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Serge Troubetzkoy

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# PERIODIC BILLIARD ORBITS IN RIGHT TRIANGLES II

SERGE TROUBETZKOY

ABSTRACT. I show a pointwise density result of periodic billiard orbits in irrational right triangles.

## 1. INTRODUCTION

A billiard ball, i.e. a point mass, moves inside a polygon  $P \subset \mathbb{R}^2$  with unit speed along a straight line until it reaches the boundary  $\partial P$ , then instantaneously changes direction according to the mirror law: “the angle of incidence is equal to the angle of reflection,” and continues along the new line. If the trajectory hits a corner of the polygon, in general it does not have a unique continuation and thus by definition it stops there.

It is an open question if there exists a periodic billiard orbit in every polygon. None the less for certain classes of polygons one can exhibit the existence of many periodic orbits. In particular one can ask how dense are the periodic orbits. Most of the known results are about rational polygons, i.e. polygons for which the angles between the side are rational multiples of  $\pi$ . The first result in this direction was that of H. Masur who showed that the directions of periodic orbits are dense in a rational polygon [Ma]. This was strengthened by Boshernitzan et. al. who showed that for a rational polygon periodic orbits are dense in the phase space [BoGaKrTr]. In this article they also showed a pointwise density result in the configuration space: in a rational polygon  $P$  there exists a dense  $G_\delta$  set  $G \subset P$  such that for each point  $p \in G$  the orbit of  $(p, \theta)$  is periodic for a dense subset of directions  $\theta \in \mathbb{S}^1$ . Vorobets strengthen this result to show that the set  $G$  is also of full measure [Vo].

Recently I showed that there is an open set  $\mathcal{O}$  of right triangles such that for each irrational  $P \in \mathcal{O}$  the set of periodic billiard orbits are dense in the phase space [Tr]. The main result of this article is a twofold strengthening of this result. First of all I extend the result to all irrational right triangles and secondly I deduce a pointwise density statement in the configuration space.

**Theorem 1.** *Suppose that  $P$  is any irrational right triangle. Then there exists an at most countable set  $B \subset P$  such that for every  $p \in$*

$P \setminus B$  the orbit of  $(p, \theta)$  is periodic for a dense subset of directions  $\theta \in \mathbb{S}^1$ .

Billiards in right triangles are well known to be equivalent to the motion of two elastic point masses on a segment (see for example [MaTa]). Theorem 1 tells us that except for an at most countable set  $B$  of initial positions  $0 \leq x_1 \leq x_2 \leq 1$  if  $(x_1, x_2) \notin B$  then the orbit of  $(x_1, v_1), (x_2, v_2)$  is periodic for a dense set of velocities  $(v_1, v_2)$ .

Somewhat surprisingly Theorem 1 is stronger than the result of Vorobets for rational polygons. There is a special class of rational polygons known as Veech polygons which are well studied, see for example [MaTa] for the definition. Combining known results on Veech polygons with the arguments of this article I can prove

**Proposition 2.** *In  $P$  is a Veech polygon there exists an at most countable set  $B \subset P$  such that for every  $p \in P \setminus B$  the orbit of  $(p, \theta)$  is periodic for a dense subset of directions  $\theta \in \mathbb{S}^1$ .*

Arithmetic or square tiled polygons form a subclass of Veech polygons [Zo]. Let  $V(P)$  be the set of corners of  $P$ . By definition there are no periodic orbits passing through  $p \in V(P)$ .<sup>1</sup> Thus Theorem 1 and Proposition 2  $B$  is nonempty since  $V(P) \subset B$ . A simple geometric argument shows

**Proposition 3.** *If  $P$  is arithmetic then for every  $p \in P \setminus V(P)$  the orbit of  $(p, \theta)$  is periodic for a dense subset of directions  $\theta \in \mathbb{S}^1$ .*

## 2. STRATEGY

An orbit segment which begins and ends at a vertex of the polygon is called a generalized diagonal. In [Tr] I proved the following theorem.  
**Theorem A.** *Suppose  $P$  is an irrational right triangle. Fix a direction  $\theta$  which is perpendicular to one of the legs of the triangle. Suppose the perpendicular orbit of the nonregularisable endpoint of the leg is twice perpendicular, and the second perpendicular hit is at an interior point. Then the invariant surface  $S_\theta$  is foliated by periodic orbits except for an at most countable collection of orbits/generalized diagonals  $O_i$ .*

The strategy to prove Theorem 1 is as follows. First remark that Theorem A holds for the direction perpendicular to the hypotenuse of  $P$ .

**Theorem A'.** *Suppose  $P$  is an irrational right triangle. Fix a direction  $\theta$  which is perpendicular to the hypotenuse of  $P$ . Suppose the*

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<sup>1</sup>There is a unique continuation of the billiard orbit through vertices with angle  $\pi/n$ . Such a vertex is called regularisable. If we consider such orbits as defined by this continuation then Proposition 3 holds for  $V(P)$  defined as the nonregularisable vertices.

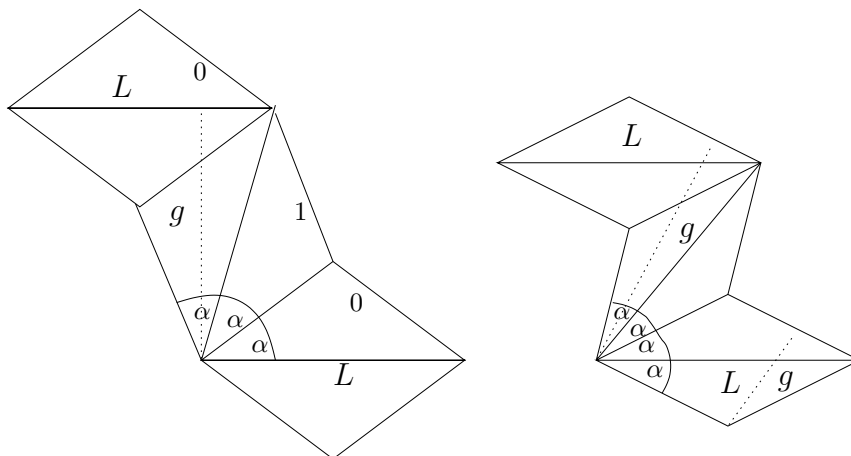


FIGURE 1.  $g$  hits perpendicularly at an interior point.

*perpendicular orbit of the endpoints of the hypotenuse are twice perpendicular, and the second perpendicular hits are at an interior points. Then the the invariant surface  $S_\theta$  is foliated by periodic orbits except for a countable collection of orbits/generalized diagonals  $O_i$ .*

The proof of Theorem A' is identical to the proof of Theorem A. Next I verify that

**Lemma 4.** *The assumption of Theorem A or Theorem A' hold for any irrational right triangle  $P$ .*

Finally I prove that the foliation of the invariant surface by periodic orbits implies the density statement of Theorem 1.

**Lemma 5.** *Suppose  $P$  is an irrational polygon. Suppose that there exists a direction  $\theta \in \mathbb{S}^1$  such that the invariant surface  $S_\theta$  is foliated by periodic orbits except for a countable collection of orbits/generalized diagonals  $O_i$ . Then there exists an at most countable set  $B \subset P$  such that for every  $p \in P \setminus B$  the orbit of  $(p, \theta)$  is periodic for a dense subset of directions  $\theta \in \mathbb{S}^1$ .*

### 3. PROOFS

The proofs follow the notation of [Tr].

**Proof of Lemma 4.** Consider the right triangle and its related rhombus so that the longer leg is horizontal. Suppose that the angle  $\alpha$  between the hypotenuse and this leg satisfies  $2n\alpha < \frac{\pi}{2}$  and  $(2n+1)\alpha > \frac{\pi}{2}$  for some  $n \geq 1$ . Then the vertical orbit  $g$  starting at the left endpoint of this leg hits the leg again perpendicularly at an interior point (the case  $n = 1$ ,  $\alpha \in (\frac{\pi}{6}, \frac{\pi}{4})$  is illustrated in figure 1a). Thus triangles for which  $\alpha \in \cup_{n \geq 1} (\frac{\pi}{4n+2}, \frac{\pi}{4n})$  satisfy the assumption of Theorem A.

Next consider the orbits perpendicular to the hypotenuse. If the angle  $\alpha$  satisfies  $2n\alpha > \frac{\pi}{2}$  and  $(2n-1)\alpha < \frac{\pi}{2}$  for some  $n \geq 2$  then

the perpendicular orbits starting at the endpoints of the hypotenuse hit the hypotenuse perpendicularly again at an interior point (the case  $n = 2$  is illustrated in Figure 1b). Thus right triangles for which  $\alpha \in \cup_{n \geq 2} (\frac{\pi}{4n}, \frac{\pi}{4n-2})$  satisfy the assumption of Theorem A'.  $\square$

**Proof of Lemma 5.** The surface  $S_\theta$  is obtained by the usual unfolding procedure [MaTa]. Thus we think of it as being tiled by copies of  $P$  which can be enumerated  $P_i$ . Fix  $i$  and consider  $P_i \cap \{O_j\}$ . This set is an at most countable collection of parallel line segments  $U_{i,j}$ . Let  $U_i = \cup_j U_{i,j}$ . Consider the projection  $\pi : S_\theta \rightarrow P$ . This is a countable to one map. Enumerate the points in  $\pi^{-1}(p)$ ,  $p_1, p_2, \dots$ . For each  $p \in P$  the directions of the  $p_k$  are dense in  $\mathbb{S}^1$ . By Lemma 4 each of these directions is the direction of a periodic orbit through  $p$ , unless  $p_k$  is in some  $U_{i,j}$ . If the periodic directions through  $p$  are not dense then there exists infinitely many  $U_i$  such that  $p \in \pi(U_i)$ . Suppose there exists  $i, i'$  such that  $p \in \pi(U_i) \cap \pi(U_{i'})$ . Since the line segments in  $\pi(U_i)$  and  $\pi(U_{i'})$  are transverse this intersection is at most countable. One completes the proof by taking a union over  $i, i'$ .  $\square$

**Proof of Theorem 1.** The theorem follows immediately by combining Lemmas 4 and 5.  $\square$

**Proof of Proposition 2.** In Veech polygons there are a dense set of periodic directions, and each periodic direction is completely foliated by periodic orbits except for a finite number of generalized diagonals. Apply the argument of Lemma 5 to finish the proof.  $\square$

**Proof of Proposition 3.** Consider the square. Fix a rational direction. Every point in this direction is periodic except for generalized diagonals. Suppose that the square unfolds to a torus. Consider the tiling of the plane by squares representing the torus. The intersection points of two generalized diagonals have rational coordinates. Thus we conclude that any point in the square for which at least one coordinate is irrational lies in at most one generalized diagonal and thus satisfies the conclusions of the proposition.

Consider a rational point  $(p_1/q, p_2/q)$  with  $\gcd(p_1, p_2, q) = 1$ . Tile the plane by  $(1/q, 1/q)$  squares. Consider the orbits of slope  $a/b$  with  $\gcd(a, b) = 1$  starting at the point  $(p_1, p_2)$ . Such an orbit either correspond to a periodic generalized diagonal or a periodic billiard orbit. It is a generalized diagonal if and only if there exists an  $i \in \mathbb{Z}$  such that  $q|(p_1 + ia)$  and  $q|(p_2 + ib)$  (see Figure 2). Note that  $p_i \in \{0, 1, \dots, q\}$  and not being in the set  $V(P)$  is equivalent to at least one of the  $p_i$  differing from 0 or  $q$ . If  $p_1 \notin \{0, q\}$  and  $q|a$  then  $q$  never divides  $p_1 + ia$  and thus applying the above condition yields that the orbit is periodic. Thus the periodic directions through the given point include the set

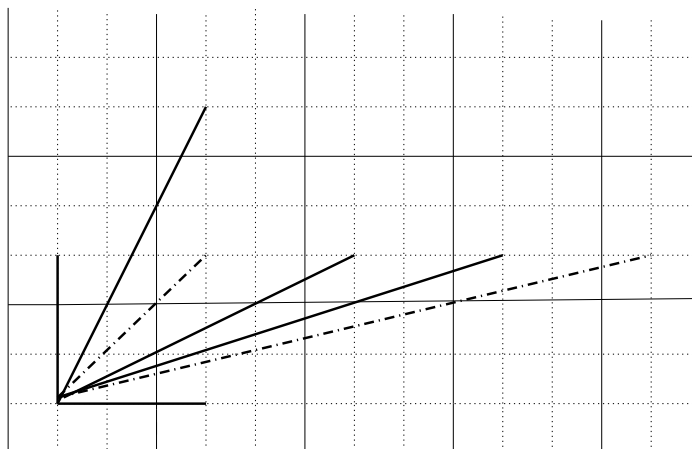


FIGURE 2. Solid lines are periodic directions while dashed lines are generalized diagonals starting at the point  $(1/3, 1/3)$ .

$\{qa'/b : qa' \wedge b = 1\}$ . Similarly applying the condition in the case  $p_2 \notin \{0, q\}$  and  $q|b$  yields that the periodic directions include the set  $\{a/qb' : a \wedge qb' = 1\}$ . Both of these sets of directions are dense in  $\mathbb{S}^1$ .

Now if  $P$  is arithmetic then the unfolded surface is a torus cover. The periodic orbits constructed above for the square lift to periodic orbits in  $P$ .  $\square$

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CENTRE DE PHYSIQUE THÉORIQUE, FEDERATION DE RECHERCHES DES UNITES DE MATHÉMATIQUE DE MARSEILLE, INSTITUT DE MATHÉMATIQUES DE LUMINY AND, UNIVERSITÉ DE LA MÉDITERRANÉE, LUMINY, CASE 907, F-13288 MARSEILLE CEDEX 9, FRANCE

*E-mail address:* [troubetz@iml.univ-mrs.fr](mailto:troubetz@iml.univ-mrs.fr)

*URL:* <http://iml.univ-mrs.fr/~troubetz/>