Symplectic aspects of Aubry-Mather theory
Patrick Bernard

Symplectic aspects of Aubry-Mather theory [1]

Patrick Bernard [2]

Résumé : On montre que les ensembles d'Aubry et de Mañé introduits par Mather en dynamique Lagrangienne sont des invariants symplectiques. On introduit pour ceci une barriere dans l'espace des phases. Ceci est aussi l'occasion d'ébaucher une théorie d'Aubry-Mather pour des Hamiltoniens non convexes.

Abstract : We prove that the Aubry and Mañé sets introduced by Mather in Lagrangian dynamics are symplectic invariants. In order to do so, we introduce a barrier on phase space. This is also an occasion to suggest an Aubry Mather theory for non convex Hamiltonians.

In Lagrangian dynamics, John Mather has defined several invariant sets, now called the Mather set, the Aubry set, and the Mañé set. These invariant sets provide obstructions to the existence of orbits wandering in phase space. Conversely, the existence of interesting orbits have been proved under some assumptions on the topology of these sets. Such results were first obtained by John Mather in [11], and then in several papers, see [1, 3, 4, 5, 16, 17] as well as recent unpublished works of John Mather.

In order to apply these results on examples one has to understand the topology of the Aubry and Mañé set, which is a very difficult task. In many perturbative situations, averaging methods appear as a promising tool in that direction. In order to use these methods, one has to understand how the averaging transformations modify the Aubry-Mather sets. In the present paper, we answer this question and prove that the Mather set, the Aubry set and the Mañé set are symplectic invariants.

In order to do so, we define a barrier on phase space, which is some symplectic analogue of the function called the Peierl's barrier by Mather in [11]. We then propose definitions of Aubry and Mañé sets for general Hamiltonian systems. We hope that these definitions may also serve as the starting point of an Aubry-Mather theory for some classes of non-convex Hamiltonians. We develop the first steps of such a theory.

Several anterior works gave hints towards the symplectic nature of Aubry-Mather theory, see [2, 13, 14, 15] for example. These works prove the symplectic invariance of the α function of Mather, and one may consider that the symplectic invariance of the Aubry set is not a surprising result after them. However, the symplectic invariance of the Mañé set is, to my point of view, somewhat unexpected. It is possible that the geometric methods introduced in [13] may also be used to obtain symplectic definitions of the Aubry and Mañé set.

[1] June 1, 2005

[2] Institut Fourier, BP 74, 38402, Saint Martin d'Hères cedex, France, patrick.bernard@ujf-grenoble.fr, http://www-fourier.ujf-grenoble.fr/~pbernard/

1 Mather theory in Lagrangian dynamics

We recall the basics of Mather theory and state our main result, Theorem 1.10. The original references for most of the material presented in this section are Mather's papers [10] and [11]. The central object is the Peierl's barrier, introduced by Mather in [11]. Our presentation is also influenced by the work of Fathi [7].

1.1   In this section, we consider a C^2 Hamiltonian function $H : T^*M \times \mathbb{T} \longrightarrow \mathbb{R}$, where M is a compact connected manifold without boundary, and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We denote by $P = (q, p)$ the points of T^*M. The cotangent bundle is endowed with its canonical one-form $\eta = pdq$, and with its canonical symplectic form $\omega = -d\eta$. Following a very standard device, we reduce our non-autonomous Hamiltonian function H to an autonomous one by considering the extended phase space $T^*(M \times \mathbb{T}) = T^*M \times T^*\mathbb{T}$. We denote by (P, t, E), $P \in T^*M$, $(t, E) \in T^*\mathbb{T}$ the points of this space. We consider the canonical one-form $\lambda = pdq + Edt$ and the associated symplectic form $\Omega = -d\lambda$. We define the new Hamiltonian $G : T^*(M \times \mathbb{T}) \longrightarrow \mathbb{R}$ be the expression

$$G(P, t, E) = E + H(P, t).$$

We denote by $V_G(P, t, E)$ the Hamiltonian vector-field of G, which is defined by the relation

$$\Omega_{(P,t,E)}(V_G, .) = dG_{(P,t,E)}.$$

We fix once and for all a Riemannian metric on M, and use it to define norms of tangent vectors and tangent covectors of M. We will denote this norm indifferently by $|P|$ or by $|p|$ when $P = (q, p) \in T_q^*M$. We denote by π the canonical projections $T^*M \longrightarrow M$ or $T^*(M \times \mathbb{T}) \longrightarrow M \times \mathbb{T}$. The theory of Mather relies on the following standard set of hypotheses.

1. COMPLETENESS.   The Hamiltonian vector-field V_G on $T^*(M \times \mathbb{T})$ generates a complete flow, denoted by Φ_t. The flow Φ_t preserves the level sets of G.

2. CONVEXITY.   For each $(q, t) \in M \times \mathbb{T}$, the function $p \longmapsto H(q, p, t)$ is convex on T_q^*M, with positive definite Hessian. Shortly, $\partial_p^2 H > 0$.

3. SUPER-LINEARITY.   For each $(q, t) \in M \times \mathbb{T}$, the function $p \longmapsto H(q, p, t)$ is super-linear, which means that $\lim_{|p| \longrightarrow \infty} H(t, x, p)/|p| = \infty$.

1.2   We associate to the Hamiltonian H a Lagrangian function $L : TM \times \mathbb{T} \longrightarrow \mathbb{R}$ defined by

$$L(t, q, v) = \sup_{p \in T_q^*M} p(v) - H(t, q, p).$$

The Lagrangian satisfies:

1. CONVEXITY.   For each $(q, t) \in M \times \mathbb{T}$, the function $v \longmapsto L(q, v, t)$ is a convex function on T_qM, with positive definite Hessian. Shortly, $\partial_v^2 L > 0$.

2. SUPER-LINEARITY.   For each $(q, t) \in M \times \mathbb{T}$, the function $v \longmapsto L(q, v, t)$ is super-linear on T_qM.

Let $X(t) = (P(t), s + t, E(t))$ be a Hamiltonian orbit of G, and let $q(t) = \pi(P(t))$. Then we have the identities

$$\lambda_{X(t)}(\dot{X}(t)) - G(X(t)) = \eta_{P(t)}(\dot{P}(t)) - H(P(t), s + t) = L(q(t), \dot{q}(t), s + t).$$

2

1.3 Following John Mather, we define the function $F : M \times \mathbb{T} \times M \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ by

$$F(q_0, t; q_1, s) = \min_\gamma \int_0^s L(\gamma(\sigma), \dot{\gamma}(\sigma), t + \sigma) d\sigma,$$

where the minimum is taken on the set of absolutely continuous curves $\gamma : [0, s] \longrightarrow M$ which satisfy $\gamma(0) = q_0$ and $\gamma(1) = q_1$. We also define the Peierl's barrier $h : M \times \mathbb{T} \times M \times \mathbb{T} \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ by

$$h(q_0, t_0; q_1, t_1) := \liminf_{n \in \mathbb{N}} F(q_0, t_0; q_1, s_1 + n),$$

where $t_0 + s_1 \bmod 1 = t_1$. This barrier is the central object in Mather's study of globally minimizing orbits.

1.4 Let us set $m(H) = \inf_{(q,t) \in M \times \mathbb{T}} h(q, t; q, t)$. It follows from [10], see also [12], that $m(H) \in \{-\infty, 0, +\infty\}$. In addition, for each Hamiltonian H satisfying the hypotheses 1.1, there exists one and only one real number $\alpha(H)$ such that $m(H - \alpha(H)) = 0$. As a consequence, there is no loss of generality in assuming that $m(H) = 0$, or equivalently that $\alpha(H) = 0$. We will make this assumption from now on in this section. Let us mention the terminology of Mañé, who called super-critical the Hamiltonians H satisfying $m(H) = +\infty$, sub-critical the Hamiltonians satisfying $m(H) = -\infty$, and critical the Hamiltonians satisfying $m(H) = 0$.

1.5 If $m(H) = 0$, the function h is a real valued Lipschitz function on $M \times \mathbb{T} \times M \times \mathbb{T}$, which satisfies the triangle inequality

$$h(q_0, t_0; q_2, t_2) \leqslant h(q_0, t_0; q_1, t_1) + h(q_1, t_1; q_2, t_2)$$

for all (q_0, t_0), (q_1, t_1) and (q_2, t_2) in $M \times \mathbb{T}$. In addition, for each $(q, t) \in M \times \mathbb{T}$, the function $h(q, t; ., .)$ is a weak KAM solution in the sense of Fathi, which means that, for $\tau \geqslant \theta$ in $\mathbb{R}$, and $x \in M$, we have

$$h(q, t; x, \tau \bmod 1) = \min \left(h(q, t; q(\theta), \theta \bmod 1) + \int_\theta^\tau L(q(s), \dot{q}(s), s) ds \right)$$

where the minimum is taken on the set of absolutely continuous curves $q(s) : [\theta, \tau] \longrightarrow M$ such that $q(\tau) = x$. Similarly, we have, for $\tau \geqslant \theta$ in $\mathbb{R}$, and $x \in M$,

$$h(x, \theta \bmod 1; q, t) = \min \left(h(q(\tau), \tau \bmod 1; q, t) + \int_\theta^\tau L(q(s), \dot{q}(s), s) ds \right)$$

where the minimum is taken on the set of absolutely continuous curves $q(s) : [\theta, \tau] \longrightarrow M$ such that $q(\theta) = x$.

1.6 The projected Aubry set $\mathcal{A}(H)$ is the set of points $(q, t) \in M \times \mathbb{T}$ such that $h(q, t; q, t) = 0$. Albert Fathi proved that, for each point $(q, t) \in \mathcal{A}(H)$, the function $h(q, t; ., .)$ is differentiable at (q, t). Let us denote by $X(q, t)$ the differential $\partial_3 h(q, t; q, t) \in T_q^* M$ of the function $h(q, t; ., t)$ at point q. The Aubry set $\tilde{\mathcal{A}}(H)$ is defined as

$$\tilde{\mathcal{A}}(H) = \{(X(q, t), t, -H(X(q, t), t)); (q, t) \in \mathcal{A}(H)\} \subset T^*(M \times \mathbb{T}).$$

The Aubry set is compact, Φ-invariant, and it is a Lipschitz graph over the projected Aubry set $\mathcal{A}(H)$. These are results of John Mather, see [11]. In our presentation, which follows Fathi, this amounts to say that the function $(q, t) \longmapsto X(q, t)$ is Lipschitz on $\mathcal{A}(H)$.

1.7   The Mather set $\tilde{\mathcal{M}}(H)$ is defined as the union of the supports of all Φ-invariant probability measures on $T^*(M \times \mathbb{T})$ concentrated on $\tilde{\mathcal{A}}(H)$. This set was first defined by Mather, but our definition is due to Mañé.

1.8   The projected Mañé set $\mathcal{N}(H)$ is the set of points $(q,t) \in M \times \mathbb{T}$ such that there exist points (q_0, t_0) and (q_1, t_1) in $\mathcal{A}(H)$, satisfying

$$h(q_0, t_0; q_1, t_1) = h(q_0, t_0; q, t) + h(q, t; q_1, t_1).$$

Let us denote by $\mathcal{I}(q_0, t_0; q_1, t_1)$ the set of points $(q,t) \in M \times \mathbb{T}$ which satisfy this relation. If $(q_0, t_0) \in \mathcal{A}(H)$ and $(q_1, t_1) \in \mathcal{A}(H)$ are given, and if $(q,t) \in \mathcal{I}(q_0, t_0; q_1, t_1)$, then the function $h(q_0, t_0; ., t)$ is differentiable at q, as well as the function $h(., t; q_1, t_1)$, and $\partial_3 h(q_0, t_0, q, t) + \partial_1 h(q, t; q_1, t_1) = 0$. This is proved in [3] following ideas of Albert Fathi. We define

$$\tilde{\mathcal{I}}(q_0, t_0; q_1, t_1) := \left\{ \left(\partial_3 h(q_0, t_0, q, t), t, -H\big(\partial_3 h(q_0, t_0, q, t), t\big)\right), (q,t) \in \mathcal{I}(q_0, t_0; q_1, t_1)\right\}.$$

The set $\tilde{\mathcal{I}}(q_0, t_0; q_1, t_1)$ is a compact Φ-invariant subset of $T^*(M \times \mathbb{T})$, and it is a Lipschitz Graph. The Mañé set $\tilde{\mathcal{N}}(H)$ is the set

$$\tilde{\mathcal{N}}(H) = \bigcup_{(q_0, t_0), (q_1, t_1) \in \mathcal{A}(H)} \tilde{\mathcal{I}}(q_0, t_0; q_1, t_1) \subset T^*(M \times \mathbb{T}).$$

The Mañé set was first introduced by Mather in [11], it is compact and Φ-invariant, and it contains the Aubry set. In other words, we have the important inclusions

$$\tilde{\mathcal{M}}(H) \subset \tilde{\mathcal{A}}(H) \subset \tilde{\mathcal{N}}(H).$$

The Mañé set is usually not a graph. However, it satisfies

$$\tilde{\mathcal{N}}(H) \cap \pi^{-1}\big(\mathcal{A}(H)\big) = \tilde{\mathcal{A}}(H).$$

This follows from the fact, proved by Albert Fathi, that, for each $(x, \theta) \in M \times \mathbb{T}$ and each $(q,t) \in \mathcal{A}(H)$, the function $h(x, \theta; ., t)$ is differentiable at q and satisfies $\partial_3 h(x, \theta; q, t) = X(q,t)$.

1.9   Mather introduced the function $d(q, t; q', t') = h(q, t; q', t') + h(q', t'; q, t)$ on $M \times \mathbb{T}$. When restricted to $\mathcal{A}(H) \times \mathcal{A}(H)$, it is a pseudo-metric. This means that this function is symmetric, non-negative, satisfies the triangle inequality, and $d(q, t; q, t) = 0$ for $(q,t) \in \mathcal{A}(H)$. We shall also denote by d the pseudo-metric $d(P, t, -H(P, t); P', t', -H(P', t')) = d(\pi(P), t; \pi(P'), t')$ on $\tilde{\mathcal{A}}(H)$. The relation $d(P, t, E; P', t', E') = 0$ is an equivalence relation on $\tilde{\mathcal{A}}(H)$. The classes of equivalence are called the static classes. Let us denote by $\dot{\mathcal{A}}(H)$ the set of static classes. The pseudo-metric d gives rise to a metric $\dot{d}$ on $\dot{\mathcal{A}}(H)$. The compact metric space $(\dot{\mathcal{A}}(H), \dot{d})$ is called the quotient Aubry set. It was introduced by John Mather.

1.10   The diffeomorphism $\Psi : T^*(M \times \mathbb{T}) \longrightarrow T^*(M \times \mathbb{T})$ is called exact if the form $\Psi^* \lambda - \lambda$ is exact.

THEOREM *Let H be a Hamiltonian satisfying the hypotheses 1.1, and let $\Psi : T^*(M \times \mathbb{T}) \longrightarrow T^*(M \times \mathbb{T})$ be an exact diffeomorphism such that the Hamiltonian*

$$\Psi^* H := G \circ \Psi(P, t, E) - E$$

4

is independent of E and satisfies the hypotheses 1.1 when considered as a function on $T^*M \times \mathbb{T}$. Then $m(\Psi^*H) = m(H)$ hence $\alpha(H) = \alpha(\Psi^*H)$. If $m(H) = 0$, then we have

$$\Psi(\tilde{\mathcal{M}}(\Psi^*H)) = \tilde{\mathcal{M}}(H), \quad \Psi(\tilde{\mathcal{A}}(\Psi^*H)) = \tilde{\mathcal{A}}(H), \quad \Psi(\tilde{\mathcal{N}}(\Psi^*H)) = \tilde{\mathcal{N}}(H).$$

In addition, Ψ sends the static classes of Ψ^*H onto the static classes of H, and the induced mapping

$$\dot{\Psi} : \dot{\mathcal{A}}(\Psi^*H) \longrightarrow \dot{\mathcal{A}}(H)$$

is an isometry for the quotient metrics.

1.11 We prove this result in the sequel. In section 2, we set the basis of a symplectic Aubry-Mather theory for general Hamiltonian systems. We prove that the analogue of Theorem 1.10 holds in this general setting. We also continue the theory a bit further than would be necessary to prove Theorem 1.10. In section 3, we prove that, under the hypotheses of Theorem 1.10, the symplectic Aubry-Mather sets coincide with the standard Aubry-Mather sets, which ends the proof of Theorem 1.10.

2   A barrier in phase space

We propose general definitions for a Mather theory of Hamiltonian systems. Of course, the definitions given below provide relevant objects only for some specific Hamiltonian systems. It would certainly be interesting to give natural conditions on H implying non-triviality of the theory developed in this section. We shall only check, in the next section, that our definitions coincide with the standard ones in the convex case, obtaining non-triviality in this special case. Let us mention once again that it might be possible and interesting to find more geometric definition using the methods of [13].

2.1 In this section, we work in a very general setting. We consider a manifold N, not necessarily compact, and an autonomous Hamiltonian function $G : T^*N \longrightarrow \mathbb{R}$. We assume that G generates a complete Hamiltonian flow Φ_t. We make no convexity assumption. We denote by λ the canonical one-form of T^*N, and by $V_G(P)$ the Hamiltonian vector-field of G. Let $D(P, P')$ be a distance on T^*N induced from a Riemannian metric. We identify N with the zero section of T^*N, so that D is also a distance on N. We assume that $D(\pi(X), \pi(X')) \leqslant D(X, X')$ for X and X' in T^*N.

2.2 Let X_0 and X_1 be two points of T^*N. A pre-orbit between X_0 and X_1 is the data of a sequence $\underline{Y} = (Y_n)$ of curves $Y_n(s) : [0, T_n] \longrightarrow T^*N$ such that:

1. For each n, the curve Y_n has a finite number N_n of discontinuity points $T_n^i \in]0, T_n[, 1 \leqslant i \leqslant N_n$ such that $T_n^{i+1} > T_n^i$. We shall also often use the notations $T_n^0 = 0$ and $T_n^{N_n+1} = T_n$.

2. The curve Y_n satisfies $Y_n(T_n^i + s) = \Phi_s(Y_n(T_n^i))$ for each $s \in [0, T_n^{i+1} - T_n^i[$. We denote by $Y_n(T_n^i-)$ the point $\Phi_{T_n^i - T_n^{i-1}}(Y(T_n^{i-1}))$ and impose that $Y_n(T_n) = Y_n(T_n-)$.

3. We have $T_n \longrightarrow \infty$ as $n \longrightarrow \infty$.

4. We have $Y_n(0) \longrightarrow X_0$ and $Y_n(T_n) \longrightarrow X_1$. In addition, we have $\lim_{n \rightarrow \infty} \Delta(Y_n) = 0$, where we denote by $\Delta(Y_n)$ the sum $\sum_{i=1}^{N_n} D(Y_n(T_n^i-), Y_n(T_n^i))$.

5. There exists a compact subset $K \subset T^*N$ which contains the images of all the curves Y_n.

The pre-orbits do not depend on the metric which has been used to define the distance D. In a standard way, we call action of the curve $Y_n(t)$ the value

$$A(Y_n) = \int_0^{T_n} \lambda_{Y_n(t)}(\dot{Y}_n(t)) - G(Y_n(t)) \, dt.$$

The action of the pre-orbit $\underline{Y}$ is

$$A(\underline{Y}) := \liminf_{n \longrightarrow \infty} A(Y_n).$$

2.3 LEMMA *If there exists a pre-orbit between X_0 and X_1, then $G(X_0) = G(X_1)$.*

PROOF. This follows easily from the fact that the Hamiltonian flow Φ preserves the Hamiltonian function G. $\qquad\square$

2.4 We define the barrier $\tilde{h} : T^*M \times T^*M \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ by the expression

$$\tilde{h}(X_0, X_1) = \inf_{\underline{Y}} A(\underline{Y})$$

where the infimum is taken on the set of pre-orbits between X_0 and X_1. As usual, we set $\tilde{h}(X_0, X_1) = +\infty$ if there does not exist any pre-orbit between X_0 and X_1. If $\tilde{h}(X_0, X_1) < +\infty$, then the forward orbit of X_0 and the backward orbit of X_1 are bounded. As a consequence, if $\tilde{h}(X, X) < +\infty$, then the orbit of X is bounded.

2.5 PROPERTY *For each $t > 0$, we have the equality*

$$\tilde{h}(X_0, X_1) = \tilde{h}(\Phi_t(X_0), X_1) + \int_0^t \lambda_{\Phi_s(X_0)}\big(V_G(\Phi_s(X_0))\big) - G(\Phi_s(X_0)) \, ds$$

and

$$\tilde{h}(X_0, \Phi_t(X_1)) = \tilde{h}(X_0, X_1) + \int_0^t \lambda_{\Phi_s(X_1)}\big(V_G(\Phi_s(X_1))\big) - G(\Phi_s(X_1)) \, ds$$

PROOF. We shall prove the first equality, the proof of the second one is similar. To each pre-orbit $\underline{Y}$ between X_0 and X_1, we associate the pre-orbit $\underline{Z}$ between $\Phi_t(X_0)$ and X_1 defined by $Z_n(s) : [0, T_n - t] \longmapsto Y_n(s + t)$. We have

$$A(\underline{Y}) = A(\underline{Z}) + \int_0^t \lambda_{\Phi_s(X_0)}\big(V_G(\Phi_s(X_0))\big) - G(\Phi_s(X_0)) \, ds$$

This implies that

$$\tilde{h}(\Phi_t(X_0), X_1) \leqslant \tilde{h}(X_0, X_1) - \int_0^t \lambda_{\Phi_s(X_0)}\big(V_G(\Phi_s(X_0))\big) - G(\Phi_s(X_0)) \, ds.$$

In a similar way, we associate to each pre-orbit $\underline{Z} = Z_n(s) : [0, T_n] \longrightarrow T^*M$ between $\Phi_t(X_0)$ and X_1 the pre-orbits $\underline{Y} : [0, T_n + t] \longrightarrow T^*M$ between X_0 and X_1 defined by $Y_n(s) = \Phi_{s-t}(Z_n(0))$ for $s \in [0, t]$ and $Y_n(s) = Z_n(s - t)$ for $s \in [t, T_n + t]$. We have

$$A(\underline{Y}) = A(\underline{Z}) + \int_0^t \lambda_{\Phi_s(X_0)}\big(V_G(\Phi_s(X_0))\big) - G(\Phi_s(X_0)) \, ds.$$

This implies that

$$\tilde{h}(X_0, X_1) \leqslant \tilde{h}(\Phi_t(X_0), X_1) + \int_0^t \lambda_{\Phi_s(X_0)}\big(V_G(\Phi_s(X_0))\big) - G(\Phi_s(X_0))\, ds.$$

□

2.6   PROPERTY   *The function $\tilde{h}$ satisfies the triangle inequality. More precisely, the relation*

$$\tilde{h}(X_1, X_3) \leqslant \tilde{h}(X_1, X_2) + \tilde{h}(X_2, X_3)$$

holds for each points X_1, X_2 and X_3 such that the right hand side has a meaning.

PROOF. If one of the values $\tilde{h}(X_1, X_2)$ or $\tilde{h}(X_2, X_3)$ is $+\infty$, then there is nothing to prove. If they are both different from $+\infty$, then, for each $\epsilon > 0$ there exists a pre-orbits $\underline{Y} = Y_n$: $[0, T_n] \longrightarrow T^*N$ between X_1 and X_2 such that $A(\underline{Y}) \leqslant \tilde{h}(X_1, X_2) + \epsilon$ (resp. $A(\underline{Y}) \leqslant -1/\epsilon$ in the case where $\tilde{h}(X_1, X_2) = -\infty$) and a pre-orbits $\underline{Y}' = Y_n' : [0, S_n] \longrightarrow T^*N$ between X_2 and X_3 such that $A(\underline{Y}') \leqslant \tilde{h}(X_2, X_3) + \epsilon$ (resp. $A(\underline{Y}') \leqslant -1/\epsilon$ in the case where $\tilde{h}(X_1, X_2) = -\infty$). Let us consider the sequence of curves $Z_n(t) : [0, T_n + S_n] \longrightarrow T^*N$ such that $Z_n = X_n$ on $[0, T_n[$ and $Z_n(t + T_n) = Y_n(t)$ for $t \in [0, S_n]$. It is clear that the sequence $\underline{Z} = Z_n$ is a pre-orbit between X_1 and X_3, and that its action satisfies

$$A(\underline{Z}) = A(\underline{X}) + A(\underline{Y}) \leqslant \tilde{h}(X_1, X_2) + \tilde{h}(X_2, X_3) + 2\epsilon.$$

As a consequence, for all $\epsilon > 0$, we have $\tilde{h}(X_1, X_3) \leqslant \tilde{h}(X_1, X_2) + \tilde{h}(X_2, X_3) + 2\epsilon$ hence the triangle inequality holds.   □

2.7   PROPERTY   *Let $\Psi : T^*N \longrightarrow T^*N$ be an exact diffeomorphism. We have the equality*

$$\tilde{h}_{G \circ \Psi}(X_0, X_1) = \tilde{h}_G(\Psi(X_0), \Psi(X_1)) + S(X_0) - S(X_1),$$

where $S : T^*N \longrightarrow \mathbb{R}$ is a function such that $\Psi^*\lambda - \lambda = dS$.

PROOF. Observe first that $\underline{Y} = Y_n$ is a pre-orbit for the Hamiltonian $G \circ \Psi$ between points X_0 and X_1 if and only if $\Psi(\underline{Y}) = \Psi(Y_n)$ is a pre-orbit for the Hamiltonian G between $\Psi(X_0)$ and $\Psi(X_1)$. As a consequence, it is enough to prove that

$$A_{G \circ \Psi}(\underline{Y}) = A_G(\Psi(\underline{Y})) + S(X_0) - S(X_1).$$

Let us denote by $\underline{Z} = Z_n$ the pre-orbit $\Psi(Y_n)$. Setting $T_n^0 = 0$ and $T_n^{N_n+1} = T_n$, we have

$$A_G(Z_n) = \sum_{i=0}^{N_n} \int_{T_n^i}^{T_n^{i+1}} \lambda_{Z_n(t)}(\dot{Z}_n(t)) - G(Z_n(t))dt$$

$$= \sum_{i=0}^{N_n} \int_{T_n^i}^{T_n^{i+1}} (\Psi^*\lambda)_{Y_n(t)}(\dot{Y}_n(t)) - G \circ \Psi(Y_n(t))dt$$

$$= \sum_{i=0}^{N_n} \left(\int_{T_n^i}^{T_n^{i+1}} \lambda_{Y_n(t)}(\dot{Y}_n(t)) - G \circ \Psi(Y_n(t))dt + S(Y_n(T_n^{i+1}-)) - S(Y_n(T_n^i)) \right)$$

7

$$= A_{G \circ \Psi}(Y_n) - S(Y_n(0)) + S(Y_n(T_n)) + \sum_{i=1}^{N_n} \big(S(Y_n(T_n^i -)) - S(Y_n(T_n^i)) \big).$$

Since the function S is Lipschitz on the compact set K which contains the image of the curves Y_n, we obtain at the limit

$$A_G(\underline{Z}) = A_{G \circ \Psi}(\underline{Y}) - S(X_0) + S(X_1).$$

$\square$

2.8 PROPOSITION *Let us set $\tilde{m}(H) := \inf_{X \in T^*N} \tilde{h}(X, X)$. We have $\tilde{m}(H) \in \{-\infty, 0, +\infty\}$. In addition, if $\tilde{m}(H) = 0$, then there exists a point X in T^*N such that $\tilde{h}(X, X) = 0$.*

PROOF. It follows from the triangle inequality that, for each $X \in T^*N$, $\tilde{h}(X, X) \geqslant 0$ or $\tilde{h}(X, X) = -\infty$. As a consequence, $\tilde{m}(H) \geqslant 0$ or $\tilde{m}(H) = -\infty$. Let us assume that $\tilde{m}(H) \in [0, \infty[$. Then there exists a point $X_0 \in T^*N$ and a pre-orbits $\underline{Y} = Y_n : [0, T_n] \longrightarrow T^*N$ between X_0 and X_0 such that $A(\underline{Y}) \in [0, \infty[$. Let K be a compact subset of T^*N which contains the image of all the curves Y_n. Let S_n be a sequence of integers such that $T_n/S_n \longrightarrow \infty$ and $S_n \longrightarrow \infty$. Let b_n be the integer part of T_n/S_n. Note that $b_n \longrightarrow \infty$. Let d_n be a sequence of integers such that $d_n \longrightarrow \infty$ and $d_n/b_n \longrightarrow 0$. Since the set K is compact, there exists a sequence $\epsilon_n \longrightarrow 0$ such that, whenether b_n points are given in K, then at least d_n of them lie in a same ball of radius ϵ_n. So there exists a point $X_n \in K$ such that at least d_n of the points $Y_n(S_n), Y_n(2S_n), \ldots, Y_n(b_n S_n)$ lie in the ball of radius ϵ_n and center X_n. Let us denote by $Y_n(t_n^1), Y_n(t_n^2) \ldots, Y_n(t_n^{d_n})$ these points, where $t_n^{i+1} \geqslant t_n^i + S_n$. We can assume, taking a subsequence, that the sequence X_n has a limit X in K. It is not hard to see that $\underline{Y}^i = Y_{n|[t_n^i, t_n^{i+1}]}$ is a pre-orbit between X and X. On the other hand, for each $k \in \mathbb{N}$, we define the sequence of curves $Z_n^k : [0, T_n + t_n^1 - t_n^k] \longrightarrow T^*N$ by $Z_n^k(t) = Y_n(t)$ for $t \in [0, t_n^1[$, and $Z_n^k(t) = Y_n(t + t_n^k - t_n^1)$ for $t \in [t_n^1, T_n + t_n^1 - t_n^k]$. For each k, the sequence Z_n^k is a pre-orbit between X_0 and X_0. We have

$$A(Y_n) = A(Z_n^k) + \sum_{i=1}^{k-1} A(Y_n^i)$$

hence

$$A(\underline{Y}) \geqslant \tilde{h}(X_0, X_0) + (k-1)\tilde{h}(X, X).$$

Since $A(\underline{Y})$ is a real number, and since this inequality holds for all $k \in \mathbb{N}$, this implies that $\tilde{h}(X, X) = 0$. $\square$

2.9 Let us define the symplectic Aubry set of G as the set

$$\tilde{\mathcal{A}}_s(G) := \{X \in T^*N \text{ such that } \tilde{h}(X, X) = 0 \text{ and } G(X) = 0\} \subset T^*N.$$

The symplectic Mather set $\tilde{\mathcal{M}}_s(G)$ of G is the union of the supports of the compactly supported Φ-invariant probability measures concentrated on $\tilde{\mathcal{A}}_s(G)$. Note that, in general, it is not clear that the symplectic Aubry set should be closed. The symplectic Mather set, then, may not be contained in the symplectic Aubry set, but only in its closure. The Mather set

and the Aubry set are Φ-invariant, as follows directly from 2.5. If $\tilde{m}(H) = 0$, then the symplectic Aubry set is not empty, and all its orbits are bounded, hence the symplectic Mather set $\tilde{\mathcal{M}}_s(G)$ is not empty.

2.10 For each pair X_0, X_1 of points in $\tilde{\mathcal{A}}_s(G)$, we define the set $\tilde{\mathcal{I}}_s(X_0, X_1)$ of points $P \in T^*N$ such that

$$\tilde{h}(X_0, X_1) = \tilde{h}(X_0, X) + \tilde{h}(X, X_1)$$

if $\tilde{h}(X_0, X_1) \in \mathbb{R}$, and $\tilde{\mathcal{I}}_s(X_0, X_1) = \emptyset$ otherwise. Note that the sets $\tilde{\mathcal{I}}_s(X_0, X_1)$ are all contained in the level $\{G = 0\}$. Indeed, the finiteness of $\tilde{h}(X_0, X)$ implies that $G(X_0) = G(X)$, while $G(X_0) = 0$ by definition of $\tilde{\mathcal{A}}_s(G)$. It follows from 2.5 that the set $\tilde{\mathcal{I}}_s(X_0, X_1)$ is Φ-invariant. We now define the symplectic Mañé set as

$$\tilde{\mathcal{N}}_s(G) := \bigcup_{X_0, X_1 \in \tilde{\mathcal{A}}_s(G)} \tilde{\mathcal{I}}_s(X_0, X_1).$$

The Mañé set is Φ-invariant, all its orbits are bounded. We have the inclusion

$$\tilde{\mathcal{A}}_s(G) \subset \tilde{\mathcal{N}}_s(G).$$

In order to prove this inclusion, just observe that $X_0 \in \tilde{\mathcal{I}}(X_0, X_0)$ for each $X_0 \in \tilde{\mathcal{A}}_s(G)$.

2.11 If $\Psi : T^*N \longrightarrow T^*N$ is an exact diffeomorphism, then we have

$$\Psi(\tilde{\mathcal{M}}_s(G \circ \Psi)) = \tilde{\mathcal{M}}_s(G), \quad \Psi(\tilde{\mathcal{A}}_s(G \circ \Psi)) = \tilde{\mathcal{A}}_s(G), \quad \Psi(\tilde{\mathcal{N}}_s(G \circ \Psi)) = \tilde{\mathcal{N}}_s(G),$$

this follows obviously from 2.7, and from the fact that Ψ conjugates the Hamiltonian flow of G and the Hamiltonian flow of $G \circ \Psi$.

2.12 Let us assume that $\tilde{m}(G) = 0$, and set

$$\tilde{d}(X, X') = \tilde{h}(X, X') + \tilde{h}(X', X).$$

We have $\tilde{d}(X, X') \geqslant 0$, and the function $\tilde{d}$ satisfies the triangle inequality, and is symmetric. In addition, we obviously have $\tilde{d}(X, X) = 0$ if and only if $X \in \tilde{\mathcal{A}}_s(G)$. The restriction of the function $\tilde{d}$ to the set $\tilde{\mathcal{A}}_s(G)$ is a pseudo-metric with $+\infty$ as a possible value. We define an equivalence relation on $\tilde{\mathcal{A}}_s(G)$ by saying that the points X and X' are equivalent if and only if $\tilde{d}(X, X') = 0$. The equivalence classes of this relation are called the static classes. Let us denote by $(\dot{\mathcal{A}}_s(G), \dot{d}_s)$ the metric space obtained from $\tilde{\mathcal{A}}_s$ by identifying points X and X' when $\tilde{d}(X, X') = 0$. In other words, the set $\dot{\mathcal{A}}_s(G)$ is the set of static classes of H. We call $(\dot{\mathcal{A}}_s(G), \dot{d}_s)$ the quotient Aubry set. Note that the metric $\dot{d}_s$ can take the value $+\infty$. The quotient Aubry set is also well behaved under exact diffeomorphisms. More precisely, if Ψ is an exact diffeomorphism of T^*N, then the image of a static class of $G \circ \Psi$ is a static class of G. This defines a map

$$\dot{\Psi} : \dot{\mathcal{A}}_s(G \circ \Psi) \longrightarrow \dot{\mathcal{A}}_s(G)$$

which is an isometry for the quotient metrics.

2.13 PROPOSITION, *Assume that $\tilde{m}(G) = 0$, and in addition that the function $\tilde{h}$ is bounded from below. Then the orbits of $\tilde{\mathcal{N}}_s(G)$ are bi-asymptotic to $\tilde{\mathcal{A}}_s(G)$. In addition, for each orbit $X(s)$ in $\tilde{\mathcal{N}}_s(G)$, there exists a static class $S-$ in $\tilde{\mathcal{A}}_s(G)$ and a static class $S+$ such that the orbit $X(s)$ is α-asymptotic to $S-$ and ω-asymptotic to $S+$.*

PROOF. Let ω and ω' be two points in the ω-limit of the orbits $X(t) = \Phi_t(X)$. We have to prove that ω and ω' belong to the symplectic Aubry set, and to the same static class. It is enough to prove that $\tilde{d}(\omega, \omega') = 0$. In order to do so, we consider two increasing sequences t_n and s_n, such that $t_n - s_n \longrightarrow \infty$, $s_n - t_{n-1} \longrightarrow \infty$, $X(t_n) \longrightarrow \omega$ and $X(s_n) \longrightarrow \omega'$. Let $\underline{Y} = Y_n : [0, t_n - s_n] \longrightarrow T^*N$ be the pre-orbit between ω' and ω defined by $Y_n(t) = X(t - s_n)$. Similarly, we consider the pre-orbit $\underline{Z} = Z_n : [0, s_{n+1} - t_n] \longrightarrow T^*N$ between ω and ω' defined by $Z_n(t) = X(t - t_n)$. Since X belongs to $\tilde{\mathcal{N}}_s(G)$, there exist points X_0 and X_1 in $\tilde{\mathcal{A}}_s(G)$ such that $X \in \tilde{\mathcal{I}}(X_0, X_1)$. In view of 2.5, we have

$$\tilde{h}(X(t_n), X_1) = \tilde{h}(X(t_m), X_1) + \int_{t_n}^{t_m} \lambda_{X(t)}(\dot{X}(t)) - G(X(t))dt$$

for all $m \geqslant n$. Since the function $\tilde{h}$ is bounded from below, we conclude that the double sequence $\int_{t_n}^{t_m} \lambda_{X(t)}(\dot{X}(t)) - G(X(t))dt, m \geqslant n$ is bounded from above, so that

$$\liminf \int_{t_n}^{t_{n+1}} \lambda_{X(t)}(\dot{X}(t)) - G(X(t))dt \leqslant 0.$$

As a consequence, we have $\liminf A(Y_{n+1}) + A(Z_n) \leqslant 0$ hence $A(\underline{Y}) + A(\underline{Z}) = 0$, and $\tilde{d}(\omega, \omega') = 0$. The proof is similar for the α-limit. $\qquad\square$

It is useful to finish with section with a technical remark.

2.14 LEMMA *Let $\underline{Y} = Y_n : [0, T_n] \longrightarrow T^*N$ be a pre-orbit between between X_0 and X_1. There exists a pre-orbit $\underline{Z}$ between X_0 and X_1 which has the same action as $\underline{Y}$, and has discontinuities only at times $1, 2, \ldots, [T_n] - 1$, where $[T_n]$ is the integer part of T_n.*

PROOF. We set $Z_n(k + s) = \Phi_s(Y_n(k))$ for each $k = 0, 1, \ldots, [T_n] - 2$, and $s \in [0, 1[$, and $Z_n([T_n] - 1 + s) = \Phi_s(Y_n([T_n] - 1))$ for each $s \in [0, 1 + T_n - [T_n][$. It is not hard to see that $A(Z_n) - A(Y_n) \longrightarrow 0$, hence $A(\underline{Y}) = A(\underline{Z})$. $\qquad\square$

3 The case of convex Hamiltonian systems

We assume the hypotheses 1.1, and prove that the symplectic definitions of section 2 agree with the standard definitions of section 1. This proves that the theory of section 2 is not trivial at least in this case. This also ends the proof of Theorem 1.10.

3.1 In this section, we consider a Hamiltonian function $H : T^*M \times \mathbb{T} \longrightarrow \mathbb{R}$ satisfying the hypotheses 1.1. We set $N = M \times \mathbb{T}$. We denote by (P, t, E) the points of T^*N and set $G(P, t, E) = E + H(P, t) : T^*N \longrightarrow \mathbb{R}$. We denote by $h(q, t; q', t')$ the Peierl's barrier associated to H in section 1 and by $\tilde{h}(P, t, E; P', t', E')$ the barrier associated to G in section 2.

3.2 Before we state the main result of this section, some terminology is necessary. If $u : M \longrightarrow \mathbb{R}$ is a continuous function, we say that $P \in T_q^*M$ is a proximal super-differential of u

at point q (or simply a super-differential) if there exists a smooth function $f : M \longrightarrow \mathbb{R}$ such that $f - u$ has a minimum at q and $df_q = P$. Clearly, if u is differentiable at q and if P is a proximal super-differential of u at q, then $P = du_q$.

3.3 PROPOSITION *We have the relation*

$$h(q,t;q',t') = \min_{P \in T_q^* M, P' \in T_{q'}^* M} \tilde{h}(P,t,-H(P,t);P',t',-H(P',t')).$$

In addition, if the minimum is reached at (P,P') then P is a super-differential of the function $h(.,t;q',t')$ at point q and $-P'$ is a super-differential of the function $h(q,t;.,t')$ at point q'.

PROOF. Let us fix two points (q,t) and (q',t') in $N = M \times \mathbb{T}$. We claim that the inequality

$$\tilde{h}(P,t,E;P',t',E') \geqslant h(q,t;q',t')$$

holds for each $(P,t,E) \in T_{(q,t)}^* N$ and each $(P',t',E') \in T_{(q',t')}^* N$. If $\tilde{h}(P,t,E;P',t',E') = +\infty$, then there is nothing to prove. Else, let us fix $\epsilon > 0$. There exists a pre-orbit $\underline{Y} = Y_n(s)$: $[0,T_n] \longrightarrow T^* N$ between (P,t,E) and (P',t',E') such that $A(\underline{Y}) \leqslant \tilde{h}(P,t,E;P',t',E') + \epsilon$ (resp. $A(\underline{Y}) \leqslant -1/\epsilon$ in the case where $\tilde{h}(P,t,E;P',t',E') = -\infty$). In view of 2.14, it is possible to assume that the discontinuity points T_n^i of Y_n satisfy $T_n^{i+1} \geqslant T_n^i + 1$. Let us write

$$Y_n(s) = (P_n(s), \tau_n(s), E_n(s)),$$

and $q_n(s) = \pi(P_n(s))$. Let δ_n^i be the real number closest to $T_n^{i+1} - T_n^i$ among those which satisfy $\tau_n(T_n^i) + \delta_n^i = \tau_n(T_n^{i+1})$.

We have

$$A(Y_n) = \sum_{i=0}^{N_n} \int_{T_n^i}^{T_n^{i+1}} L(q_n(s), \dot{q}_n(s), s + \tau_n(T_n^i)) dt \geqslant \sum_{i=0}^{N_n} F(q(T_n^i), \tau_n(T_n^i); q(T_n^{i+1}-), T_n^{i+1} - T_n^i).$$

It is known that the functions $F(q,t;q',s)$ is Lipschitz on $\{s \geqslant 1\}$, see for example [1], 3.2. We have

$$\sum_{i=0}^{N_n} \left| F(q_n(T_n^i), \tau_n(T_n^i); q_n(T_n^{i+1}-), T_n^{i+1} - T_n^i) - F(q_n(T_n^i), \tau_n(T_n^i); q_n(T_n^{i+1}), \delta_n^i) \right|$$

$$\leqslant C \sum_{i=0}^{N_n-1} D(q_n(T_n^{i+1}-), \tau_n(T_n^{i+1}-); q_n(T_n^{i+1}), \tau_n(T_n^{i+1}))$$

$$\leqslant C \sum_{i=0}^{N_n-1} D(Y_n(T_n^{i+1}-), Y_n(T_n^{i+1})) \longrightarrow 0.$$

As a consequence, we have

$$A(\underline{Y}) \geqslant \liminf \sum_{i=0}^{N_n} F(q(T_n^i), \tau_n(T_n^i); q(T_n^{i+1}), \delta_n^i)$$

$$\geqslant \liminf F\left(q_n(0), \tau_n(0); q_n(T_n), \sum_{i=0}^{N_n} \delta_n^i \right) \geqslant h(q,t;q',t'),$$

11

hence $\epsilon + \tilde{h}(P, t, E; P', t', E') \geqslant h(q, t; q', t')$ (resp. $-1/\epsilon \geqslant h(q, t; q', t')$). Since this holds for all $\epsilon > 0$, we have $\tilde{h}(P, t, E; P', t', E') \geqslant h(q, t; q', t')$ as desired.

Conversely, let us consider a sequence T_n such that $T_n \longrightarrow \infty$, $t + T_n \bmod 1 = t'$, and

$$h(q, t; q', t') = \lim_{n \longrightarrow \infty} F(q, t; q', T_n).$$

Let $q_n(s) : [0, T_n] \longrightarrow M$ be a curve such that

$$\int_0^{T_n} L(q_n(s), \dot{q}_n(s), s + t)ds = F(q, t; q', T_n).$$

Since the curve q_n is minimizing the action, there exists a Hamiltonian trajectory

$$Y_n(s) = (P_n(s), t + s, E_n(s) = -H(X_n(s), t + s)) : [0, T_n] \longrightarrow T^*N$$

whose projection on M is the curve q_n. In addition, by well known results on minimizing orbits, see [10], there exists a compact subset of T^*M which contains the images of all the curves $P_n(s)$. As a consequence, we can assume, taking a subsequence if necessary, that the sequences $P_n(0)$ and $P_n(T_n)$ have limits $P \in T_q^*M$ and $P' \in T_{q'}^*M$. The sequence $\underline{Y} = Y_n$ is then a pre-orbit between $(P, t, -H(P, t))$ and $(P', t', -H(P', t'))$, and its action is

$$A(\underline{Y}) = \lim A(Y_n) = \lim \int_0^{T_n} L(q_n(s), \dot{q}_n(s), t + s)ds = h(q, t; q', t').$$

As a consequence, we have

$$\tilde{h}(P, t, -H(P, t)); P', t', -H(P', t')) \leqslant h(q, t; q', t').$$

This ends the proof of the first part of the Proposition.

Let now $Y = (P, t, E) \in T_q^*M \times T^*\mathbb{T}$ and $Y' = (P', t', E') \in T_{q'}^*M \times T^*\mathbb{T}$ be points such that $h(q, t; q', t') = \tilde{h}(Y; Y')$. Let $q(s)$ be the projection on M of the orbit $\Phi_s(Y)$. Using 2.5 and 1.5, we get

$$\tilde{h}(Y, Y') = \tilde{h}(\Phi_s(Y), Y') + \int_0^s \lambda_{\Phi_\sigma(Y)}(V_G(\Phi_\sigma(Y)) - G(\Phi_\sigma(Y)))d\sigma$$

$$\geqslant h(q(s), t + s; q', t') + \int_0^s L(q(\sigma), \dot{q}(\sigma), t + \sigma)dt \geqslant h(q, t; q', t') = \tilde{h}(Y, Y').$$

As a consequence, all the inequalities are equalities. We obtain that the curve $q(s)$ is minimizing in the expression

$$h(q, t; q', t') = \min\left(h(q(s), t + s; q', t') + \int_0^s L(q(\sigma), \dot{q}(\sigma), t + \sigma)dt\right).$$

Fathi has proved that $-P$ is then a super-differential of the function $h(., t; q', t')$ at q. The properties at (q', t') are treated in a similar way. $\qquad\square$

3.4 COROLLARY *If H satisfies the hypotheses of 1.1, then $m(H) \leqslant \tilde{m}(H)$.*

3.5 COROLLARY *If H satisfies the hypotheses of 1.1, and if $m(H) = 0$, then $\tilde{m}(H) = 0$, and we have have $\tilde{\mathcal{A}}_s(G) = \tilde{\mathcal{A}}(H)$. In addition, we have*

$$\tilde{h}(X_0, t_0, E_0; X_1, t_1, E_1) = h(\pi(P_0), t_0; \pi(P_1), t_1)$$

for each (P_0, t_0, E_0) and (P_1, t_1, E_1) in $\tilde{\mathcal{A}}(H)$.

PROOF. Let (P, t, E) be a point of T^*N and $q = \pi(P)$. If $(P, t, E) \in \tilde{\mathcal{A}}_s(G)$, then

$$\tilde{h}(P, t, E; P, t, E) = 0,$$

so that $h(q, t; q, t) \leqslant 0$. Since, on the other hand, we have $h(q, t; q, t) \geqslant m(H) = 0$, we conclude that $h(q, t; q, t) = 0$ hence $(q, t) \in \mathcal{A}(H)$. As a consequence, the function $h(q, t; ., t)$ is differentiable at q, see 1.6, and $\left(\partial_3 h(q, t; q, t), t - H(\partial_3 h(q, t; q, t), t)\right) \in \tilde{\mathcal{A}}(H)$. Since $\tilde{h}(P, t, E; P, t, E) = h(q, t; q, t)$, the point P is a super-differential of $h(q, t; ., t)$ at q, and we must have $P = \partial_3 h(q, t; q, t)$. Moreover, we have $G(P, t, E) = H(P, t) + E = 0$, hence $(P, t, E) \in \tilde{\mathcal{A}}(H)$.

Conversely, assume that $(P, t, E) \in \tilde{\mathcal{A}}(H)$. We then have $E = -H(P, t)$. In addition, $h(q, t; q, t) = 0$, the functions $h(q, t; ., t)$ and $h(., t; q, t)$ are differentiable at q, and we have $P = \partial_3 h(q, t; q, t) = -\partial_1 h(q, t; q, t)$. Now let $X \in T_q^*M$ and $X' \in T_q^*M$ be such that

$$\tilde{h}(X, t, -H(X, t); X', t', -H(X', t')) = h(q, t; q, t).$$

Then $-X$ is a super-differential at q of $h(., t; q, t)$, and X' is a super-differential at q of $h(q, t; ., t)$. It follows that $X = P = X'$. Hence we have $\tilde{h}(P, t, E; P, t, E) = h(q, t; q, t) = 0$. This proves that $\tilde{m}(H) = 0$, and that $(P, t, E) \in \tilde{\mathcal{A}}_s(G)$.

Finally, let $(P_0, t_0, E_0) \in T_{q_0}^*M \times T^*\mathbb{T}$ and $(P_1, t_1, E_1) \in T_{q_1}^*M \times T^*\mathbb{T}$ be two points of $\tilde{\mathcal{A}}(H)$. We have $E_0 = -H(P_0, t_0)$ and $E_1 = -H(P_1, t_1)$. Furthermore, the function $h(q_0, t_0; ., t_1)$ is differentiable at q_1, with $\partial_3 h(q_0, t_0; q_1, t_1) = P_1$, and that the function $h(., t_0; q_1, t_1)$ is differentiable at q_0, with $\partial_1 h(q_0, t_0; q_1, t_1) = -P_0$. Since $-P_0$ and P_1 are then the only super-differentials of $h(., t_0; q_1, t_1)$ and $h(q_0, t_0; ., t_1)$, we conclude that $\tilde{h}(P_0, t_0, E_0; P_1, t_1, E_1) = h(q_0, t_0; q_1, t_1)$. $\square$

3.6 COROLLARY *If H satisfies the hypotheses of 1.1, and if $m(H) = 0$, then $\tilde{\mathcal{M}}_s(G) = \tilde{\mathcal{M}}(H)$.*

3.7 COROLLARY *If H satisfies the hypotheses of 1.1, and if $m(H) = 0$, then $\tilde{\mathcal{N}}_s(G) = \tilde{\mathcal{N}}(H)$.*

PROOF. It is enough to prove that, if (P_0, t_0, E_0) and (P_1, t_1, E_1) belong to $\tilde{\mathcal{A}}_s(G)$, and $q_0 = \pi(P_0), q_1 = \pi(P_1)$, then

$$\tilde{\mathcal{I}}_s(P_0, t_0, E_0; P_1, t_1, E_1) = \tilde{\mathcal{I}}(q_0, t_0, q_1, t_1).$$

Let (P, t, E) be a point of $\tilde{\mathcal{I}}_s(P_0, t_0, E_0; P_1, t_1, E_1)$. We then have $G(P_0, t_0, E_0) = G(P, t, E) = 0$ hence $E = -H(P, t)$. Furthermore, the inequalities

$$h(q_0, t_0; q_1, t_1) = \tilde{h}(P_0, t_0, -H(P_0, t_0); P_1, t_1, -H(P_1, t_1))$$

$$= \tilde{h}(P_0, t_0, -H(P_0, t_1); P, t, E) + \tilde{h}(P, t, E; P_1, t_1, -H(P_1, t_1))$$

$$\geqslant h(q_0, t_0; q, t) + h(q, t; q_1, t_1) \geqslant h(q_0, t_0; q_1, t_1)$$

are all equalities. As a consequence, the point (q, t) belongs to the set $\mathcal{I}(q_0, t_0; q_1, t_1)$, and the differentials $\partial_3 h(q_0, t_0; q, t)$ and $\partial_1 h(q, t; q_1, t_1)$ exist, we have $\partial_3 h(q_0, t_0; q, t) = -\partial_1 h(q, t; q_1, t_1)$, and the point

$$(X, t, e) = (\partial_3 h(q_0, t_0; q, t), t, -H(\partial_3 h(q_0, t_0; q, t), t))$$

belongs to $\tilde{\mathcal{I}}(q_0, t_0; q_1, t_1)$, as follows from our definition of the Mañé set. Since

$$\tilde{h}(P_0, t_0, -H(P_0, t_0); P, t, -H(P, t)) = h(q_0, t_0; q, t),$$

the point P must be a super-differential of $h(q_0, t_0; ., t)$ at q, hence $P = X$. We have proved that $(P, t, E) \in \tilde{\mathcal{I}}(q_0, t_0; q_1, t_1)$.

Conversely, assume that $(P, t, E) \in \tilde{\mathcal{I}}(q_0, t_0; q_1, t_1)$, so that $E = -H(P, t)$. Then

$$h(q_0, t_0; q, t) + h(q, t; q_1, t_1) = h(q_0, t_0; q_1, t_1)$$

and

$$P = \partial_3 h(q_0, t_0; q, t) = -\partial_1 h(q, t; q_{1,1}).$$

In addition, since (q_0, t_0) and (q_1, t_1) belong to $\mathcal{A}(H)$, the differential $P_0 = \partial_1 h(q_0, t_0; q, t)$ exists for all q, and satisfies $(P_0, t_0, -H(P_0, t_0)) \in \tilde{\mathcal{A}}(H))$. Similarly, setting $P_1 = \partial_3 h(q, t; q_1, t_1)$, we have $(P_1, t_1, -H(P_1, t_1)) \in \tilde{\mathcal{A}}(H))$. We conclude that

$$\tilde{h}(P_0, t_0, -H(P_0, t_0); P, t, E) = h(q_0, t_0; q, t)$$

and

$$\tilde{h}(P, t, E; P_1, t_1, -H(P_1, t_1)) = h(q, t; q_1, t_1).$$

As a consequence, setting $E_0 = -H(P_0, t_0)$ and $E_1 = -H(P_1, t_1)$, we have

$$\tilde{h}(P_0, t_0, E_0; P, t, E) + \tilde{h}(P, t, E; P_1, t_1, E_1) = \tilde{h}(P_0, t_0, E_0; P_1, t_1, E_1).$$

$\square$

References

[1] P. BERNARD, Connecting orbits of time dependent Lagrangian systems. *Ann. Inst. Fourier* **52** (2002), 1533–1568.

[2] P. BERNARD, The action spectrum near positive definite invariant tori. Bull. Soc. Math. France 131 (2003), no. 4, 603–616.

[3] P. BERNARD, The dynamics of pseudographs in convex Hamiltonian systems, preprint.

[4] C.-Q. CHENG, J. YAN, Existence of diffusion orbits in a priori unstable Hamiltonian systems, to appear in J. Diff. Geom.

[5] C.-Q. CHENG, J. YAN, Arnold Diffusion in Hamiltonian systems: the a priori unstable case, préprint.

[6] G. CONTRERAS, J. DELGADO, R. ITURRIAGA, Lagrangian flows: the dynamics of globally minimizing orbits. II. *Bol. Soc. Brasil. Mat. (N.S.)* **28** (1997), no. 2, 155–196.

[7] A. FATHI, Book in preparation.

[8] A. FATHI, Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens, (French) [A weak KAM theorem and Mather's theory of Lagrangian systems] *C. R. Acad. Sci. Paris Ser. I Math.* **324** (1997), no. 9, 1043–1046.

[9] A. FATHI, Solutions KAM faibles conjuguées et barrières de Peierls. (French) [Weakly conjugate KAM solutions and Peierls's barriers] *C. R. Acad. Sci. Paris Ser. I Math.* **325** (1997), no. 6, 649–652.

[10] J. N. MATHER Action minimizing invariant measures for positive definite Lagrangian systems, Math. Z. **207** 169–207 (1991)

[11] J. N. MATHER Variational construction of connecting orbits, Ann. Inst. Fourier, **43** (1993), 1349-1368.

[12] R. MAÑÉ Lagrangian flows: The dynamics of globally minimizing orbits, Bol. Soc. Bras. Mat, **28** (1997) 141-153

[13] G. P. PATERNAIN; L. POLTEROVICH; K. F. SIBURG, Boundary rigidity for Lagrangian submanifolds, non-removable intersections, and Aubry-Mather theory. Dedicated to Vladimir I. Arnold on the occasion of his 65th birthday. Mosc. Math. J. 3 (2003), no. 2, 593–619, 745.

[14] K. F. SIBURG, Symplectic invariants of elliptic fixed points. Comment. Math. Helv. 75 (2000), no. 4, 681–700

[15] K. F. SIBURG, *The principle of least action in geometry and dynamics.* Lecture Notes in Mathematics, **1844**, Springer-Verlag, Berlin, 2004.

[16] Z. XIA, Arnold diffusion: a variational construction, in *Proceedings of the International Congress of Mathematicians* , Vol. II (Berlin, 1998), Extra Vol. II, 867-877 (electronic), (1998).

[17] Z. XIA, Arnold Diffusion and instabilities in Hamiltonian dynamics, preprint.