



**HAL**  
open science

# A characterization of Markov processes enjoying the time-inversion property

Stephan Lawi

► **To cite this version:**

Stephan Lawi. A characterization of Markov processes enjoying the time-inversion property. 2005.  
hal-00005074v2

**HAL Id: hal-00005074**

**<https://hal.science/hal-00005074v2>**

Preprint submitted on 1 Jun 2005

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A characterization of Markov processes enjoying the time-inversion property

S. Lawi\*

1st June 2005

## Abstract

We give a necessary and sufficient condition for a homogeneous Markov process taking values in  $\mathbb{R}^n$  to enjoy the time-inversion property of degree  $\alpha$ . The condition sets the shape for the semigroup densities of the process and allows to further extend the class of known processes satisfying the time-inversion property. As an application we recover the result of Watanabe in [24] for continuous and conservative Markov processes on  $\mathbb{R}_+$ . As new examples we generalize Dunkl processes and construct a matrix-valued process with jumps related to the Wishart process by a skew-product representation.

**Keywords:** homogeneous Markov processes; time-inversion property; Bessel processes; Dunkl processes; Wishart processes; semi-stable processes.

**Mathematics Subject Classification (2000):** 60J25; 60J60; 60J65; 60J99.

## 1 Introduction

Let  $\{(X_t, t \geq 0); (P_x)_{x \in \mathbb{R}^n}\}$  be a homogeneous Markov process with semigroup densities (assumed to exist):

$$P_t(x, dy) = p_t(x, y)dy. \quad (1.1)$$

For all  $x \in \mathbb{R}^n$  and some  $\alpha > 0$ , the process  $\{(t^\alpha X_{\frac{1}{t}}, t > 0); P_x\}$  is Markov and in general inhomogeneous.

**Definition 1.1.** The process  $\{(X_t, t \geq 0); P_x\}$  is said to enjoy the *time-inversion property of degree  $\alpha$*  if the Markov process  $\{(t^\alpha X_{\frac{1}{t}}, t > 0); P_x\}$  is homogeneous.

Celebrated examples of Markov processes, known to enjoy this property for  $\alpha = 1$ , are Brownian motions with drift in  $\mathbb{R}^n$  and Bessel processes with drift (see [21, 24]). Gallardo and Yor [12] recently worked out a sufficient condition on the semigroup densities for a Markov process to enjoy the time-inversion property. Their argument extended the class of processes to processes with jumps such as the Dunkl process [23] and matrix-valued processes such as the Wishart process [3]. The aim of the paper is to show that the former condition is actually necessary and sufficient and to provide some new examples.

Section 2 contains the main theorem of the paper, which is proved in section 3 using straightforward analytical arguments. Section 4 considers an application of the theorem to Markov processes on  $\mathbb{R}_+$ . The result is shown to be strong enough to entirely characterize the class of diffusion processes on  $\mathbb{R}_+$  that enjoy the time-inversion property and thus provide a different proof than that

---

\*Laboratoire de Probabilités et Modèles Aléatoires, CNRS-UMR 7599, Université Paris VI & Université Paris VII, 4 Place Jussieu, 75252 Paris Cedex 05, France. *Email:* lawi@proba.jussieu.fr.

of Watanabe in [24]. Section 5 gives new examples of processes that enjoy the time-inversion property. We review how the generalized Dunkl process fits the requirements of the theorem and then introduce a matrix-valued process with jumps. The relation of the latter process to the Wishart process mimics the relation between the one-dimensional Dunkl and Bessel processes.

## 2 Markov processes which enjoy time-inversion

Fix  $x \in \mathbb{R}^n$ . Recall that under  $P_x$ , the process  $(t^\alpha X_{\frac{1}{t}}, t > 0)$  is Markov, inhomogeneous, with transitional probability densities  $q_{s,t}^{(x)}(z, y)$ , ( $s < t; z, y \in \mathbb{R}^n$ ), which satisfy the following relation:

$$E_x \left[ f(t^\alpha X_{\frac{1}{t}}) \mid s^\alpha X_{\frac{1}{s}} = z \right] = \int dy f(y) q_{s,t}^{(x)}(z, y) \quad (2.1)$$

where

$$q_{s,t}^{(x)}(a, b) = t^{-n\alpha} \frac{p_{\frac{1}{t}} \left( x, \frac{b}{t^\alpha} \right) p_{\frac{1}{s} - \frac{1}{t}} \left( \frac{b}{t^\alpha}, \frac{a}{s^\alpha} \right)}{p_{\frac{1}{s}} \left( x, \frac{a}{s^\alpha} \right)}. \quad (2.2)$$

Throughout the paper, we assume the semigroup densities  $p_t(x, y)$  to be positive over the domain and regular enough to be at least twice differentiable in the space and time variables.

The process  $(t^\alpha X_{\frac{1}{t}}, t > 0)$  is not uniquely defined from the knowledge of the semigroup densities  $p_t(x, y)$ . Actually, there exists at least one transformation that leaves the semigroup densities  $q_{s,t}^{(x)}(a, b)$  unchanged: Doob's  $h$ -transform.

**Definition 2.1.** Doob's  $h$ -transform is the transformation

$$T_h : P_x |_{\mathcal{F}_t} \mapsto \frac{h(X_t)}{h(x)} e^{-\nu t} P_x |_{\mathcal{F}_t}. \quad (2.3)$$

for some function  $h$  and some constant  $\nu > 0$ .

This remark leads to our first two assertions.

**Proposition 2.2.** *Two processes related by  $h$ -transforms yield the same process by time-inversion.*

**Proposition 2.3.**  *$T_h$  is an equivalence relation over the class of homogeneous Markov processes.*

It is hence legitimate to research for a criterium to classify all processes that enjoy the time-inversion property up to  $h$ -transforms. The following theorem gives a concise statement of our main result:

**Theorem 2.4.** *The Markov process  $(t^\alpha X_{\frac{1}{t}}, t > 0)$  is homogeneous if and only if the semigroup densities of  $(X_t, t \geq 0)$  are of the form:*

$$p_t(x, y) = t^{-\frac{n\alpha}{2}} \Phi \left( \frac{x}{t^{\frac{\alpha}{2}}}, \frac{y}{t^{\frac{\alpha}{2}}} \right) \theta \left( \frac{y}{t^{\frac{\alpha}{2}}} \right) \exp \left\{ \rho \left( \frac{x}{t^{\frac{\alpha}{2}}} \right) + \rho \left( \frac{y}{t^{\frac{\alpha}{2}}} \right) \right\} \quad (2.4)$$

or if they are in  $h$ -transform relationship with it. The functions  $\Phi, \theta, \rho$  have the following properties for  $\lambda > 0$ :

1.  $\Phi(\lambda x, y) = \Phi(x, \lambda y)$ ;
2.  $\theta(\lambda y) = \lambda^\beta \theta(y)$  for some  $\beta \in \mathbb{R}$ ;
3.  $\rho(\lambda x) = \lambda^{\frac{2}{\alpha}} \rho(x)$ ;

Moreover, if the symmetry condition  $\Phi(x, y) = \Phi(y, x)$  is satisfied, then the semigroup densities are related as follows:

$$q_t^{(x)}(a, b) = \frac{\Phi(x, b)}{\Phi(x, a)} \exp \left\{ t\rho(x) \right\} p_t(a, b). \quad (2.5)$$

**Definition 2.5.** A Markov process  $(X_t, t \geq 0)$  is called semi-stable with index  $\gamma$  in the sense of Lamperti [18] if:

$$\left\{ (X_{ct}, t \geq 0); P_x \right\} \stackrel{(d)}{=} \left\{ (c^\gamma X_t, t \geq 0); P_{x/c^\gamma} \right\}. \quad (2.6)$$

As a consequence, the semigroup densities of a semi-stable process with index  $\gamma$  have the following property:

$$p_t(x, y) = t^{-n\gamma} p_1\left(\frac{x}{t^\gamma}, \frac{y}{t^\gamma}\right). \quad (2.7)$$

Remarking that the expression for the semigroup densities in Theorem 2.4 satisfies this property for  $\gamma = \alpha/2$  yields the following corollary:

**Corollary 2.6.** *A Markov process that enjoys the time-inversion property of degree  $\alpha$  is a semi-stable process with index  $\alpha/2$ , or is in h-transform relationship with it. The converse is not true.*

### 3 Proof of the Theorem

#### 3.1 Sufficiency

If  $p_t(x, y)$  satisfies the condition (2.4), then from formula (2.2) the semigroup densities  $q_{s,t}^{(x)}(a, b)$  can be written as

$$q_{s,t}^{(x)}(a, b) = (t-s)^{-\frac{n\alpha}{2}} \prod_{i=1}^3 R_{s,t}^{(i)}$$

where

$$\begin{aligned} R_{s,t}^{(1)} &= \Phi\left(x t^{\frac{\alpha}{2}}, \frac{b}{t^\alpha} t^{\frac{\alpha}{2}}\right) \Phi\left(\frac{b}{t^\alpha} \left(\frac{st}{t-s}\right)^{\frac{\alpha}{2}}, \frac{a}{s^\alpha} \left(\frac{st}{t-s}\right)^{\frac{\alpha}{2}}\right) \Phi\left(x s^{\frac{\alpha}{2}}, \frac{a}{s^\alpha} s^{\frac{\alpha}{2}}\right)^{-1} \\ R_{s,t}^{(2)} &= \theta\left(\frac{b}{t^\alpha} t^{\frac{\alpha}{2}}\right) \theta\left(\frac{a}{s^\alpha} \left(\frac{st}{t-s}\right)^{\frac{\alpha}{2}}\right) \theta\left(\frac{a}{s^\alpha} s^{\frac{\alpha}{2}}\right)^{-1} \\ R_{s,t}^{(3)} &= \exp\left\{ \rho\left(x t^{\frac{\alpha}{2}}\right) + \rho\left(\frac{b}{t^\alpha} t^{\frac{\alpha}{2}}\right) + \rho\left(\frac{b}{t^\alpha} \left(\frac{st}{t-s}\right)^{\frac{\alpha}{2}}\right) + \rho\left(\frac{a}{s^\alpha} \left(\frac{st}{t-s}\right)^{\frac{\alpha}{2}}\right) \right. \\ &\quad \left. - \rho\left(x s^{\frac{\alpha}{2}}\right) - \rho\left(\frac{a}{s^\alpha} s^{\frac{\alpha}{2}}\right) \right\} \end{aligned}$$

Using the properties of  $\Phi, \theta, \rho$  described in Theorem 2.4, we obtain:

$$\begin{aligned} R_{s,t}^{(1)} &= \frac{\Phi(x, b)}{\Phi(x, a)} \Phi\left(\frac{b}{(t-s)^{\frac{\alpha}{2}}}, \frac{a}{(t-s)^{\frac{\alpha}{2}}}\right) \\ R_{s,t}^{(2)} &= \theta\left(\frac{b}{(t-s)^{\frac{\alpha}{2}}}\right) \\ R_{s,t}^{(3)} &= \exp\left\{ (t-s)\rho(x) + \rho\left(\frac{a}{(t-s)^{\frac{\alpha}{2}}}\right) + \rho\left(\frac{b}{(t-s)^{\frac{\alpha}{2}}}\right) \right\}. \end{aligned}$$

Hence there is no separate dependence on  $s$  and  $t$ , but only on the difference  $t-s$ , which allows to conclude that

$$q_{s,t}^{(x)}(a, b) = q_{t-s}^{(x)}(a, b)$$

and proves homogeneity for the process  $(t^\alpha X_{\frac{1}{t}}, t > 0)$ . If in addition  $\Phi(x, y) = \Phi(y, x)$ , we obtain (2.5).

## 3.2 Necessity

For simplicity of notation, we prove the necessity of condition (2.4) in the case  $\alpha = 1$ . The extension to  $\alpha > 0$  is immediate by change of variables  $X_t \mapsto X_t^\alpha$ . Recall first the following definition of homogeneous functions:

**Definition 3.1.** A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  that satisfies for  $x = (x_1, \dots, x_n)$ ,  $\forall \lambda \in \mathbb{R}_+$ ,

$$f(\lambda x) = \lambda^\beta f(x)$$

is called *homogeneous of degree*  $\beta \in \mathbb{R}$ . If  $f \in C^1(\mathbb{R}^n)$ , Euler's Homogeneous Function Theorem gives a necessary and sufficient condition for the function  $f(x)$  to be homogeneous:

$$\sum_{i=1}^n x_i \frac{\partial}{\partial x_i} f(x) = \beta f(x).$$

Consider the function of  $2n + 1$  variables:  $l(x, y, t) = \ln(p_t(x, y))$ ,  $x, y \in \mathbb{R}^n$ . From (2.2),  $l$  must satisfy for  $s = t - h$ :

$$\frac{\partial}{\partial t} \left[ n \ln \frac{1}{t} + l \left( x, \frac{b}{t}, \frac{1}{t} \right) + l \left( \frac{b}{t}, \frac{a}{t-h}, \frac{1}{t-h} - \frac{1}{t} \right) - l \left( x, \frac{a}{t-h}, \frac{1}{t-h} \right) \right] = 0. \quad (3.1)$$

### 3.2.1 The kernel $\Phi(x, y)$

Taking derivatives with respect to  $b_i$  and  $a_j$  for some  $0 \leq i, j \leq n$  yields:

$$\frac{\partial}{\partial b_i} \frac{\partial}{\partial a_j} \frac{\partial}{\partial t} l \left( \frac{b}{t}, \frac{a}{t-h}, \frac{1}{t-h} - \frac{1}{t} \right) = 0.$$

If we set  $\phi(x, y, t) = \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j} l(x, y, t)$ , the latter becomes

$$-\frac{1}{t^2(t-h)} \left( \phi + \frac{b}{t} \cdot \nabla_1 \phi \right) - \frac{1}{t(t-h)^2} \left( \phi + \frac{a}{t-h} \cdot \nabla_2 \phi \right) + \frac{1}{t(t-h)} \left( \frac{1}{t^2} - \frac{1}{(t-h)^2} \right) \partial \phi = 0,$$

with the notation  $\nabla_1 = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})^T$ ,  $\nabla_2 = (\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n})^T$  and  $\phi = \phi \left( \frac{b}{t}, \frac{a}{t-h}, \frac{1}{t-h} - \frac{1}{t} \right)$ . For clarity, we change variables to

$$z_1 = \frac{b}{t}, \quad z_2 = \frac{a}{t-h}, \quad t_1 = -\frac{1}{t}, \quad t_2 = \frac{1}{t-h}.$$

Then  $\phi = \phi(z_1, z_2, t_1 + t_2)$  and

$$t_1 \left( \phi + z_1 \cdot \nabla_1 \phi + t_1 \partial \phi \right) = t_2 \left( \phi + z_2 \cdot \nabla_2 \phi + t_2 \partial \phi \right),$$

or equivalently

$$t_1 \left( \phi + z_1 \cdot \nabla_1 \phi + (t_1 + t_2) \partial \phi \right) = t_2 \left( \phi + z_2 \cdot \nabla_2 \phi + (t_1 + t_2) \partial \phi \right). \quad (3.2)$$

We change variables once more:  $u = \frac{t_1}{t_2}, v = t_1 + t_2$  to get:

$$\phi + z_1 \cdot \nabla_1 \phi + v \partial \phi = u \left( \phi + z_2 \cdot \nabla_2 \phi + v \partial \phi \right). \quad (3.3)$$

This gives the following proposition.

**Proposition 3.2.** Equation (3.3) is satisfied if and only if

$$\phi(\lambda x, \mu y, \lambda \mu t) = \frac{1}{\lambda \mu} \phi(x, y, t). \quad (3.4)$$

*Proof.* As the LHS of equation (3.3) is independent of  $u$ , one can readily take the equivalent condition:

$$\phi + z_1 \cdot \nabla_1 \phi + v \partial \phi = 0, \quad (3.5)$$

$$\phi + z_2 \cdot \nabla_2 \phi + v \partial \phi = 0. \quad (3.6)$$

Let  $g(\lambda) = \phi(\lambda z_1, z_2, \lambda v)$  and  $h(\mu) = \phi(z_1, \mu z_2, \mu v)$ . Equation (3.5) implies  $g(\lambda) + \lambda g'(\lambda) = 0$ , which solves to  $g(\lambda) = \lambda^{-1} g(1)$  and hence

$$\phi(\lambda z_1, z_2, \lambda v) = \frac{1}{\lambda} \phi(z_1, z_2, v).$$

Equation (3.6) implies  $h(\mu) + \mu h'(\mu) = 0$ , which solves to  $h(\mu) = \mu^{-1} h(1)$  and hence

$$\phi(z_1, \mu z_2, \mu v) = \frac{1}{\mu} \phi(z_1, z_2, v).$$

Combining the latter two equations yields (3.4).

Conversely, if  $g$  is homogeneous of degree  $-1$  and  $h$  is homogeneous of degree  $-1$ , we get:

$$g(\lambda) + \lambda g'(\lambda) = h(\mu) + \mu h'(\mu) = 0,$$

which is equivalent to (3.5) and (3.6) and concludes the proof.  $\square$

Note that the previous result remains valid for all second derivatives of  $l(x, y, t)$  with respect to  $x_i$  and  $y_j$ ,  $\forall i, j \in \{1, \dots, n\}$ . By integration over  $x_i$  and  $y_j$ , one can thus already make an assumption on the general shape of the function  $l$ :

$$l(x, y, t) = k(x, y, t) + \varphi_1(x, t) + \varphi_2(y, t) + h_1(x) + h_2(y) + \tau(t), \quad (3.7)$$

where  $k : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}$ ,  $\varphi_1, \varphi_2 : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ ,  $\tau : \mathbb{R} \rightarrow \mathbb{R}$  and  $h_1, h_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ . Moreover, the kernel  $k(x, y, t)$  must satisfy the following property:

$$k(\lambda x, \mu y, \lambda \mu t) = k(x, y, t).$$

This is immediate from (3.4) and

$$\begin{aligned} \phi(\lambda x, \mu y, \lambda \mu t) &= \frac{\partial}{\partial \lambda x_i} \frac{\partial}{\partial \mu y_j} l(\lambda x, \mu y, \lambda \mu t) = \frac{\partial}{\partial \lambda x_i} \frac{\partial}{\partial \mu y_j} k(\lambda x, \mu y, \lambda \mu t) = \frac{1}{\lambda \mu} \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j} k(\lambda x, \mu y, \lambda \mu t), \\ \frac{1}{\lambda \mu} \phi(x, y, t) &= \frac{1}{\lambda \mu} \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j} l(x, y, t) = \frac{1}{\lambda \mu} \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j} k(x, y, t). \end{aligned}$$

This property is actually equivalent to

$$k\left(\lambda \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}, 1\right) = k\left(\frac{x}{\sqrt{t}}, \lambda \frac{y}{\sqrt{t}}, 1\right)$$

Setting  $\Phi\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) = \exp k(x, y, t)$  completes the proof of the first condition of the theorem, i.e.

$$\Phi(\lambda x, y) = \Phi(x, \lambda y).$$

### 3.2.2 The function $\rho(x)$

We now focus on the functions  $\varphi_1(x, t), \varphi_2(y, t)$ . From (3.1), they must satisfy

$$\begin{aligned} \frac{\partial}{\partial t} \left[ \varphi_1 \left( x, \frac{1}{t} \right) + \varphi_2 \left( \frac{b}{t}, \frac{1}{t} \right) + \varphi_1 \left( \frac{b}{t}, \frac{1}{t-h} - \frac{1}{t} \right) + \varphi_2 \left( \frac{a}{t-h}, \frac{1}{t-h} - \frac{1}{t} \right) \right. \\ \left. - \varphi_1 \left( x, \frac{1}{t-h} \right) - \varphi_2 \left( \frac{a}{t-h}, \frac{1}{t-h} \right) \right] = 0. \end{aligned} \quad (3.8)$$

Recall that the variables  $a, b, x$  are independent from each other and that  $\varphi_1, \varphi_2$  are defined to be explicitly dependent on their variables. So taking the derivative of (3.8) with respect to  $x_i$  gives

$$\frac{1}{t^2} \partial_t \psi_1 \left( x, \frac{1}{t} \right) = \frac{1}{(t-h)^2} \partial_t \psi_1 \left( x, \frac{1}{t-h} \right)$$

where  $\psi_1(x, t) = \frac{\partial}{\partial x_i} \varphi_1(x, t)$ . Hence  $\psi_1(x, t)$  must be of the form  $\psi_1(x, t) = \frac{1}{t} \psi_{11}(x) + \psi_{12}(x)$ , which implies

$$\varphi_1(x, t) = \frac{1}{t} \varphi_{11}(x) + \varphi_{12}(x).$$

We omit the second term as it can always be recast as part of  $h(x)$  in (3.7). There are so far no further conditions to add on the function  $\varphi_{11}(x)$ .

Taking the derivative of (3.8) with respect to  $b_i$  leads to

$$\frac{\partial}{\partial b_i} \frac{\partial}{\partial t} \left[ \varphi_2 \left( \frac{b}{t}, \frac{1}{t} \right) + \frac{t(t-h)}{h} \varphi_{11} \left( \frac{b}{t} \right) \right] = 0,$$

which develops to

$$-\frac{1}{t^2} \partial_i \varphi_2 - \frac{b}{t^3} \cdot \nabla \partial_i \varphi_2 - \frac{1}{t^3} \partial_t \partial_i \varphi_2 + \frac{2t-h}{ht} \partial_i \varphi_{11} - \frac{t(t-h)}{h} \frac{1}{t^2} \partial_i \varphi_{11} - \frac{t(t-h)}{h} \frac{b}{t^3} \cdot \nabla \partial_i \varphi_{11} = 0.$$

Recall that  $\psi_{11}(x) = \frac{\partial}{\partial x_i} \varphi_{11}(x)$  and set  $\psi_2(x, t) = \frac{\partial}{\partial x_i} \varphi_2(x, t)$ . In this notation, we get

$$-\frac{1}{t} \psi_2 - \frac{b}{t^2} \cdot \nabla \psi_2 - \frac{1}{t^2} \partial_t \psi_2 + \frac{2t-h}{h} \psi_{11} - \frac{t(t-h)}{h} \frac{1}{t} \psi_{11} - \frac{t(t-h)}{h} \frac{b}{t^2} \cdot \nabla \psi_{11} = 0.$$

Let  $z = \frac{b}{t}$ ,  $t_1 = \frac{1}{t}$ ,  $t_2 = \frac{1}{t-h}$ , then we have

$$\psi_2(z, t_1) + z \cdot \nabla \psi_2(z, t_1) + t_1 \partial_t \psi_2(z, t_1) = \frac{t_2}{t_1(t_2 - t_1)} \psi_{11}(z) - \frac{1}{t_2 - t_1} z \cdot \nabla \psi_{11}(z).$$

As the LHS is independent of  $t_2$ , so must be the RHS, which implies

$$z \cdot \nabla \psi_{11}(z) = \psi_{11}(z).$$

Hence  $\psi_{11}$  is homogeneous of degree 1. In that case,

$$\psi_2(z, t_1) + z \cdot \nabla \psi_2(z, t_1) + t_1 \partial_t \psi_2(z, t_1) = \frac{1}{t_1} \psi_{11}(z).$$

We solve the equation for  $g(\lambda) = \psi_2(\lambda z, \lambda t_1)$  to get  $g(\lambda) = \frac{1}{t_1} \psi_{11}(z) + \frac{1}{\lambda} c(z, t_1)$ , where  $c(z, t_1)$  turns out to be a homogeneous function of degree  $-1$ . Integration over  $z_i$  yields some conditions for  $\varphi_2(z, t)$ . Once again, since the index  $i \in \{1, \dots, n\}$  was chosen arbitrarily, we obtain the following characteristics:

$$\varphi_1(\lambda z, \lambda t) = \frac{\varphi_{11}(\lambda z)}{\lambda t} \quad \text{with} \quad \varphi_{11}(\lambda z) = \lambda^2 \varphi_{11}(z)$$

and

$$\varphi_2(\lambda z, \lambda t) = \frac{\varphi_{21}(\lambda z)}{\lambda t} + \hat{c}(\lambda z, \lambda t),$$

where  $\varphi_{21}(\lambda z) = \lambda^2 \varphi_{21}(z)$  and even  $\varphi_{21}(z) = \varphi_{11}(z)$  without loss of generality. Let  $\rho(z) = \varphi_{11}(z) = \varphi_{12}(z)$ , then we recover the third condition of the theorem, that is

$$\rho(\lambda z) = \lambda^2 \rho(z).$$

### 3.2.3 The function $\theta(y)$

In order to further investigate the properties of  $\hat{c}(z, t)$ , we derive (3.8) by  $a_i$ :

$$\frac{\partial}{\partial a_i} \frac{\partial}{\partial t} \left[ \frac{1}{h} \varphi(a) + \hat{c} \left( \frac{a}{t-h}, \frac{1}{t-h} - \frac{1}{t} \right) - \hat{c} \left( \frac{a}{t-h}, \frac{1}{t-h} \right) \right] = 0.$$

This leads to

$$c(z, t_2 - t_1) + z \cdot \nabla c(z, t_2 - t_1) + \frac{t_1^2 - t_2^2}{t_2} \partial_t c(z, t_2 - t_1) = c(z, t_2) + z \cdot \nabla c(z, t_2) + t_2 \partial_t c(z, t_2).$$

Since  $c(z, t)$  is homogeneous of degree  $-1$ , the RHS is zero. This only proves that  $\frac{\partial}{\partial t} c(z, t) = \frac{\partial}{\partial t} \frac{\partial}{\partial z_i} \hat{c}(z, t) = 0, \forall i$ . Hence,  $\hat{c}(z, t) = \hat{c}_1(z) + \hat{c}_2(t)$ . Going back to equation (3.8) derived by  $b_i$  gives this time for  $\hat{c}(z, t)$ ,

$$\frac{\partial}{\partial b_i} \frac{\partial}{\partial t} \left[ \hat{c} \left( \frac{b}{t}, \frac{1}{t} \right) \right] = \frac{\partial}{\partial b_i} \frac{\partial}{\partial t} \left[ \hat{c}_1 \left( \frac{b}{t} \right) \right] = 0,$$

which clearly implies  $\hat{c}_1(\frac{b}{t}) = \hat{c}_{11}(b) + \hat{c}_{12}(\frac{1}{t})$ . Using equation (3.8) once more gives

$$\frac{\partial}{\partial t} \left[ \hat{c}_{12} \left( \frac{1}{t} \right) + \hat{c}_2 \left( \frac{1}{t} \right) + \hat{c}_2 \left( \frac{1}{t-h} - \frac{1}{t} \right) - \hat{c}_2 \left( \frac{1}{t-h} \right) \right] = 0, \quad (3.9)$$

which is equivalent to

$$t_1^2 \hat{c}'_{12}(t_1) + t_1^2 \hat{c}'_2(t_1) + (t_1 + t_2)(t_2 - t_1) \hat{c}'_2(t_2 - t_1) - t_2^2 \hat{c}'_2(t_2) = 0.$$

We set  $g(t) = t \hat{c}'_2(t)$  so that

$$t_1^2 \hat{c}'_{12}(t_1) + t_1 g(t_1) = t_2 (g(t_2) - g(t_2 - t_1)) - t_1 g(t_2 - t_1).$$

Assuming  $\lim_{t_1 \rightarrow 0} \frac{LHS}{t_1} = \frac{\beta}{2} \in \mathbb{R}$  with  $|\beta| < \infty$ , we divide the equation by  $t_1$  and take the limit as  $t_1 \rightarrow 0$ . This yields  $\frac{1}{2}\beta = t_2 g'(t_2) - g(t_2)$ , which solves for  $g(t) = -\frac{1}{2}\beta + \gamma t$ ,  $\gamma \in \mathbb{R}$ . Hence,

$$\hat{c}_2(t) = -\frac{1}{2}\beta \ln t + \gamma t \quad \text{and} \quad \hat{c}_{12}(t) = \beta \ln t,$$

where any additional constant term has been set to 0. It will be convenient to set  $\gamma = 0$ . A term  $\gamma \neq 0$  corresponds to the independent case of an affine function of  $t$  in the sum (3.7) and is included in the discussion for  $\tau(t)$ . To summarize, the function  $\hat{c}(z, t)$  must be expressed as

$$\hat{c}(z, t) = \hat{c}_1(z) - \frac{\beta}{2} \ln t,$$

where  $c_1(z)$  has the following property:  $\hat{c}_1(\lambda z) = \hat{c}_1(z) + \beta \ln \lambda$ . Let  $\theta(z) = \exp \hat{c}_1(z)$ . We then recover condition 2 of the theorem, i.e.

$$\theta(\lambda z) = \lambda^\beta \theta(z).$$

### 3.2.4 The functions $h_1(x), h_2(y)$

The obvious solution to (3.1) in this case is  $h(z) = h_1(z) = -h_2(z)$ .  $\exp h$  hence defines an  $h$ -transform, which is neglected since we classify all processes up to  $h$ -transforms.



### 3.2.5 The function $\tau(t)$

It remains to explicitly formulate the form of the function  $\tau : \mathbb{R}_+ \rightarrow \mathbb{R}$  that is defined to satisfy

$$\frac{\partial}{\partial t} \left[ -n \ln t + \tau \left( \frac{1}{t} \right) + \tau \left( \frac{1}{t-h} - \frac{1}{t} \right) - \tau \left( \frac{1}{t-h} \right) \right] = 0.$$

But this equation is similar to (3.9), so we immediately have the solution ( $\gamma \in \mathbb{R}$ ):

$$\tau(t) = -\frac{n}{2} \ln t + \gamma t.$$

Once again  $e^{\gamma t}$  can be included in an  $h$ -transform, so  $\gamma$  is set to 0 and  $e^{\tau(t)} = t^{-\frac{n}{2}}$  gives the first factor in the semigroup densities (2.4).

### 3.2.6 The general case $\alpha > 0$

For  $\alpha > 0$ , we use the change of variables  $X_t \mapsto X_t^\alpha$ . The semigroup densities for  $X_t^\alpha$  become

$$p_t^\alpha(x, y) = p_t \left( x^{\frac{1}{\alpha}}, y^{\frac{1}{\alpha}} \right) J(y)$$

where  $J(y)$  is the Jacobian of the inverse transformation, that is  $J(y) = \alpha^{-n} y_1^{\frac{1}{\alpha}-1} \cdots y_n^{\frac{1}{\alpha}-1}$ , and we use the slight abuse of notation  $x^{\frac{1}{\alpha}} = \left( x_1^{\frac{1}{\alpha}}, \dots, x_n^{\frac{1}{\alpha}} \right)$ . The semigroup densities can be recast into

$$p_t^\alpha(x, y) = t^{-\frac{n\alpha}{2}} \Phi^\alpha \left( \frac{x}{t^{\frac{\alpha}{2}}}, \frac{y}{t^{\frac{\alpha}{2}}} \right) \theta^\alpha \left( \frac{y}{t^{\frac{\alpha}{2}}} \right) \exp \left\{ \rho^\alpha \left( \frac{x}{t^{\frac{\alpha}{2}}} \right) + \rho^\alpha \left( \frac{y}{t^{\frac{\alpha}{2}}} \right) \right\}$$

where  $\Phi^\alpha(x, y) = \Phi \left( x^{\frac{1}{\alpha}}, y^{\frac{1}{\alpha}} \right)$  satisfies condition 1,  $\theta^\alpha(y) = \theta \left( y^{\frac{1}{\alpha}} \right) J(y)$  satisfies condition 2 for  $\bar{\beta} = \frac{\beta+n}{\alpha} - n$  and  $\rho^\alpha(x) = \rho \left( x^{\frac{1}{\alpha}} \right)$  satisfies condition 3 of equation (2.4).

## 4 Application to diffusions on $\mathbb{R}_+$

The case of the diffusions on  $\mathbb{R}_+$  was entirely characterized by Watanabe in [24]. It was shown that only Bessel processes in the wide sense (which we recall the definition below) enjoy the time-inversion property of degree 1.

**Definition 4.1.** For some  $\nu > -1$  and  $c \geq 0$ , the diffusion process generated by

$$\mathcal{L} = \frac{1}{2} \frac{\partial^2}{\partial x^2} + \left( \frac{2\nu+1}{2x} + \frac{h'_c(x)}{h_c(x)} \right) \frac{\partial}{\partial x} \tag{4.1}$$

is called *Bessel process in the wide sense*. The function  $h_c(x)$  is given by

$$h_c(x) = 2^\nu \Gamma(\nu+1) (\sqrt{2c} x)^{-\nu} I_\nu(\sqrt{2c} x), \tag{4.2}$$

where  $I_\nu$  is the modified Bessel function.

*Remark 4.2.* The Bessel process in the wide sense is in  $h$ -transform relationship, for  $h \equiv h_c$ , with the Bessel process.

We show that the result in [24] is a consequence of Theorem 2.4, which has the following one-dimensional formulation:

**Theorem 4.3.** *The Markov process  $(t^\alpha X_{\frac{1}{t}}, t > 0)$  on  $\mathbb{R}_+$  is homogeneous if and only if the semigroup densities of  $(X_t, t \geq 0)$  are of the form:*

$$p_t(x, y) = t^{-\frac{\alpha}{2}(1+\beta)} \phi\left(\frac{xy}{t^\alpha}\right) y^\beta \exp\left\{-\frac{k^2}{2}\left(\frac{x^{\frac{2}{\alpha}}}{t} + \frac{y^{\frac{2}{\alpha}}}{t}\right)\right\} \quad (4.3)$$

for  $k > 0$ , or if it is in  $h$ -transform relationship with it. Moreover, the semigroup densities are related as follows:

$$q_t^{(x)}(a, b) = \frac{\phi(xb)}{\phi(xa)} \exp\left(-t \frac{k^2}{2} x^{\frac{2}{\alpha}}\right) p_t(a, b). \quad (4.4)$$

*Proof.* Theorem 2.4 formulated on  $\mathbb{R}_+$  implies  $\Phi(x, y) = \phi(xy)$ ,  $\theta(y) = y^\beta$  and  $\rho(x) = -\frac{k^2}{2} x^{\frac{2}{\alpha}}$ ,  $k > 0$ , for the condition 1-3 to be satisfied in one dimension.  $\square$

We identify further the class of diffusion processes and provide a different proof for the result in [24].

**Proposition 4.4.** *If  $(X_t, t \geq 0)$  is a diffusion process and  $(tX_{\frac{1}{t}}, t > 0)$  is homogeneous and conservative, then both are necessarily (possibly time-scaled) Bessel processes in the wide sense.*

*Proof.* If  $(X_t, t \geq 0)$  is a diffusion process, then its infinitesimal generator has the following general structure:

$$\mathcal{L} = s(y) \frac{\partial^2}{\partial y^2} + \mu(y) \frac{\partial}{\partial y}$$

where it remains to identify the functions  $s(y)$  and  $\mu(y)$ . For a fixed  $x > 0$ , let  $\mathcal{L}^{(x)}$  be the infinitesimal generator of  $(tX_{\frac{1}{t}}, t > 0)$ . From equation (2.5),  $\mathcal{L}^{(x)}$  has the following relationship with  $\mathcal{L}$ :

$$\mathcal{L}^{(x)} : f(b) \mapsto \frac{1}{\phi(xb)} \mathcal{L}(\phi(xb)f(b)) - \frac{k^2}{2} x^2 f(b),$$

which develops to

$$\mathcal{L}^{(x)} f(b) = s(b) f''(b) + \left\{ \mu(b) + 2x s(b) \frac{\phi'(xb)}{\phi(xb)} \right\} f'(b) + U(x, b) f(b).$$

For the process to be conservative, we require  $U(x, b) = 0$ , which implies no killing in the interior of the domain, that is

$$s(b) x^2 \frac{\phi''(xb)}{\phi(xb)} + \mu(b) x \frac{\phi'(xb)}{\phi(xb)} - \frac{k^2}{2} x^2 = 0.$$

We change variables to  $z = xb$  to obtain

$$s(z/x) \phi''(z) + \frac{\mu(z/x)}{x} \phi'(z) - \frac{k^2}{2} \phi(z) = 0.$$

Since the latter must be valid for all  $x > 0$ , we are led to set:  $s(b) = \frac{\sigma^2}{2}$  for  $\sigma > 0$  and  $\mu(b) = \frac{\sigma^2}{2} \frac{2\nu+1}{b}$  for  $\nu > -1$ . This yields the following equation

$$\frac{1}{2} \phi''(z) + \frac{2\nu+1}{2z} \phi'(z) - \frac{k^2}{2\sigma^2} \phi(z) = 0.$$

The general solution (non-singular at 0 and up to a constant factor) is expressed through the modified Bessel function of the first kind as follows:

$$\phi(z) = \left(\frac{kz}{\sigma}\right)^{-\nu} I_\nu\left(\frac{kz}{\sigma}\right).$$

Gathering the different factors in (4.3) leads to

$$p_t(x, y) = N t^{-\frac{1+\beta}{2}} \left(\frac{xy}{t}\right)^{-\nu} I_\nu\left(\frac{k xy}{\sigma t}\right) y^\beta \exp\left\{-\frac{k^2}{2}\left(\frac{x^2}{t} + \frac{y^2}{t}\right)\right\},$$

where  $N$  is a normalization factor. The additional condition that  $\lim_{t \rightarrow 0} p_t(x, y) = \delta(x - y)$  implies  $\beta = 2\nu + 1$  and  $k = \frac{1}{\sigma}$ , which leads to the semigroup densities of a time-scaled Bessel process of dimension  $\nu$ :

$$p_t(x, y) = \frac{y}{\sigma^2 t} \left(\frac{y}{x}\right)^\nu I_\nu\left(\frac{xy}{\sigma^2 t}\right) \exp\left\{-\frac{x^2 + y^2}{2\sigma^2 t}\right\}.$$

The infinitesimal generator for  $(tX_{\frac{1}{t}}, t > 0)$  is given by

$$\mathcal{L}^{(x)} = \frac{\sigma^2}{2} \frac{\partial^2}{\partial b^2} + \sigma^2 \left\{ \frac{2\nu + 1}{2b} + x \frac{\phi'(xb)}{\phi(xb)} \right\} \frac{\partial}{\partial b},$$

where one recognizes expression (4.1) for  $h_c(b) = \phi(xb)$  and a time-scale  $t \mapsto \sigma^2 t$ .  $\square$

Proposition 4.4 smoothly extends to any power  $\alpha > 0$  of the Bessel process through the mapping  $X_t \mapsto X_t^\alpha$ . In particular, it is worth remarking that the case  $\alpha = 2$  gives rise to squares of Bessel processes, which leads to the following result:

**Proposition 4.5.** *If  $(X_t, t \geq 0)$  is a diffusion process and  $(t^2 X_{\frac{1}{t}}, t > 0)$  is homogeneous and conservative, then both are necessarily (possibly time-scaled) squares of Bessel processes in the wide sense.*

## 5 Examples

### 5.1 Generalized Dunkl processes and Jacobi-Dunkl processes

#### 5.1.1 Multidimensional Dunkl processes

We briefly review the construction of the Dunkl process in  $\mathbb{R}^n$  (see [22, 23]).

**Definition 5.1.** The Dunkl process in  $\mathbb{R}^n$  is the Markov càdlàg process with infinitesimal generator

$$\frac{1}{2} \mathcal{L}^{(k)} = \frac{1}{2} \sum_{i=1}^n T_i^2 \tag{5.1}$$

where  $T_i$ ,  $1 \leq i \leq n$ , is a one-dimensional differential-difference operator defined for  $u \in C^1(\mathbb{R}^n)$  by

$$T_i u(x) = \frac{\partial u(x)}{\partial x_i} + \sum_{\alpha \in R_+} k(\alpha) \alpha_i \frac{u(x) - u(\sigma_\alpha x)}{\langle \alpha, x \rangle}. \tag{5.2}$$

$\langle \cdot, \cdot \rangle$  is the usual scalar product.  $R$  is a root system in  $\mathbb{R}^n$  and  $R_+$  a positive subsystem.  $k(\alpha)$  is a non-negative multiplicity function defined on  $R$  and invariant by the finite reflection group  $W$  associated with  $R$ .  $\sigma_\alpha$  is the reflection operator with respect to the hyperplane  $H_\alpha$  orthogonal to  $\alpha$  such that  $\sigma_\alpha x = x - \langle \alpha, x \rangle \alpha$  and for convenience  $\langle \alpha, \alpha \rangle = 2$  (see [8, 9]).

A result obtained by M. Rösler [22] yields the semigroup densities as follows:

$$p_t^{(k)}(x, y) = \frac{1}{c_k t^{\gamma+n/2}} \exp\left(-\frac{|x|^2 + |y|^2}{2t}\right) D_k\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) \omega_k(y) \tag{5.3}$$

where  $D_k(x, y) > 0$  is the Dunkl kernel,  $\omega_k(y) = \prod_{\alpha \in R_+} |\langle \alpha, y \rangle|^{2k(\alpha)}$  the weight function which is homogeneous of degree  $2\gamma = 2 \sum_{\alpha \in R_+} k(\alpha)$  and  $c_k = \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} \omega_k(x) dx$ .

Following a thorough study of the properties of the one-dimensional Dunkl process in [13], it was remarked in [12] that the Dunkl process in  $\mathbb{R}^n$  enjoys the time-inversion property of degree 1. Considering that the Dunkl kernel satisfies

$$D_k(x, y) = D_k(y, x) \quad \text{and} \quad D_k(\mu x, y) = D_k(x, \mu y), \quad (5.4)$$

the proof is straightforward with

$$\Phi(x, y) \equiv D_k(x, y), \quad \theta(y) \equiv \omega_k(y), \quad \rho(x) \equiv -\frac{|x|^2}{2}. \quad (5.5)$$

By equation (2.5), the semigroup densities of the time-inverted process is even in  $h$ -transform relationship with the semigroup densities of the original Dunkl process:

$$q_t^{(x)}(a, b) = \frac{D_k(x, b)}{D_k(x, a)} \exp\left(-\frac{|x|^2}{2}t\right) p_t^{(k)}(a, b). \quad (5.6)$$

### 5.1.2 Generalized Dunkl processes

In an attempt to generalize the Dunkl process, we extend the definition of the infinitesimal generator to

$$\mathcal{L}^{(k, \lambda)} f(x) = \frac{1}{2} \Delta f(x) + \sum_{\alpha \in R_+} k(\alpha) \frac{\langle \nabla f(x), \alpha \rangle}{\langle x, \alpha \rangle} + \sum_{\alpha \in R_+} \lambda(\alpha) \frac{f(\sigma_\alpha x) - f(x)}{\langle x, \alpha \rangle^2} \quad (5.7)$$

where  $\Delta$  is the usual Laplacian,  $f \in C^2(\mathbb{R}^n)$  and  $\lambda(\alpha)$  is a non-negative multiplicity function defined on  $R$  and invariant by the finite reflection group  $W$ , similarly as  $k(\alpha)$ . We retrieve the Dunkl process for  $\lambda(\alpha) = k(\alpha)$ . Note that these processes are no longer martingales for  $\lambda(\alpha) \neq k(\alpha)$ .

The one-dimensional case was introduced in [13], where the semigroup densities were explicitly derived,

$$p_t^{(k, \lambda)}(x, y) = \frac{1}{t^{k-\frac{1}{2}}} y^{k-\frac{1}{2}} \exp\left(-\frac{x^2 + y^2}{2t}\right) D_{k, \lambda}\left(\frac{xy}{t}\right) \quad (5.8)$$

with the generalized Dunkl kernel, ( $\nu = k - \frac{1}{2}$ ,  $\mu = \sqrt{\nu^2 + 4\lambda}$ ),

$$D_{k, \lambda}(z) = 1_{\{y \in \mathbb{R}_-\}} \frac{1}{2z^\nu} (I_\nu - I_\mu)(-z) + 1_{\{y \in \mathbb{R}_+\}} \frac{1}{2z^\nu} (I_\nu + I_\mu)(z), \quad (5.9)$$

for  $z = \frac{xy}{t}$ . From (5.4), the generalized Dunkl kernel satisfies

$$D_{k, \lambda}(x, y) = D_{k, \lambda}(y, x) \quad \text{and} \quad D_{k, \lambda}(\mu x, y) = D_{k, \lambda}(x, \mu y), \quad (5.10)$$

which readily implies that the generalized Dunkl process also enjoys the time-inversion property of degree 1. The semigroup densities were derived as an application of the skew-product representation of the generalized Dunkl process ( $X_t, t \geq 0$ ) in terms of its absolute value (a Bessel process) and an independent Poisson process  $N_t^{(\lambda)}$ :

$$X_t \stackrel{(d)}{=} |X_t| (-1)^{N_{A_t}^{(\lambda)}} \quad (5.11)$$

where  $A_t = \int_0^t \frac{ds}{X_s^2}$ .

In the  $n$ -dimensional case, the application of the skew-product representation derived by Chybiryakov [6] shows that the generalized Dunkl process enjoys time-inversion for some specific root systems. We first recall one of the main results of [6].

**Proposition 5.2.** *Let  $(X_t, t \geq 0)$  be the generalized Dunkl process generated by (5.7), with  $(X_t^W, t \geq 0)$  its radial part, i.e. the process confined to a Weyl chamber. Let  $R_+ \equiv \{\alpha_1, \dots, \alpha_l\}$  for some  $l \in \mathbb{N}$  be the corresponding positive root system and let  $(N_t^i, t \geq 0)$ ,  $i = 1, \dots, l$  be independent Poisson processes of respective intensities  $\lambda(\alpha_i)$ . Then  $X_t$  may be represented as  $Y_t^l$ , which is defined by induction as follows:*

$$Y_t^0 = X_t^W \quad \text{and} \quad Y_t^i = \sigma_{\alpha_i}^{N_t^i} Y_t^{i-1}, \quad i = 1, \dots, l,$$

$$\text{where } A_t^i = \int_0^t \frac{ds}{\langle Y_s^{i-1}, \alpha_i \rangle^2}.$$

The proof follows the argument in [6], while replacing  $k(\alpha_i)$  by  $\lambda(\alpha_i)$  appropriately.

From now on, let  $R_+ \equiv \{\alpha_1, \dots, \alpha_l\}$  for some  $l \leq n$  be an orthogonal positive root system, that is  $\langle \alpha_i, \alpha_j \rangle = 2\delta_{ij}$ . For this particular root system, one can show that the generalized Dunkl process enjoys time-inversion of degree 1. We first prove the following absolute continuity relation:

**Lemma 5.3.** *Let  $(X_t^W, t \geq 0)$  be the radial Dunkl process with infinitesimal generator*

$$\mathcal{L}_k^W f(x) = \frac{1}{2} \Delta f(x) + \sum_{i=1}^l k(\alpha_i) \frac{\langle \alpha_i, \nabla f(x) \rangle}{\langle \alpha_i, x \rangle}. \quad (5.12)$$

*Fixed  $\nu \in \{1, \dots, l\}$ . Let  $k'(\alpha)$  be another coefficient function on the root system  $R_+$  such that  $k'(\alpha_\nu) > k(\alpha_\nu)$  and  $k'(\alpha_i) = k(\alpha_i)$  for  $i \neq \nu$ . Then, denoting  $\mathbb{P}_x^{(k)}$  the law of the radial Dunkl process  $X_t^W$  starting from  $x$ , we have*

$$\mathbb{P}_x^{(k')} |_{\mathcal{F}_t} = \left( \frac{\langle \alpha_\nu, X_t^W \rangle}{\langle \alpha_\nu, x \rangle} \right)^{k'(\alpha_\nu) - k(\alpha_\nu)} \exp \left[ - \frac{(k'(\alpha_\nu) - \frac{1}{2})^2 - (k(\alpha_\nu) - \frac{1}{2})^2}{2} \int_0^t \frac{ds}{\langle \alpha_\nu, X_s^W \rangle^2} \right] \cdot \mathbb{P}_x^{(k)} |_{\mathcal{F}_t}. \quad (5.13)$$

*Proof.* Let  $k_0(\alpha)$  be a coefficient such that  $k_0(\alpha_\nu) = \frac{1}{2}$  for some fixed  $i \in \{1, \dots, l\}$ .  $X_t^W$  has the following martingale decomposition (see [11]):

$$X_t^W = x + B_t^{(k_0)} + \sum_{i=1}^l k_0(\alpha_i) \int_0^t \frac{ds}{\langle \alpha_i, X_s^W \rangle} \alpha_i$$

where  $B_t^{(k_0)}$  is a  $(\mathbb{P}_x^{(k_0)}, \mathcal{F}_t)$ -Brownian motion. Consider the local martingale

$$L_t^{(k')} = \exp \left( \left( k'(\alpha_\nu) - \frac{1}{2} \right) \int_0^t \frac{\langle \alpha_\nu, dB_s^{(k_0)} \rangle}{\langle \alpha_\nu, X_s^W \rangle} - \frac{(k'(\alpha_\nu) - \frac{1}{2})^2}{2} \int_0^t \frac{ds}{\langle \alpha_\nu, X_s^W \rangle^2} \right),$$

for some coefficient function  $k'(\alpha)$  such that  $k'(\alpha_\nu) > \frac{1}{2}$  and  $k'(\alpha_i) = k_0(\alpha_i)$  for  $i \neq \nu$ . The Itô formula for  $\ln(\langle \alpha_\nu, X_t^W \rangle)$  combined with the orthogonality of the roots yields

$$L_t^{(k')} = \left( \frac{\langle \alpha_\nu, X_t^W \rangle}{\langle \alpha_\nu, x \rangle} \right)^{k'(\alpha_\nu) - \frac{1}{2}} \exp \left( - \frac{(k'(\alpha_\nu) - \frac{1}{2})^2}{2} \int_0^t \frac{ds}{\langle \alpha_\nu, X_s^W \rangle^2} \right).$$

Define the new law  $\mathbb{P}_x^{(k')} |_{\mathcal{F}_t} = L_t^{(k')} \cdot \mathbb{P}_x^{(k_0)} |_{\mathcal{F}_t}$ . By the Girsanov theorem,

$$B_t^{(k')} = B_t^{(k_0)} - \left( k'(\alpha_\nu) - \frac{1}{2} \right) \int_0^t \frac{ds}{\langle \alpha_\nu, X_s^W \rangle}$$

is a  $(\mathbb{P}_x^{(k')}, \mathcal{F}_t)$ -Brownian motion and hence,

$$X_t^W = x + B_t^{(k')} + \sum_{i=1}^l k'(\alpha_i) \int_0^t \frac{ds}{\langle \alpha_i, X_s^W \rangle} \alpha_i$$

is a radial Dunkl process under  $(P_x^{(k')}, \mathcal{F}_t)$ .

Define  $k$  as another coefficient on the root system that satisfies the conditions enunciated in the lemma. The absolute continuity relation is then a consequence of the successive application of the latter result to the indices  $k'$  and  $k$ .  $\square$

Now as an application of Proposition 5.2, we prove the following:

**Proposition 5.4.** *Let  $R_+ \equiv \{\alpha_1, \dots, \alpha_l\}$  be an orthogonal positive root system for  $l \leq n$ . Then the generalized Dunkl process  $(X_t, t \geq 0)$  generated by (5.7) enjoys the time-inversion property of degree 1.*

*Proof.* Using orthogonality of the roots, remark that

$$\langle \alpha_i, \sigma_j x \rangle^2 = \langle \alpha_i, x - \langle \alpha_j, x \rangle \alpha_j \rangle^2 = \langle \alpha_i, x \rangle^2,$$

which implies in particular

$$\langle \alpha_i, Y_t^i \rangle^2 = \langle \alpha_i, X_t^W \rangle^2,$$

so that the inductive representation of  $X_t$  in Proposition 5.2 becomes

$$X_t = \prod_{i=1}^l \sigma_{\alpha_i}^{N_{A_i^i}} X_t^W \quad \text{for} \quad A_t^i = \int_0^t \frac{ds}{\langle X_s^W, \alpha_i \rangle^2}.$$

The radial part of a Dunkl process enjoys the time-inversion property of degree 1. We need to show that the semigroup densities of the generalized Dunkl process are related to the semigroup densities of its radial parts. For  $f \in C^2(\mathbb{R}^n)$ ,

$$\mathbb{E}_x[f(Y_t^i)] = \mathbb{E}_x[f(Y_t^{i-1})1_{\{N_{A_i^i} \text{ is even}\}}] + \mathbb{E}_x[f(\sigma_{\alpha_i} Y_t^{i-1})1_{\{N_{A_i^i} \text{ is odd}\}}].$$

Since  $P(N_u^i \text{ is even}) = \frac{1}{2}(1 + \exp(-2\lambda(\alpha_i)u))$ , we obtain

$$\mathbb{E}_x[f(Y_t^i)] = \mathbb{E}_x \left[ f(Y_t^{i-1}) \frac{1}{2} (1 + \exp(-2\lambda(\alpha_i)A_t^i)) \right] + \mathbb{E}_x \left[ f(\sigma_{\alpha_i} Y_t^{i-1}) \frac{1}{2} (1 - \exp(-2\lambda(\alpha_i)A_t^i)) \right].$$

The expectation  $\mathbb{E}_x[f(X_t)]$  can thus be evaluated by induction on  $i \in \{1, \dots, l\}$ . It follows that the semigroup densities of  $X_t$  can be expressed as the product of the semigroup densities of its radial parts times a function involving expectations of the form

$$\mathbb{E}_x^{(k)} \left[ \exp(-2\lambda(\alpha_\nu)A_t^\nu) \middle| X_t^W = y \right],$$

for  $\nu \in \{1, \dots, l\}$ . From Lemma 5.3,

$$\mathbb{E}_x^{(k)} \left[ \exp(-2\lambda(\alpha_\nu)A_t^\nu) \middle| X_t^W = y \right] = \frac{p_t^{(k')}(x, y)}{p_t^{(k)}(x, y)} \left( \frac{\langle \alpha_\nu, y \rangle}{\langle \alpha_\nu, x \rangle} \right)^{k(\alpha_\nu) - k'(\alpha_\nu)},$$

where  $k'(\alpha_i) = k(\alpha_i)$  for  $i \neq \nu$  and  $k'(\alpha_\nu) = \frac{1}{2} + \sqrt{(k(\alpha_\nu) - \frac{1}{2})^2 + 4\lambda(\alpha_\nu)}$ . The form of the semigroup densities in (5.3) implies that the expectation is a ratio of Dunkl kernels,

$$\mathbb{E}_x^{(k)} \left[ \exp(-2\lambda(\alpha_\nu)A_t^\nu) \middle| X_t^W = y \right] = \frac{c_k}{c_{k'}} \frac{D_{k'} \left( \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}} \right)}{D_k \left( \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}} \right)} \frac{w_{k'} \left( \frac{y}{\sqrt{t}} \right)}{w_k \left( \frac{y}{\sqrt{t}} \right)} \left( \frac{\langle \alpha_\nu, \frac{y}{\sqrt{t}} \rangle}{\langle \alpha_\nu, \frac{x}{\sqrt{t}} \rangle} \right)^{k(\alpha_\nu) - k'(\alpha_\nu)},$$

which reduces to

$$\mathbb{E}_x^{(k)} \left[ \exp(-2\lambda(\alpha_\nu)A_t^\nu) \middle| X_t^W = y \right] = \frac{c_k}{c_{k'}} \frac{D_{k'} \left( \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}} \right)}{D_k \left( \frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}} \right)} \left( \langle \alpha_\nu, \frac{y}{\sqrt{t}} \rangle \langle \alpha_\nu, \frac{x}{\sqrt{t}} \rangle \right)^{k'(\alpha_\nu) - k(\alpha_\nu)},$$

by definition of  $k'(\alpha)$ . The conditional expectation thus satisfies condition 1 of Theorem 2.4. As a consequence, the conditions of Theorem 2.4 are satisfied for

$$\Phi(x, y) \equiv D_{k, \lambda}(x, y), \quad \theta(y) \equiv \omega_k(y), \quad \rho(x) \equiv -\frac{|x|^2}{2}$$

where  $D_{k, \lambda}(x, y)$  is a generalized Dunkl kernel given explicitly in terms of radial Dunkl kernels and satisfying equivalent conditions (see (5.4)).  $\square$

### 5.1.3 Jacobi-Dunkl processes

Gallardo et al. [5] derived the Jacobi-Dunkl process as the hyperbolic analog of the one-dimensional Dunkl process. It is defined as the process generated by

$$\mathcal{L}^{(\alpha, \beta)} f(x) = \frac{\partial^2 f(x)}{\partial x^2} + \frac{A'(x)}{A(x)} \frac{\partial f(x)}{\partial x} + \frac{\partial}{\partial x} \left( \frac{A'(x)}{A(x)} \right) \left( \frac{f(x) - f(-x)}{2} \right), \quad (5.14)$$

where  $A(x) = (\sinh^2(x))^{\alpha + \frac{1}{2}} (\cosh^2(x))^{\beta + \frac{1}{2}}$ . From the expression of the semigroup densities developed in [5], this process does not enjoy the time-inversion property. Its radial part however corresponds to the Jacobi process of index  $(\alpha, \beta)$  on  $\mathbb{R}_+$  (see [15, 16]). The infinitesimal generator of the Jacobi process, expressed by

$$\mathcal{L}^{(\alpha, \beta)} f(x) = \frac{1}{2} \frac{\partial^2 f(x)}{\partial x^2} + \frac{A'(x)}{A(x)} \frac{\partial f(x)}{\partial x}, \quad (5.15)$$

is in  $h$ -transform relationship with the Laplacian operator for  $h(x) = \sqrt{A(x)}$ . Since the Brownian motion enjoys the time-inversion property of degree 1, so does the Jacobi process by Theorem 2.4.

## 5.2 Matrix-valued processes

### 5.2.1 Eigenvalue processes

Dyson in [10] described the eigenvalues of a Hermitian Brownian motion as the joint evolution of independent Brownian motions conditioned never to collide (see also [14] and [4]). It was further remarked that the process version of the Gaussian orthogonal ensemble does not admit such a representation for its eigenvalues. This work was extended by König and O'Connell in [17] to the process version of the Laguerre ensemble, denominated the Laguerre process and defined as follows:

**Definition 5.5.** Let  $B_t$  be an  $n \times m$  matrix with independent standard complex Brownian entries. The Laguerre process is the matrix-valued process defined by  $\{X_t = B_t' B_t, t \geq 0\}$ , where  $B_t'$  is the transpose of  $B_t$ .

From [17], the eigenvalues of the Laguerre process evolve like  $m$  independent squared Bessel processes conditioned never to collide. No such representation however exists for the case where the entries of  $B_t$  are real Brownian motions, i.e. the Wishart process considered by Bru in [3].

The main result of [17] is that both of the above mentioned eigenvalues processes (in the complex Brownian case) can be obtained as the  $h$ -transform of processes with  $m$  independent components. The joint eigenvalues process is thus in  $h$ -transform relationship with a process that enjoys the time inversion property of degree 1 in the case of the  $m$ -dimensional Brownian motion and degree 2 in the case of the  $m$ -dimensional squared Bessel process, as made explicit in the following proposition:

**Proposition 5.6.** Let  $p_t(x_i, y_i)$  ( $i = 1, \dots, m$ ) be the semigroup densities of squared Bessel processes (respectively Brownian motions), and let

$$h(x) = \prod_{i < j}^m (x_j - x_i) \quad (5.16)$$

for  $x = (x_1, \dots, x_m)$ . Then, the semigroup densities of the joint eigenvalues process of the Laguerre process (respectively the Hermitian Brownian motion) are given by

$$\tilde{p}_t(x, y) = \frac{h(y)}{h(x)} \prod_{i=1}^m p_t(x_i, y_i) \quad (5.17)$$

with respect to the Lebesgue measure  $dy = \prod_{i=1}^m dy_i$ .

It follows immediately by Theorem 2.4 that the eigenvalues processes enjoy the time-inversion property. Moreover by Proposition 2.2, they yield the same process under time-inversion as the  $m$ -dimensional Brownian motion or the  $m$ -dimensional squared Bessel process respectively.

### 5.2.2 Wishart processes

The Wishart process  $\text{WIS}(\delta, tI_m, \frac{1}{t}x)$ , introduced by Bru in [2], is a continuous Markov process taking values in the space of real symmetric positive definite  $m \times m$  matrices  $S_m^+$ . It is solution to the following stochastic differential equation:

$$dX_t = \sqrt{X_t} dB_t + dB'_t \sqrt{X_t} + \delta I_m dt, \quad X_0 = x, \quad (5.18)$$

where  $B_t$  is an  $m \times m$  matrix with Brownian entries and  $I_m$  the identity matrix. Further results have been obtained in [1] and [3]. In [7], among other major findings about the Wishart process, the transition probability densities expressed with respect to the Lebesgue measure  $dy = \prod_{i \leq j} (dy_{ij})$

were derived in terms of generalized Bessel functions (we refer to the appendix for the definition):

$$p_t(x, y) = \frac{1}{(2t)^{\frac{m(m+1)}{2}}} \exp\left(-\frac{1}{2t} \text{Tr}(x+y)\right) \left(\frac{\det(y)}{\det(x)}\right)^{\frac{\delta-m-1}{4}} \tilde{\mathbf{I}}_{\frac{\delta-m-1}{2}}\left(\frac{xy}{4t^2}\right), \quad (5.19)$$

for  $x, y \in S_m^+$  and  $\delta > m - 1$ . From the shape of its densities, the Wishart process was stated in [12] as an example of Markov processes enjoying the time-inversion property of degree 2. The hypothesis of Theorem 2.4 is indeed satisfied for  $n = \frac{1}{2}m(m+1)$  and

$$\Phi(x, y) \equiv (\det(x) \det(y))^{-\frac{\delta-m-1}{4}} \tilde{\mathbf{I}}_{\frac{\delta-m-1}{2}}\left(\frac{xy}{4}\right), \quad \theta(y) \equiv \frac{1}{2^n} (\det(y))^{\frac{\delta-m-1}{2}}, \quad \rho(x) \equiv -\frac{1}{2} \text{Tr}(x). \quad (5.20)$$

Next we use a skew-product representation, as for the Dunkl process, to elaborate on the Wishart process and derive a matrix-valued process with jumps. The skew-product representation allows the expression of the semigroup densities in terms of the Wishart transition probability densities.

**Definition 5.7.** Let  $(N_t^{(\lambda)}, t \geq 0)$  be a Poisson process with intensity  $\lambda$ . Let  $(X_t, t \geq 0)$  be a Wishart process  $\text{WIS}(\delta, tI_m, \frac{1}{t}x)$  independent of the Poisson process. The skew-Wishart process  $(X_t^{(\lambda)}, t \geq 0)$  is defined through the skew-product

$$X_t^{(\lambda)} = X_t (-1)^{N_{A_t}^{(\lambda)}} \quad (5.21)$$

where  $A_t = \int_0^t \text{Tr}(X_s^{-1}) ds$ .

**Proposition 5.8.** The transition probability densities of the skew-Wishart process are related to the semigroup densities  $p_t(x, y)$  of the Wishart process  $X_t$  as follows

$$p_t^{(\lambda)}(x, y) = p_t(x, |y|) \left\{ 1_{\{y \in S_m^+\}} \frac{1}{2} \left( 1 + \left( \frac{\tilde{\mathbf{I}}_{\nu'}}{\tilde{\mathbf{I}}_{\nu}} \right) \left( \frac{xy}{4t^2} \right) \right) + 1_{\{y \in S_m^-\}} \frac{1}{2} \left( 1 - \left( \frac{\tilde{\mathbf{I}}_{\nu'}}{\tilde{\mathbf{I}}_{\nu}} \right) \left( \frac{-xy}{4t^2} \right) \right) \right\}. \quad (5.22)$$

for  $\nu = \frac{\delta-m-1}{2}$ ,  $\nu' = \sqrt{\nu^2 + 4\lambda}$  and  $|y| = y(1_{\{y \in S_m^+\}} - 1_{\{y \in S_m^-\}})$ .



*Proof.* Let  $(\mathbf{P}_t)_{t>0}$  be the semigroup of the Dunkl-Wishart process. For  $x > 0$  and  $f \in C_c(M_m(\mathbb{R}))$ ,

$$\mathbf{P}_t f(x) = \mathbf{E}_x \left[ f(X_t^{(\lambda)}) \right] = \mathbf{E}_x \left[ f(X_t) 1_{\{N_{A_t}^{(\lambda)} \text{ is even}\}} \right] + \mathbf{E}_x \left[ f(-X_t) 1_{\{N_{A_t}^{(\lambda)} \text{ is odd}\}} \right].$$

With  $\mathbf{P}(N_u^{(\lambda)} \text{ is even}) = \frac{1}{2}(1 + \exp(-2\lambda u))$ , we have

$$\mathbf{P}_t f(x) = \mathbf{E}_x \left[ f(X_t) \frac{1}{2}(1 + \exp(-2\lambda A_t)) \right] + \mathbf{E}_x \left[ f(-X_t) \frac{1}{2}(1 - \exp(-2\lambda A_t)) \right]. \quad (5.23)$$

Let  $\mathbf{Q}_x^{(\nu')}$  with  $\nu' = \frac{\delta' - m - 1}{2}$  denote the probability law of a Wishart process  $\text{WIS}(\delta', tI, \frac{1}{t}x)$ , and  $\mathbf{Q}_x^{(\nu)}$  with  $\nu = \frac{\delta - m - 1}{2}$  the probability law of  $X_t$ . According to Theorem 1.2 (Remark 2.3) in [7], the probability laws are related as follows:

$$\mathbf{Q}_x^{(\nu')}|_{\mathcal{F}_t} = \left( \frac{\det X_t}{\det x} \right)^{\frac{\nu' - \nu}{2}} \exp \left( -\frac{\nu'^2 - \nu^2}{2} \int_0^t \text{Tr}(X_s^{-1}) ds \right) \cdot \mathbf{Q}_x^{(\nu)}|_{\mathcal{F}_t},$$

from which we deduce

$$\frac{\mathbf{p}_t^{(\nu')}(x, y)}{\mathbf{p}_t^{(\nu)}(x, y)} = \left( \frac{\det y}{\det x} \right)^{\frac{\nu' - \nu}{2}} \mathbf{Q}_x^{(\nu)} \left[ \exp \left( -\frac{\nu'^2 - \nu^2}{2} \int_0^t \text{Tr}(X_s^{-1}) ds \right) \middle| X_t = y \right].$$

Thus, from the expression of the semigroup densities in (5.19), we have

$$\mathbf{E}_x^{(\nu)} \left[ \exp \left( -2\lambda \int_0^t \text{Tr}(X_s^{-1}) ds \right) \middle| X_t = y \right] = \left( \frac{\tilde{\mathbf{I}}_{\sqrt{\nu^2 + 4\lambda}}}{\tilde{\mathbf{I}}_\nu} \right) \left( \frac{xy}{4t^2} \right).$$

Combining the latter with (5.23) yields the semigroup densities for the skew-Wishart process.  $\square$

The skew-Wishart is an example of matrix-valued process with jumps that enjoys the time-inversion property of degree 2. Indeed, by setting

$$\begin{aligned} \Phi(x, y) &\equiv (\det(x) \det(|y|))^{-\frac{\nu}{2}} \left\{ 1_{\{y \in S_m^+\}} \frac{1}{2} \left( \tilde{\mathbf{I}}_\nu + \tilde{\mathbf{I}}_{\nu'} \right) \left( \frac{xy}{4} \right) + 1_{\{y \in S_m^-\}} \frac{1}{2} \left( \tilde{\mathbf{I}}_\nu - \tilde{\mathbf{I}}_{\nu'} \right) \left( \frac{-xy}{4} \right) \right\}, \\ \theta(y) &\equiv \frac{1}{2^n} (\det(|y|))^\nu, \quad \rho(x) \equiv -\frac{1}{2} \text{Tr}(|x|), \end{aligned} \quad (5.24)$$

the conditions of Theorem 2.4 are satisfied for  $\alpha = 2$ .

# A Generalized hypergeometric functions

Using the notation in Muirhead [20], hypergeometric functions of matrix arguments are defined for a real symmetric  $m \times m$  matrix  $X$ ,  $a_i \in \mathbb{C}$  and  $b_j \in \mathbb{C} \setminus \{0, \frac{1}{2}, 1, \dots, \frac{m-1}{2}\}$  by

$${}_p\mathbf{F}_q(a_1, \dots, a_p; b_1, \dots, b_q; X) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_{\kappa} \cdots (a_p)_{\kappa}}{(b_1)_{\kappa} \cdots (b_q)_{\kappa}} \frac{C_{\kappa}(X)}{k!} \quad (\text{A.1})$$

where the second summation is over all partitions  $\kappa = (k_1, \dots, k_m)$ ,  $k_1 \geq \dots \geq k_m \geq 0$ , of  $k = \sum_{i=1}^m k_i$ ,  $k! = k_1! \cdots k_m!$  and the generalized Pochhammer symbols are given by

$$(a)_{\kappa} = \prod_{i=1}^m \left( a - \frac{i-1}{2} \right)_{k_i}, \quad (a)_k = a(a+1) \cdots (a+k-1), \quad (a)_0 = 1.$$

$C_{\kappa}(X)$  is the zonal polynomial corresponding to  $\kappa$ , which is a symmetric, homogeneous polynomial of degree  $k$  in the eigenvalues of  $X$  that satisfies

$$C_{\kappa}(YX) = C_{\kappa}(\sqrt{Y}X\sqrt{Y}) \quad (\text{A.2})$$

for some  $Y \in S_m^+$ . The function  ${}_p\mathbf{F}_q(a_1, \dots, a_p; b_1, \dots, b_q; YX)$  thus makes sense. Finally, we define the generalized modified Bessel function by

$$\tilde{\mathbf{I}}_{\nu}(X) = \frac{(\det(X))^{\frac{\nu}{2}}}{\mathbf{\Gamma}_m\left(\nu + \frac{m+1}{2}\right)} {}_0\mathbf{F}_1\left(\nu + \frac{m+1}{2}; X\right) \quad (\text{A.3})$$

where the generalized gamma function is given as a product of the usual gamma functions,

$$\mathbf{\Gamma}_m(\alpha) = \pi^{\frac{m(m-1)}{4}} \prod_{i=1}^m \Gamma\left(\alpha - \frac{i-1}{2}\right) \quad (\text{A.4})$$

for  $\text{Re}(\alpha) > \frac{m-1}{2}$ . Note that the generalized modified Bessel for  $m = 1$  relates to the usual modified Bessel function  $I_{\nu}(x)$  by  $\tilde{\mathbf{I}}_{\nu}(x) = I_{\nu}(2\sqrt{x})$  (see [19]).

## References

- [1] M.F. Bru. Diffusions of perturbed principal component analysis. *J. Multivariate Anal.*, 29:127–136, 1989.
- [2] M.F. Bru. Processus de Wishart. *C.R. Acad. Sci. Paris, Série I*, t.308:29–32, 1989.
- [3] M.F. Bru. Wishart processes. *J. Theo. Probab.*, 4:725–751, 1991.
- [4] E. Cépa and D. Lépingle. Brownian particles with electrostatic repulsion on the circle: Dyson’s model for unitary random matrices revisited. *ESAIM Probability and Statistics*, 5:203–224, 2001.
- [5] F. Chouchene, L. Gallardo, and M. Mili. The heat semigroup for the Jacobi-Dunkl operator and applications. *Preprint*, 2005.
- [6] O. Chybiryakov. Skew-product representations of Dunkl Markov processes. *Preprint*, 2005.
- [7] C. Donati-Martin, Y. Doumerc, H. Matsumoto, and M. Yor. Some properties of the Wishart processes and a matrix extension of the Hartman-Watson laws. *Publ. RIMS, Kyoto Univ.*, 40:1385–1412, 2004.

- [8] C. Dunkl. Differential-difference operators associated to reflection groups. *Trans. Amer. Math. Soc.*, 311(1):167–183, 1989.
- [9] C. Dunkl. Hankel transforms associated to finite reflection groups. *Contemp. Math.*, 138:123–138, 1992.
- [10] F.J. Dyson. A Brownian-motion model for the eigenvalues of a random matrix. *J. Math. Phys.*, 3:1191–1198, 1962.
- [11] L. Gallardo and M. Yor. A chaotic representation property of the multidimensional Dunkl processes. *To appear in Ann. Prob.*, 2005.
- [12] L. Gallardo and M. Yor. Some new examples of Markov processes which enjoy the time-inversion property. *Probab. Theory Relat. Fields*, 132:150–162, 2005.
- [13] L. Gallardo and M. Yor. Some remarkable properties of the Dunkl martingales. *To appear in Sém. Proba. 39, dedicated to Paul André Meyer*, Springer LNM 2005.
- [14] D. Grabiner. Brownian motion in a Weyl chamber, non-colliding particles, and random matrices. *Annales de l'I. H. P. Probabilités et Statistiques*, 35:177–204, 1999.
- [15] J.C. Gruet. Windings of hyperbolic Brownian motion, in Exponential functionals and principal values related to Brownian motion. *Bibl. Rev. Math. Ibroamericana, Rev. Math. Ibroamericana, Madrid*, pages 35–72, 1997.
- [16] J.C. Gruet. Jacobi radial stable processes. *Ann. Math. Blaise Pascal*, 5:39–48, 1998.
- [17] W. König and N. O'Connell. Eigenvalues of the Laguerre process as non-colliding squared Bessel processes. *Elect. Comm. in Probab.*, 6:107–114, 2001.
- [18] J. Lamperti. Semi-stable Markov processes I. *Zeit. für Wahr.*, 22:205–255, 1972.
- [19] N.N. Lebedev. *Special Functions and Their Applications*. Dover, New York, 1972.
- [20] R.J. Muirhead. *Aspects of Multivariate Statistic Theory*. Wiley, New York, 1982.
- [21] J. Pitman and M. Yor. Bessel processes and infinitely divisible laws. *In: Stochastic Integrals. ed. D. Williams. Lect. Notes in Math*, 851, 1981.
- [22] M. Rösler. Generalized Hermite polynomials and the heat equations for Dunkl operators. *Comm. Math. Phys.*, 192:519–542, 1998.
- [23] M. Rösler and M. Voit. Markov processes related with Dunkl operators. *Adv. in App. Math.*, 21(4):575–643, 1998.
- [24] S. Watanabe. On time inversion of one-dimensional diffusion processes. *Z. Wahrsch. Verw. Gebiete*, 31:115–124, 1975.