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A Step in Supercritical WKB Analysis and some Applications

CHRISTOPHE CHEVERRY ¹, OLIVIER GUÈS ²

Abstract. We study the existence and the asymptotic behavior of large amplitude high-frequency oscillating waves subjected to the 2D Burger equation. This program is achieved by developing tools related to supercritical WKB analysis. By selecting solutions which are divergence free, we show that incompressible or compressible 2D Euler equations are not locally closed for the weak L^2 topology.

1 Introduction.

This article is devoted to the study of the two dimensional incompressible Euler equation

$$(1.1) \quad \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_x) \mathbf{u} + \nabla_x \mathbf{p} = 0, \quad \operatorname{div}_x \mathbf{u} = 0$$

as well as to the study of the two dimensional Burger equation

$$(1.2) \quad \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_x) \mathbf{u} + \mathbf{f} = 0.$$

The time, space and state variables are respectively

$$t \in \mathbb{R}, \quad x = {}^t(x_1, x_2) \in \mathbb{R}^d, \quad \mathbf{u} = {}^t(\mathbf{u}^1, \mathbf{u}^2) \in \mathbb{R}^d, \quad d = 2.$$

The equations (1.1) and (1.2) are completed with some initial data having locally finite kinetic energy

$$(1.3) \quad \mathbf{u}(0, x) = h(x) = {}^t(h^1(x), h^2(x)) \in L_{loc}^2(\mathbb{R}^2; \mathbb{R}^2).$$

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The analysis of the Cauchy problems (1.1)-(1.3) and (1.2)-(1.3) depends strongly on the regularity assumptions imposed on the function h . For instance, consider (1.1)-(1.3). When

$$h \in E_s := \{ v \in L^2_{loc}(\mathbb{R}^2; \mathbb{R}^2); \operatorname{curl} v := \partial_1 v^2 - \partial_2 v^1 \in (L^1 \cap L^\infty)(\mathbb{R}^2; \mathbb{R}) \}$$

the solution \mathbf{u} of (1.1)-(1.3) is global in time and is unique [1]-[3]-[21]. Now, since the equation (1.1) can be put in the conservative form

$$\partial_t \mathbf{u} + \operatorname{div}_x (\mathbf{u} \otimes \mathbf{u}) + \nabla_x \mathbf{p} = 0, \quad \operatorname{div}_x \mathbf{u} = 0, \quad \mathbf{u} \otimes \mathbf{u} := (\mathbf{u}^j \mathbf{u}^i)_{1 \leq i, j \leq 2},$$

one is tempted to work in a more general functional framework. This means to enter the field of *weak* solutions to (1.1). In the case of vortex-sheet initial data with vorticity of distinguished sign

$$h \in E_w := \{ v \in L^2_{loc}(\mathbb{R}^2; \mathbb{R}^2); 0 \leq \operatorname{curl} v \in \mathcal{M}(\mathbb{R}^2; \mathbb{R}) \},$$

existence results hold whereas the question of uniqueness is still open. The first proof is due to J.-M. Delort [9]. Then, further informations have been obtained. For background and expository accounts on this subject, the reader may consult [1], [20] and the related references.

The usual way to derive existence results of weak solutions to (1.1) is based on two steps. First, construct *approximate-solution* sequences $\{\mathbf{u}^\varepsilon\}_{\varepsilon \in]0,1]}$ of (1.1), either by smoothing the initial data h or by adding to (1.1) a small viscosity (to get Navier-Stokes equations). Secondly, exhibit a property of *concentration-cancellation* which means that the family $\{\mathbf{u}^\varepsilon\}_\varepsilon$ does not converge strongly in L^2_{loc} yet all the extracted L^2_{loc} weak limits satisfy (1.1).

This approach is well presented and clearly explained in the recent book of L. Bertozzi and A. Majda [1]. Read especially the surveys given in chapters 10, 11 and 12. Up to the present, when $d = 2$, the weak limits of approximate solutions to (1.1) have always been observed to be solutions to (1.1). Thus, one can ask if the property of concentration-cancellation does generalize to the space L^2 of energy estimates. This question is explicitly raised in [1]-p. 479. It makes sense also in the less restrictive framework L^2_{loc} .

However, such a program comes again a deep objection because :

Theorem 1.1. *There is a bounded open domain $\Omega \subset \mathbb{R} \times \mathbb{R}^2$ and a family of functions $\{\mathbf{u}^\varepsilon\}_\varepsilon$ such that*

$$i) \quad \mathbf{u}^\varepsilon \in C^1(\Omega), \quad \sup \{ \|\mathbf{u}^\varepsilon\|_{L^2(\Omega)}; \varepsilon \in]0,1] \} < \infty,$$

$$ii) \quad \mathbf{u}^\varepsilon \text{ is a solution to (1.1) on } \Omega,$$

$$iii) \quad \mathbf{u}^\varepsilon \text{ converges weakly (as } \varepsilon \text{ goes to zero) to } \mathbf{u}^0 \in C^1(\Omega).$$

But \mathbf{u}^0 is not a solution to (1.1).

When the space dimension is three ($d = 3$), the corresponding result can easily be proved, see [10]-p. 674 or [1]-p. 478. It suffices to look at *simple waves* which undergo rapid variations with respect to a *linear* phase $k \cdot x$ where $k \in \mathbb{R}^3$. Note $\theta \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ a fast variable. Build the oscillation

$$(1.4) \quad \mathbf{u}^\varepsilon(t, x) = \mathbf{H}(t, x, k \cdot x/\varepsilon), \quad \mathbf{H}(t, x, \theta) \in C^\infty(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{T}; \mathbb{R}^3).$$

The profile \mathbf{H} can be adjusted so that \mathbf{u}^ε is a solution of (1.1) and (1.2) with respectively \mathbf{p} constant and $\mathbf{f} \equiv 0$. But the associated weak limit \mathbf{u}^0 is not a solution to (1.1). Observe that

$$\mathbf{u}^0(t, x) = \bar{\mathbf{H}}(t, x) := \int_0^1 \mathbf{H}(t, x, \theta) d\theta = \mathbf{H}(t, x, \theta) - \mathbf{H}^*(t, x, \theta).$$

When $d = 2$, such a basic procedure does not apply. Of course, there are solutions to (1.1) of the form (1.4) with $x \in \mathbb{R}^2$ and $k \in \mathbb{R}^2$. However, all known examples of such solutions produce weak limits which satisfy (1.1). Thus, to push further the investigations, it is necessary to take into account more general structures including *nonlinear* phases and *perturbations* of simple waves. On this way, new difficulties appear.

In fact, the study of solution sequences $\{\mathbf{u}^\varepsilon\}_{\varepsilon \in]0,1]}$ to (1.1) can reveal very complex phenomena. The asymptotic behavior of \mathbf{u}^ε when ε goes to zero can involve both concentrations and oscillations. Certainly, this is a current challenge to understand what happens in the limiting process. Our aim here is precisely to bring informations of this type. This is achieved by following an original strategy which is presented below.

- Section 2 is devoted to the Cauchy problem (1.2)-(1.3) which recently raised new interests [2]-[13]-[19]. The initial data h is defined on some open set $\omega \subset \mathbb{R}^2$. It is smooth and bounded $h \in C_b^1(\omega; \mathbb{R}^2)$. The source term \mathbf{f} is globally defined on $\mathbb{R} \times \mathbb{R}^2$ and is also smooth $\mathbf{f} \in C^1(\mathbb{R} \times \mathbb{R}^2; \mathbb{R}^2)$. Lemma 2.1 provides with a domain of determinacy $\Omega_{h,\mathbf{f}}$ whose size is controled by explicit constraints imposed on h and \mathbf{f} , say $(h, \mathbf{f}) \in \mathcal{B}_g^\tau(\omega)$.

Three special cases $\mathbf{f}_\iota \equiv \nabla_x \mathbf{p}_\iota$ with $\iota \in \{-1, 0, 1\}$ for the choice of \mathbf{f} are then distinguished. When $\mathbf{f} \equiv \mathbf{f}_\iota$, the conditions mentioned above concern only h and are noted in abbreviated form $h \in \mathcal{B}_\iota^\tau(\omega)$. In fact, they reduce to some control on the quantities h , $\operatorname{div}_x h$ and $\det D_x h$. This is Lemma 2.2.

Moreover, to each ι corresponds a nonlinear functional set $\mathcal{V}_\iota^\tau(\omega) \subset \mathcal{B}_\iota^\tau(\omega)$ which is defined through a *Monge-Ampère* equation and which is kept invariant (Lemma 2.3) under both flows issued from (1.1) and (1.2). Therefore, when restricted to $\mathcal{V}_\iota^\tau(\omega)$, all the discussion concerning (1.1) can be transferred at the level of (1.2). This argument will be used repeatedly.

The condition $h \in \mathcal{B}_l^r(\omega)$ does not mean a uniform control on *all* derivatives of h . In particular, the function h can carry oscillations. The sequel to section 2 is devoted to such oscillations, in a monophasic context.

- In subsection 2.2, we identify (Lemma 2.4) the necessary and sufficient conditions which must be imposed on the phase ϕ and the profile H to have

$$(1.5) \quad h^\varepsilon(x) := H(x, \phi(x)/\varepsilon) + O(\varepsilon) \in \mathcal{B}_l^r(\omega), \quad \forall \varepsilon \in]0, 1].$$

- In subsection 2.3, we extract (Lemma 2.6) supplementary constraints (on ϕ and H) which allow to get the more restrictive condition

$$(1.6) \quad h^\varepsilon(x) := H(x, \phi(x)/\varepsilon) + O(\varepsilon) \in \mathcal{V}_l^r(\omega), \quad \forall \varepsilon \in]0, 1].$$

Couples (H, ϕ) coming from (1.5) and (1.6) are called *compatible* with respectively (1.2) and (1.1). The corresponding families $\{h^\varepsilon\}_\varepsilon$ are said *well prepared* for respectively the Burger equation and the Euler equation.

• The chapter 3 tackles the evolution problem through the study of simple waves. It considers expressions having the form

$$(1.7) \quad \tilde{\mathbf{u}}^\varepsilon(t, x) = \mathbf{H}(t, x, \Phi(t, x)/\varepsilon), \quad \varepsilon \in]0, 1]$$

where \mathbf{H} and Φ do not depend on ε .

- In the subsection 3.1, we seek solutions to (1.2) which are as in (1.7). The functions \mathbf{H} and Φ must be subjected to the overdetermined system

$$(1.8) \quad \begin{cases} \partial_t \mathbf{H} + (\mathbf{H} \cdot \nabla_x) \mathbf{H} + \mathbf{f} = 0, \\ \partial_t \Phi + (\tilde{\mathbf{H}} \cdot \nabla_x) \Phi = 0, \\ \nabla_x \Phi(t, x) \cdot \mathbf{H}^*(t, x, \theta) = 0. \end{cases}$$

Nothing guarantees that the third condition in (1.8) is propagated by the two first equations. In fact, for general (well polarized) initial data

$$(1.9) \quad \mathbf{H}(0, x, \theta) = H(x, \theta), \quad \Phi(0, x) = \phi(x), \quad \nabla_x \phi(x) \cdot H^*(x, \theta) = 0$$

the Cauchy problem (1.8)-(1.9) has no solution. But it becomes well posed (Lemma 3.1) when (H, ϕ) is compatible with (1.2) and when \mathbf{f} is subjected to conditions which will be explicitated. The incoming profile \mathbf{H} can be put (Lemma 3.2) in the form

$$\mathbf{H}(t, x, \theta) = \tilde{H}(\Phi(t, x), \mathbf{r}(t, x, \theta)), \quad \tilde{H} \in C^1(\mathbb{R}^2; \mathbb{R}^2)$$

where \tilde{H} is fixed whereas \mathbf{r} is the *underlying scalar profile*.

- In subsection 3.2, we consider solutions to (1.1) having the form (1.7). This time, the functions \mathbf{H} and Φ must be subjected to

$$(1.10) \quad \begin{cases} \partial_t \mathbf{H} + (\mathbf{H} \cdot \nabla_x) \mathbf{H} + \nabla_x \mathbf{p} = 0, & \operatorname{div}_x \mathbf{H} = 0, \\ \partial_t \Phi + (\bar{\mathbf{H}} \cdot \nabla_x) \Phi = 0, & \nabla_x \Phi \cdot \mathbf{H}^* = 0. \end{cases}$$

Again this is an overdetermined system. When $\mathbf{p} \equiv \mathbf{p}_\iota$ with $\iota \in \{-1, 0, 1\}$, a complete analysis (Lemma 3.3) is available. The phase Φ must be constant on parallel hyperplanes which means that

$$\Phi(t, x) = at + bx_1 + x_2, \quad (a, b) \in \mathbb{R}^2.$$

At this stage, weak limits of solutions to (1.1) of the form (1.7) are still solutions to (1.1). However, this picture does not resist under the influence of small perturbations.

• Precisely, the chapter 4 is faced with the problem of *stability*. Seek solutions to (1.1) or to (1.2) having the form

$$(1.11) \quad \mathbf{u}^\varepsilon(t, x) = \mathbf{U}^\varepsilon(t, x, \Phi(t, x)/\varepsilon) = \mathbf{H}(t, x, \Phi(t, x)/\varepsilon) + O(\varepsilon).$$

The equation (1.2) written in the variables (t, x, θ) becomes

$$(1.12) \quad \partial_t \mathbf{U}^\varepsilon + (\mathbf{U}^\varepsilon \cdot \nabla_x) \mathbf{U}^\varepsilon + \varepsilon^{-1} (\partial_t \Phi + \mathbf{U}^\varepsilon \cdot \nabla_x \Phi) \partial_\theta \mathbf{U}^\varepsilon + \mathbf{f} = 0.$$

Observe that the profile \mathbf{H} is a special solution to (1.12). The linearized equation of (1.12) along \mathbf{H} is

$$(1.13) \quad \partial_t \dot{\mathbf{U}}^\varepsilon + (\mathbf{H} \cdot \nabla_x) \dot{\mathbf{U}}^\varepsilon + (\dot{\mathbf{U}}^\varepsilon \cdot \nabla_x) \mathbf{H} + \varepsilon^{-1} (\dot{\mathbf{U}}^\varepsilon \cdot \nabla_x \Phi) \partial_\theta \mathbf{H} = 0.$$

The factor ε^{-1} expresses the presence of a *singularity*. It prevents from deriving energy estimates which involve the state variable $\dot{\mathbf{U}}^\varepsilon$ and which are uniform in $\varepsilon \in]0, 1]$.

Large amplitude oscillations on the velocity field, like \mathbf{u}^ε , are known to be strongly unstable. This observation has been noticed for over a century. It was already mentioned in works of Kelvin and Rayleigh. Since, it has motivated many studies till for instance the recent contributions of S. Friedlander, W. Strauss and M. Vishik (see [11] and the related references).

The amplification phenomena under question appear also in the field of nonlinear geometric optics [7]-[12]. There, they are often called up to explain why classical methods do not allow to construct non trivial perturbations \mathbf{u}^ε of $\tilde{\mathbf{u}}^\varepsilon$, with \mathbf{u}^ε defined on some open domain of determinacy $\Omega \subset \mathbb{R} \times \mathbb{R}^2$ which does not shrink to the empty set as ε goes to zero.

Now, supposing that (1.5) involves a couple (H, ϕ) which is compatible with (1.2), the analysis of chapter 2 guarantees *a priori* that the oscillating Cauchy problem (1.2)-(1.5) is well posed on such fixed domain Ω . On the other hand, it says nothing concerning the asymptotic behavior when ε goes to zero of the corresponding solution family $\{\mathbf{u}^\varepsilon\}_\varepsilon$.

At this stage, the discussion falls under the scope of *supercritical* WKB analysis. This approach has been initiated by G. Lebeau [17]. Some small advancements have also been achieved in [4]-[5]-[6].

The articles [5]-[6] are specifically devoted to the propagation of oscillations which are solutions to (1.1). However, the situations studied in [5]-[6] differ from those under consideration here.

On the one hand, in [5]-[6], the data ϕ and H are selected arbitrarily leading generically to a cascade of phases. On the contrary, we adjust here the phase ϕ and the profile H so that such phenomena are avoided. This is a consequence of the relations (2.19) and (2.20) imposed on ϕ and H .

On the other hand, in [5]-[6], the WKB analysis is just formal or justified in the presence of a small anisotropic viscosity. On the contrary, our aim here is to face the problem of stability in the hyperbolic context.

To do that, the difficulty is the following. The descriptions of Sections 2 and 3 rely on ϕ and H , and then on Φ and \mathbf{H} leading to $\tilde{\mathbf{u}}^\varepsilon$. But the structure of $\tilde{\mathbf{u}}^\varepsilon$ is too rigid to be compatible with the influence of a perturbation, say $\varepsilon \ ^t(\mathbf{V}^{\varepsilon 2}, \mathbf{V}^{\varepsilon 3})$, of size $O(\varepsilon)$. Supplementary flexibility is required on \mathbf{H} .

- This is obtained in subsection 4.1 by introducing a new state variable, say $\mathbf{V}^{\varepsilon 1}$. The scalar function $\mathbf{V}^{\varepsilon 1}$ modifies the principal term of the oscillation since it acts by translation on the fast variable θ of \mathbf{H} . On the other hand, it interferes with $\ ^t(\mathbf{V}^{\varepsilon 2}, \mathbf{V}^{\varepsilon 3})$ in a nonlinear way.

The implementation of $\mathbf{V}^{\varepsilon 1}$ can be put in concrete form by way of a *blow up* procedure. This manipulation corresponds to a *dependent change* of state variables which substitutes for \mathbf{U}^ε the function

$$\mathbf{V}^\varepsilon(t, x, \theta) = \ ^t(\mathbf{V}^{\varepsilon 1}, \mathbf{V}^{\varepsilon 2}, \mathbf{V}^{\varepsilon 3})(t, x, \theta) \in C^1(\Omega \times \mathbb{T}; \mathbb{R}^3), \quad \Omega \subset \mathbb{R} \times \mathbb{R}^2.$$

More precisely, Proposition 4.1 says that the singular equation (1.12) can be replaced with some well posed hyperbolic quasilinear system (4.5) which gives rise to \mathbf{V}^ε . The construction of \mathbf{V}^ε on a domain $\Omega \times \mathbb{T}$ independent of ε is made possible because (4.5) involves no more coefficients which are singular with respect to $\varepsilon \in]0, 1]$. Observe by the way that θ plays at the level of (4.5) the part of a *slow* variable.

- Now, it remains to perform the analysis at the level of (4.5) and then to interpret the results in terms of the original variables. This is the frame of subsection 4.2 which is devoted to applications.

Seek solutions \mathbf{V}^ε to (4.5) having the form

$$(1.14) \quad \mathbf{V}^\varepsilon(t, x, \theta) = \sum_{j=0}^{\infty} \varepsilon^j \mathbf{V}_j(t, x, \theta, \vec{\Psi}(t, x, \theta)/\varepsilon)$$

where $\vec{\Psi}$ contains m phases

$$\vec{\Psi} = (\Psi_1, \dots, \Psi_m) \in C^\infty(\Omega \times \mathbb{T}; \mathbb{R}^m), \quad m \in \mathbb{N}^*$$

and where the profiles \mathbf{V}_j are smooth

$$\mathbf{V}_j(t, x, \theta, \tilde{\theta}) \in C^\infty(\Omega \times \mathbb{T} \times \mathbb{T}^m; \mathbb{R}^3), \quad \forall j \in \mathbb{N}.$$

The analysis depends strongly on the structure hypothesis imposed on the profiles \mathbf{V}_j . The elementary case is when all the \mathbf{V}_j do not depend on $\tilde{\theta}$. Then, to identify \mathbf{V}_0 , it suffices to retain the principal term in the Taylor expansions with respect to ε of the coefficients which appear in (4.5). This yields a transport equation (4.6) on $\mathbf{V}_0 = {}^t(\mathbf{V}_0^1, \mathbf{V}_0^2, \mathbf{V}_0^3)$. The related oscillation \mathbf{u}^ε can be recovered through the formula

$$(1.15) \quad \mathbf{u}^\varepsilon(t, x) = \mathbf{U}_0(t, x, \Phi(t, x)/\varepsilon) + O(\varepsilon)$$

where \mathbf{U}_0 is given by

$$(1.16) \quad \mathbf{U}_0(t, x, \theta) = \mathbf{H}(t, x, \theta + \mathbf{V}_0^1(t, x, \theta)).$$

Of special interest is the study of such \mathbf{u}^ε issued from data h^ε which are well prepared for (1.1) because these \mathbf{u}^ε are solutions to (1.1). On the one hand, the examination of the associated Φ , \mathbf{H} and \mathbf{V}_0^1 indicates that caustics phenomena can occur for incompressible Euler equations. On the other hand, the influence of \mathbf{V}_0^1 induces the objection previously mentioned (Theorem 1.1) to the concentration-cancellation property.

Finally, choose to work with a \mathbf{V}_0 which is only a function of (t, x, θ) whereas the other contributions \mathbf{V}_j with $j \geq 1$ depend on $\tilde{\theta}$. Suppose moreover that the trace of $\vec{\Psi}$ at time $t = 0$ does not see θ . In other words

$$\vec{\Psi}(0, x, \theta) = \vec{\psi}(x), \quad \vec{\psi}(x) = (\psi_1(x), \dots, \psi_m(x)) \in C^\infty(\omega; \mathbb{R}^m).$$

Under these assumptions, the discussion of (4.5)-(1.14) rests on the theory of multidimensional weakly nonlinear geometric optics [14]-[15]. Now, coming back to \mathbf{u}^ε , we find

$$\mathbf{u}^\varepsilon(0, x) = H(x, \phi(x)/\varepsilon) + \varepsilon H_0(x, \phi(x)/\varepsilon, \vec{\psi}(x)/\varepsilon) + O(\varepsilon^2).$$

This means that, interpreted at the level of (1.2) or (1.1), the matter is to describe the *interaction* of some large amplitude high frequency monophasic wave with small amplitude oscillations which can involve phases ψ_j transversal to ϕ . It turns out that the data can be adjusted (Proposition 4.2) so that the asymptotic behavior of $\{\mathbf{u}^\varepsilon\}_\varepsilon$ on Ω is governed by

$$\mathbf{u}^\varepsilon(t, x) = \mathbf{H}(t, x, \Phi(t, x)/\varepsilon) + \varepsilon \mathbf{H}_0(t, x, \vec{\psi}(t, x, \Phi(t, x)/\varepsilon)/\varepsilon) + O(\varepsilon^2).$$

This contains a *rigorous justification* that kinetic energy of solutions to (1.1) can be transferred from “low” wave numbers modes (namely of size ε^{-1}) to “high” wave numbers modes (of size ε^{-2}). Such a transfert could also affect the principal term of the oscillation.

The basic mechanism thus revealed could be repeated. It can be conceived as a starting point in order to study the propagation of more complicated oscillations. It is a convenient way to tackle phenomena (possibly infinite) of cascade of energy from large-scale motions to small-scale motions. Obviously, the modern tools of nonlinear geometric optics offer means of exploring some *turbulent features* in fluid mechanics.

Contents

1	Introduction.	1
2	Well prepared initial data.	9
2.1	Preliminaries.	9
2.1.1	About the life span of solutions to Burger equations.	9
2.1.2	Three special cases.	12
2.1.3	Invariant sets given by Monge-Ampère equations.	13
2.1.4	Large amplitude monophasic oscillations.	16
2.2	Well prepared families for the Burger equation.	17
2.3	Well prepared families for the Euler equation.	21
3	Simple waves.	25
3.1	The case of Burger equations.	25
3.2	The case of incompressible equations.	29
4	The problem of stability.	32
4.1	A blow up procedure.	33
4.2	Applications.	40
4.2.1	Various caustics phenomena for Euler equations.	40
4.2.2	On the concentration-cancellation property.	42
4.2.3	Interaction with small amplitude transversal oscillations.	44

2 Well prepared initial data.

This Section is devoted to general considerations concerning the dynamics of a two dimensional vector field $\mathbf{u} = {}^t(\mathbf{u}^1, \mathbf{u}^2) \in \mathbb{R}^2$ described by the Burger equation (1.2).

Let ω be a bounded open domain of \mathbb{R}^2 . If necessary, make a translation in x to be sure that ${}^t(0,0) \in \omega$. Note C_b^k with $k \in \mathbb{N} \cup \{+\infty\}$ the space of functions with bounded continuous derivatives up to the order k . Select data h and \mathbf{f} such that

$$(2.1) \quad \mathbf{u}(0, x) = h(x) \in C_b^1(\omega), \quad \mathbf{f}(t, x) \in C^1(\mathbb{R} \times \mathbb{R}^2).$$

2.1 Preliminaries.

We start by looking at the life span of solutions to (1.2)-(2.1).

2.1.1 About the life span of solutions to Burger equations.

Look at the ordinary differential equation

$$(2.2) \quad \frac{d^2}{dt^2} \Xi(t, a, b) + \mathbf{f}(t, \Xi(t, a, b)) = 0$$

completed with

$$\Xi(0, a, b) = a \in \mathbb{R}^2, \quad \frac{d}{dt} \Xi(0, a, b) = b \in \mathbb{R}^2.$$

Suppose that the source term \mathbf{f} has at most a linear growth with respect to the space variable, uniformly in the time variable

$$(2.3) \quad \exists C \in \mathbb{R}_+; \quad |\mathbf{f}(t, x)| \leq C(1 + |x|), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^2.$$

Then, the application Ξ is globally defined and is smooth $\Xi \in C^1(\mathbb{R} \times \mathbb{R}^4)$. Introduce the graphs

$$G_h := \{ (x, h(x)); x \in \omega \} \subset \mathbb{R}^2 \times \mathbb{R}^2,$$

$$DG_h := \{ (x, h(x), D_x h(x)); x \in \omega \} \subset \mathbb{R}^2 \times \mathbb{R}^2 \times M^2(\mathbb{R}),$$

where $M^d(\mathbb{R})$ is the space of $d \times d$ matrices with real coefficients. Note \bar{G}_h and \bar{DG}_h the closures of respectively G_h and DG_h . By hypothesis, these are compact sets. The speed of propagation up to time T of a solution \mathbf{u} to (1.2) coming from h and \mathbf{f} is bounded by

$$c_{h,\mathbf{f}}(T) := \|h\|_{L^\infty(\omega)} + T \sup \{ |\mathbf{f}(t, \Xi(t, a, b))|; (t, a, b) \in [0, T] \times \bar{G}_h \}.$$

This allows to define the domain of determinacy

$$\Omega_{h,\mathbf{f}}(T) := \{ (t, x) \in [0, T[\times \mathbb{R}^2; B(x, t c(T)) \subset \omega \}$$

where $B(x, r[$ with $r \in \mathbb{R}_+$ is the ball

$$B(x, r[:= \{ \tilde{x} = {}^t(\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^2; |\tilde{x} - x|^2 = (\tilde{x}_1 - x_1)^2 + (\tilde{x}_2 - x_2)^2 < r^2 \}.$$

Let $(t, a, b, M) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{M}^2(\mathbb{R})$. Introduce the 2×2 matrix

$$\Gamma(t, a, b, M) := D_a \Xi(t, a, b) + D_b \Xi(t, a, b) M.$$

Compute

$$R_g(a, b, M) := \sup \{ T \in \mathbb{R}_+; \det \Gamma(t, a, b, M) > 0, \forall t \in [0, T[\}.$$

Lemma 2.1. *The Cauchy problem (1.2)-(2.1) has a C^1 solution \mathbf{u} which is defined on the domain*

$$\Omega_{h,\mathbf{f}} := \cup_{\{T; 0 < T < T_{h,\mathbf{f}}\}} \Omega_{h,\mathbf{f}}(T), \quad T_{h,\mathbf{f}} := \inf \{ R_g(z); z \in D\bar{G}_h \}.$$

Remark 2.1.1 - introduction of $\mathcal{B}_g^\tau(\omega)$. Since Γ is a continuous function satisfying $\Gamma(0, \cdot) \equiv \text{Id}$, we have $T_{h,\mathbf{f}} \in \mathbb{R}_+ \cup \{+\infty\}$ so that $\Omega_{h,\mathbf{f}} \neq \emptyset$. Let $\tau \in]0, 1]$. Consider

$$\mathcal{B}_g^\tau(\omega) := \left\{ (h, \mathbf{f}) \in C_b^1 \times C^1; |h(x)| \leq \tau^{-1}, |\mathbf{f}(t, x)| \leq \tau^{-1} (1 + |x|), \right. \\ \left. R_g(x, h(x), D_x h(x)) \in [\tau, +\infty], \forall (t, x) \in \mathbb{R} \times \omega \right\}.$$

It is sure that

$$c(\tau) := \sup \{ c_{h,\mathbf{f}}(T); h \in \mathcal{B}_g^\tau(\omega) \} < \infty.$$

Applying Lemma 2.1, the solution \mathbf{u} of (1.2)-(2.1) issued from $h \in \mathcal{B}_g^\tau(\omega)$ is defined on

$$\Omega(\tau) := \{ (t, x) \in [0, \tau[\times \mathbb{R}^2; B(x, c(\tau)t \subset \omega \}. \quad \diamond$$

Remark 2.1.2 - a more general situation. Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a C^1 diffeomorphism. Look at the Burgers type equation

$$\partial_t \mathbf{u} + (g(\mathbf{u}) \cdot \nabla_x) \mathbf{u} = 0.$$

In fact, this situation is equivalent to (1.2) with $\mathbf{f} \equiv 0$. Indeed, it suffices to take $g(\mathbf{u})$ as the new unknown. \diamond

Proof of Lemma 2.1. Since unicity is clear in the context of C^1 functions, it remains to show that a solution \mathbf{u} of (1.2)-(2.1) exists on $\Omega_{h,\mathbf{f}}(T)$ when $0 < T < T_{h,\mathbf{f}}$. Classical results yield a time $\tilde{T} \in \mathbb{R}_+$ such that a solution \mathbf{u} of (1.2)-(2.1) exists on $\Omega_{h,\mathbf{f}}(\tilde{t})$ for all $\tilde{t} < \tilde{T}$. Take \tilde{T} as large as possible.

If $\tilde{T} \geq T$, there is nothing to do. Thus, suppose that $\tilde{T} < T$. The continuation principle (exposed in Majda [18] paragraph 2.2) says that

$$(2.4) \quad \sup \{ \| D_{\tilde{x}} \mathbf{u}(\tilde{t}, \tilde{x}) \| ; (\tilde{t}, \tilde{x}) \in \Omega_{h,\mathbf{f}}(\tilde{T}) \} = +\infty.$$

To simplify the notations, define $\check{\Xi}(t, x) := \Xi(t, x, h(x))$ and

$$\check{\Gamma}(t, x) := \Gamma(t, x, h(x), D_x h(x)), \quad \check{R}(x) := R_g(x, h(x), D_x h(x)).$$

Since \mathbf{u} must be constant along characteristics, for each $(\tilde{t}, \tilde{x}) \in \Omega_{h,\mathbf{f}}(\tilde{T})$, there is $x \in \omega$ such that

$$(2.5) \quad \mathbf{u}(\tilde{t}, \tilde{x}) = h(x) - \int_0^{\tilde{t}} \mathbf{f}(t, \check{\Xi}(t, x)) dt, \quad \tilde{x} = \check{\Xi}(\tilde{t}, x).$$

It follows that

$$(2.6) \quad D_{\tilde{x}} \mathbf{u}(\tilde{t}, \tilde{x}) \check{\Gamma}(\tilde{t}, x) = D_x h(x) - \int_0^{\tilde{t}} D_x \mathbf{f}(t, \check{\Xi}(t, x)) \check{\Gamma}(t, x) dt.$$

On the one hand

$$\tilde{t} < \tilde{T} < T < T_{h,\mathbf{f}} \leq \check{R}(x), \quad \forall x \in \omega.$$

On the other hand

$$\det \check{\Gamma}(t, x) > 0, \quad \forall (t, x) \in [0, \check{R}(x)[\times \omega.$$

In particular

$$\det \check{\Gamma}(t, x) > 0, \quad \forall (t, x) \in [0, \tilde{T}[\times \omega.$$

Suppose that there is a sequence $\{(t_n, x_n)\}_n \in ([0, \tilde{T}[\times \omega)^{\mathbb{N}}$ which satisfies

$$\lim_n \rightarrow \infty \check{\Gamma}(t_n, x_n) = 0.$$

Extract a subsequence (given by $\ell : \mathbb{N} \rightarrow \mathbb{N}$) such that

$$\lim_n \rightarrow \infty (t_{\ell(n)}, x_{\ell(n)}, h(x_{\ell(n)}), D_x h(x_{\ell(n)})) = (\bar{t}, \bar{z}) \in [0, \tilde{T}] \times \bar{D}G_h.$$

The continuity of Γ guarantees that $\det \Gamma(\bar{t}, \bar{z}) = 0$. This implies that

$$R_g(\bar{z}) \leq \bar{t} \leq \tilde{T} < T_{h,\mathbf{f}} \leq R_g(\bar{z})$$

which clearly is not possible. It means that

$$(2.7) \quad \inf \{ \det \check{\Gamma}(t, x) ; (t, x) \in [0, \tilde{T}[\times \omega \} = c > 0.$$

Note $\text{Co}(M)$ the co-matrix of M . Remember that

$$\check{\Gamma}(\tilde{t}, x)^{-1} = (\det \check{\Gamma}(\tilde{t}, x))^{-1} \text{Co}(\check{\Gamma}(\tilde{t}, x)).$$

Use this and the relation (2.6) to get

$$\sup \{ \| D_{\tilde{x}} \mathbf{u}(\tilde{t}, \tilde{x}) \| ; (\tilde{t}, \tilde{x}) \in \Omega_{h,\mathbf{f}}(\tilde{T}) \} < \infty$$

which is a contradiction with (2.4). Therefore $\tilde{T} \geq T$. □

2.1.2 Three special cases.

This paragraph deals with

$$(2.8) \quad \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_x) \mathbf{u} + \mathbf{f}_\iota = 0, \quad \mathbf{f}_\iota(t, x) := \iota x, \quad \iota \in \{-1, 0, 1\}.$$

First, consider the case $\iota = 0$. When $\iota = 0$, simplifications occur in the preceding discussion. Introduce the domain of determinacy

$$\Omega_h := \{ (t, x) \in [0, +\infty[\times \mathbb{R}^2; B(x, t \parallel h \parallel_{L^\infty(\omega)} [\subset \omega \}.$$

Define the application

$$\begin{aligned} I_h : \omega &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto I_h(x) = {}^t(I_h^1(x), I_h^2(x)) := {}^t(\operatorname{div}_x h(x), \det D_x h(x)). \end{aligned}$$

The image of ω by I_h is

$$I_h(\omega) := \{ I_h(x); x \in \omega \}.$$

Note $\bar{I}_h(\omega)$ the closure of $I_h(\omega)$. This is a compact set. Compute

$$R(a, b) := \sup \{ T \in \mathbb{R}_+; 1 + at + bt^2 > 0, \forall t \in [0, T[\} \in \mathbb{R}_+ \cup \{+\infty\}.$$

The result 2.2 below is true even if ω is not bounded.

Lemma 2.2. *Take $\iota = 0$. The Cauchy problem (2.8)-(2.1) has a C^1 solution \mathbf{u} which is defined on the truncated cone*

$$\Omega_h(T_h) := \{ (t, x) \in \Omega_h; t < T_h \}, \quad T_h := \inf \{ R(y); y \in \bar{I}_h(\omega) \}.$$

Remark 2.1.3 - introduction of $\mathcal{B}_\iota^r(\omega)$. The condition (2.3) is obviously satisfied by the functions \mathbf{f}_ι . Instead of $\mathcal{B}_g^r(\omega)$, we can consider

$$\begin{aligned} \mathcal{B}_\iota^r(\omega) := \{ h \in C_b^1(\omega; \mathbb{R}^2); |h(x)| \leq \tau^{-1}, \\ R(I_h(x)) \in [\tau, +\infty], \quad \forall x \in \omega \}. \quad \diamond \end{aligned}$$

Proof of Lemma 2.2. When ω is bounded, just apply the procedure of Lemma 2.1 to find

$$c(T) = \parallel h \parallel_{L^\infty(\omega)}, \quad \Xi(t, a, b) = a + tb, \quad \Gamma(t, a, b, M) = \operatorname{Id} + tM.$$

It follows that

$$\begin{aligned} \Omega_{h, \mathbf{f}_0}(T) \subset \Omega_{h, \mathbf{f}_0}(\tilde{T}), \quad \forall T \in]0, \tilde{T}[, \\ \det D_x \tilde{\Xi}(t, x) = \det (\operatorname{Id} + t D_x h(x)) = 1 + I_h^1(x) t + I_h^2(x) t^2. \end{aligned}$$

Now, it is easy to see that $T_{h, \mathbf{f}_0} \equiv T_h$ and $\Omega_{h, \mathbf{f}_0} \equiv \Omega_h$.

When ω is not bounded, it suffices to remark that the time T_h^r associated with the restriction of h to the open domain $\omega \cap B(0, r[$ is such that

$$T_h \leq T_h^r, \quad \forall r \in \mathbb{R}_+, \quad \lim_{r \rightarrow +\infty} T_h^r = T_h.$$

Since the speed of propagation is bounded by $\|h\|_{L^\infty(\omega)}$, the expected result can be obtained by passing to the limit ($r \rightarrow +\infty$). \square

Remark 2.1.4 - comparison with other results. Observe that

$$R^{-1}(+\infty) = \{(a, b) \in \mathbb{R}^2; a \geq 0, b \geq 0\} \cup \{(a, b) \in \mathbb{R}^2; a < 0, b \geq \frac{a^2}{4}\}.$$

Suppose that $\omega = \mathbb{R}^2$. Then, the solution of (2.8)-(2.1) with $\iota = 0$ is global in time if and only if

$$(2.9) \quad R(I_h(x)) = +\infty, \quad \forall x \in \mathbb{R}^2.$$

Simple computations indicate that this criterion is equivalent with the condition imposed in Theorem 2.2 of [19]. On the other hand, when (2.9) is violated, our analysis of the local in time existence is more refined than what can be extracted from [19]. \diamond

Remark 2.1.5 - the other cases. When $\iota = -1$ or $\iota = 1$, the solutions of (2.2) are respectively

$$\Xi_{-1}(t, a, b) = a \operatorname{ch} t + b \operatorname{sh} t, \quad \Xi_1(t, a, b) = a \cos t + b \sin t.$$

It follows that

$$\Gamma_{-1}(t, a, b, M) = \operatorname{ch} t (\operatorname{Id} + \operatorname{th} t M), \quad \Gamma_1(t, a, b, M) = \cos t (\operatorname{Id} + \operatorname{tg} t M).$$

Thus, the stopping times T_{h, \mathbf{f}_ι} which are associated with the functions \mathbf{f}_ι are

$$T_{h, \mathbf{f}_{-1}} = \begin{cases} \operatorname{arcth} T_h & \text{if } T_h < 1, \\ +\infty & \text{if } T_h \geq 1, \end{cases} \quad T_{h, \mathbf{f}_1} = \operatorname{arctg} T_h. \quad \diamond$$

The reason why the functions \mathbf{f}_ι are distinguished is the following. When h is conveniently adjusted, the Cauchy problem (2.8)-(2.1) furnishes a solution of (1.1)-(2.1). This fact is proved in the next subsection.

2.1.3 Invariant sets given by Monge-Ampère equations.

Observe that

$$\mathbf{f}_\iota(t, x) = \nabla_x \mathbf{p}_\iota(x), \quad \mathbf{p}_\iota(x) := \iota \frac{1}{2} |x|^2 + C.$$

Therefore, the solutions of (2.8)-(2.1) which are incompressible

$$(2.10) \quad \operatorname{div}_x \mathbf{u}(t, x) = 0, \quad \forall (t, x) \in \Omega_{h, \mathbf{f}}$$

can be interpreted as solutions of incompressible Euler equations. When $\iota = 0$, they satisfy (1.1) with \mathbf{p} constant. Thus, they are also solutions of compressible Euler equations. In fact, they are subjected to the pressureless gas dynamics system.

Suppose to simplify the discussion that ω is simply connected in \mathbb{R}^2 . Then, look at the nonlinear functional set

$$\mathcal{V}_\iota^\tau(\omega) := \left\{ h = {}^t(-\partial_2 \varrho, \partial_1 \varrho) \in C_b^1(\omega; \mathbb{R}^2); |h(x)| \leq \tau^{-1}, \right. \\ \left. \varrho \in C^2(\omega; \mathbb{R}), \det D_{xx}^2 \varrho \equiv \iota \right\}.$$

According to the usual terminology, this definition involves a *Monge-Ampère* equation which is called hyperbolic, degenerate or elliptic when respectively $\iota = -1$, $\iota = 0$ or $\iota = 1$. Obviously, the subset $\mathcal{V}_\iota^\tau(\omega)$ is not empty.

Lemma 2.3. *The C^1 solution \mathbf{u} of the Cauchy problem (2.8)-(2.1) is subjected to (2.10) if and only if $h \in \mathcal{V}_\iota^\tau(\omega)$ for some $\tau \in]0, 1]$.*

Remark 2.1.6 - another interpretation of $\mathcal{V}_\iota^\tau(\omega)$. Introduce

$$\tilde{\mathcal{V}}_\iota^\tau(\omega) := \left\{ h \in C_b^1(\omega; \mathbb{R}^2); |h(x)| \leq \tau^{-1}, I_h(x) = {}^t(0, \iota), \forall x \in \omega \right\}.$$

Let $h = {}^t(-\partial_2 \varrho, \partial_1 \varrho) \in \mathcal{V}_\iota^\tau(\omega)$. Obviously

$$I_h^1(x) = 0, \quad I_h^2(x) = \det D_{xx}^2 \varrho(x) = \iota, \quad R(I_h(x)) \geq 1, \quad \forall x \in \omega.$$

It follows that

$$\mathcal{V}_\iota^\tau(\omega) \subset \tilde{\mathcal{V}}_\iota^\tau(\omega) \subset \tilde{\mathcal{B}}_\iota^\tau(\omega), \quad \forall \tau \in]0, 1].$$

Now, if $h \in \tilde{\mathcal{V}}_\iota^\tau(\omega)$, the restriction $I_h^1 \equiv 0$ means that h coincide with ${}^t(-\partial_2 \varrho, \partial_1 \varrho)$ for some scalar function $\varrho \in C^2(\omega; \mathbb{R})$. Then, the condition $I_h^2 \equiv \iota$ is equivalent to $\det D_{xx}^2 \varrho \equiv \iota$. Therefore $\tilde{\mathcal{V}}_\iota^\tau(\omega) \equiv \mathcal{V}_\iota^\tau(\omega)$.

If $h \in \mathcal{V}_0^\tau(\mathbb{R}^2)$, the solution \mathbf{u} of (1.2)-(2.1) or (1.1)-(2.1) is global in time. Moreover, the set $\mathcal{V}_0^\tau(\mathbb{R}^2)$ is invariant under the flow

$$h \in \mathcal{V}_0^\tau(\mathbb{R}^2) \implies \mathbf{u}(t, \cdot) \in \mathcal{V}_0^\tau(\mathbb{R}^2), \quad \forall t \in \mathbb{R}_+. \quad \diamond$$

Proof of Lemma 2.3. When $\iota = 0$, the result 2.3 can be deduced from Theorem 2.6 in [8]. The proof given below is different and more direct. It is also more general since it allows to incorporate the cases $\iota = \pm 1$.

◦ Suppose that \mathbf{u} is a solution of (2.8)-(2.1) which is subjected to (2.10). On the one hand, the relation (2.10) means that the two (complex) eigenvalues of the matrix $D_x \mathbf{u}(t, x)$ are opposite, say $\pm \lambda(t, x)$. On the other hand, the equation (2.8) implies that

$$(\partial_t + \mathbf{u} \cdot \nabla_x) D_x \mathbf{u} + (D_x \mathbf{u})^2 + \iota \text{Id} = 0, \quad \forall (t, x) \in \Omega_{h, \mathbf{f}_\iota}.$$

Take the trace to get

$$\text{Tr} (D_x \mathbf{u})^2 + 2 \iota = 2 (\lambda^2 + \iota) = 0, \quad \forall (t, x) \in \Omega_{h, \mathbf{f}_\iota}.$$

Look at what happens when $t = 0$. Since ω is supposed to be simply connected in \mathbb{R}^2 , there is $\varrho \in C^2(\omega; \mathbb{R})$ such that

$$\mathbf{u}(0, x) = h(x) = {}^t \nabla_x \varrho^\perp(x) := {}^t (-\partial_2 \varrho(x), \partial_1 \varrho(x)), \quad \forall x \in \omega$$

and the preceding condition reduces to

$$-\lambda(0, x)^2 = \det D_x \mathbf{u}(0, x) = \det D_x h(x) = \det D_{xx}^2 \varrho(x) = \iota.$$

Take τ small enough to be sure that $h \in \mathcal{V}_\iota^\tau(\omega)$.

◦ Conversely, suppose that $h \in \mathcal{V}_\iota^\tau(\omega)$. Apply Lemma 2.1 to find a solution \mathbf{u} of (2.8)-(2.1) on $\Omega_{h, \mathbf{f}_\iota}$. Note λ_1 and λ_2 the two eigenvalues of $D_x \mathbf{u}$. Deduce from (2.8) that

$$(\partial_t + \mathbf{u} \cdot \nabla_x) (D_x \mathbf{u})^2 + 2 (D_x \mathbf{u})^3 + 2 \iota D_x \mathbf{u} = 0, \quad \forall (t, x) \in \Omega_{h, \mathbf{f}_\iota}.$$

Observe that

$$\text{Tr} (D_x \mathbf{u})^3 = \lambda_1^3 + \lambda_2^3 = (\lambda_1 + \lambda_2) (\lambda_1^2 - \lambda_1 \lambda_2 + \lambda_2^2).$$

It follows that

$$\begin{cases} (\partial_t + \mathbf{u} \cdot \nabla_x) \text{div}_x \mathbf{u} + [\text{Tr} (D_x \mathbf{u})^2 + 2 \iota] = 0, \\ (\partial_t + \mathbf{u} \cdot \nabla_x) [\text{Tr} (D_x \mathbf{u})^2 + 2 \iota] + 2 (\lambda_1^2 - \lambda_1 \lambda_2 + \lambda_2^2 + \iota) \text{div}_x \mathbf{u} = 0. \end{cases}$$

By hypothesis, at time $t = 0$, we start with

$$\text{div}_x h(x) = 0, \quad \text{Tr} D_x h(x)^2 + 2 \iota = 2 (-\det D_{xx}^2 \varrho(x) + \iota) = 0.$$

Combining these informations with Gronwall's Lemma gives rise to

$$\text{div}_x \mathbf{u}(t, x) = 0, \quad \text{Tr} D_x \mathbf{u}(t, x)^2 + 2 \iota = 0, \quad \forall (t, x) \in \Omega_{h, \mathbf{f}_\iota}.$$

In particular, the divergence free condition (2.10) is verified. \square

Remark 2.1.7 - about the choice of ι . In the preceding proof, the value of ι can be fixed arbitrarily in \mathbb{R} . What is important is only that ι does not depend on (t, x) . Nevertheless, when $\iota \neq 0$, the change of time-space variables $(t, x) / |\iota|^{-1/2} (t, x)$ reduces to the situation $\iota \in \{-1, 1\}$. \diamond

2.1.4 Large amplitude monophasic oscillations.

According to the preceding construction, solutions \mathbf{u} of (2.8)-(2.1) issued from $h \in \mathcal{B}_l^\tau(\omega)$ are defined on $\Omega(\tau)$. When $\iota = 0$, we can take

$$\Omega(\tau) := \{ (t, x) \in [0, \tau[\times \mathbb{R}^2; B(x, t\tau^{-1}) \subset \omega \}.$$

Now, the constraint $h \in \mathcal{B}_l^\tau(\omega)$ does not imply a bound on all derivatives contained in $D_x h$. In particular, with τ fixed, it is possible to find families $\{h^\varepsilon\}_{\varepsilon \in]0,1]}$ which satisfy

$$(2.11) \quad h^\varepsilon \in \mathcal{B}_l^\tau(\omega), \quad \forall \varepsilon \in]0, 1]$$

but for which

$$(2.12) \quad \lim_{\varepsilon \rightarrow 0} \|D_x h^\varepsilon\|_{L^\infty(\omega)} = +\infty.$$

The aim of this paper is precisely to consider such families $\{h^\varepsilon\}_\varepsilon$, having a specific behaviour when ε goes to zero. Some material is needed to describe the oscillating structure of h^ε . Fix $J \in \mathbb{N}$. Introduce :

2.a) a phase $\phi \in C^\infty(\omega; \mathbb{R})$ with $\nabla_x \phi \in C_b^\infty(\omega; \mathbb{R})$. Assume that ϕ is not stationary. More precisely, impose

$$(2.13) \quad \exists c \in \mathbb{R}_+, \quad \partial_2 \phi(x) \geq c, \quad \forall x \in \omega,$$

2.b) a profile $H \equiv H_{-1} = {}^t(H^1, H^2) \in C_b^\infty(\omega \times \mathbb{T}; \mathbb{R}^2)$ which is non trivial

$$(2.14) \quad \exists (x, \theta) \in \omega \times \mathbb{T}; \quad \partial_\theta H(x, \theta) \neq 0,$$

2.c) other profiles

$$H_j = {}^t(H_j^1, H_j^2) \in C_b^\infty(\omega \times \mathbb{T}; \mathbb{R}^2), \quad j \in \{0, \dots, J-1\},$$

2.d) a function $rh^\varepsilon \in C_b^\infty(\omega; \mathbb{R}^2)$ which is controlled by

$$(2.15) \quad \sup \{ \|rh^\varepsilon\|_{L^\infty(\omega)} + \varepsilon^{J-1} \|D_x rh^\varepsilon\|_{L^\infty(\omega)}; \varepsilon \in]0, 1] \} < \infty.$$

With all these ingredients, build the asymptotic expansion

$$(2.16) \quad h^\varepsilon(x) = \sum_{j=-1}^{J-1} \varepsilon^{j+1} H_j(x, \phi(x)/\varepsilon) + \varepsilon^{J+1} rh^\varepsilon(x).$$

Observe that h^ε is a large amplitude oscillating wave

$$h^\varepsilon(x) = H(x, \phi(x)/\varepsilon) + O(\varepsilon), \quad \partial_\theta H \neq 0.$$

In the next subsection, we identify necessary and sufficient constraints to impose on ϕ and the H_j in order to have (2.11) for some fixed $\tau \in]0, 1]$.

2.2 Well prepared families for the Burger equation.

The hypothesis 2.a), \dots , 2.d) guarantee (2.12) and the fact that

$$\exists \tau \in \mathbb{R}_+; \quad \sup \{ |h^\varepsilon(x)|; (\varepsilon, x) \in]0, 1] \times \omega \} \leq \tau^{-1} < \infty.$$

Now, to obtain a family $\{h^\varepsilon\}_\varepsilon \in \mathcal{B}_l^\tau(\omega)^{]0,1]}$ with corresponding solutions $\{\mathbf{u}^\varepsilon\}_\varepsilon$ defined on a domain $\Omega(\tau)$ (which does not shrink to the empty set as ε goes to 0), it suffices to check the condition on I_{h^ε} which is stated below.

Definition 2.1. *We say that the family $\{h^\varepsilon\}_\varepsilon$, where h^ε is defined as in (2.16) and is made of ingredients satisfying 2.a), \dots , 2.d), is well prepared on ω for the Burger equation (1.2) if*

$$(2.17) \quad \exists \tau \in \mathbb{R}_+; \quad R(I_{h^\varepsilon}(x)) \in [\tau, +\infty], \quad \forall (x, \varepsilon) \in \omega \times]0, 1].$$

Any function $u \in L^1(\omega \times \mathbb{T})$ can be decomposed according to

$$u(x, \theta) = \langle u \rangle(x) + u^*(x, \theta) = \bar{u}(x) + u^*(x, \theta)$$

where $\langle u \rangle \equiv \bar{u}$ is the mean value

$$\langle u \rangle(x) = \bar{u}(x) := \int_{\mathbb{T}} u(x, \theta) d\theta.$$

Introduce the oscillating support of u which is

$$\text{osup } u := \{ x \in \omega; u^*(x, \cdot) \not\equiv 0 \}.$$

Define also

$$(\partial_\theta^{-1} u^*)(x, \theta) := \int_0^\theta u^*(x, \tilde{\theta}) d\tilde{\theta} - \int_0^1 \left(\int_0^\theta u^*(x, \tilde{\theta}) d\tilde{\theta} \right) d\theta.$$

In view of (2.13), for each $x_1 \in \mathbb{R}$, the application

$$\begin{aligned} \omega(x_1) &:= \{ x_2; (x_1, x_2) \in \omega \} \longrightarrow \mathbb{R} \\ x_2 &\longmapsto \phi(x_1, x_2) \end{aligned}$$

is strictly increasing. The image set

$$\mathcal{I}(x_1) := \{ \phi(x_1, x_2); x_2 \in \omega(x_1) \}$$

is open. Introduce the other open set

$$\mathcal{I} := \cup_{x_1 \in \mathbb{R}} \mathcal{I}(x_1) \subset \mathbb{R}.$$

Seek a family $\{h^\varepsilon\}_\varepsilon$ which is well prepared on ω for (1.2). Easy computations indicate that the analysis can be reduced to the case $\omega = \text{osup } H$. Thus, from now on, we can suppose that $\omega = \text{osup } H^*$. We suppose moreover that the curve $\{x \in \omega; \phi(x) = z\}$ is connected for all $z \in \mathcal{I}$. These are technical assumptions which simplify the following statements.

Lemma 2.4. *Under the assumptions mentioned above, the family $\{h^\varepsilon\}_\varepsilon$ is subjected to the condition (2.17) if and only if the three following conditions are satisfied :*

i) the profile H^ is polarized according to*

$$(2.18) \quad \nabla_x \phi(x) \cdot H^*(x, \theta) = 0, \quad \forall (x, \theta) \in \omega \times \mathbb{T}.$$

ii) there exists f in $C_b^\infty(\mathcal{I}; \mathbb{R})$ such that

$$(2.19) \quad \partial_1 \phi(x) = f(\phi(x)) \partial_2 \phi(x), \quad \forall x \in \omega.$$

iii) there exists g in $C_b^\infty(\mathcal{I}; \mathbb{R})$ such that

$$(2.20) \quad f(\phi(x)) \bar{H}^1(x) + \bar{H}^2(x) = g(\phi(x)), \quad \forall x \in \omega.$$

Couples (H, ϕ) which satisfy (2.18), (2.19) and (2.20) for some f and g are called *compatible* with (1.2).

Proof of Lemma 2.4. Note that

$$R(a, b) := \begin{cases} +\infty & \text{if } a < 0 \quad \text{and} \quad b > \frac{1}{4} a^2, \\ +\infty & \text{if } a \geq 0 \quad \text{and} \quad b \geq 0, \\ -\frac{1}{a} & \text{if } a < 0 \quad \text{and} \quad b = 0. \end{cases}$$

For all other values of (a, b) , one has

$$R(a, b) = -\frac{a}{2b} - \frac{(a^2 - 4b)^{\frac{1}{2}}}{2b}.$$

It follows that

$$R^{-1}([\tau, +\infty]) := \left\{ (a, b) \in \mathbb{R}^2; a \leq -2\tau^{-1}, b > \frac{1}{4}a^2 \right\} \\ \cup \left\{ (a, b) \in \mathbb{R}^2; a \geq -2\tau^{-1}, b \geq -\tau^{-2}(\tau a + 1) \right\}.$$

◦ Suppose that $\{h^\varepsilon\}_\varepsilon$ is subjected to (2.17). Observe that

$$I_{h^\varepsilon}(x) = \varepsilon^{-1} I_{-1}(x, \varepsilon^{-1} \phi(x)) + O(1), \quad I_{-1}(x, \theta) = {}^t(I_{-1}^1, I_{-1}^2)(x, \theta)$$

with $I_{-1}^1 = \nabla_x \phi \cdot \partial_\theta H^*$ and

$$I_{-1}^2 = \partial_1 \phi (\partial_\theta H^1 \partial_2 H^2 - \partial_\theta H^2 \partial_2 H^1) + \partial_2 \phi (\partial_\theta H^2 \partial_1 H^1 - \partial_\theta H^1 \partial_1 H^2).$$

Select any $(x, \theta) \in \omega \times \mathbb{T}$ such that $\phi(x) \neq 0$. Introduce

$$\varepsilon_k := \phi(x) (\theta + \iota k)^{-1}, \quad \iota := \text{sgn } \phi(x), \quad k \in \mathbb{N} \setminus \{0, 1\}.$$

By construction

$$(2.21) \quad I_{h^{\varepsilon_k}}(x) = \varepsilon_k^{-1} I_{-1}(x, \theta) + O(1).$$

This indicates that the sequence $\{I_{h^{\varepsilon_k}}(x)\}_k$ is asymptotic when k goes to ∞ with the half line

$$\mathcal{D} := \{ \lambda I_{-1}(x, \theta); \lambda \in \mathbb{R}_+ \}.$$

In view of the geometry of the set $R^{-1}([\tau, +\infty])$, this is compatible with (2.17) only if $I_{-1}^1(x, \theta) \geq 0$. Because of (2.13) and the continuity of the function I_{-1}^1 , it leads to

$$I_{-1}^1(x, \theta) \geq 0, \quad \forall (x, \theta) \in \omega \times \mathbb{T}.$$

The foregoing shows that I_{-1}^1 is a positive function. On the other hand, it is obviously a periodic function with mean zero. Therefore, it must be zero which is exactly the polarization condition (2.18). There is some scalar function $s \equiv s^* \in C^\infty(\omega \times \mathbb{T})$ which is such that

$$(2.22) \quad H(x, \theta) = \bar{H}(x) + s^*(x, \theta) \nabla_x \phi(x)^\perp, \quad \nabla_x \phi^\perp := \begin{pmatrix} -\partial_2 \phi \\ \partial_1 \phi \end{pmatrix}.$$

The information (2.22) allows to simplify the expression of

$$\begin{aligned} I_{-1}^2 &= [\partial_1 \phi \partial_2 \phi (\partial_1 \bar{H}^1 - \partial_2 \bar{H}^2) - (\partial_1 \phi)^2 \partial_2 \bar{H}^1 + (\partial_2 \phi)^2 \partial_1 \bar{H}^2] \partial_\theta s^* \\ &\quad + [-2 \partial_1 \phi \partial_2 \phi \partial_{12}^2 \phi + (\partial_1 \phi)^2 \partial_{22}^2 \phi + (\partial_2 \phi)^2 \partial_{11}^2 \phi] s^* \partial_\theta s^*. \end{aligned}$$

Since now $I_{-1}^1(x, \theta) = 0$, the sequence $\{I_{h^{\varepsilon_k}}(x)\}_k$ is asymptotic when k goes to ∞ with

$$\mathcal{D}_\pm := \{ (0, b); \pm b > 0 \} \quad \text{if} \quad \pm I_{-1}^2(x, \theta) > 0.$$

In view of the geometry of the set $R^{-1}([\tau, +\infty])$, this is compatible with (2.17) on condition that $I_{-1}^2(x, \theta) \geq 0$. Because of (2.13) and the continuity of the function I_{-1}^2 , it leads to

$$I_{-1}^2(x, \theta) \geq 0, \quad \forall (x, \theta) \in \omega \times \mathbb{T}.$$

Again I_{-1}^2 is a positive function. It is periodic and the formula given above indicates that it is with mean zero. Therefore, it must be zero. Since by hypothesis $\omega = \text{osup } s^*$, the condition $I_{-1}^2 \equiv 0$ amounts to the same thing as imposing for all $x \in \omega$ the two relations

$$(2.23) \quad \partial_1 \phi \partial_2 \phi (\partial_1 \bar{H}^1 - \partial_2 \bar{H}^2) - (\partial_1 \phi)^2 \partial_2 \bar{H}^1 + (\partial_2 \phi)^2 \partial_1 \bar{H}^2 = 0,$$

$$(2.24) \quad 2 \partial_1 \phi \partial_2 \phi \partial_{12}^2 \phi - (\partial_1 \phi)^2 \partial_{22}^2 \phi - (\partial_2 \phi)^2 \partial_{11}^2 \phi = 0.$$

In view of (2.13), the identity (2.24) is equivalent to

$$(2.25) \quad (-\partial_2 \phi \partial_1 + \partial_1 \phi \partial_2) (\partial_1 \phi / \partial_2 \phi) = 0, \quad \forall x \in \omega.$$

This differential equation implies that the quotient $\partial_1\phi/\partial_2\phi$ is locally constant on each level curve of ϕ . In fact, it is constant on the whole curve because by hypothesis the curve is connected. Since moreover the application $x_2 \mapsto \phi(x_1, x_2)$ is a local C^∞ diffeomorphism, the property (2.19) must be achieved for some $f \in C_b^\infty(\mathcal{I}; \mathbb{R})$.

Since ϕ is now subjected to the restriction (2.19), the relation (2.23) can be simplified according to

$$(2.26) \quad (-\partial_2\phi \partial_1 + \partial_1\phi \partial_2) (\bar{H}^2 + f(\phi) \bar{H}^1) = 0.$$

The same argument as above gives (2.20).

◦ Conversely, choose any couple (\bar{H}, ϕ) satisfying (2.19) and (2.20). Select any profile s^* . Define H as in (2.22). It implies that $I_{-1} \equiv 0$ so that

$$\exists C > 0; \quad |I_{h^\varepsilon}^1(x)| \leq C, \quad |I_{h^\varepsilon}^2(x)| \leq C, \quad \forall (\varepsilon, x) \in]0, 1] \times \omega.$$

In particular, we find (2.17) for some $\tau \in]0, 1]$. □

Remark 2.2.1 - existence of compatible couples. Take any $f \in C_b^\infty(\mathbb{R}; \mathbb{R})$ with $f' \leq 0$ and any $g \in C_b^\infty(\mathbb{R}; \mathbb{R})$. Select any function ϕ_0 with

$$\phi_0 \in C^\infty(\mathbb{R}), \quad \phi_0' \in C_b^\infty(\mathbb{R}), \quad \phi_0'(x_2) \geq 2c > 0, \quad \forall x_2 \in \mathbb{R}.$$

Solve the conservation law (2.19) where x_1 is interpreted as a time variable and the Cauchy data is $\phi(0, x_2) = \phi_0(x_2)$. It furnishes a solution of (2.19) which is defined on a domain of the form

$$\check{\omega} = \{ x = {}^t(x_1, x_2) \in \mathbb{R}^2; |x_1| < \eta \}, \quad \eta > 0.$$

If necessary, restrict η to be sure that (2.13) is verified. Take any s^* and any \bar{H}^1 . Extract \bar{H}^2 through (2.20). Then, piece together these ingredients to obtain a compatible couple (H, ϕ) which is defined on $\check{\omega}$.

When f' is not constant, the construction of a compatible couple (H, ϕ) which is globally defined on \mathbb{R}^2 is not possible. This is due to (2.19). When $f'' \equiv 0$, the characteristics are straight lines which cross. ◇

Introduce the family of lines

$$E(z) := \{ \lambda {}^t(-1, f(z)); \lambda \in \mathbb{R} \} \subset \mathbb{R}^2, \quad z \in \mathcal{I}.$$

The proof of Lemma 2.4 is based on a decomposition of the profile H in its oscillating part H^* and its mean value \bar{H} . Another point of view consists in looking at the vector valued function H in the basis $({}^t\nabla_x\phi, \nabla\phi^\perp)$. This changes the presentation of H .

Lemma 2.5. *Let $\phi \in C_b^\infty(\omega; \mathcal{I})$ satisfying (2.19). The profile H is subjected to (2.18) and (2.20) for some function $g \in C_b^\infty(\mathcal{I}; \mathbb{R})$ if and only if there is a scalar function $r \in C_b^\infty(\omega \times \mathbb{T}; \mathbb{R})$ (called the underlying scalar profile) and there are two smooth vector fields*

$$H_\triangleleft \in C_b^\infty(\mathcal{I}; \mathbb{R}^2), \quad H_\triangleright \in C_b^\infty(\mathcal{I} \times \mathbb{R}; \mathbb{R}^2)$$

which are polarized according to

$$(2.27) \quad H_\triangleleft(z) \subset E(z)^\perp, \quad H_\triangleright(z, y) \subset E(z), \quad \forall (z, y) \in \mathcal{I} \times \mathbb{R}$$

and which are such that

$$(2.28) \quad H(x, \theta) = H_\triangleleft(\phi(x)) + H_\triangleright(\phi(x), r(x, \theta)), \quad \forall (x, \theta) \in \omega \times \mathbb{T}.$$

Proof of Lemma 2.5. Decompose H according to $H = H_\triangleleft^c + H_\triangleright^c$ with

$$\begin{aligned} H_\triangleleft^c(x, \theta) &= |\nabla_x \phi(x)|^{-2} (\nabla_x \phi(x) \cdot H(x, \theta)) \nabla_x \phi(x), \\ H_\triangleright^c(x, \theta) &= |\nabla_x \phi(x)|^{-2} (\nabla_x \phi(x)^\perp \cdot H(x, \theta)) \nabla_x \phi(x)^\perp. \end{aligned}$$

◦ Suppose (2.18) and (2.20). Then

$$\begin{aligned} H_\triangleleft^c(x, \theta) &= H_\triangleleft(\phi(x)), & H_\triangleleft(z) &:= [1 + f(z)^2]^{-1} g(z) {}^t(f(z), 1), \\ H_\triangleright^c(x, \theta) &= H_\triangleright(\phi(x), r(x, \theta)), & H_\triangleright(z, y) &:= y {}^t(-1, f(z)). \end{aligned}$$

This can be achieved with the choice

$$r(x, \theta) = [1 + f(\phi)^2]^{-1} {}^t(-1, f(\phi)) \cdot H(x, \theta).$$

With these definitions, we recover (2.27) and (2.28).

◦ Conversely, suppose (2.27) and (2.28). Then

$$\nabla_x \phi(x) \cdot H^*(x, \theta) = \nabla_x \phi(x) \cdot [H_\triangleright(\phi(x), r(x, \theta))]^* = 0$$

and we obtain (2.20) with $g(z) = {}^t(f(z), 1) \cdot H_\triangleleft(z)$. □

2.3 Well prepared families for the Euler equation.

Our aim in this subsection is to identify the constraints to impose on ϕ and H in order to have

$$(2.29) \quad h^\varepsilon \in \mathcal{V}_0^\tau(\omega), \quad \forall \varepsilon \in]0, 1].$$

We work with $\mathbf{f}_0 \equiv 0$. The other cases $\iota = 1$ and especially $\iota = -1$ are interesting but they will not be considered here.

Definition 2.2. We say that the family $\{h^\varepsilon\}_\varepsilon$, where h^ε is defined as in (2.16) and is made of ingredients satisfying 2.a), \dots , 2.d), is well prepared on ω for the incompressible Euler equation (1.1) if

$$(2.30) \quad I_{h^\varepsilon}^1(x) = \operatorname{div}_x h^\varepsilon(x) = 0, \quad \forall (x, \varepsilon) \in \omega \times]0, 1],$$

$$(2.31) \quad I_{h^\varepsilon}^2(x) = \det(D_x h^\varepsilon(x)) = 0, \quad \forall (x, \varepsilon) \in \omega \times]0, 1].$$

In other words, a well prepared family $\{h^\varepsilon\}_\varepsilon$ for (1.1) is well prepared for (1.2) and is made of functions h^ε which are contained in $\mathcal{V}_0^T(\omega)$.

Suppose that $\{h^\varepsilon\}_\varepsilon$ is well prepared on ω for (1.1). According to Lemma 2.4, we have (2.16) with a phase ϕ and a principal profile H adjusted according to (2.18), (2.19) and (2.20). It is interesting to identify the other constraints satisfied by such ϕ and H . In fact, these supplementary conditions concern only the choice of g and of the function s^* defined in (2.22).

Lemma 2.6. Select any functions $\phi \in C_b^\infty(\omega; \mathbb{R})$ and $f \in C_b^\infty(\mathcal{I}; \mathbb{R})$ satisfying (2.19). Select any functions $K \in C_b^\infty(\mathcal{I} \times \mathbb{T}; \mathbb{R})$ and $f_0 \in C_b^\infty(\mathcal{I}; \mathbb{R})$. There exist some open set $\tilde{\omega} \subset \omega$ and a family $\{\check{h}^\varepsilon\}_\varepsilon$ which is given by an asymptotic expansion like (2.16) with $J \geq 0$, which is well prepared on $\tilde{\omega}$ for (1.1) and which is associated with the phase ϕ and the profile

$$(2.32) \quad H(x, \theta) = K(\phi(x), \theta + \phi_0(x, \theta)) \begin{pmatrix} 1 \\ -f \circ \phi(x) \end{pmatrix} + \begin{pmatrix} 0 \\ g \circ \phi(x) \end{pmatrix}$$

where

$$(2.33) \quad g(z) := K(0, 0) f(0) + \int_0^z \bar{K}(y) f'(y) dy, \quad z \in \mathcal{I}$$

and where ϕ_0 is subjected to the scalar conservation law

$$(2.34) \quad \begin{aligned} \partial_1 \phi_0 - f(\phi) \partial_2 \phi_0 + (-f'(\phi) \phi_0 + f_0(\phi)) \partial_2 \phi \partial_\theta \phi_0 \\ + (-f'(\phi) \phi_0 + f_0(\phi)) \partial_2 \phi = 0. \end{aligned}$$

Proof of Lemma 2.6. The restriction (2.31) means that we can find

$$\tilde{h}^\varepsilon = {}^t(\tilde{h}^{\varepsilon 1}, \tilde{h}^{\varepsilon 2}) \in C_b^\infty(\mathcal{I}; \mathbb{R}^2), \quad \varphi^\varepsilon \in C^\infty(\omega; \mathcal{I}), \quad \varepsilon \in]0, 1]$$

such that

$$(2.35) \quad \check{h}^\varepsilon(x) = {}^t(\check{h}^{\varepsilon 1}(x), \check{h}^{\varepsilon 2}(x)) = \tilde{h}^\varepsilon(\varphi^\varepsilon(x)), \quad \forall (x, \varepsilon) \in \omega \times]0, 1].$$

Choose any sequence

$$K_j \in C^\infty(\mathcal{I} \times \mathbb{T}; \mathbb{R}), \quad j \in \{0, \dots, J-1\}.$$

Define

$$\begin{aligned}\tilde{h}^{\varepsilon 1}(z) &= K^\varepsilon\left(z, \frac{z}{\varepsilon}\right) := K\left(z, \frac{z}{\varepsilon}\right) + \varepsilon \sum_{j=0}^{J-1} \varepsilon^j K_j\left(z, \frac{z}{\varepsilon}\right), \quad (z, \varepsilon) \in \mathcal{I} \times]0, 1], \\ \tilde{h}^{\varepsilon 2}(z) &= \int_0^z (-f(y) + \varepsilon f^\varepsilon(y)) (\tilde{h}^{\varepsilon 1})'(y) dy, \quad (z, \varepsilon) \in \mathcal{I} \times]0, 1],\end{aligned}$$

where

$$f^\varepsilon(z) = \sum_{j=0}^{J-1} \varepsilon^j f_j(z), \quad f_j \in C_b^\infty(\mathbb{R}; \mathbb{R}), \quad j \in \{0, \dots, J-1\}.$$

Two integrations by parts lead to

$$\begin{aligned}\tilde{h}^{\varepsilon 2}(z) &= (-f(z) + \varepsilon f^\varepsilon(z)) \tilde{h}^{\varepsilon 1}(z) - (-f(0) + \varepsilon f^\varepsilon(0)) \tilde{h}^{\varepsilon 1}(0) \\ &\quad - \int_0^z (-f'(y) + \varepsilon (f^\varepsilon)'(y)) \bar{K}^\varepsilon(y) dy \\ &\quad - \varepsilon (-f'(z) + \varepsilon (f^\varepsilon)'(z)) (\partial_\theta^{-1} K^{\varepsilon*})(z, \frac{z}{\varepsilon}) \\ &\quad + \varepsilon (-f'(0) + \varepsilon (f^\varepsilon)'(0)) (\partial_\theta^{-1} K^{\varepsilon*})(0, 0) \\ &\quad + \varepsilon \int_0^z (-f''(y) + \varepsilon (f^\varepsilon)''(y)) (\partial_\theta^{-1} K^{\varepsilon*})(y, \frac{y}{\varepsilon}) dy \\ &\quad + \varepsilon \int_0^z (-f'(y) + \varepsilon (f^\varepsilon)'(y)) (\partial_y \partial_\theta^{-1} K^{\varepsilon*})(y, \frac{y}{\varepsilon}) dy.\end{aligned}$$

The procedure can be repeated since for instance

$$(\partial_\theta^{-1} K^{\varepsilon*})(y, \frac{y}{\varepsilon}) = \varepsilon \partial_y [(\partial_\theta^{-1} \partial_\theta^{-1} K^{\varepsilon*})(y, \frac{y}{\varepsilon})] - \varepsilon (\partial_y \partial_\theta^{-1} \partial_\theta^{-1} K^{\varepsilon*})(y, \frac{y}{\varepsilon}).$$

It furnishes

$$\tilde{h}^{\varepsilon 2}(z) = \tilde{H}^2(z, \frac{z}{\varepsilon}) + \varepsilon \sum_{j=0}^{J-1} \varepsilon^j \tilde{H}_j^2(z, \frac{z}{\varepsilon}) + O(\varepsilon^{J+1})$$

with in particular

$$(2.36) \quad \tilde{H}^2(z, \theta) := -f(z) K(z, \theta) + f(0) K(0, 0) + \int_0^z \bar{K}(y) f'(y) dy,$$

$$(2.37) \quad \begin{aligned}\tilde{H}_0^2(z, \theta) &:= -f(z) K_0(z, \theta) + f_0(z) K(z, \theta) + f(0) K_0(0, 0) \\ &\quad - f_0(0) K(0, 0) + \int_0^z f'(y) \bar{K}_0(y) dy - \int_0^z f'_0(y) \bar{K}(y) dy \\ &\quad + f'(z) (\partial_\theta^{-1} K^*)(z, \theta) - f'(0) (\partial_\theta^{-1} K^*)(0, 0).\end{aligned}$$

The expression \tilde{h}^ε is adjusted so that (2.30) becomes the consequence of

$$(2.38) \quad \partial_1 \varphi^\varepsilon + (-f(\varphi^\varepsilon) + \varepsilon f^\varepsilon(\varphi^\varepsilon)) \partial_2 \varphi^\varepsilon = 0.$$

Seek solutions φ^ε of (2.38) in the form

$$(2.39) \quad \varphi^\varepsilon(x) = \phi(x) + \varepsilon \phi^\varepsilon(x, \phi(x)/\varepsilon), \quad \phi^\varepsilon \in C_b^\infty(\check{\omega} \times \mathbb{T}; \mathbb{R})$$

where $\phi(x)$ is subjected to (2.19) and $\phi^\varepsilon(x, \theta)$ can be expanded according to

$$\phi^\varepsilon(x, \theta) = \sum_{j=0}^{J+1} \varepsilon^j \phi_j(x, \theta) + \varepsilon^{J+2} r\phi^\varepsilon(x, \theta).$$

The expression ϕ^ε must satisfy

$$(2.40) \quad \partial_1 \phi^\varepsilon + a(\varepsilon, t, x, \phi^\varepsilon) \partial_2 \phi^\varepsilon + b(\varepsilon, t, x, \phi^\varepsilon) \partial_\theta \phi^\varepsilon + c(\varepsilon, t, x, \phi^\varepsilon) = 0$$

with

$$\begin{aligned} a(\varepsilon, t, x, z) &:= -f(\phi + \varepsilon z) + \varepsilon f^\varepsilon(\phi + \varepsilon z), \\ b(\varepsilon, t, x, z) &:= \left[-z \int_0^1 f'(\phi + \varepsilon s z) ds + f^\varepsilon(\phi + \varepsilon z) \right] \partial_2 \phi. \end{aligned}$$

Look at (2.40) as an evolution equation in the variable x_1 . Remark that $\mathcal{J} := \omega(0) \subset \mathbb{R}$ is an open interval containing 0. Choose functions

$$\phi_{0j} \in C_b^\infty(\mathcal{J} \times \mathbb{T}; \mathbb{R}), \quad r\phi_0^\varepsilon \in C_b^\infty(\mathcal{J} \times \mathbb{T}; \mathbb{R}), \quad j \in \{0, \dots, J+1\}$$

where the family $\{r\phi_0^\varepsilon\}_\varepsilon$ is such that

$$\sup \left\{ \|r\phi_0^\varepsilon\|_{L^\infty(\mathcal{J} \times \mathbb{T})} + \varepsilon^{J-2} \|D_{x_2, \theta} r\phi_0^\varepsilon\|_{L^\infty(\mathcal{J} \times \mathbb{T})}; \varepsilon \in]0, 1] \right\} < \infty.$$

Complete (2.40) with the initial data

$$(2.41) \quad \phi^\varepsilon(0, x_2, \theta) = \phi_0^\varepsilon(x_2, \theta), \quad \phi_0^\varepsilon = \sum_{j=0}^{J+1} \varepsilon^j \phi_{0j} + \varepsilon^{J+2} r\phi_0^\varepsilon.$$

Under these conditions, it is possible to find some open set $\check{\omega} \subset \omega \subset \mathbb{R}^2$ with $\check{\omega}(0) = \mathcal{J}$ and, for all $\varepsilon \in]0, 1]$, a solution ϕ^ε of (2.40)-(2.41) on $\check{\omega}$. The main term ϕ_0 is subjected to (2.34) and the remainder $r\phi^\varepsilon$ is controlled by

$$\sup \left\{ \|r\phi^\varepsilon\|_{L^\infty(\check{\omega} \times \mathbb{T})} + \varepsilon^{J-2} \|D_{x, \theta} r\phi^\varepsilon\|_{L^\infty(\check{\omega} \times \mathbb{T})}; \varepsilon \in]0, 1] \right\} < \infty.$$

By way of formula (2.39), this furnishes a solution φ^ε of (2.38). Plug φ^ε in \check{h}^ε as in (2.35) to get with $K_{-1} \equiv K$ and $\check{H}_{-1}^2 \equiv \check{H}^2$

$$\begin{aligned} \check{h}^{\varepsilon 1}(x) &= \sum_{j=-1}^{J-1} \varepsilon^{j+1} K_j(\phi(x) + \varepsilon \phi^\varepsilon(x), \phi(x)/\varepsilon + \phi^\varepsilon(x)), \\ \check{h}^{\varepsilon 2}(x) &= \sum_{j=-1}^{J-1} \varepsilon^{j+1} \check{H}_j^2(\phi(x) + \varepsilon \phi^\varepsilon(x), \phi(x)/\varepsilon + \phi^\varepsilon(x)) + O(\varepsilon^{J+1}). \end{aligned}$$

Use a Taylor formula to recover (2.16) with ingredients satisfying 2.a), \dots , 2.d). The relation (2.36) leads to (2.32) with g as indicated. Obviously, the family $\{\check{h}^\varepsilon\}_\varepsilon$ is well prepared on $\check{\omega}$ for (1.1). \square

We can also deduce from the preceding construction the following more refined informations.

Lemma 2.7. *Select any functions $K_0 \in C_b^\infty(\mathcal{I} \times \mathbb{T}; \mathbb{R})$ and $f_1 \in C_b^\infty(\mathcal{I}; \mathbb{R})$. In the framework of Lemma 2.6 applied with $J \geq 1$, the second profile H_0 can be put in the form*

$$H_0(x, \theta) = \check{H}_0(x, \phi(x), \theta + \phi_0(x, \theta)), \quad \check{H}_0 \in C_b^\infty(\check{\omega} \times \mathbb{R} \times \mathbb{T}; \mathbb{R}^2).$$

Above, the function $\check{H}_0 = {}^t(\check{H}_0^1, \check{H}_0^2)$ can be adjusted so that

$$\begin{aligned}\check{H}_0^1(x, z, \theta) &= \phi_0(x, \theta) (\partial_z K)(z, \theta) + \phi_1(x, \theta) (\partial_\theta K)(z, \theta) + K_0(z, \theta), \\ \check{H}_0^2(x, z, \theta) &= \phi_0(x, \theta) (\partial_z \check{H}^2)(z, \theta) + \phi_1(x, \theta) (\partial_\theta \check{H}^2)(z, \theta) + \check{H}_0^2(z, \theta),\end{aligned}$$

with \check{H}^2 and \check{H}_0^2 defined according to (2.36) and (2.37), and with ϕ_1 subjected to the scalar conservation law

$$(2.42) \quad \begin{aligned}\partial_1 \phi_1 - f(\phi) \partial_2 \phi_1 + \partial_2 \phi (-f'(\phi) \phi_0 + f_0(\phi)) \partial_\theta \phi_1 \\ - \partial_2 \phi (1 + \partial_\theta \phi_0) f'(\phi) \phi_1 + \partial_2 \phi_0 (-f'(\phi) \phi_0 + f_0(\phi)) \\ + \partial_2 \phi (1 + \partial_\theta \phi_0) \left(-\frac{1}{2} f''(\phi) \phi_0^2 + f'_0(\phi) \phi_0 + f_1(\phi)\right) = 0.\end{aligned}$$

Remark 2.3.2 - more general constructions. The preceding description of families which are well prepared for (1.1) is not exhaustive. For instance, for all $j \in \{0, \dots, J-1\}$, the function $f_j(z)$ can be replaced by $f_j(z, \frac{z}{\varepsilon})$ which yields more complicated formulae. \diamond

3 Simple waves.

A simple wave is a solution of (1.1) or (1.2) having the form (1.7) where the profile \mathbf{H} and the phase Φ do not depend on ε . The functions \mathbf{H} and Φ are chosen smooth on some domain of determinacy $\Omega \subset \mathbb{R} \times \mathbb{R}^2$, say

$$\mathbf{H} \in C_b^\infty(\Omega \times \mathbb{T}; \mathbb{R}^2), \quad \Phi \in C^\infty(\Omega; \mathbb{R}).$$

3.1 The case of Burger equations.

The aim of this subsection 3.1 is to construct *all* simple waves which are associated with (1.2). Suppose that a family $\{\tilde{\mathbf{u}}^\varepsilon\}_{\varepsilon \in]0,1]}$ is made of C^1 solutions on Ω to the Burger equation (1.2) and is given by a formula like (1.7). Then, the corresponding initial data is

$$\mathbf{H}(0, x, \Phi(0, x)/\varepsilon) = H(x, \phi(x)/\varepsilon)$$

with

$$H(x, \theta) := \mathbf{H}(0, x, \theta), \quad \phi(x) := \Phi(0, x).$$

To avoid inside Ω the crossing of characteristics, necessarily the couple (H, ϕ) must be compatible. It means that the expressions H and ϕ are adjusted as in Lemma 2.4, with ingredients f , g and s^* yielding (2.19), (2.20) and (2.22). Now, there is a natural way to associate with such (H, ϕ) a simple wave. Define

$$s(x, \theta) := -\bar{H}^1(x) / \partial_2 \phi(x) + s^*(x, \theta).$$

Introduce the eiconal equation

$$(3.1) \quad \partial_t \Phi + g(\Phi) \partial_2 \Phi = 0$$

and the scalar conservation law

$$(3.2) \quad \begin{aligned} \partial_t \mathbf{s} + g(\Phi) \partial_2 \mathbf{s} + \mathbf{s} (-\partial_2 \Phi \partial_1 + \partial_1 \Phi \partial_2) \mathbf{s} \\ = \partial_2 \Phi \mathbf{s} (g'(\Phi) + f'(\Phi) \partial_2 \Phi \mathbf{s}) - |\nabla_x \Phi|^{-2} {}^t \nabla_x \Phi^\perp \cdot \mathbf{f}. \end{aligned}$$

Complete these equations with the initial data

$$(3.3) \quad \Phi(0, x) = \phi(x), \quad \mathbf{s}(0, x, \theta) = s(x, \theta).$$

Lemma 3.1. *Select any compatible couple (H, ϕ) . There is a domain of determinacy Ω of the form*

$$\Omega(\tau) := \{ (t, x) \in [0, \tau[\times \mathbb{R}^2; B(x, \tau^{-1} t) \subset \omega \}, \quad \tau \in]0, 1]$$

such that the Cauchy problems (3.1)-(3.3) and (3.2)-(3.3) have solutions Φ and \mathbf{s} respectively on $\Omega(\tau)$ and $\Omega(\tau) \times \mathbb{T}$. With these ingredients, build the large amplitude wave

$$(3.4) \quad \begin{aligned} \tilde{\mathbf{u}}^\varepsilon(t, x) &= \mathbf{H}(t, x, \Phi(t, x)/\varepsilon) \\ &:= \mathbf{s}(t, x, \Phi(t, x)/\varepsilon) \begin{pmatrix} -\partial_2 \Phi(t, x) \\ \partial_1 \Phi(t, x) \end{pmatrix} + \begin{pmatrix} 0 \\ g(\Phi(t, x)) \end{pmatrix}. \end{aligned}$$

At the time $t = 0$, one has

$$\tilde{\mathbf{u}}^\varepsilon(0, x) = \tilde{h}^\varepsilon(x) := H(x, \phi(x)/\varepsilon).$$

Assume that the source term \mathbf{f} is adjusted so that $\nabla_x \Phi \cdot \mathbf{f} \equiv 0$. Then, the expression $\tilde{\mathbf{u}}^\varepsilon$ is a simple wave which is a solution on $\Omega(\tau)$ of (1.2). Moreover, the relations (2.19) and (2.20) are conserved during the evolution.

Proof of Lemma 3.1. Start by checking the initial data. Use (2.19), (2.20) and (2.22) to obtain

$$\begin{aligned} \mathbf{H}(0, x, \theta) &= \mathbf{s}(0, x, \theta) \nabla_x \phi(x)^\perp + {}^t(0, g \circ \phi(x)) \\ &= \bar{H}(x) + s^*(x, \theta) \nabla_x \phi(x)^\perp = H(x, \theta). \end{aligned}$$

The definition (3.4) clearly implies that

$$\mathbf{H}^*(t, x, \theta) \cdot \nabla_x \Phi(t, x) = 0, \quad \forall (t, x, \theta) \in \Omega(\tau) \times \mathbb{T}.$$

Plug (3.4) at the level of (1.2). Say that the expressions with ε^{-1} and ε^0 in factor are separately equal to zero. Then, collect all the preceding informations to get the system (1.8) which is overdetermined.

For general choices of (H, ϕ) , the system (1.8) has no solution. In fact, the matter is to show that the Cauchy problem (1.8) is (locally in time) well-posed once the couple (H, ϕ) is compatible with (1.2). Lemma 3.1 proposes to solve (1.8) by using the formula

$$(3.5) \quad \mathbf{H}(t, x, \theta) = \mathbf{s}(t, x, \theta) \begin{pmatrix} -\partial_2 \Phi(t, x) \\ \partial_1 \Phi(t, x) \end{pmatrix} + \begin{pmatrix} 0 \\ g(\Phi(t, x)) \end{pmatrix}.$$

This formulation (3.5) contains obviously the polarization condition on \mathbf{H}^* . It is adjusted so that the second equation of (1.8) gives rise to (3.1). From (3.1), it is also possible to extract

$$[\partial_t + g(\Phi) \partial_2 + \partial_2 \Phi g'(\Phi)] (\partial_1 \Phi - f(\Phi) \partial_2 \Phi) = 0.$$

It follows that the relations (2.19) and (2.20) are propagated during the evolution. In other words

$$(3.6) \quad \partial_1 \Phi(t, x) = f(\Phi(t, x)) \partial_2 \Phi(t, x), \quad \forall (t, x) \in \Omega(\tau),$$

$$(3.7) \quad \bar{\mathbf{H}}^2(t, x) + f(\Phi(t, x)) \bar{\mathbf{H}}^1(t, x) = g(\Phi(t, x)), \quad \forall (t, x) \in \Omega(\tau).$$

Consider now the first equation in (1.8). Because of (3.2), it reduces to

$$(3.8) \quad \begin{aligned} \partial_t \nabla_x \Phi^\perp + g(\Phi) \partial_2 \nabla_x \Phi^\perp + \partial_2 \Phi g'(\Phi) \nabla_x \Phi^\perp \\ + \mathbf{s} [(\nabla_x \Phi^\perp \cdot \nabla_x) \nabla_x \Phi^\perp + f'(\Phi) (\partial_2 \Phi)^2 \nabla_x \Phi^\perp] = 0. \end{aligned}$$

On the one hand, the equation (3.6) guarantees that

$$(-\partial_2 \Phi \partial_1 + \partial_1 \Phi \partial_2) \nabla_x \Phi^\perp = -f'(\Phi) (\partial_2 \Phi)^2 \nabla_x \Phi^\perp.$$

On the other hand, the derivation of (3.1) furnishes

$$\partial_t \nabla_x \Phi^\perp + g(\Phi) \partial_2 \nabla_x \Phi^\perp + \partial_2 \Phi g'(\Phi) \nabla_x \Phi^\perp = 0.$$

Thus, the identity (3.8) is verified. \square

There is another way to consider \mathbf{H} . This point of view consists in appealing to the framework of Lemma 2.5.

Lemma 3.2. *The profile \mathbf{H} can be written*

$$\mathbf{H}(t, x, \theta) = \tilde{H}(\Phi(t, x), \mathbf{r}(t, x, \theta)), \quad \tilde{H}(z, y) = H_{\triangleleft}(z) + H_{\triangleright}(z, y)$$

where the phase Φ is given by

$$(3.9) \quad \partial_t \Phi + (H_{\triangleleft}(\Phi) \cdot \nabla_x) \Phi = 0, \quad \Phi(0, x) = \phi(x)$$

whereas the function \mathbf{r} is obtained by solving the scalar conservation law

$$(3.10) \quad \begin{cases} \partial_t \mathbf{r} + (\tilde{H}(\Phi, \mathbf{r}) \cdot \nabla_x) \mathbf{r} + (1 + f(\Phi)^2)^{-1} {}^t(-1, f(\Phi)) \cdot \mathbf{f} = 0, \\ \mathbf{r}(0, x, \theta) = r(x, \theta). \end{cases}$$

Remark 3.1.1 - about \mathbf{r} . To solve (1.8), it suffices first to consider the eiconal equation (3.9) and then to look at the transport equation (3.10). For obvious reasons, the function \mathbf{r} is called the *underlying scalar profile*. \diamond

Proof of Lemma 3.2. In view of Lemma 2.5, we have

$$\mathbf{H}(0, x, \theta) = H(x, \theta) = \tilde{H}(\phi(x), r(x, \theta)).$$

The solution of (1.8), once it exists, is unique. Therefore, it suffices to show that the expressions Φ and \mathbf{H} which can be extracted from (3.9) and (3.10) satisfy the system (1.8). The condition (3.9) implies that

$$[\partial_t + H_{\triangleleft}(\Phi) \cdot \nabla_x + H'_{\triangleleft}(\Phi) \cdot \nabla_x \Phi] (\partial_1 \Phi - f(\Phi) \partial_2 \Phi) = 0.$$

One has (3.6) which means that $\nabla_x \Phi \perp E(\Phi)$. Using (2.27), it gives rise to

$$\bar{\mathbf{H}} \cdot \nabla_x \Phi = H_{\triangleleft}(\Phi) \cdot \nabla_x \Phi, \quad \nabla_x \Phi \cdot \mathbf{H}^* = \nabla_x \Phi \cdot H_{\triangleright}(\Phi, \mathbf{r})^* = 0.$$

Thus, (3.9) is the same as the second equation of (1.8) whereas the third equation of (1.8) is verified. It remains to compute

$$\partial_t \mathbf{H} + (\mathbf{H} \cdot \nabla_x) \mathbf{H} = [\partial_t \mathbf{r} + (\mathbf{H} \cdot \nabla_x) \mathbf{r}] \partial_y H_{\triangleright}(\Phi, \mathbf{r}) + \mathbf{f}$$

which is equal to zero because of (3.10). \square

Remark 3.1.2 - weak convergence. Look at the special choices

$$\mathbf{s}(t, x, \theta) = x_1 \cos(2\Pi\theta) / (1 + t \cos(2\Pi\theta)), \quad \Phi(t, x) = -x_2,$$

which satisfy all the required conditions (with $f \equiv 0$ and $g \equiv 0$). The weak limit $\bar{\mathbf{u}}$ of the corresponding family $\{\bar{\mathbf{u}}^\varepsilon\}_\varepsilon$ is

$$\bar{\mathbf{u}}(t, x) = x_1 \int_0^1 [\cos(2\Pi\theta) / (1 + t \cos(2\Pi\theta))] d\theta \quad t(1, 0).$$

Observe that $\bar{\mathbf{u}}$ is not a solution of (1.2) since

$$(\partial_t \bar{\mathbf{u}}_1 + (\bar{\mathbf{u}} \cdot \nabla_x) \bar{\mathbf{u}}_1)(0, x) = -x_1 \int_0^1 (\cos(2\Pi\theta))^2 d\theta \neq 0.$$

It means that (1.2) is not closed for the weak topology of L^2 . Now, this is a very natural expectation since (1.2) has no conservative form. \diamond

Remark 3.1.3 - special diffeomorphisms. Select again a couple (H, ϕ) which is compatible with (1.2). The application

$$\Xi_t^\varepsilon : x \longmapsto x + t H(x, \phi(x)/\varepsilon)$$

is a local diffeomorphism whose inverse can be explicitated. It is

$$(\Xi_t^\varepsilon)^{-1} : x \longmapsto x - t \mathbf{H}(t, x, \Phi(t, x)/\varepsilon)$$

where \mathbf{H} and Φ are obtained by solving the nonlinear system (1.8). \diamond

Remark 3.1.4 - caustics of the first type. There is no constraint (except the regularity) on the choice of g . When the function g is non increasing the solution Φ of (3.1) develops shocks which correspond to the formation of caustics for the Burger equation. \diamond

3.2 The case of incompressible equations.

The study of the overdetermined system (1.10) is more delicate than the one of (1.8). We restrict here our attention to the case $\mathbf{p} \equiv \mathbf{p}_\iota$. In this particular context, the construction of simple waves which are incompressible needs to restrict the choice of the phase.

Lemma 3.3. *Among the solutions to (1.2) given by Lemma 3.1, those who satisfy also (2.8) are plane waves*

$$(3.11) \quad \tilde{\mathbf{u}}^\varepsilon(t, x) = \left[k\left(z, \frac{\psi(z)}{\varepsilon}\right) \psi'(z) \right]_{|z=at+bx_1+x_2} \begin{pmatrix} -1 \\ b \end{pmatrix} - \begin{pmatrix} 0 \\ a \end{pmatrix}$$

built with $k \in C_b^\infty(\mathbb{R} \times \mathbb{T}; \mathbb{R})$, $\psi \in C^\infty(\mathbb{R}; \mathbb{R})$ and $(a, b) \in \mathbb{R}^2$.

Proof of Lemma 3.3. Select a general simple wave $\tilde{\mathbf{u}}^\varepsilon$ as in (3.4), involving ingredients Φ , \mathbf{s} and \mathbf{H} as in (3.1), (3.2) and (3.5). In fact, the matter is to solve (1.8) where $\mathbf{f} \equiv \mathbf{f}_\iota$ and where the condition $\operatorname{div}_x \mathbf{H} \equiv 0$ is added. This amounts to the same thing as looking at the scalar conservation law (3.2) completed with the constraints

$$(3.12) \quad \nabla_x \Phi^\perp \cdot \nabla_x \mathbf{s}^* = 0, \quad \nabla_x \Phi^\perp \cdot \nabla_x \bar{\mathbf{s}} + g'(\Phi) \partial_2 \Phi = 0.$$

The first condition means that $\mathbf{s}^*(t, \cdot)$ is a function of $\Phi(t, \cdot)$. The second condition, when imposed at time $t = 0$, must be preserved during the evolution induced by (3.2). The initial data ϕ and s must be adjusted to this end. However, it is delicate to identify at the level of (3.2) the constraints to impose on ϕ and s . The difficulties are due to the interplay between the oscillating part \mathbf{s}^* and the mean value $\bar{\mathbf{s}}$.

Below, we adopt another point of view. Our argument consists in appealing to the criterion of the paragraph 2.1.3. The expression

$$\tilde{h}^\varepsilon(x) = \tilde{\mathbf{u}}^\varepsilon(0, x) = \mathbf{H}(0, x, \Phi(0, x)/\varepsilon) = H(x, \phi(x)/\varepsilon)$$

must satisfy

$$(3.13) \quad \operatorname{div}_x \tilde{h}^\varepsilon(x) = 0, \quad \det D_x \tilde{h}^\varepsilon(x) = \iota, \quad \forall x \in \omega.$$

The divergence free condition yields

$$\operatorname{div}_x \tilde{h}^\varepsilon(x) = \nabla_x s(x, \phi(x)/\varepsilon) \cdot \nabla_x \phi(x)^\perp + g'(\phi(x)) \partial_2 \phi(x) = 0.$$

Pass to the weak limit ($\varepsilon \rightarrow 0$) to find

$$\nabla_x s(x, \theta) \cdot \nabla_x \phi(x)^\perp + g'(\phi(x)) \partial_2 \phi(x) = 0, \quad \forall (x, \theta) \in \omega \times \mathbb{T}.$$

Separate the mean value

$$(3.14) \quad -\partial_2 \phi(x) \partial_1 \bar{s}(x) + \partial_1 \phi(x) \partial_2 \bar{s}(x) + g'(\phi(x)) \partial_2 \phi(x) = 0$$

from the oscillating part

$$(3.15) \quad \nabla_x s^*(x, \theta) \cdot \nabla_x \phi(x)^\perp = 0.$$

As expected, the conditions (3.14) and (3.15) are the same as (3.12). In particular, there is $k^* \in C_b^\infty(\mathcal{I} \times \mathbb{T}; \mathbb{R})$ such that

$$(3.16) \quad s^*(x, \theta) = k^*(\phi(x), \theta), \quad k^* \not\equiv 0, \quad \forall (x, \theta) \in \omega \times \mathbb{T}.$$

For $j = 1$ or for $j = 2$, one has

$$\begin{aligned} \partial_j \tilde{h}^\varepsilon(x) &= \partial_j \left[\partial_2 \phi(x) s\left(x, \frac{\phi(x)}{\varepsilon}\right) \right]^t (-1, f(\phi(x))) \\ &\quad + \partial_j \phi(x) \left[\partial_2 \phi(x) f'(\phi(x)) s\left(x, \frac{\phi(x)}{\varepsilon}\right) + g'(\phi(x)) \right]^t (0, 1). \end{aligned}$$

Mark the coefficients

$$\begin{aligned} a^\varepsilon(x) &:= \partial_2 \phi(x) f'(\phi(x)) s\left(x, \frac{\phi(x)}{\varepsilon}\right) + g'(\phi(x)) \\ &= \bar{a}(x) + \bar{m}_a(x) k^*\left(\phi(x), \frac{\phi(x)}{\varepsilon}\right). \\ b^\varepsilon(x) &:= \partial_1 \left[\partial_2 \phi(x) s\left(x, \frac{\phi(x)}{\varepsilon}\right) \right] - f(\phi(x)) \partial_2 \left[\partial_2 \phi(x) s\left(x, \frac{\phi(x)}{\varepsilon}\right) \right] = 0. \\ &= \partial_2 \phi(x) \partial_1 \bar{s}(x) - \partial_1 \phi(x) \partial_2 \bar{s}(x) + f'(\phi(x)) (\partial_2 \phi(x))^2 s\left(x, \frac{\phi(x)}{\varepsilon}\right) \\ &= \bar{b}(x) + \bar{m}_b(x) k^*\left(\phi(x), \frac{\phi(x)}{\varepsilon}\right). \end{aligned}$$

Now, examine the second condition in (3.13) which is equivalent to

$$\partial_2 \phi(x) a^\varepsilon(x) b^\varepsilon(x) + \iota = 0, \quad \forall \varepsilon \in]0, 1].$$

Pass to the weak limit ($\varepsilon \rightarrow 0$) to find

$$\begin{aligned} \bar{a}(x) \bar{b}(x) + \iota / \partial_2 \phi(x) + (\bar{m}_a(x) + \bar{m}_b(x)) k^*(\phi(x), \theta) \\ + \bar{m}_a(x) \bar{m}_b(x) k^*(\phi(x), \theta)^2 = 0. \end{aligned}$$

Since this must be true for all $(x, \theta) \in \omega \times \mathbb{T}$, this is the same as

$$\bar{a}(x) \bar{b}(x) = 0, \quad \bar{m}_a(x) + \bar{m}_b(x) = 0, \quad \bar{m}_a(x) \bar{m}_b(x) = 0.$$

In particular

$$\bar{m}_a(x) \bar{m}_b(x) = (\partial_2 \phi(x))^3 f'(\phi(x))^2 = 0.$$

Necessarily f is a constant, say $f = b \in \mathbb{R}$. In view of (2.19), one has

$$\exists \psi \in C^\infty(\mathbb{R}; \mathbb{R}); \quad \pm \psi' > 0, \quad \phi(x) = \psi(bx_1 + x_2).$$

Now, remember that the construction of Lemma 3.1 requires a condition on \mathbf{f} . This condition concerns here \mathbf{f}_ι and, at time $t = 0$, it is

$$\nabla_x \Phi(0, x) \cdot \mathbf{f}_\iota(0, x) = \iota (bx_1 + x_2) \psi'(bx_1 + x_2) = 0, \quad \forall x \in \omega.$$

Since $\psi' \not\equiv 0$, this is possible only if $\iota = 0$. Therefore, the hyperbolic and elliptic cases ($\iota = -1$ and $\iota = 1$) are excluded. From now on, take $\iota = 0$. Use (3.14) to extract

$$\bar{a}(x) \bar{b}(x) = \partial_2 \phi(x) g'(\phi(x))^2 = 0.$$

It means that g is a constant, say $g = -a \in \mathbb{R}$. In view of (3.14), one has

$$\exists \bar{k} \in C^\infty(\mathbb{R}; \mathbb{R}); \quad \bar{s}(x) = \bar{k}(bx_1 + x_2).$$

Solve (3.1) to get

$$\Phi(t, x) = \psi(at + bx_1 + x_2).$$

It remains to look at (3.2) that is

$$\partial_t \mathbf{s} - a \partial_2 \mathbf{s} + \mathbf{s} \psi' (-\partial_1 + b \partial_2) \mathbf{s} = 0.$$

Observe that the choice

$$\mathbf{s}(t, x, \theta) = \bar{k}(z) + k^*(\psi(z), \theta) \psi'(z) {}^t(-1, b), \quad z = at + bx_1 + x_2$$

is convenient. This is the expected result. \square

Remark 3.2.1 - intuitive derivation of Lemma 3.3. The restriction on Φ contained in Lemma 3.3 can be guessed by looking at the construction underlying Lemma 2.6. To find simple waves, small amplitude terms (of size ε^j with $j \geq 1$) must be suppressed. Now, to get

$$\mathbf{u}^\varepsilon(0, x) = \check{h}^\varepsilon(x) = {}^t(\check{h}^{\varepsilon 1}(x), \check{h}^{\varepsilon 2}(x)) = H(x, \phi(x)/\varepsilon)$$

it is convenient to take

$$r\phi^\varepsilon \equiv 0, \quad \phi_j \equiv 0, \quad K_j \equiv 0, \quad \forall j \in \{0, \dots, J+1\}.$$

Then, consider $\check{h}^{\varepsilon 2}$. We must have

$$\partial_\theta \check{H}_0^2(x, z, \theta) = \partial_\theta \check{H}_0^2(z, \theta) = f_0(z) \partial_\theta K^*(z, \theta) + f'(z) K^*(z, \theta) = 0.$$

This is possible only if $f' \equiv 0$ on the set $\check{\omega} = \text{osup } K^* \neq \emptyset$. It follows that $g \equiv -a$ for some $a \in \mathbb{R}$ and, therefore, that $\Phi(t, x) = \psi(at + bx_1 + x_2)$ for some function $\psi \in C^\infty(\mathbb{R}; \mathbb{R})$. \diamond

Remark 3.2.2 - weak convergence. The weak limit of the family $\{\check{\mathbf{u}}^\varepsilon\}_\varepsilon$ is

$$\check{\mathbf{u}}(t, x) = (\bar{k} \psi')(at + bx_1 + x_2) \begin{matrix} t \\ -1, b \end{matrix} - \begin{matrix} t \\ 0, a \end{matrix}.$$

Adjust the profile k so that

$$\bar{k}(z)^2 \neq \int_0^1 k(z, \theta)^2 d\theta.$$

Then, the weak limit of the product $\{\check{\mathbf{u}}^{\varepsilon_1} \check{\mathbf{u}}^{\varepsilon_2}\}_\varepsilon$ differs from the product of the weak limits, that is $\check{\mathbf{u}}^1 \check{\mathbf{u}}^2$. A similar remark is made in [1] (see p. 495, example 12.9). Observe however that $\check{\mathbf{u}}(t, x)$ is still a solution to (1.1). Therefore, this argument does not bring any contradiction with the property of concentration-cancellation. \diamond

4 The problem of stability.

Let (H, ϕ) be a compatible couple. Lemma 3.1 describes the evolution of the large amplitude wave $\tilde{\mathbf{u}}^\varepsilon$ which is issued from

$$\tilde{\mathbf{u}}^\varepsilon(0, x) = \tilde{h}^\varepsilon(x) := H(x, \phi(x)/\varepsilon).$$

Now, consider the perturbed initial data $h^\varepsilon(x) = H^\varepsilon(x, \phi(x)/\varepsilon)$ built with a profile H^ε defined according to

$$(4.1) \quad H^\varepsilon(x, \theta) := H(x, \theta) + \varepsilon \sum_{j=0}^{J-1} \varepsilon^j H_j(x, \theta) + \varepsilon^{J+1} rh^\varepsilon(x).$$

Above, the profiles H_j and the remainder rh^ε are adjusted as in 2.c) and 2.d) of paragraph 2.1.4. By virtue of subsection 2.2, the family $\{h^\varepsilon\}_\varepsilon$ is still well prepared on ω for (1.2) so that the oscillating Cauchy problem

$$(4.2) \quad \partial_t \mathbf{u}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla_x) \mathbf{u}^\varepsilon + \mathbf{f} = 0, \quad \mathbf{u}^\varepsilon(0, x) = h^\varepsilon(x), \quad \varepsilon \in]0, 1]$$

is well posed on $\Omega(\tau)$ for some $\tau > 0$. In other words, as explained in the introduction, the existence of a solution \mathbf{u}^ε to (4.2) on a domain $\Omega(\tau)$ which does not shrink to the empty set as ε goes to zero is guaranteed *a priori*.

On the other hand, the analysis of chapters 2 and 3 says nothing about the asymptotic behavior on $\Omega(\tau)$ of the family $\{\mathbf{u}^\varepsilon\}_\varepsilon$. To see where the difficulty is, look at the linearized equation along $\tilde{\mathbf{u}}^\varepsilon$, that is

$$\partial_t \dot{\mathbf{u}} + (\tilde{\mathbf{u}}^\varepsilon \cdot \nabla_x) \dot{\mathbf{u}} + (\dot{\mathbf{u}} \cdot \nabla_x) \mathbf{H} + \varepsilon^{-1} (\dot{\mathbf{u}} \cdot \nabla_x \Phi) \partial_\theta \mathbf{H} = 0.$$

Multiplying by ${}^t\dot{\mathbf{u}}$ and integrating over \mathbb{R}^2 yields the rough control

$$\frac{d}{dt} \|\dot{\mathbf{u}}(t)\|_{L^2(\mathbb{R}^2)} \leq C \varepsilon^{-1} \|\dot{\mathbf{u}}(t)\|_{L^2(\mathbb{R}^2)} .$$

This inequality does not provide with energy estimates which are uniform in $\varepsilon \in]0, 1]$. It allows the presence of instabilities. Nothing guarantees that \mathbf{u}^ε remains closed to $\tilde{\mathbf{u}}^\varepsilon$ and, in fact, what follows shows that this is not the case. More precisely, the difference $\mathbf{u}^\varepsilon - \tilde{\mathbf{u}}^\varepsilon$ is of size $O(\varepsilon)$ when $t = 0$ but it can become of size $O(1)$ for some $t \in]0, 1]$.

The possibility of such a mechanism of amplification is already hidden behind the discussion of chapters 2 and 3. Indeed, there is a contrast between the weak conditions made on (H, ϕ) in Lemma 2.6 and the strong restrictions imposed in Lemma 3.3. The first conditions (imposed on the *main* term of the oscillation) are weaker *because* small perturbations of size $O(\varepsilon)$ are allowed. This remark seems alleviating but it announces in fact the following more general principle.

When dealing with the propagation of large amplitude high frequency oscillating waves, the shape of the main contribution which is of size $O(1)$ is generically coupled with what happens at the level of smaller terms, for instance of size $O(\varepsilon)$.

The aim of this chapter 4 is to clarify this assertion. The links between large and small terms is cleared up in the next subsection.

4.1 A blow up procedure.

From now on, fix \mathbf{H} , Φ and \mathbf{f} as in Lemma 3.1. Work with both slow and fast variables (t, x, θ) . Use the representation (1.11) with

$$\mathbf{U}^\varepsilon(t, x, \theta) \in C_b^1(\Omega(\tau) \times \mathbb{T}; \mathbb{R}^2), \quad \forall \varepsilon \in]0, 1] .$$

The equation (1.2) becomes (1.12). It is completed with

$$(4.3) \quad \mathbf{U}^\varepsilon(0, x, \theta) = H^\varepsilon(x, \theta) .$$

At first sight, nothing guarantees that the Cauchy problem (1.12)-(4.3) is well posed on $\Omega(\tau) \times \mathbb{T}$. On the one hand, the analysis of chapter 2 brings no information since it is not adapted to (1.12)-(4.3). On the other hand, the usual methods do not apply at the level of (1.12).

Indeed, they rely on energy estimates related to the linearized equation along $\mathbf{H}(t, x, \theta)$, that is (1.13). Now, as before, this equation (1.13) leads to

$$\frac{d}{dt} \|\dot{\mathbf{U}}(t)\|_{L^2(\mathbb{R}^2 \times \mathbb{T})} \leq C \varepsilon^{-1} \|\dot{\mathbf{U}}(t)\|_{L^2(\mathbb{R}^2 \times \mathbb{T})} .$$

This differential equation is not sufficient to obtain some control which is uniform with respect to the parameter $\varepsilon \in]0, 1]$. In fact, the structure of $H^\varepsilon(x, \theta)$ is too rigid to absorb a perturbation of size $O(\varepsilon)$. More flexibility is needed. Observe that a translation on the fast variable

$$\begin{aligned} T_{\mathbf{V}^1} : C^1(\Omega(\tau) \times \mathbb{T}) &\longrightarrow C^1(\Omega(\tau) \times \mathbb{T}) \\ \mathbf{H}(t, x, \theta) &\longmapsto \mathbf{H}(t, x, \theta + \mathbf{V}^1) \end{aligned}, \quad \mathbf{V}^1 \in \mathbb{R}$$

alters the profile \mathbf{H} without changing the energy $\langle \mathbf{H}^2 \rangle$ of the oscillation. From this point of view, the application $T_{\mathbf{V}^1}$ plays the part of a *gauge transformation*. Now, the idea is to incorporate \mathbf{V}^1 as a new state variable. As explained in the introduction, this can be achieved through a *dependant change of variables*.

Proposition 4.1. *Select data \mathbf{f} and H^ε as in (2.1) and (4.1). There is $\tau > 0$ such that, for all $\varepsilon \in]0, 1]$, the Cauchy problem (1.12)-(4.3) has a solution \mathbf{U}^ε on a fixed domain $\Omega(\tau) \times \mathbb{T}$ (which does not shrink to the empty set as ε goes to zero). Moreover, there are smooth functions*

$$\begin{aligned} X^\varepsilon &\equiv X(\varepsilon, t, x, \theta; \mathbf{V}) \in C^\infty([0, 1] \times \Omega(\tau) \times \mathbb{T} \times \mathbb{R}^3; \mathbb{R}^3), \\ M^\varepsilon &\equiv M(\varepsilon, t, x, \theta; \mathbf{V}) \in C^\infty([0, 1] \times \Omega(\tau) \times \mathbb{T} \times \mathbb{R}^3; \mathbb{M}^3(\mathbb{R})), \\ \Gamma^\varepsilon &\equiv \Gamma(\varepsilon, t, x, \theta; \mathbf{V}) \in C^\infty([0, 1] \times \Omega(\tau) \times \mathbb{T} \times \mathbb{R}^3; \mathbb{R}^2), \end{aligned}$$

such that the solution \mathbf{U}^ε of (1.12) can be represented as

$$(4.4) \quad \mathbf{U}^\varepsilon(t, x, \theta) = \Gamma(\varepsilon, t, x, \theta; \mathbf{V}^\varepsilon(t, x, \theta)), \quad \mathbf{V}^\varepsilon = {}^t(\mathbf{V}^{\varepsilon 1}, \mathbf{V}^{\varepsilon 2}, \mathbf{V}^{\varepsilon 3})$$

where \mathbf{V}^ε is subjected to the non singular Burgers type equation

$$(4.5) \quad \partial_t \mathbf{V}^\varepsilon + (X(\varepsilon, t, x, \theta; \mathbf{V}^\varepsilon) \cdot \nabla_{x, \theta}) \mathbf{V}^\varepsilon = M(\varepsilon, t, x, \theta; \mathbf{V}^\varepsilon) \mathbf{V}^\varepsilon.$$

We have $\mathbf{V}^\varepsilon = \mathbf{V}_0 + O(\varepsilon)$ with $\mathbf{V}_0 = {}^t(\mathbf{V}_0^1, 0, \mathbf{V}_0^3)$. The scalar components \mathbf{V}_0^1 and \mathbf{V}_0^3 are determined by solving the non linear system

$$(4.6) \quad \begin{cases} \partial_t \mathbf{V}_0^1 + (\tilde{X}^0 \cdot \nabla_{x, \theta}) \mathbf{V}_0^1 = -|\nabla_x \Phi|^2 \mathbf{V}_0^3, \\ \partial_t \mathbf{V}_0^3 + (\tilde{X}^0 \cdot \nabla_{x, \theta}) \mathbf{V}_0^3 = 0, \end{cases}$$

associated with the initial data

$$\mathbf{V}_0^1(0, x, \theta) = 0, \quad \mathbf{V}_0^3(0, x, \theta) = |\nabla_x \phi(x)|^{-2} \nabla_x \phi(x) \cdot H_0(x, \theta)$$

and involving

$$\begin{aligned} \tilde{X}^0 \cdot \nabla_{x, \theta} &= \tilde{X}^0(t, x, \theta; \mathbf{V}_0^1, \mathbf{V}_0^3) \cdot \nabla_{x, \theta} \\ &:= H(t, x, \theta + \mathbf{V}_0^1) \cdot \nabla_x + |\nabla_x \Phi|^2 \mathbf{V}_0^3 \partial_\theta. \end{aligned}$$

Remark 4.1.1 - identification of the main term in the oscillation. The application Γ can be explicitated. Noting $\mathbf{V} = {}^t(\mathbf{V}^1, \mathbf{V}^2, \mathbf{V}^3)$, it is

$$\Gamma(\varepsilon, t, x, \theta; \mathbf{V}) := \mathbf{H}(t, x, \theta + \mathbf{V}^1) + \mathbf{V}^2 \nabla_x \Phi(t, x)^\perp + \varepsilon \mathbf{V}^3 \nabla_x \Phi(t, x).$$

It follows that

$$(4.7) \quad \mathbf{U}^\varepsilon(t, x, \theta) = \mathbf{H}(t, x, \theta + \mathbf{V}_0^1(t, x, \theta)) + O(\varepsilon).$$

By construction, at time $t = 0$, the contribution \mathbf{V}_0^1 does not occur. Suppose that $\nabla_x \phi \cdot H_0 \neq 0$. Then $\mathbf{V}_0^3(0, \cdot) \neq 0$ and \mathbf{V}_0^1 becomes non trivial when $t > 0$, due to the coupling in system (4.6). We see here that an information of size $O(\varepsilon)$ at time $t = 0$ (namely $\varepsilon \mathbf{V}_0^3(0, \cdot) \nabla_x \phi$) can influence the shape $\mathbf{H}(t, x, \theta + \mathbf{V}_0^1)$ of the large amplitude wave when $t > 0$. \diamond

Remark 4.1.2 - other formulation of (4.6). Suppose that $\mathbf{f} \equiv 0$. Introduce the vector valued function $\mathbf{W}_0 = {}^t(\mathbf{W}_0^1, \mathbf{W}_0^2, \mathbf{W}_0^3)$ defined according to

$${}^t(\mathbf{W}_0^1, \mathbf{W}_0^2) := \mathbf{H}(t, x, \theta + \mathbf{V}_0^1), \quad \mathbf{W}_0^3 := (1 + f(\Phi)^2) \mathbf{V}_0^3.$$

The informations contained in (1.8) and (3.6) allow to extract from (4.6) the constraint

$$(4.8) \quad \partial_t \mathbf{W}_0 + \mathbf{W}_0^1 \partial_1 \mathbf{W}_0 + \mathbf{W}_0^2 \partial_2 \mathbf{W}_0 + (\partial_2 \Phi)^2 \mathbf{W}_0^3 \partial_\theta \mathbf{W}_0 = 0$$

completed with

$$\mathbf{W}_0(0, x, \theta) = {}^t(H^1(x, \theta), H^2(x, \theta), \partial_2 \phi(x)^{-2} \nabla_x \phi \cdot H_0(x, \theta)).$$

The equation (4.8) is a three dimensional *Burgers type* equation. Suppose that $\partial_2 \Phi$ is a function of Φ , say $\partial_2 \Phi = h(\Phi)$. This means that $f' \equiv g' \equiv 0$ or that, for all $t \in [0, \tau]$, the function $\Phi(t, \cdot)$ is constant on parallel lines (as in subsection 3.2). In this special case, the equation (4.8) can be further reduced. Just replace the component \mathbf{W}_0^3 by $h(\Phi)^2 \mathbf{W}_0^3$ to recover exactly a three dimensional Burger equation on \mathbf{W}_0 . \diamond

Proof of Proposition 4.1. First, recall that

$$\nabla_x \Phi(t, x) = {}^t(\partial_1 \Phi, \partial_2 \Phi)(t, x), \quad \nabla_x \Phi(t, x)^\perp = {}^t(-\partial_2 \Phi, \partial_1 \Phi)(t, x).$$

For each $(\varepsilon, t, x, \theta) \in]0, 1] \times \Omega(\tau) \times \mathbb{T}$, the application

$$\mathbb{R}^3 \ni \mathbf{V} \longmapsto \Gamma(\varepsilon, t, x, \theta; \mathbf{V}) \in \mathbb{R}^2$$

is surjective (but not injective). In particular, the initial data H^ε can be achieved as the image of some function V^ε . In other words

$$H^\varepsilon(x, \theta) = \Gamma(\varepsilon, 0, x, \theta; V^\varepsilon(x, \theta)), \quad V^\varepsilon = {}^t(V^{\varepsilon 1}, V^{\varepsilon 2}, V^{\varepsilon 3}).$$

There are several possible choices for V^ε . Take $V^{\varepsilon 1} \equiv 0$ and seek

$$(V^{\varepsilon 2}, V^{\varepsilon 3}) = \sum_{j=0}^J \varepsilon^j (V_j^2, V_j^3) + \varepsilon^{J+1} (r^{\varepsilon 2}, r^{\varepsilon 3}), \quad (r^{\varepsilon 2}, r^{\varepsilon 3}) = O(1).$$

The contribution due to V_0^2 can be absorbed inside H . Thus, impose $V_0^2 \equiv 0$. Then, it suffices to set

$$\begin{aligned} V_j^2 &:= |\nabla_x \Phi|^{-2} \nabla_x \Phi^\perp \cdot H_{j-1}, & \forall j \in \{1, \dots, J\}, \\ V_j^3 &:= |\nabla_x \Phi|^{-2} \nabla_x \Phi \cdot H_j, & \forall j \in \{0, \dots, J-1\}, \end{aligned}$$

and then to adjust $(r^{\varepsilon 2}, r^{\varepsilon 3})$ conveniently. Observe that $V^\varepsilon \in C^1(\omega \times \mathbb{T}; \mathbb{R}^3)$ and that by construction the family $\{V^\varepsilon\}_\varepsilon$ is subjected to the uniform control

$$(4.9) \quad \sup \left\{ \|V^\varepsilon\|_{L^\infty(\omega \times \mathbb{T})} + \|D_x V^\varepsilon\|_{L^\infty(\omega \times \mathbb{T})}; \varepsilon \in]0, 1] \right\} < \infty.$$

At time $t = 0$, impose

$$(4.10) \quad \mathbf{V}^\varepsilon(0, x, \theta) = V^\varepsilon(x, \theta), \quad \forall \varepsilon \in]0, 1].$$

Seek a solution to (1.12) which can be expressed like in (4.4). This manipulation corresponds to a dependent change of state variable, the vector valued function $\mathbf{V}^\varepsilon(t, x, \theta)$ being the new unknown. Introduce the vector field X^ε which is such that

$$\begin{aligned} X^\varepsilon \cdot \nabla_{x, \theta} &= X(\varepsilon, t, x, \theta; \mathbf{V}) \cdot \nabla_{x, \theta} = X_x^\varepsilon \cdot \nabla_x + X_\theta^\varepsilon \partial_\theta \\ &= \mathbf{H}(t, x, \theta + \mathbf{V}^1) \cdot \nabla_x + \mathbf{V}^2 \nabla_x \Phi(t, x)^\perp \cdot \nabla_x \\ &\quad + \varepsilon \mathbf{V}^3 \nabla_x \Phi(t, x) \cdot \nabla_x + |\nabla_x \Phi(t, x)|^2 \mathbf{V}^3 \partial_\theta. \end{aligned}$$

For $\varepsilon = 0$, it remains

$$\begin{aligned} X^0 \cdot \nabla_{x, \theta} &\equiv X(0, t, x, \theta; \mathbf{V}) \cdot \nabla_{x, \theta} \\ &= \mathbf{H}(t, x, \theta + \mathbf{V}^1) \cdot \nabla_x + \mathbf{V}^2 \nabla_x \Phi(t, x)^\perp \cdot \nabla_x + |\nabla_x \Phi(t, x)|^2 \mathbf{V}^3 \partial_\theta. \end{aligned}$$

Use (1.8) to interpret (1.12) according to

$$(\partial_t + X(\varepsilon, t, x, \theta; \mathbf{V}^\varepsilon) \cdot \nabla_{x, \theta}) \Gamma(\varepsilon, t, x, \theta; \mathbf{V}^\varepsilon(t, x, \theta)) + \mathbf{f} = 0.$$

Remember that \mathbf{f} does not depend on θ . This fact and again (1.8) allow to reduce the preceding equation to

$$\begin{aligned} &\varepsilon^0 \{ (\partial_t + X^\varepsilon \cdot \nabla_{x, \theta}) \mathbf{V}^{\varepsilon 1} + |\nabla_x \Phi|^2 \mathbf{V}^{\varepsilon 3} \} \partial_\theta \mathbf{H} \\ &\quad + \varepsilon^0 \{ (\partial_t + X^\varepsilon \cdot \nabla_{x, \theta}) \mathbf{V}^{\varepsilon 2} \} \nabla_x \Phi^\perp + \mathfrak{X}^{\varepsilon 2} \mathbf{V}^{\varepsilon 2} \\ &\quad + \varepsilon^1 \{ (\partial_t + X^\varepsilon \cdot \nabla_{x, \theta}) \mathbf{V}^{\varepsilon 3} \} \nabla_x \Phi + \mathfrak{X}^{\varepsilon 3} \mathbf{V}^{\varepsilon 3} \} = 0. \end{aligned}$$

Here the notation $\mathfrak{X}^{\varepsilon 2}$ is for the vector

$$\mathfrak{X}^{\varepsilon 2} := \partial_t \nabla_x \Phi^\perp + (X_x^\varepsilon \cdot \nabla_x) \nabla_x \Phi^\perp + ((\nabla_x \Phi^\perp \cdot \nabla_x) \mathbf{H})(t, x, \theta + \mathbf{V}^{\varepsilon 1}).$$

On the other hand, the notation $\mathfrak{X}^{\varepsilon 3}$ is for the vector

$$\mathfrak{X}^{\varepsilon 3} := \partial_t \nabla_x \Phi + (X_x^\varepsilon \cdot \nabla_x) \nabla_x \Phi + ((\nabla_x \Phi \cdot \nabla_x) \mathbf{H})(t, x, \theta + \mathbf{V}^{\varepsilon 1}).$$

There are three unknowns $\mathbf{V}^{\varepsilon 1}$, $\mathbf{V}^{\varepsilon 2}$ and $\mathbf{V}^{\varepsilon 3}$ for two equations so that the preceding constraint on \mathbf{V}^ε is underdetermined. To remedy to this, just impose the supplementary condition

$$X^\varepsilon \mathbf{V}^{\varepsilon 1} + |\nabla_x \Phi|^2 \mathbf{V}^{\varepsilon 3} = 0.$$

Then, it remains (4.5) with

$$\begin{aligned} M^\varepsilon &= M(\varepsilon, t, x, \theta; \mathbf{V}^\varepsilon) = (M_{ij}(\varepsilon, t, x, \theta; \mathbf{V}^\varepsilon))_{1 \leq i, j \leq 3} \\ &:= -\frac{1}{|\nabla_x \Phi|^2} \begin{pmatrix} 0 & 0 & |\nabla_x \Phi|^4 \\ 0 & \mathfrak{X}^{\varepsilon 2} \cdot \nabla_x \Phi^\perp & \varepsilon \mathfrak{X}^{\varepsilon 3} \cdot \nabla_x \Phi^\perp \\ 0 & \varepsilon^{-1} \mathfrak{X}^{\varepsilon 2} \cdot \nabla_x \Phi & \mathfrak{X}^{\varepsilon 3} \cdot \nabla_x \Phi \end{pmatrix}. \end{aligned}$$

We have to show that this matrix M^ε does not involve coefficients which are singular with respect to ε . In particular, we must pay attention in $M_{32}(\varepsilon, \cdot)$. Recall that

$$\mathbf{H} = \bar{\mathbf{H}} + \mathbf{s}^* \nabla_x \Phi(t, x)^\perp = \bar{\mathbf{H}} + \mathbf{s}^* \partial_2 \Phi \ ^t(-1, f(\Phi)).$$

Remark that

$$\begin{aligned} \partial_t \nabla_x \Phi + (\bar{\mathbf{H}} \cdot \nabla_x) \nabla_x \Phi + (\nabla_x \Phi \cdot \nabla_x) \bar{\mathbf{H}} &= (\partial_1 \bar{\mathbf{H}}^2 - \partial_2 \bar{\mathbf{H}}^1) \nabla_x \Phi^\perp, \\ \partial_t \nabla_x \Phi^\perp + (\bar{\mathbf{H}} \cdot \nabla_x) \nabla_x \Phi^\perp + (\nabla_x \Phi^\perp \cdot \nabla_x) \bar{\mathbf{H}} &= S \nabla_x \Phi, \end{aligned}$$

where S is the symmetric matrix

$$S = \begin{pmatrix} 2 \partial_2 \bar{\mathbf{H}}^1 & \partial_2 \bar{\mathbf{H}}^2 - \partial_1 \bar{\mathbf{H}}^1 \\ \partial_2 \bar{\mathbf{H}}^2 - \partial_1 \bar{\mathbf{H}}^1 & -2 \partial_1 \bar{\mathbf{H}}^2 \end{pmatrix}.$$

The relation (3.7) gives rise to

$$\begin{aligned} \ ^t(f(\Phi), 1) \cdot (\nabla_x \Phi^\perp \cdot \nabla_x) \bar{\mathbf{H}} &= (\nabla_x \Phi^\perp \cdot \nabla_x) [\ ^t(f(\Phi), 1) \cdot \bar{\mathbf{H}}] \\ &= (\nabla_x \Phi^\perp \cdot \nabla_x) g(\Phi) = 0. \end{aligned}$$

It follows that

$$\begin{aligned} \ ^t \nabla_x \Phi S \nabla_x \Phi &= 2 [(\partial_1 \Phi)^2 \partial_2 \bar{\mathbf{H}}^1 + \partial_1 \Phi \partial_2 \Phi (\partial_2 \bar{\mathbf{H}}^2 - \partial_1 \bar{\mathbf{H}}^1) - (\partial_2 \Phi)^2 \partial_1 \bar{\mathbf{H}}^2] \\ &= 2 \nabla_x \Phi \cdot (\nabla_x \Phi^\perp \cdot \nabla_x) \bar{\mathbf{H}} = 0. \end{aligned}$$

Observe that

$$(\nabla_x \Phi^\perp \cdot \nabla_x) \nabla_x \Phi^\perp = -f'(\Phi) (\partial_2 \Phi)^2 \nabla_x \Phi^\perp.$$

The preceding informations can be collected in order to simplify the expressions given for $\mathfrak{X}^{\varepsilon^2}$ and $\mathfrak{X}^{\varepsilon^3}$. We find

$$\begin{aligned}\mathfrak{X}^{\varepsilon^2} &= \varepsilon \mathbf{V}^{\varepsilon^3} (\nabla_x \Phi \cdot \nabla_x) \nabla_x \Phi^\perp \\ &\quad + \{ -f'(\Phi) (\partial_2 \Phi)^2 [2 \mathbf{s}^*(t, x, \theta + \mathbf{V}^{\varepsilon^1}) + \mathbf{V}^{\varepsilon^2}] \\ &\quad \quad + (\nabla_x \Phi^\perp \cdot \nabla_x) \mathbf{s}^* + |\nabla_x \Phi|^{-2} {}^t \nabla_x \Phi^\perp S \nabla_x \Phi \} \nabla_x \Phi^\perp, \\ \mathfrak{X}^{\varepsilon^3} &= \varepsilon \mathbf{V}^{\varepsilon^3} (\nabla_x \Phi \cdot \nabla_x) \nabla_x \Phi + (\partial_1 \bar{\mathbf{H}}^2 - \partial_2 \bar{\mathbf{H}}^1) \nabla_x \Phi^\perp \\ &\quad + \mathbf{s}^*(t, x, \theta + \mathbf{V}^{\varepsilon^1}) [(\nabla_x \Phi^\perp \cdot \nabla_x) \nabla_x \Phi + (\nabla_x \Phi \cdot \nabla_x) \nabla_x \Phi^\perp] \\ &\quad + \mathbf{V}^{\varepsilon^2} (\nabla_x \Phi^\perp \cdot \nabla_x) \nabla_x \Phi + (\nabla_x \Phi \cdot \nabla_x) \mathbf{s}^* \nabla_x \Phi^\perp.\end{aligned}$$

It implies that

$$\varepsilon^{-1} \mathfrak{X}^{\varepsilon^2} \cdot \nabla_x \Phi = \mathbf{V}^{\varepsilon^3} (\partial_2 \Phi)^2 |\nabla_x \Phi|^2 f'(\Phi).$$

Therefore

$$M^\varepsilon = (M_{ij}^\varepsilon)_{1 \leq i, j \leq 3} = M^0 + \varepsilon M^1 + \varepsilon^2 M^2, \quad M^k = (M_{ij}^k)_{1 \leq i, j \leq 3}$$

where the coefficients M_{ij}^k are smooth functions on $\Omega(\tau) \times \mathbb{T} \times \mathbb{R}^3$. It means that M is a smooth function of $(t, x, \theta, \mathbf{V})$ and also of $\varepsilon \in [0, 1]$. In particular, it satisfies uniform estimates with respect to $\varepsilon \in]0, 1]$.

Consequently, the quasilinear symmetric system (4.5) is a nonlinear transport equation involving coefficients and a source term which all are non singular with respect to $\varepsilon \in [0, 1]$. Therefore, it can be solved in the context of C^1 regularity by the usual method of characteristics, in a way similar to what has been done in paragraph 2.1.1.

Taking into account (4.9), there is a domain of determinacy $\Omega(\tau) \times \mathbb{T}$ such that, for all $\varepsilon \in]0, 1]$, the Cauchy problem (4.5)-(4.10) has a solution \mathbf{V}^ε on $\Omega(\tau) \times \mathbb{T}$ (where τ does not depend on ε). By way of formula (4.4), we recover a solution \mathbf{U}^ε to (1.12)-(4.3) on $\Omega(\tau) \times \mathbb{T}$.

By construction, at time $t = 0$, the expression V^ε is given by the Taylor expansion

$$V^\varepsilon = \sum_{j=0}^J \varepsilon^j V_j + \varepsilon^{J+1} r^\varepsilon, \quad \forall \varepsilon \in]0, 1]$$

where

$$V_j(x, \theta) = {}^t(V_j^1, V_j^2, V_j^3)(x, \theta) \in C^1(\omega \times \mathbb{T}; \mathbb{R}^3), \quad 0 \leq j \leq J.$$

Now, seek profiles

$$\mathbf{V}_j(t, x, \theta) = {}^t(\mathbf{V}_j^1, \mathbf{V}_j^2, \mathbf{V}_j^3)(t, x, \theta) \in C^1(\Omega(\tau) \times \mathbb{T}; \mathbb{R}^3), \quad 0 \leq j \leq J$$

adjusted so that

$$(4.11) \quad \mathbf{V}_a^\varepsilon(t, x, \theta) = \sum_{j=0}^J \varepsilon^j \mathbf{V}_j(t, x, \theta), \quad \varepsilon \in]0, 1]$$

is an approximate solution of (4.5) in the following sense

$$\sup \{ \varepsilon^{-J} \|\mathbf{V}^\varepsilon - \mathbf{V}_a^\varepsilon\|_{C^0(\Omega(\tau) \times \mathbb{T})}; \varepsilon \in]0, 1] \} < \infty.$$

The vector valued functions \mathbf{V}_j are determined as follows. Substitute the sum (4.11) into (4.5) to get a formal expansion. The term with ε^j in factor can be put in the form

$$(\mathcal{M}_j) \quad \partial_t \mathbf{V}_j + (X^0 \cdot \nabla_{x,\theta}) \mathbf{V}_j + \mathcal{F}_j(t, x, \theta; \mathbf{V}_0, \dots, \mathbf{V}_j) = 0$$

where $X^0 = X(0, t, x, \theta; \mathbf{V}_0)$. Complete (\mathcal{M}_j) with the initial data

$$(4.12) \quad \mathbf{V}_j(0, x, \theta) = V_j(x, \theta).$$

In particular, to identify the main contribution \mathbf{V}_0 in \mathbf{V}^ε , it suffices to solve

$$\partial_t \mathbf{V}_0 + (X^0 \cdot \nabla_{x,\theta}) \mathbf{V}_0 = M^0 \mathbf{V}_0, \quad \mathbf{V}_0(0, x, \theta) = {}^t(0, 0, V_0^3).$$

Since $M_{23}^0 \equiv 0$, the condition $\mathbf{V}_0^2(0, \cdot) \equiv 0$ is propagated which means that

$$\mathbf{V}_0(t, x, \theta) = {}^t(\mathbf{V}_0^1(t, x, \theta), 0, \mathbf{V}_0^3(t, x, \theta)), \quad \forall (t, x, \theta) \in \Omega(\tau) \times \mathbb{T}.$$

and which implies that $X^0 \equiv \tilde{X}^0$. It remains to compute

$$M_{33}^0(t, x, \theta; {}^t(\mathbf{V}^1, 0, \mathbf{V}^3)) = \mathbf{s}^*(t, x, \theta + \mathbf{V}^{\varepsilon 1}) \\ \times \nabla_x \Phi \cdot [(\nabla_x \Phi^\perp \cdot \nabla_x) \nabla_x \Phi + (\nabla_x \Phi \cdot \nabla_x) \nabla_x \Phi^\perp].$$

On the one hand

$$\nabla_x \Phi \cdot (\nabla_x \Phi^\perp \cdot \nabla_x) \nabla_x \Phi = -(\partial_2 \Phi)^2 |\nabla_x \Phi|^2 f'(\Phi).$$

On the other hand

$$\nabla_x \Phi \cdot (\nabla_x \Phi \cdot \nabla_x) \nabla_x \Phi^\perp = \partial_2 \Phi \nabla_x \Phi \cdot {}^t(0, f'(\Phi) |\nabla_x \Phi|^2).$$

Therefore

$$M_{33}^0(t, x, \theta; {}^t(\mathbf{V}^1, 0, \mathbf{V}^3)) = 0$$

and the validity of (4.6) is established. Once \mathbf{V}_0 has been identified, the systems (\mathcal{M}_j) with $0 < j$ are made of *linear* transport equations. Therefore, the Cauchy problems (\mathcal{M}_j) -(4.12) with $0 < j$ can be solved inductively on the whole domain of determinacy $\Omega(\tau) \times \mathbb{T}$ where \mathbf{V}_0 is defined. \square

Remark 4.1.3 - the incidence of the procedure on θ . At the level of (4.4), the status of θ is changed. There, it plays the part of a *slow* variable. \diamond

Remark 4.1.4 - come back to the original state variables. The expression

$$\mathbf{u}^\varepsilon(t, x) = \mathbf{U}^\varepsilon(t, x, \Phi(t, x)/\varepsilon), \quad \varepsilon \in]0, 1]$$

where \mathbf{U}^ε is given by Proposition 4.1 is a solution to (4.2). \diamond

4.2 Applications.

Subsection 4.1 emphasizes the system (4.5). The equation (4.5) being non singular with respect to ε , it falls under the scope of classical theories in nonlinear geometric optics. Applying usual tools, we can construct solutions \mathbf{V}^ε to (4.5) having the form (1.14).

The most elementary case is when all the \mathbf{V}_j do not depend on $\tilde{\theta}$. Then, the expressions $\mathbf{V}_j(t, x, \theta)$ can be identified just by looking at the Taylor expansions with respect to ε of the coefficients which appear in (4.5). This is exactly what has been done to produce and to justify (4.7).

The paragraphs 4.2.1 and 4.2.2 below are devoted to this simple situation, when \mathbf{u}^ε is determined through (1.15) and (1.16). The difficulty there is not the asymptotic analysis. It is due to the fact that we deal with (1.1). Therefore, all the data ϕ , H and H_0 are not authorized which means that special profiles \mathbf{U}_0 are involved.

4.2.1 Various caustics phenomena for Euler equations

Lemmas 2.6 and 2.7 explain how to adjust the data H^ε and ϕ and therefore V^ε and ϕ in order to have

$$\check{\mathbf{u}}^\varepsilon(0, x) = H^\varepsilon(x, \phi(x)/\varepsilon) \in \mathcal{V}_0^\tau(\omega), \quad \forall \varepsilon \in]0, 1].$$

Lemma 2.3 says that the corresponding solution $\check{\mathbf{u}}^\varepsilon(t, x)$ to (1.2) where $\mathbf{f} \equiv 0$ is also a solution to (1.1) with \mathbf{p} constant. Then, Proposition 4.1 exhibits for the subsequent family $\{\check{\mathbf{U}}^\varepsilon\}_\varepsilon$ the asymptotic behaviour (4.7).

Note however that Lemmas 2.6 and 2.7 require conditions on H and H_0 . These conditions can influence \mathbf{H} and \mathbf{V}_0^1 . In other words, the formula (4.7), when applied in the context (1.1), involves special expressions \mathbf{H} and \mathbf{V}_0 . The discussion starts by examining and commenting the constraints thus retrieved on \mathbf{H} and \mathbf{V}_0^1 .

In contrast with the situation of Lemma 3.3, the Lemma 2.6 demands no particular restriction on the phase ϕ , except (2.19). Moreover, the function Φ is obtained through (3.1) where g is given by (2.33). As in remark 3.1.4, the phase Φ can develop shocks.

This formation of shocks corresponds to caustics phenomena of the *first* type for (1.1). This can surprise since Euler equations are often regarded as being linearly degenerate. But, it is so. Note however that the analysis of subsection 4.1 stops before the formation of these caustics.

The profile H must be subjected to (2.32). The expression \mathbf{H} can be deduced from (3.2) or (3.10). We find

$$(4.13) \quad \mathbf{H} \equiv \check{\mathbf{H}} := K(\Phi, \theta + \Phi_0) \ ^t(1, -f \circ \Phi) + \ ^t(0, g \circ \Phi)$$

where the scalar function $\Phi_0(t, x, \theta)$ is determined by

$$(4.14) \quad \begin{cases} \partial_t \Phi_0 + K(\Phi, \theta + \Phi_0) (\partial_1 - f \circ \Phi \partial_2) \Phi_0 + g \circ \Phi \partial_2 \Phi_0 = 0, \\ \Phi_0(0, \cdot) = \phi_0. \end{cases}$$

The expression Φ_0 which appears here plays an ambiguous part. Of course, it is linked with the underlying scalar profile \mathbf{r} and it influences H^* (since it can depend on θ in a non trivial way). But also, it acts on K as a phase shift. To insist on this second aspect, Φ_0 is called the *second phase*.

Now, consider the following specific situation. Take

$$\omega = \mathbb{R}^2, \quad f \equiv g \equiv 0, \quad \phi(x) = \Phi(t, x) = x_2.$$

Work in the framework of Lemma 2.6. Impose $\phi_0|_{x_1=0} \equiv 0$ but $f_0 \not\equiv 0$ so that $\phi_0(x) = -x_1 f_0(x_2)$. Select a profile K satisfying $K(0, 0) = 0$, $\bar{K} \equiv 0$ but $K^* \not\equiv 0$. It remains

$$H(x, \theta) = K^*(x_2, \theta - x_1 f_0(x_2)) \ ^t(1, 0), \quad 0 \not\equiv f_0 \in C_b^\infty(\mathbb{R}; \mathbb{R}).$$

Under these conditions, the Cauchy problem (4.14) reduces to

$$\partial_t \Phi_0 + K^*(x_2, \theta + \Phi_0) \partial_1 \Phi_0 = 0, \quad \Phi_0(0, \cdot) = -x_1 f_0(x_2).$$

The variables x_2 and θ can be regarded as parameters. Since by construction

$$\exists (x_1, x_2, \theta) \in \mathbb{R}^2 \times \mathbb{T}, \quad -f_0(x_2) \partial_\theta K^*(x_2, \theta - x_1 f_0(x_2)) < 0$$

shocks do appear at the level of Φ_0 . In other words

$$\exists \tilde{T} \in \mathbb{R}_+; \quad \lim_{t \rightarrow \tilde{T}_-} \|\partial_1 \Phi_0(t, \cdot)\|_{L^\infty(\mathbb{R}^2 \times \mathbb{T})} = +\infty.$$

By virtue of Lemma 2.6, the couple (H, ϕ) which is selected above is associated with the *main term* of oscillations $\{\check{h}^\varepsilon\}_\varepsilon$ which are well prepared on \mathbb{R}^2 for (1.1). More precisely

$$\check{h}^\varepsilon(x) = H(x, \phi(x)/\varepsilon) + \varepsilon r h^\varepsilon(x) \in \mathcal{V}_0^\tau(\mathbb{R}^2), \quad r h^\varepsilon \not\equiv 0, \quad \forall \varepsilon \in]0, 1].$$

Note $\check{\mathbf{u}}^\varepsilon(t, x)$ the solution issued from \check{h}^ε by solving (1.2) with $\mathbf{f} \equiv 0$. Applying Lemma 2.3 (see also remark 2.1.6), the function $\check{\mathbf{u}}^\varepsilon$ is a *global* (in both time and space variables) C^1 solution to (1.1).

Applying Lemma 3.1, the couple (H, ϕ) gives rise also to a simple wave $\tilde{\mathbf{u}}^\varepsilon$ which is a *local* solution to (1.2) with $\mathbf{f} \equiv 0$, defined on the domain $[0, \tilde{T}[\times\mathbb{R}^2$, with *breakdown* when $t < \tilde{T}$ goes to \tilde{T} . We have seen in subsection 4.1 that the asymptotic behaviour of the family $\{\check{\mathbf{u}}^\varepsilon\}_\varepsilon$ is modeled by the one of $\{\tilde{\mathbf{u}}^\varepsilon\}_\varepsilon$ (modulo some translation in the fast variable θ).

There is no contradiction between the two preceding assertions.

On the one hand, looking at Φ_0 as a second phase, the presence of shocks can be interpreted as *caustics* phenomena of the *second* type for Euler equations. It does not preclude the solution $\check{\mathbf{u}}^\varepsilon$ to remain smooth. In fact, only from this point of view, the situation can be compared to what is observed in the framework of semilinear equations [16].

On the other hand, the simple wave $\tilde{\mathbf{u}}^\varepsilon$ is a solution of (1.2) but it is not a solution of (1.1). Indeed, the profile $H(x, \theta)$ does not depend only on x_2 , as is required in Lemma 3.3. Thus, nothing guarantees that $\tilde{\mathbf{u}}^\varepsilon$ remains a smooth function.

4.2.2 On the concentration-cancellation property.

The preceding analysis produces special families $\{\check{\mathbf{u}}^\varepsilon\}_\varepsilon$ made of C^1 solutions on $\Omega(\tau)$ of incompressible equations

$$\partial_t \check{\mathbf{u}}^\varepsilon + (\check{\mathbf{u}}^\varepsilon \cdot \nabla_x) \check{\mathbf{u}}^\varepsilon = 0, \quad \operatorname{div}_x \check{\mathbf{u}}^\varepsilon = 0, \quad \varepsilon \in]0, 1].$$

It allows also to identify the corresponding weak limits $\check{\mathbf{u}}$ which are

$$\check{\mathbf{u}}(t, x) = \langle \check{\mathbf{U}} \rangle(t, x), \quad \check{\mathbf{U}}(t, x, \theta) = \check{\mathbf{H}}(t, x, \theta + \mathbf{V}_0^1 t, x, \theta)$$

where $\check{\mathbf{H}}$ is of the form (4.13) whereas \mathbf{V}_0^1 is given by (4.5). Since the divergence free relation is obviously preserved when passing to the weak limit, the expression $\check{\mathbf{u}}$ is subjected to

$$(4.15) \quad \partial_t \check{\mathbf{u}} + (\check{\mathbf{u}} \cdot \nabla_x) \check{\mathbf{u}} + \check{\mathbf{f}} = 0, \quad \operatorname{div}_x \check{\mathbf{u}} = 0.$$

It remains to determine the function $\check{\mathbf{f}}$. In general, as already noticed in Remark 3.2.2, we find $\check{\mathbf{f}} \neq 0$. In fact, we have even the following more precise information.

Lemma 4.1. *The construction can be adjusted so that $\operatorname{curl} \check{\mathbf{f}} \neq 0$.*

Proof of Lemma 4.1. Recall (4.8) which contains

$$\partial_t \check{\mathbf{U}} + (\check{\mathbf{U}} \cdot \nabla_x) \check{\mathbf{U}} + (\partial_2 \Phi)^2 \mathbf{W}_0^3 \partial_\theta \check{\mathbf{U}} = 0.$$

Extract the mean value to get

$$\check{\mathbf{f}} = \langle (\check{\mathbf{U}}^* \cdot \nabla_x) \check{\mathbf{U}}^* \rangle + (\partial_2 \Phi)^2 \langle \mathbf{W}_0^3 \partial_\theta \check{\mathbf{U}} \rangle.$$

At time $t = 0$, it remains

$$\check{\mathbf{f}}|_{t=0} = \langle (\check{H}^* \cdot \nabla_x) \check{H}^* \rangle + \langle (\nabla_x \phi \cdot H_0) \partial_\theta \check{H} \rangle$$

where $\check{H} := \check{\mathbf{H}}|_{t=0}$ is defined as in (2.32). In Lemma 2.6, select $f_0 \equiv 0$ and take $\phi_0 \equiv 0$ so that

$$\check{H}(x, \theta) = K(\phi(x), \theta) \ ^t(1, -f \circ \phi(x)) + \ ^t(0, g \circ \phi(x)).$$

In particular, we have

$$\check{H}^*(x, \theta) = K^*(\phi(x), \theta) \ ^t(1, -f \circ \phi(x)).$$

With (2.19), it follows immediately that

$$(\check{H}^* \cdot \nabla_x) \check{H}^* = -K^*(\phi, \theta) \partial_2 \phi^{-1} (\nabla \phi^\perp \cdot \nabla_x) [K^*(\phi, \theta) \ ^t(1, -f \circ \phi)] \equiv 0.$$

In Lemma 2.7, select $f_1 \equiv 0$ and take $\phi_1 \equiv 0$. We have to deal with

$$H_0 = \bar{H}_0(\phi) + K_0(\phi, \theta) \ ^t(1, -f \circ \phi) + f' \circ \phi (\partial_\theta^{-1} K^*)(\phi, \theta) \ ^t(0, 1).$$

Now, compute

$$\check{\mathbf{f}}|_{t=0} = f' \circ \phi \langle K^*(\phi, \theta)^2 \rangle \nabla \phi^\perp$$

which leads to

$$\text{curl } \check{\mathbf{f}}(0, x) = \Delta(l \circ \phi)(x), \quad l(z) := \int_0^z f'(y) K^*(y, \theta)^2 dy.$$

In general, this quantity is non trivial. For instance, choose

$$f(z) = z, \quad \phi(x) = x_2/(1+x_1), \quad K(z, \theta) = \sqrt{2} \cos(\pi\theta).$$

In this special case, we find

$$\text{curl } \check{\mathbf{f}}(0, x) = 2 x_2/(1+x_1)^3 \neq 0. \quad \square$$

Proof of Theorem 1.1. The family $\{\check{\mathbf{u}}^\varepsilon\}_\varepsilon$ satisfies the conditions i) and ii) of Theorem 1.1. It converges weakly (as ε goes to zero) to $\mathbf{u}^0 \equiv \check{\mathbf{u}} \in C^1(\Omega(\tau))$. The function $\check{\mathbf{u}}$ is subjected to (4.15) which involves the source term $\check{\mathbf{f}}$. If $\check{\mathbf{f}}$ can be expressed as the gradient of some scalar function \mathbf{p} , one has $\text{curl } \check{\mathbf{f}} \equiv 0$. But, in view of Lemma 4.1, this condition is not always verified. Therefore, it can happen that \mathbf{u}^0 is not a solution to (1.1). \square

Remark 4.2.1 - beyond Theorem 1.1. An improvement would be to replace in Theorem 1.1 the bounded domain Ω by \mathbb{R}^2 and the Sobolev space L^2 by L^p with any $p \in [1, +\infty]$. This would express that the operator solution associated with (1.1) is not closed for the weak L^p topology, replying to the question raised by L. Bertozzi and A. Majda [1]-p. 479.

Our approach does not give yet access to such a result. Indeed, our key argument relies on the use of the nonlinear functional set $\mathcal{V}_0^r(\omega)$. At this level, two difficulties arise. On the one hand, the families $\{\tilde{h}^\varepsilon\}_\varepsilon$ which are well prepared on \mathbb{R}^2 for (1.1) satisfy (2.35) which is not compatible with $L^p(\mathbb{R}^2)$ except if $p = +\infty$. On the other hand, the hypothesis $f' \not\equiv 0$ is crucial in the proof of Lemma 4.1. However, when $f' \not\equiv 0$ or when $(f^\varepsilon)' \not\equiv 0$, there is no way to define on \mathbb{R}^2 a C^1 solution φ^ε to (2.39).

Of course, one is tempted to *localize* the analysis. But, this forces into being faced with the problem of stability in the setting of (1.1). Now, this is a more delicate matter than what we did in subsection 4.1. \diamond

Remark 4.2.2 - interpretation related to the strong topology of L^p . Many mechanisms of amplifications concerning (1.1) and involving the L^p -norm have recently been exhibited [7], [11], [12], [17], see also Proposition 5.1 in [5]. The preceding method provides an alternative way to obtain such results. However, in so far as this subject has much been studied, this aspect will not be underlined here. \diamond

4.2.3 Interaction with small amplitude transversal oscillations.

From now on, consider (1.2) with $\mathbf{f} \equiv 0$ or (1.1) with \mathbf{p} constant. Fix two integers J and m with $J \geq 2$ and $m \geq 1$. Choose any couple (H, ϕ) which is compatible with (1.2). Note \mathbf{H} and Φ the expressions obtained through (3.1) and (3.2). Select m smooth phases

$$\psi_j \in C^\infty(\omega; \mathbb{R}), \quad \nabla_x \psi_j \in C_b^\infty(\omega; \mathbb{R}^2), \quad \forall j \in \{1, \dots, m\}.$$

Note $\vec{\psi} := (\psi_1, \dots, \psi_m)$. Suppose that

$$(4.16) \quad \exists j \in \{1, \dots, m\}; \quad \nabla_x \phi^\perp \cdot \nabla_x \psi_j \not\equiv 0.$$

Take profiles \hat{H}_j such that

$$\hat{H}_j(x, \theta, \tilde{\theta}) \in C_b^\infty(\omega \times \mathbb{T} \times \mathbb{T}^m; \mathbb{R}^2), \quad \forall j \in \{0, \dots, J-1\}.$$

Assume that \hat{H}_0 is a non trivial function of $\tilde{\theta}$. More precisely, impose

$$(4.17) \quad \nabla_x \phi^\perp \cdot \partial_{\tilde{\theta}_j} \hat{H}_0 \not\equiv 0.$$

On the contrary, prescribe

$$(4.18) \quad \nabla_x \phi \cdot \partial_{\hat{\theta}_j} \hat{H}_0 \equiv 0, \quad \forall j \in \{1, \dots, m\}.$$

With these ingredients, build the multidimensional oscillatory wave

$$(4.19) \quad \hat{h}^\varepsilon(x) := H(x, \phi(x)/\varepsilon) + \sum_{j=0}^{J-1} \varepsilon^{j+1} \hat{H}_j(x, \phi(x)/\varepsilon, \vec{\psi}(x)/\varepsilon) + \varepsilon^{J+1} \hat{r}h^\varepsilon(x).$$

Require that the remainder $\hat{r}h^\varepsilon$ is controlled by

$$(4.20) \quad \sup \{ \| \hat{r}h^\varepsilon \|_{L^\infty(\omega)} + \varepsilon^{J-1} \| D_x \hat{r}h^\varepsilon \|_{L^\infty(\omega)}; \varepsilon \in]0, 1] \} < \infty.$$

Easy computations using (4.18) indicate that

$$\operatorname{div}_x \hat{h}^\varepsilon(x) = O(1), \quad \det(D_x \hat{h}^\varepsilon(x)) = O(1).$$

Complete (1.2) with the initial data

$$(4.21) \quad \hat{\mathbf{u}}^\varepsilon(0, x) = \hat{h}^\varepsilon(x).$$

According to Lemma 2.2, the oscillating Cauchy problem (1.2)-(4.21) yields a family $\{\hat{\mathbf{u}}^\varepsilon\}_\varepsilon$ made of C^1 solutions $\hat{\mathbf{u}}^\varepsilon$ to (1.2). The functions $\hat{\mathbf{u}}^\varepsilon$ are defined on some open domain of determinacy $\Omega \subset \mathbb{R} \times \mathbb{R}^2$ which does not depend on ε . The structure of $\hat{\mathbf{u}}^\varepsilon$ encodes the interaction of the large amplitude wave $\mathbf{H}(t, x, \Phi(t, x)/\varepsilon)$ with small amplitude oscillations which can involve phases ψ_j transversal to ϕ . Here is what happens.

Proposition 4.2. *Assume (4.16), (4.17) and (4.18). The solution $\hat{\mathbf{u}}^\varepsilon$ to the Cauchy problem (1.2)-(4.21) inherits on Ω the asymptotic behavior*

$$(4.22) \quad \hat{\mathbf{u}}^\varepsilon(t, x) = \mathbf{H}(t, x, \Phi(t, x)/\varepsilon) + \varepsilon \hat{\mathbf{H}}_0(t, x, \Phi(t, x)/\varepsilon, \vec{\Psi}(t, x, \Phi(t, x)/\varepsilon)/\varepsilon) + O(\varepsilon^2)$$

where $\nabla_{\hat{\theta}} \hat{\mathbf{H}}_0 \not\equiv 0$ and $\partial_{\hat{\theta}} \vec{\Psi} \not\equiv 0$. Moreover, the family of initial data $\{\hat{h}^\varepsilon\}_\varepsilon$ can be adjusted so that

$$\hat{h}^\varepsilon \in \mathcal{V}_0^\top(\omega), \quad \forall \varepsilon \in]0, 1].$$

It follows that the transformation of (4.19) into (4.22) actually occurs at the level of incompressible Euler equations.

Proof of Proposition 4.2. Define the following functions of $(x, \theta, \tilde{\theta})$

$$\begin{aligned} \hat{V}_{j+1}^2 &:= |\nabla_x \phi|^{-2} \nabla_x \phi^\perp \cdot \hat{H}_j, & \forall j \in \{0, \dots, J-1\}, \\ \hat{V}_j^3 &:= |\nabla_x \phi|^{-2} \nabla_x \phi \cdot \hat{H}_j, & \forall j \in \{0, \dots, J-1\}. \end{aligned}$$

Consider also

$$r^{\varepsilon 2}(x) := |\nabla_x \phi|^{-2} \nabla_x \phi^\perp \cdot \hat{r} h^\varepsilon, \quad r^{\varepsilon 3}(x) := |\nabla_x \phi|^{-2} \nabla_x \phi \cdot \hat{r} h^\varepsilon.$$

Introduce

$$\begin{aligned} \hat{V}^{\varepsilon 2}(x, \theta) &:= \sum_{j=0}^{J-1} \varepsilon^{j+1} \hat{V}_{j+1}^2(x, \theta, \vec{\psi}(x)/\varepsilon) + \varepsilon^{J+1} \hat{r}^{\varepsilon 2}(x), \\ \hat{V}^{\varepsilon 3}(x, \theta) &:= \sum_{j=0}^{J-1} \varepsilon^j \hat{V}_j^3(x, \theta, \vec{\psi}(x)/\varepsilon) + \varepsilon^J \hat{r}^{\varepsilon 3}(x). \end{aligned}$$

At time $t = 0$, impose

$$(4.23) \quad \hat{\mathbf{V}}^\varepsilon(0, x, \theta) = \hat{V}^\varepsilon(x, \theta) := {}^t(0, \hat{V}^{\varepsilon 2}(x, \theta), \hat{V}^{\varepsilon 3}(x, \theta)).$$

Observe that

$$\hat{\mathbf{u}}^\varepsilon(0, x) = \hat{h}^\varepsilon(x) = \hat{U}^\varepsilon(x, \phi(x)/\varepsilon), \quad \hat{U}^\varepsilon(x, \theta) := \Gamma(\varepsilon, 0, x, \theta; \hat{V}^\varepsilon(x, \theta)).$$

The definition of $\hat{V}^{\varepsilon 2}$ and the condition (4.18) imply that

$$(4.24) \quad \hat{V}^\varepsilon(x, \theta) = \hat{V}_0(x, \theta) + \sum_{j=1}^{J-1} \varepsilon^j \hat{V}_j(x, \theta, \vec{\psi}(x)/\varepsilon) + \varepsilon^J r^\varepsilon(x)$$

where by convention $\hat{V}_0 := {}^t(0, 0, \hat{V}_0^3)$ and

$$\hat{V}_j(x, \theta, \tilde{\theta}) := {}^t(0, \hat{V}_j^2(x, \theta, \tilde{\theta}), \hat{V}_j^3(x, \theta, \tilde{\theta})), \quad \forall j \in \{0, \dots, J-1\}.$$

Associate with the system (4.6) the initial data ${}^t(0, \hat{V}_0^3)$. This furnishes a solution ${}^t(\hat{\mathbf{V}}_0^1, \hat{\mathbf{V}}_0^3)$ to (4.6). Now, for all $j \in \{1, \dots, m\}$, solve on $\Omega \times \mathbb{T}$ the eiconal equation

$$(4.25) \quad \begin{cases} \partial_t \Psi_j + (\mathbf{H}(t, x, \theta + \hat{\mathbf{V}}_0^1) \cdot \nabla_x) \Psi_j + |\nabla_x \Phi|^2 \hat{\mathbf{V}}_0^3 \partial_\theta \Psi_j = 0, \\ \Psi_j(0, x, \theta) = \psi_j(x). \end{cases}$$

The description of the propagation of solutions $\hat{\mathbf{V}}^\varepsilon$ to (4.5) issued from oscillating initial data \hat{V}^ε as in (4.24) is the matter of what is called multidimensional weakly nonlinear geometric optics [14]-[15]. Assumptions are needed to use the related results.

On the one hand, the real vector space spanned by the function $\Psi_0(t, x) = t$ and the other functions $\Psi_j(t, x)$ must be *coherent*. On the other hand, some *small divisor property* must be satisfied. It turns out that all theses prerequisites can be verified in the present context.

In fact, the corresponding hypothesis are checked in [8] (where a very similar situation is examined). Therefore, Theorem 6.4 of [14] can be applied. The solution $\hat{\mathbf{V}}^\varepsilon$ is such that

$$\hat{\mathbf{V}}^\varepsilon(t, x, \theta) = \hat{\mathbf{V}}_0(t, x, \theta) + \varepsilon \hat{\mathbf{V}}_1(t, x, \theta, \vec{\Psi}(t, x, \theta)/\varepsilon) + O(\varepsilon^2)$$

where

$$\hat{\mathbf{V}}_1(0, x, \theta, \tilde{\theta}) = \hat{V}_1(x, \theta, \tilde{\theta}), \quad \vec{\Psi} = (\Psi_1, \dots, \Psi_m).$$

Recall that

$$\hat{\mathbf{u}}^\varepsilon(t, x) = \hat{\mathbf{U}}^\varepsilon(t, x, \Phi(t, x)/\varepsilon), \quad \hat{\mathbf{U}}^\varepsilon(t, x, \theta) := \Gamma(\varepsilon, t, x, \theta; \hat{\mathbf{V}}^\varepsilon(t, x, \theta)).$$

In particular

$$\hat{\mathbf{U}}^\varepsilon(t, x, \theta) = \mathbf{H}(t, x, \theta) + \varepsilon \hat{\mathbf{H}}_0(t, x, \theta, \vec{\Psi}(t, x, \theta)/\varepsilon) + O(\varepsilon^2)$$

with

$$\hat{\mathbf{H}}_0(t, x, \theta, \tilde{\theta}) := \hat{\mathbf{V}}_1^2(t, x, \theta, \tilde{\theta}) \nabla_x \Phi(t, x)^\perp + \hat{\mathbf{V}}_0^3(t, x, \theta) \nabla_x \Phi(t, x).$$

In view of (4.17), we have $\partial_{\tilde{\theta}_j} \mathbf{V}_1^2(0, \cdot) \neq 0$. Exploit (4.16) to obtain

$$\partial_t (\partial_\theta \Psi_j)(0, \cdot) = -(\partial_2 \phi)^{-1} \partial_\theta s^* \nabla_x \phi^\perp \cdot \nabla_x \psi_j \neq 0.$$

It follows that $\nabla_{\tilde{\theta}} \hat{\mathbf{H}}_0 \neq 0$ and $\partial_\theta \vec{\Psi} \neq 0$ as expected. The first part of Proposition 4.2 is shown.

Examples of families $\{\check{h}^\varepsilon\}_\varepsilon$ which are well prepared on ω for incompressible Euler equations are given in Lemma 2.6. Follow the same procedure as in the proof of Lemma 2.6 except at the level of (2.39). There, seek solutions φ^ε to (2.38) having the form

$$\varphi^\varepsilon(x) = \phi(x) + \varepsilon \phi^\varepsilon(x, \phi(x)/\varepsilon, \vec{\psi}(x)/\varepsilon)$$

where ϕ^ε can be expanded according to

$$\phi^\varepsilon(x, \theta, \tilde{\theta}) = \sum_{j=0}^{J+1} \varepsilon^j \phi_j(x, \theta, \tilde{\theta}) + \varepsilon^{J+2} r \phi^\varepsilon(x, \theta, \tilde{\theta}).$$

Impose $\nabla_{\tilde{\theta}} \phi_0 \equiv 0$ and $\partial_{\tilde{\theta}_j} \phi_1 \neq 0$ so that (Lemma 2.7)

$$\partial_{\tilde{\theta}_j} H_0(x, \theta, \tilde{\theta}) = -\partial_2 \phi^{-1} \partial_{\tilde{\theta}_j} \phi_1(x, \theta, \tilde{\theta}) \partial_\theta K \nabla_x \phi^\perp$$

Obviously, the two restrictions (4.17) and (4.18) are verified. This observation completes the proof. \square

Remark 4.2.3 - transfer of energy. Examine how the square $\mathcal{F}(\hat{\mathbf{u}}^\varepsilon)^2$ of the Fourier transform of $\hat{\mathbf{u}}^\varepsilon$ is distributed. At time $t = 0$, it is divided amongst the two characteristic wave numbers $k \simeq 1$ and $k \simeq \varepsilon^{-1}$. In view of (4.22), this situation does not persist. At a time $t > 0$, the concentration is around the three wave numbers $k \simeq 1$, $k \simeq \varepsilon^{-1}$ and $k \simeq \varepsilon^{-2}$. This spontaneous apparition of a new frequency expresses turbulent features. \diamond

Remark 4.2.4 - improvement of Proposition 4.2. In fact, the preceding analysis allows to take into account, already at time $t = 0$, oscillating phases $\psi_j(x, \phi(x)/\varepsilon)$. The presentation has been here intentionally simplified. It is to underline the creation phenomena of a new frequency (by interaction of waves having the same frequencies). \diamond

Remark 4.2.5 - this is just a first step. Of course, the situation studied here is particular since the data H and ϕ have been adjusted in a very special way. The propagation of a general large amplitude wave solution to (1.1) can reveal many other complex phenomena. For sure, the picture must be completed with the cascade of phases observed in [5], and no only. \diamond

Remark 4.2.6 - about the choice of ϕ_0 . The equation (2.38) can be solved on a domain which does not depend on ε even if $\nabla_{\tilde{\theta}}\phi_0 \neq 0$. This indicates that the method developed here can be pushed forward to incorporate the interaction of large amplitude multiphase oscillations. We are satisfied here to touch on this delicate subject through the present remark. \diamond

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