha}{2}}\ \Phi^\alpha\left(\frac{x}{t^{\frac{\alpha}{2}}}, \frac{y}{t^{\frac{\alpha}{2}}}\right)\ \theta^\alpha\left(\frac{y}{t^{\frac{\alpha}{2}}}\right)\ \exp\left\{\rho^\alpha\left(\frac{x}{t^{\frac{\alpha}{2}}}\right) + \rho^\alpha\left(\frac{y}{t^{\frac{\alpha}{2}}}\right)\right\}$$

where $\Phi^\alpha(x,y) = \Phi\left(x^{\frac{1}{\alpha}}, y^{\frac{1}{\alpha}}\right)$ satisfies condition 1, $\theta^\alpha(y) = \theta\left(y^{\frac{1}{\alpha}}\right) J(y)$ satisfies condition 2 for $\bar{\beta} = \frac{\beta+n}{\alpha} - n$ and $\rho^\alpha(x) = \rho\left(y^{\frac{1}{\alpha}}\right)$ satisfies condition 3 of equation (2.4).

4 Application to diffusions on $\mathbb{R}_+$

The case of the diffusions on $\mathbb{R}_+$ was entirely characterized by Watanabe in [24]. It was shown that only Bessel processes in the wide sense (which we recall the definition below) enjoy the time-inversion property of degree 1.

Definition 4.1. For some $\nu > -1$ and $c \geq 0$, the diffusion process generated by

$$\mathcal{L} = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{2\nu+1}{2x} + \frac{h_c'(x)}{h_c(x)}\right)\frac{\partial}{\partial x} \tag{4.1}$$

is called *Bessel process in the wide sense*. The function $h_c(x)$ is given by

$$h_c(x) = 2^\nu\Gamma(\nu+1)\ (\sqrt{2c}\ x)^{-\nu}\ I_\nu(\sqrt{2c}\ x), \tag{4.2}$$

where I_ν is the modified Bessel function.

Remark 4.2. The Bessel process in the wide sense is in h-transform relationship, for $h \equiv h_c$, with the Bessel process.

We show that the result in [24] is a consequence of Theorem 2.4, which has the following one-dimensional formulation:

Theorem 4.3. *The Markov process $(t^\alpha X_{\frac{1}{t}}, t>0)$ on $\mathbb{R}_+$ is homogeneous if and only if the semigroup densities of $(X_t, t\geq 0)$ are of the form:*

$$p_t(x,y) = t^{-\frac{\alpha}{2}(1+\beta)}\ \phi\left(\frac{xy}{t^\alpha}\right)\ y^\beta\ \exp\left\{-\frac{k^2}{2}\left(\frac{x^{\frac{2}{\alpha}}}{t}+\frac{y^{\frac{2}{\alpha}}}{t}\right)\right\} \tag{4.3}$$

for $k>0$, or if it is in h-transform relationship with it. Moreover, the semigroup densities are related as follows:

$$q_t^{(x)}(a,b) = \frac{\phi(xb)}{\phi(xa)}\ \exp\left(-t\ \frac{k^2}{2}\ x^{\frac{2}{\alpha}}\right)\ p_t(a,b). \tag{4.4}$$

Proof. Theorem 2.4 formulated on $\mathbb{R}_+$ implies $\Phi(x,y)=\phi(xy)$, $\theta(y)=y^\beta$ and $\rho(x)=-\frac{k^2}{2}\ x^{\frac{2}{\alpha}},\ k>0$, for the condition 1-3 to be satisfied in one dimension. □

We identify further the class of diffusion processes and provide a different proof for the result in [24].

Proposition 4.4. *If $(X_t, t\geq 0)$ is a diffusion process and $(tX_{\frac{1}{t}}, t>0)$ is homogeneous and conservative, then both are necessarily (possibly time-scaled) Bessel processes in the wide sense.*

Proof. If $(X_t, t\geq 0)$ is a diffusion process, then its infinitesimal generator has the following general structure:

$$\mathcal{L} = s(y)\frac{\partial^2}{\partial y^2} + \mu(y)\frac{\partial}{\partial y}$$

where it remains to identify the functions $s(y)$ and $\mu(y)$. For a fixed $x>0$, let $\mathcal{L}^{(x)}$ be the infinitesimal generator of $(tX_{\frac{1}{t}}, t>0)$. From equation (2.5), $\mathcal{L}^{(x)}$ has the following relationship with $\mathcal{L}$:

$$\mathcal{L}^{(x)} : f(b) \mapsto \frac{1}{\phi(xb)}\ \mathcal{L}\big(\phi(xb)f(b)\big) - \frac{k^2}{2}\ x^2 f(b),$$

which develops to

$$\mathcal{L}^{(x)}f(b) = s(b)f''(b) + \left\{\mu(b) + 2x\ s(b)\ \frac{\phi'(xb)}{\phi(xb)}\right\} f'(b) + U(x,b)f(b).$$

For the process to be conservative, we require $U(x,b)=0$, which implies no killing in the interior of the domain, that is

$$s(b)\ x^2\ \frac{\phi''(xb)}{\phi(xb)} + \mu(b)\ x\ \frac{\phi'(xb)}{\phi(xb)} - \frac{k^2}{2}\ x^2 = 0.$$

We change variables to $z=xb$ to obtain

$$s(z/x)\ \phi''(z) + \frac{\mu(z/x)}{x}\ \phi'(z) - \frac{k^2}{2}\ \phi(z) = 0.$$

Since the latter must be valid for all $x>0$, we are led to set: $s(b)=\frac{\sigma^2}{2}$ for $\sigma>0$ and $\mu(b)=\frac{\sigma^2}{2}\ \frac{2\nu+1}{b}$ for $\nu>-1$. This yields the following equation

$$\frac{1}{2}\phi''(z) + \frac{2\nu+1}{2z}\ \phi'(z) - \frac{k^2}{2\sigma^2}\ \phi(z) = 0.$$

The general solution (non-singular at 0 and up to a constant factor) is expressed through the modified Bessel function of the first kind as follows:

$$\phi(z) = \left(\frac{kz}{\sigma}\right)^{-\nu}\ I_\nu\left(\frac{kz}{\sigma}\right).$$

Gathering the different factors in (4.3) leads to

$$p_t(x,y) = N\ t^{-\frac{1+\beta}{2}}\ \left(\frac{xy}{t}\right)^{-\nu}\ I_\nu\left(\frac{k}{\sigma}\frac{xy}{t}\right)\ y^\beta\ \exp\left\{-\frac{k^2}{2}\left(\frac{x^2}{t}+\frac{y^2}{t}\right)\right\},$$

where N is a normalization factor. The additional condition that $\lim_{t\to 0} p_t(x,y) = \delta(x-y)$ implies $\beta = 2\nu+1$ and $k = \frac{1}{\sigma}$, which leads to the semigroup densities of a time-scaled Bessel process of dimension ν:

$$p_t(x,y) = \frac{y}{\sigma^2 t}\ \left(\frac{y}{x}\right)^\nu\ I_\nu\left(\frac{xy}{\sigma^2 t}\right)\ \exp\left\{-\frac{x^2+y^2}{2\sigma^2 t}\right\}.$$

The infinitesimal generator for $(tX_{\frac{1}{t}}, t>0)$ is given by

$$\mathcal{L}^{(x)} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial b^2} + \sigma^2\left\{\frac{2\nu+1}{2b} + x\frac{\phi'(xb)}{\phi(xb)}\right\}\frac{\partial}{\partial b},$$

where one recognizes expression (4.1) for $h_c(b) = \phi(xb)$ and a time-scale $t \mapsto \sigma^2 t$. □

Proposition 4.4 smoothly extends to any power $\alpha > 0$ of the Bessel process through the mapping $X_t \mapsto X_t^\alpha$. In particular, it is worth remarking that the case $\alpha = 2$ gives rise to squares of Bessel processes, which leads to the following result:

Proposition 4.5. *If $(X_t, t \geq 0)$ is a diffusion process and $(t^2 X_{\frac{1}{t}}, t>0)$ is homogeneous and conservative, then both are necessarily (possibly time-scaled) squares of Bessel processes in the wide sense.*

5 Examples

5.1 Generalized Dunkl processes and Jacobi-Dunkl processes

5.1.1 Multidimensional Dunkl processes

We briefly review the construction of the Dunkl process in $\mathbb{R}^n$ (see [22, 23]).

Definition 5.1. The Dunkl process in $\mathbb{R}^n$ is the Markov càdlàg process with infinitesimal generator

$$\frac{1}{2}\mathcal{L}^{(k)} = \frac{1}{2}\sum_{i=1}^n T_i^2 \tag{5.1}$$

where $T_i,\ 1 \leq i \leq n$, is a one-dimensional differential-difference operator defined for $u \in C^1(\mathbb{R}^n)$ by

$$T_i u(x) = \frac{\partial u(x)}{\partial x_i} + \sum_{\alpha \in R_+} k(\alpha)\alpha_i \frac{u(x) - u(\sigma_\alpha x)}{\langle \alpha, x\rangle}. \tag{5.2}$$

$\langle\cdot,\cdot\rangle$ is the usual scalar product. R is a root system in $\mathbb{R}^n$ and R_+ a positive subsystem. $k(\alpha)$ is a non-negative multiplicity function defined on R and invariant by the finite reflection group W associated with R. σ_α is the reflection operator with respect to the hyperplane H_α orthogonal to α such that $\sigma_\alpha x = x - \langle\alpha, x\rangle\alpha$ and for convenience $\langle\alpha,\alpha\rangle = 2$ (see [8, 9]).

A result obtained by M. Rösler [22] yields the semigroup densities as follows:

$$p_t^{(k)}(x,y) = \frac{1}{c_k t^{\gamma+n/2}}\ \exp\left(-\frac{|x|^2+|y|^2}{2t}\right)\ D_k\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right)\ \omega_k(y) \tag{5.3}$$

where $D_k(x,y) > 0$ is the Dunkl kernel, $\omega_k(y) = \prod_{\alpha \in R_+} |\langle \alpha, y \rangle|^{2k(\alpha)}$ the weight function which is homogeneous of degree $2\gamma = 2\sum_{\alpha \in R_+} k(\alpha)$ and $c_k = \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} \omega_k(x) dx$.

Following a thorough study of the properties of the one-dimensional Dunkl process in [13], it was remarked in [12] that the Dunkl process in $\mathbb{R}^n$ enjoys the time-inversion property of degree 1. Considering that the Dunkl kernel satisfies

$$D_k(x,y) = D_k(y,x) \quad \text{and} \quad D_k(\mu x, y) = D_k(x, \mu y), \tag{5.4}$$

the proof is straightforward with

$$\Phi(x,y) \equiv D_k(x,y), \quad \theta(y) \equiv \omega_k(y), \quad \rho(x) \equiv -\frac{|x|^2}{2}. \tag{5.5}$$

By equation (2.5), the semigroup densities of the time-inverted process is even in h-transform relationship with the semigroup densities of the original Dunkl process:

$$q_t^{(x)}(a,b) = \frac{D_k(x,b)}{D_k(x,a)} \exp\left(-\frac{|x|^2}{2}t\right) p_t^{(k)}(a,b). \tag{5.6}$$

5.1.2 Generalized Dunkl processes

In an attempt to generalize the Dunkl process, we extend the definition of the infinitesimal generator to

$$\mathcal{L}^{(k,\lambda)} f(x) = \frac{1}{2}\Delta f(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{\langle \nabla f(x), \alpha \rangle}{\langle x, \alpha \rangle} + \sum_{\alpha \in R_+} \lambda(\alpha) \frac{f(\sigma_\alpha x) - f(x)}{\langle x, \alpha \rangle^2} \tag{5.7}$$

where Δ is the usual Laplacian, $f \in C^2(\mathbb{R}^n)$ and $\lambda(\alpha)$ is a non-negative multiplicity function defined on R and invariant by the finite reflection group W, similarly as $k(\alpha)$. We retrieve the Dunkl process for $\lambda(\alpha) = k(\alpha)$. Note that these processes are no longer martingales for $\lambda(\alpha) \neq k(\alpha)$.

The one-dimensional case was introduced in [13], where the semigroup densities were explicitly derived,

$$p_t^{(k,\lambda)}(x,y) = \frac{1}{t^{k-\frac{1}{2}}} y^{k-\frac{1}{2}} \exp\left(-\frac{x^2+y^2}{2t}\right) D_{k,\lambda}\left(\frac{xy}{t}\right) \tag{5.8}$$

with the generalized Dunkl kernel, ($\nu = k - \frac{1}{2}$, $\mu = \sqrt{\nu^2 + 4\lambda}$),

$$D_{k,\lambda}(z) = 1_{\{y \in \mathbb{R}_-\}} \frac{1}{2z^\nu}(I_\nu - I_\mu)(-z) + 1_{\{y \in \mathbb{R}_+\}} \frac{1}{2z^\nu}(I_\nu + I_\mu)(z), \tag{5.9}$$

for $z = \frac{xy}{t}$. From (5.4), the generalized Dunkl kernel satisfies

$$D_{k,\lambda}(x,y) = D_{k,\lambda}(y,x) \quad \text{and} \quad D_{k,\lambda}(\mu x, y) = D_{k,\lambda}(x, \mu y), \tag{5.10}$$

which readily implies that the generalized Dunkl process also enjoys the time-inversion property of degree 1. The semigroup densities were derived as an application of the skew-product representation of the generalized Dunkl process $(X_t, t \geq 0)$ in terms of its absolute value (a Bessel process) and an independent Poisson process $N_t^{(\lambda)}$:

$$X_t \overset{(d)}{=} |X_t| (-1)^{N_{A_t}^{(\lambda)}} \tag{5.11}$$

where $A_t = \int_0^t \frac{ds}{X_s^2}$.

In the n-dimensional case, the application of the skew-product representation derived by Chybiryakov [6] shows that the generalized Dunkl process enjoys time-inversion for some specific root systems. We first recall one of the main results of [6].

Proposition 5.2. *Let $(X_t, t \geq 0)$ be the generalized Dunkl process generated by (5.7), with $(X_t^W, t \geq 0)$ its radial part, i.e. the process confined to a Weyl chamber. Let $R_+ \equiv \{\alpha_1, \ldots, \alpha_l\}$ for some $l \in \mathbb{N}$ be the corresponding positive root system and let $(N_t^i, t \geq 0)$, $i = 1, \ldots, l$ be independent Poisson processes of respective intensities $\lambda(\alpha_i)$. Then X_t may be represented as Y_t^l, which is defined by induction as follows:*

$$Y_t^0 = X_t^W \quad \text{and} \quad Y_t^i = \sigma_{\alpha_i}^{N^i_{A_t^i}} \, Y_t^{i-1}, \quad i = 1, \ldots, l,$$

where $A_t^i = \displaystyle\int_0^t \frac{ds}{\langle Y_s^{i-1}, \alpha_i \rangle^2}$.

The proof follows the argument in [6], while replacing $k(\alpha_i)$ by $\lambda(\alpha_i)$ appropriately.

From now on, let $R_+ \equiv \{\alpha_1, \ldots, \alpha_l\}$ for some $l \leq n$ be an orthogonal positive root system, that is $\langle \alpha_i, \alpha_j \rangle = 2\delta_{ij}$. For this particular root system, one can show that the generalized Dunkl process enjoys time-inversion of degree 1. We first prove the following absolute continuity relation:

Lemma 5.3. *Let $(X_t^W, t \geq 0)$ be the radial Dunkl process with infinitesimal generator*

$$\mathcal{L}_k^W f(x) = \frac{1}{2}\Delta f(x) + \sum_{i=1}^{l} k(\alpha_i) \frac{\langle \alpha_i, \nabla f(x) \rangle}{\langle \alpha_i, x \rangle}. \tag{5.12}$$

Fixed $\nu \in \{1, \ldots, l\}$. Let $k'(\alpha)$ be another coefficient function on the root system R_+ such that $k'(\alpha_\nu) > k(\alpha_\nu)$ and $k'(\alpha_i) = k(\alpha_i)$ for $i \neq \nu$. Then, denoting $\mathrm{P}_x^{(k)}$ the law of the radial Dunkl process X_t^W starting from x, we have

$$\mathrm{P}_x^{(k')}\big|_{\mathcal{F}_t} = \left(\frac{\langle \alpha_\nu, X_t^W \rangle}{\langle \alpha_\nu, x \rangle} \right)^{k'(\alpha_\nu) - k(\alpha_\nu)} \exp\left[- \frac{\left(k'(\alpha_\nu) - \frac{1}{2}\right)^2 - \left(k(\alpha_\nu) - \frac{1}{2}\right)^2}{2} \int_0^t \frac{ds}{\langle \alpha_\nu, X_s^W \rangle^2} \right] \cdot \mathrm{P}_x^{(k)}\big|_{\mathcal{F}_t}. \tag{5.13}$$

Proof. Let $k_0(\alpha)$ be a coefficient such that $k_0(\alpha_\nu) = \frac{1}{2}$ for some fixed $i \in \{1, \ldots, l\}$. X_t^W has the following martingale decomposition (see [11]):

$$X_t^W = x + B_t^{(k_0)} + \sum_{i=1}^{l} k_0(\alpha_i) \int_0^t \frac{ds}{\langle \alpha_i, X_s^W \rangle} \, \alpha_i$$

where $B_t^{(k_0)}$ is a $(\mathrm{P}_x^{(k_0)}, \mathcal{F}_t)$-Brownian motion. Consider the local martingale

$$L_t^{(k')} = \exp\left(\left(k'(\alpha_\nu) - \frac{1}{2} \right) \int_0^t \frac{\langle \alpha_\nu, dB_s^{(k_0)} \rangle}{\langle \alpha_\nu, X_s^W \rangle} - \frac{\left(k'(\alpha_\nu) - \frac{1}{2}\right)^2}{2} \int_0^t \frac{ds}{\langle \alpha_\nu, X_s^W \rangle^2} \right),$$

for some coefficient function $k'(\alpha)$ such that $k'(\alpha_\nu) > \frac{1}{2}$ and $k'(\alpha_i) = k_0(\alpha_i)$ for $i \neq \nu$. The Itô formula for $\ln\left(\langle \alpha_\nu, X_t^W \rangle\right)$ combined with the orthogonality of the roots yields

$$L_t^{(k')} = \left(\frac{\langle \alpha_\nu, X_t^W \rangle}{\langle \alpha_\nu, x \rangle} \right)^{k'(\alpha_\nu) - \frac{1}{2}} \exp\left(- \frac{\left(k'(\alpha_\nu) - \frac{1}{2}\right)^2}{2} \int_0^t \frac{ds}{\langle \alpha_\nu, X_s^W \rangle^2} \right).$$

Define the new law $\mathrm{P}_x^{(k')}|_{\mathcal{F}_t} = L_t^{(k')} \cdot \mathrm{P}_x^{(k_0)}|_{\mathcal{F}_t}$. By the Girsanov theorem,

$$B_t^{(k')} = B_t^{(k_0)} - \left(k'(\alpha_\nu) - \frac{1}{2} \right) \int_0^t \frac{ds}{\langle \alpha_\nu, X_s^W \rangle}$$

is a $(\mathrm{P}_x^{(k')}, \mathcal{F}_t)$-Brownian motion and hence,

$$X_t^W = x + B_t^{(k')} + \sum_{i=1}^{l} k'(\alpha_i) \int_0^t \frac{ds}{\langle \alpha_i, X_s^W \rangle} \, \alpha_i$$

is a radial Dunkl process under $(\mathrm{P}_x^{(k')}, \mathcal{F}_t)$.

Define k as another coefficient on the root system that satisfies the conditions enunciated in the lemma. The absolute continuity relation is then a consequence of the successive application of the latter result to the indices k' and k. □

Now as an application of Proposition 5.2, we prove the following:

Proposition 5.4. *Let $R_+ \equiv \{\alpha_1, \ldots, \alpha_l\}$ be an orthogonal positive root system for $l \leq n$. Then the generalized Dunkl process $(X_t, t \geq 0)$ generated by (5.7) enjoys the time-inversion property of degree 1.*

Proof. Using orthogonality of the roots, remark that

$$\langle \alpha_i, \sigma_j x \rangle^2 = \langle \alpha_i, x - \langle \alpha_j, x \rangle \alpha_j \rangle^2 = \langle \alpha_i, x \rangle^2,$$

which implies in particular

$$\langle \alpha_i, Y_t^i \rangle^2 = \langle \alpha_i, X_t^W \rangle^2,$$

so that the inductive representation of X_t in Proposition 5.2 becomes

$$X_t = \prod_{i=1}^{l} \sigma_{\alpha_i}^{N^i_{A_t^i}} \; X_t^W \quad \text{for} \quad A_t^i = \int_0^t \frac{ds}{\langle X_s^W, \alpha_i \rangle^2}.$$

The radial part of a Dunkl process enjoys the time-inversion property of degree 1. We need to show that the semigroup densities of the generalized Dunkl process are related to the semigroup densities of its radial parts. For $f \in C^2(\mathbb{R}^n)$,

$$\mathrm{E}_x\big[f(Y_t^i)\big] = \mathrm{E}_x\big[f(Y_t^{i-1}) 1_{\{N^i_{A_t^i} \text{ is even}\}}\big] + \mathrm{E}_x\big[f(\sigma_{\alpha_i} Y_t^{i-1}) 1_{\{N^i_{A_t^i} \text{ is odd}\}}\big].$$

Since $\mathrm{P}(N_u^i \text{ is even}) = \frac{1}{2}\left(1 + \exp(-2\lambda(\alpha_i)u)\right)$, we obtain

$$\mathrm{E}_x\big[f(Y_t^i)\big] = \mathrm{E}_x\left[f(Y_t^{i-1}) \; \frac{1}{2}\left(1 + \exp(-2\lambda(\alpha_i) A_t^i)\right)\right] + \mathrm{E}_x\left[f(\sigma_{\alpha_i} Y_t^{i-1}) \; \frac{1}{2}\left(1 - \exp(-2\lambda(\alpha_i) A_t^i)\right)\right].$$

The expectation $\mathrm{E}_x\big[f(X_t)\big]$ can thus be evaluated by induction on $i \in \{1, \ldots, l\}$. It follows that the semigroup densities of X_t can be expressed as the product of the semigroup densities of its radial parts times a function involving expectations of the form

$$\mathrm{E}_x^{(k)}\left[\exp\left(-2\lambda(\alpha_\nu) A_t^\nu\right) \middle| X_t^W = y\right],$$

for $\nu \in \{1, \ldots, l\}$. From Lemma 5.3,

$$\mathrm{E}_x^{(k)}\left[\exp\left(-2\lambda(\alpha_\nu) A_t^\nu\right) \middle| X_t^W = y\right] = \frac{p_t^{(k')}(x,y)}{p_t^{(k)}(x,y)} \; \left(\frac{\langle \alpha_\nu, y \rangle}{\langle \alpha_\nu, x \rangle}\right)^{k(\alpha_\nu) - k'(\alpha_\nu)},$$

where $k'(\alpha_i) = k(\alpha_i)$ for $i \neq \nu$ and $k'(\alpha_\nu) = \frac{1}{2} + \sqrt{\left(k(\alpha_\nu) - \frac{1}{2}\right)^2 + 4\lambda(\alpha_\nu)}$. The form of the semigroup densities in (5.3) implies that the expectation is a ratio of Dunkl kernels,

$$\mathrm{E}_x^{(k)}\left[\exp\left(-2\lambda(\alpha_\nu) A_t^\nu\right) \middle| X_t^W = y\right] = \frac{c_k}{c_{k'}} \; \frac{D_{k'}\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right)}{D_k\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right)} \; \frac{w_{k'}\left(\frac{y}{\sqrt{t}}\right)}{w_k\left(\frac{y}{\sqrt{t}}\right)} \left(\frac{\langle \alpha_\nu, \frac{y}{\sqrt{t}} \rangle}{\langle \alpha_\nu, \frac{x}{\sqrt{t}} \rangle}\right)^{k(\alpha_\nu) - k'(\alpha_\nu)},$$

which reduces to

$$\mathrm{E}_x^{(k)}\left[\exp\left(-2\lambda(\alpha_\nu) A_t^\nu\right) \middle| X_t^W = y\right] = \frac{c_k}{c_{k'}} \; \frac{D_{k'}\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right)}{D_k\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right)} \; \left(\langle \alpha_\nu, \frac{y}{\sqrt{t}} \rangle \langle \alpha_\nu, \frac{x}{\sqrt{t}} \rangle\right)^{k'(\alpha_\nu) - k(\alpha_\nu)},$$

by definition of $k'(\alpha)$. The conditional expectation thus satisfies condition 1 of Theorem 2.4. As a consequence, the conditions of Theorem 2.4 are satisfied for

$$\Phi(x,y) \equiv D_{k,\lambda}(x,y), \quad \theta(y) \equiv \omega_k(y), \quad \rho(x) \equiv -\frac{|x|^2}{2}$$

where $D_{k,\lambda}(x,y)$ is a generalized Dunkl kernel given explicitly in terms of radial Dunkl kernels and satisfying equivalent conditions (see (5.4)). □

5.1.3 Jacobi-Dunkl processes

Gallardo et al. [5] derived the Jacobi-Dunkl process as the hyperbolic analog of the one-dimensional Dunkl process. It is defined as the process generated by

$$\mathcal{L}^{(\alpha,\beta)} f(x) = \frac{\partial^2 f(x)}{\partial x^2} + \frac{A'(x)}{A(x)} \frac{\partial f(x)}{\partial x} + \frac{\partial}{\partial x} \left(\frac{A'(x)}{A(x)} \right) \left(\frac{f(x) - f(-x)}{2} \right), \tag{5.14}$$

where $A(x) = \left(\sinh^2(x)\right)^{\alpha+\frac{1}{2}} \left(\cosh^2(x)\right)^{\beta+\frac{1}{2}}$. From the expression of the semigroup densities developed in [5], this process does not enjoy the time-inversion property. Its radial part however corresponds to the Jacobi process of index (α,β) on $\mathbb{R}_+$ (see [15, 16]). The infinitesimal generator of the Jacobi process, expressed by

$$\mathcal{L}^{(\alpha,\beta)} f(x) = \frac{1}{2} \frac{\partial^2 f(x)}{\partial x^2} + \frac{A'(x)}{A(x)} \frac{\partial f(x)}{\partial x}, \tag{5.15}$$

is in h-transform relationship with the Laplacian operator for $h(x) = \sqrt{A(x)}$. Since the Brownian motion enjoys the time-inversion property of degree 1, so does the Jacobi process by Theorem 2.4.

5.2 Matrix-valued processes

5.2.1 Eigenvalue processes

Dyson in [10] described the eigenvalues of a Hermitian Brownian motion as the joint evolution of independent Brownian motions conditioned never to collide (see also [14] and [4]). It was further remarked that the process version of the Gaussian orthogonal ensemble does not admit such a representation for its eigenvalues. This work was extended by König and O'Connell in [17] to the process version of the Laguerre ensemble, denominated the Laguerre process and defined as follows:

Definition 5.5. Let B_t be an $n \times m$ matrix with independent standard complex Brownian entries. The Laguerre process is the matrix-valued process defined by $\{X_t = B_t' B_t, t \geq 0\}$, where B_t' is the transpose of B_t.

From [17], the eigenvalues of the Laguerre process evolve like m independent squared Bessel processes conditioned never to collide. No such representation however exists for the case where the entries of B_t are real Brownian motions, i.e. the Wishart process considered by Bru in [3].

The main result of [17] is that both of the above mentioned eigenvalues processes (in the complex Brownian case) can be obtained as the h-transform of processes with m independent components. The joint eigenvalues process is thus in h-transform relationship with a process that enjoys the time inversion property of degree 1 in the case of the m-dimensional Brownian motion and degree 2 in the case of the m-dimensional squared Bessel process, as made explicit in the following proposition:

Proposition 5.6. *Let $p_t(x_i, y_i)$ $(i = 1, \ldots, m)$ be the semigroup densities of squared Bessel processes (respectively Brownian motions), and let*

$$h(x) = \prod_{i<j}^{m} (x_j - x_i) \tag{5.16}$$

for $x = (x_1, \ldots, x_m)$. Then, the semigroup densities of the joint eigenvalues process of the Laguerre process (respectively the Hermitian Brownian motion) are given by

$$\tilde{p}_t(x,y) = \frac{h(y)}{h(x)} \prod_{i=1}^{m} p_t(x_i, y_i) \tag{5.17}$$

with respect to the Lebesgue measure $dy = \prod_{i=1}^{m} dy_i$.

It follows immediately by Theorem 2.4 that the eigenvalues processes enjoy the time-inversion property. Moreover by Proposition 2.2, they yield the same process under time-inversion as the m-dimensional Brownian motion or the m-dimensional squared Bessel process respectively.

5.2.2 Wishart processes

The Wishart process $\mathrm{WIS}(\delta, tI_m, \frac{1}{t}x)$, introduced by Bru in [2], is a continuous Markov process taking values in the space of real symmetric positive definite $m \times m$ matrices S_m^+. It is solution to the following stochastic differential equation:

$$dX_t = \sqrt{X_t} dB_t + dB_t' \sqrt{X_t} + \delta I_m dt, \quad X_0 = x, \tag{5.18}$$

where B_t is an $m \times m$ matrix with Brownian entries and I_m the identity matrix. Further results have been obtained in [1] and [3]. In [7], among other major findings about the Wishart process, the transition probability densities expressed with respect to the Lebesgue measure $dy = \prod_{i \leq j} (dy_{ij})$ were derived in terms of generalized Bessel functions (we refer to the appendix for the definition):

$$p_t(x,y) = \frac{1}{(2t)^{\frac{m(m+1)}{2}}} \exp\left(-\frac{1}{2t} Tr(x+y)\right) \left(\frac{\det(y)}{\det(x)}\right)^{\frac{\delta-m-1}{4}} \tilde{\mathbf{I}}_{\frac{\delta-m-1}{2}}\left(\frac{xy}{4t^2}\right), \tag{5.19}$$

for $x, y \in S_m^+$ and $\delta > m-1$. From the shape of its densities, the Wishart process was stated in [12] as an example of Markov processes enjoying the time-inversion property of degree 2. The hypothesis of Theorem 2.4 is indeed satisfied for $n = \frac{1}{2}m(m+1)$ and

$$\Phi(x,y) \equiv (\det(x)\det(y))^{-\frac{\delta-m-1}{4}} \tilde{\mathbf{I}}_{\frac{\delta-m-1}{2}}\left(\frac{xy}{4}\right) \quad , \quad \theta(y) \equiv \frac{1}{2^n}(\det(y))^{\frac{\delta-m-1}{2}} \quad , \quad \rho(x) \equiv -\frac{1}{2}Tr(x). \tag{5.20}$$

Next we use a skew-product representation, as for the Dunkl process, to elaborate on the Wishart process and derive a matrix-valued process with jumps. The skew-product representation allows the expression of the semigroup densities in terms of the Wishart transition probability densities.

Definition 5.7. Let $(N_t^{(\lambda)}, t \geq 0)$ be a Poisson process with intensity λ. Let $(X_t, t \geq 0)$ be a Wishart process $\mathrm{WIS}(\delta, tI_m, \frac{1}{t}x)$ independent of the Poisson process. The skew-Wishart process $(X_t^{(\lambda)}, t \geq 0)$ is defined through the skew-product

$$X_t^{(\lambda)} = X_t \, (-1)^{N_{A_t}^{(\lambda)}} \tag{5.21}$$

where $A_t = \int_0^t Tr(X_s^{-1}) ds$.

Proposition 5.8. *The transition probability densities of the skew-Wishart process are related to the semigroup densities $p_t(x,y)$ of the Wishart process X_t as follows*

$$p_t^{(\lambda)}(x,y) = p_t(x,|y|) \left\{ 1_{\{y \in S_m^+\}} \frac{1}{2}\left(1 + \left(\frac{\tilde{\mathbf{I}}_{\nu'}}{\tilde{\mathbf{I}}_{\nu}}\right)\left(\frac{xy}{4t^2}\right)\right) + 1_{\{y \in S_m^-\}} \frac{1}{2}\left(1 - \left(\frac{\tilde{\mathbf{I}}_{\nu'}}{\tilde{\mathbf{I}}_{\nu}}\right)\left(\frac{-xy}{4t^2}\right)\right)\right\}. \tag{5.22}$$

for $\nu = \frac{\delta-m-1}{2}$, $\nu' = \sqrt{\nu^2 + 4\lambda}$ and $|y| = y(1_{\{y \in S_m^+\}} - 1_{\{y \in S_m^-\}})$.

Proof. Let $(\mathbf{P}_t)_{t>0}$ be the semigroup of the Dunkl-Wishart process. For $x>0$ and $f\in C_c(M_m(\mathbb{R}))$,

$$\mathbf{P}_t f(x)=\mathbf{E}_x\left[f(X_t^{(\lambda)})\right]=\mathbf{E}_x\left[f(X_t)\ 1_{\{N^{(\lambda)}_{A_t}\text{ is even}\}}\right]+\mathbf{E}_x\left[f(-X_t)\ 1_{\{N^{(\lambda)}_{A_t}\text{ is odd}\}}\right].$$

With $\mathrm{P}(N_u^{(\lambda)}\text{ is even})=\frac{1}{2}(1+\exp(-2\lambda u))$, we have

$$\mathbf{P}_t f(x)=\mathbf{E}_x\left[f(X_t)\ \frac{1}{2}(1+\exp(-2\lambda A_t))\right]+\mathbf{E}_x\left[f(-X_t)\ \frac{1}{2}(1-\exp(-2\lambda A_t))\right]. \tag{5.23}$$

Let $\mathbf{Q}_x^{(\nu')}$ with $\nu'=\frac{\delta'-m-1}{2}$ denote the probability law of a Wishart process $\mathrm{WIS}(\delta',tI,\frac{1}{t}x)$, and $\mathbf{Q}_x^{(\nu)}$ with $\nu=\frac{\delta-m-1}{2}$ the probability law of X_t. According to Theorem 1.2 (Remark 2.3) in [7], the probability laws are related as follows:

$$\mathbf{Q}_x^{(\nu')}\big|_{\mathcal{F}_t}=\left(\frac{\det X_t}{\det x}\right)^{\frac{\nu'-\nu}{2}}\exp\left(-\frac{\nu'^2-\nu^2}{2}\int_0^t Tr(X_s^{-1})ds\right)\cdot\mathbf{Q}_x^{(\nu)}\big|_{\mathcal{F}_t},$$

from which we deduce

$$\frac{\mathbf{p}_t^{(\nu')}(x,y)}{\mathbf{p}_t^{(\nu)}(x,y)}=\left(\frac{\det y}{\det x}\right)^{\frac{\nu'-\nu}{2}}\mathbf{Q}_x^{(\nu)}\left[\exp\left(-\frac{\nu'^2-\nu^2}{2}\int_0^t Tr(X_s^{-1})ds\right)\Big|X_t=y\right].$$

Thus, from the expression of the semigroup densities in (5.19), we have

$$\mathbf{E}_x^{(\nu)}\left[\exp\left(-2\lambda\int_0^t Tr(X_s^{-1})ds\right)\Big|X_t=y\right]=\left(\frac{\tilde{\mathbf{I}}_{\sqrt{\nu^2+4\lambda}}}{\tilde{\mathbf{I}}_\nu}\right)\left(\frac{xy}{4t^2}\right).$$

Combining the latter with (5.23) yields the semigroup densities for the skew-Wishart process. □

The skew-Wishart is an example of matrix-valued process with jumps that enjoys the time-inversion property of degree 2. Indeed, by setting

$$\Phi(x,y)\equiv(\det(x)\det(|y|))^{-\frac{\nu}{2}}\ \left\{1_{\{y\in S_m^+\}}\frac{1}{2}\left(\tilde{\mathbf{I}}_\nu+\tilde{\mathbf{I}}_{\nu'}\right)\left(\frac{xy}{4}\right)+1_{\{y\in S_m^-\}}\frac{1}{2}\left(\tilde{\mathbf{I}}_\nu-\tilde{\mathbf{I}}_{\nu'}\right)\left(\frac{-xy}{4}\right)\right\},$$

$$\theta(y)\equiv\frac{1}{2^n}(\det(|y|))^\nu\quad,\quad\rho(x)\equiv-\frac{1}{2}Tr(|x|), \tag{5.24}$$

the conditions of Theorem 2.4 are satisfied for $\alpha=2$.

A Generalized hypergeometric functions

Using the notation in Muirhead [20], hypergeometric functions of matrix arguments are defined for a real symmetric $m \times m$ matrix X, $a_i \in \mathbb{C}$ and $b_j \in \mathbb{C}\backslash\{0, \frac{1}{2}, 1, \ldots, \frac{m-1}{2}\}$ by

$$ {}_p\mathbf{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; X) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa(X)}{k!} \tag{A.1} $$

where the second summation is over all partitions $\kappa = (k_1, \ldots, k_m)$, $k_1 \geq \cdots \geq k_m \geq 0$, of $k = \sum_{i=1}^m k_i$, $k! = k_1! \cdots k_m!$ and the generalized Pochhammer symbols are given by

$$ (a)_\kappa = \prod_{i=1}^{m} \left(a - \frac{i-1}{2} \right)_{k_i}, \quad (a)_k = a(a+1)\cdots(a+k-1), \quad (a)_0 = 1. $$

$C_\kappa(X)$ is the zonal polynomial corresponding to κ, which is a symmetric, homogeneous polynomial of degree k in the eigenvalues of X that satisfies

$$ C_\kappa(YX) = C_\kappa(\sqrt{Y}X\sqrt{Y}) \tag{A.2} $$

for some $Y \in S_m^+$. The function ${}_p\mathbf{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; YX)$ thus makes sense. Finally, we define the generalized modified Bessel function by

$$ \tilde{\mathbf{I}}_\nu(X) = \frac{(\det(X))^{\frac{\nu}{2}}}{\mathbf{\Gamma}_m\left(\nu + \frac{m+1}{2}\right)} \, {}_0\mathbf{F}_1\left(\nu + \frac{m+1}{2}; X\right) \tag{A.3} $$

where the generalized gamma function is given as a product of the usual gamma functions,

$$ \mathbf{\Gamma}_m(\alpha) = \pi^{\frac{m(m-1)}{4}} \prod_{i=1}^{m} \Gamma\left(\alpha - \frac{i-1}{2}\right) \tag{A.4} $$

for $\mathrm{Re}(\alpha) > \frac{m-1}{2}$. Note that the generalized modified Bessel for $m = 1$ relates to the usual modified Bessel function $I_\nu(x)$ by $\tilde{\mathbf{I}}_\nu(x) = I_\nu(2\sqrt{x})$ (see [19]).

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