Sinai's condition for real valued Lévy processes

Victor Rivero

Sinaĭ's condition for real valued Lévy processes

Víctor RIVERO *

24th May 2005

Abstract

We prove that the upward ladder height subordinator H associated to a real valued Lévy process ξ has Laplace exponent φ that varies regularly at ∞ (resp. at 0) if and only if the underlying Lévy process ξ satisfies Sinaĭ's condition at 0 (resp. at ∞). Sinaĭ's condition for real valued Lévy processes is the continuous time analogue of Sinaĭ's condition for random walks. We provide several criteria in terms of the characteristics of ξ to determine whether or not it satisfies Sinaĭ's condition. Some of these criteria are deduced from tail estimates of the Lévy measure of H, here obtained, and which are analogous to the estimates of the tail distribution of the ladder height random variable of a random walk which are due to Veraverbeke and Grübel.

Key words: Lévy processes, Fluctuation theory, Regular Variation, long tailed Lévy measures.
MSC: 60 G 30 (60 G 51).

1 Introduction and main result

Let $\xi = \{\xi_t, t \geq 0\}$ be a real valued Lévy process, $S = (S_t, t \geq 0)$ its current supremum and $L = (L_t, t \geq 0)$ the local time at 0 of the strong Markov process ξ reflected at its current supremum, that is to say $(S_t - \xi_t, t \geq 0)$. In this work we will obtain some asymptotic properties of the ascending ladder height subordinator H associated to ξ (that is, the current supremum of ξ evaluated at the inverse of the local time at 0, i.e. L^{-1}, $H \equiv (S_{L_t^{-1}}, t \geq 0)$). According to Fristedt [9] the ascending ladder process (L^{-1}, H) is a bivariate subordinator, that is, a Lévy process in $\mathbb{R}^2$ with increasing paths (coordinatewise) whose bivariate Laplace exponent κ,

$$e^{-\kappa(\lambda_1,\lambda_2)} \equiv \mathbf{E}(e^{-\lambda_1 L_1^{-1} - \lambda_2 H_1}), \qquad \lambda_1, \lambda_2 \geq 0,$$

with the assumption $e^{-\infty} = 0$, is given by

$$\kappa(\lambda_1, \lambda_2) = \mathrm{k} \exp\left\{ \int_0^\infty \frac{\mathrm{d}t}{t} \int_{[0,\infty[} (e^{-t} - e^{-\lambda_1 t - \lambda_2 x})\, \mathbf{P}(\xi_t \in \mathrm{d}x) \right\}, \qquad \lambda_1, \lambda_2 \geq 0,$$

*Équipe MODAL'X, Université Paris X Nanterre, UFR SEGMI, 200 Avuenue de la République, 92000 Nanterre CEDEX France; E–mail: rivero@ccr.jussieu.fr

with k a constant that depends on the normalization of the local time. (See Doney [6], for a survey, and Bertoin [2] VI, for a detailed exposition of the fluctuation theory of Lévy processes and Vigon [22] for a description of the Lévy measure of H.)

The fact that the ladder process (L^{-1}, H) is a bivariate subordinator is central in the fluctuation theory of Lévy processes because it enables to obtain several properties of the underlying Lévy process using results for subordinators, which are objects simpler to manipulate. Among the various properties that can be obtained using this fact, there is a well known arc-sine law in the time scale for Lévy processes, see Theorem VI.3.14 in Bertoin's book [2] for a precise statement. That result tell us that Spitzer's condition is a condition about the underlying Lévy process ξ which ensures that the Laplace exponent $\kappa(\cdot, 0)$ of the ladder time subordinator L^{-1} is regularly varying and which in turn permits to obtain an arc-sine law in the time scale for Lévy processes. Now, if we want to establish an analogous result in the space scale we have to answer the question: *What is the analogue of Spitzer's condition for the upward ladder height process H?* or put another way: *What do we need to assume about ξ to ensure that the Laplace exponent*

$$\varphi(\lambda) \equiv \kappa(0, \lambda) = \mathrm{k} \exp\left\{ \int_0^\infty \frac{\mathrm{d}t}{t} \int_{[0,\infty[} (e^{-t} - e^{-\lambda x})\, \mathbf{P}(\xi_t \in \mathrm{d}x) \right\}, \qquad \lambda \geq 0,$$

of H varies regularly?

To motivate the answer that we will provide to these questions we will make a slight digression to recall the analogue of this problem for random walks. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables and Z its associated random walk $Z_0 = 0,\ Z_n = \sum_{k=1}^n X_k, n > 0$. The analogue of the upward ladder process for Z is given by (N, Z_N) where N is the first ladder epoch of the random walk Z, $N = \min\{k : Z_k > 0\}$, and Z_N, is the position of Z at the instant N. We denote by $M_n = \sup\{Z_k, 0 \leq k \leq n\}, n \geq 0$ the current supremum of the random walk and by $T(x)$ the first instant at which the random walk passes above the level x, $T(x) = \inf\{n > 0 : Z_n \geq x\}$.

Greennwood, Omey and Teugels [13] proved that the condition,

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbf{P}(z < Z_n \leq \lambda z) \xrightarrow[z\to\infty]{} \beta \log(\lambda), \qquad \forall \lambda > 1,$$

for some $\beta \in [0, 1]$, which they called Sinaĭ's condition making reference to a work of Sinaĭ's [19], plays the same rôle, in the obtaining of limit results for the ladder height, as does Spitzer's condition for the ladder time. More precisely, when the random walk does not drift to $-\infty$, we have the following equivalences.

Theorem 1 (Dynkin [8], Rogozin [17], Greenwood et al. [13]). *Assume $\sum_n \frac{1}{n} \mathbf{P}(Z_n > 0) = \infty$. For $\beta \in [0, 1]$, the following are equivalent*

(i) $\lim_{z\to\infty} \sum_n \frac{1}{n} \mathbf{P}(z < Z_n \leq \lambda z) = \beta \log(\lambda), \qquad \forall \lambda > 1;$

(ii) $\int_0^x \mathbf{P}(Z_N > u)\mathrm{d}u \sim x^{1-\beta} l(x)$ *as* $x \to \infty$, *with l slowly varying;*

(iii) The random variables $\gamma_x = \dfrac{M_{T(x)-1}}{x}$ *converge in distribution as* $x \to \infty$*; the limit law is the generalized arc-sine law of parameter* β*, that is to say that it has a density*

$$q_\beta(y) = \frac{\sin(\beta\pi)}{\pi} y^{-\beta}(1-y)^{\beta-1}, \qquad y \in]0,1[,$$

if $\beta \in]0,1[$*; and is degenerate with unit mass at 1 or 0, according as* $\beta = 1$ *or* $\beta = 0$*.*

The equivalence between (i) and (ii) is due to Greenwood et al. [13] (see also [4] Theorem 8.9.17), that between (ii) and (iii) is due to Dynkin [8] in the case $\beta \in]0,1[$ and to Rogozin [17] in the other cases.

Thus, the previous theorem tells us that Sinaĭ's condition enables to obtain a spatial arc-sine law for random walks and, given that the fluctuation theory for Levy processes mirrors that of random walks, it is natural to hope that the answer to the questions posed above is the continuous time version of Sinaĭ's condition.

We will say that a Lévy process ξ satisfies Sinaĭ's condition at ∞ (resp. at 0) if

(Sinaĭ) There exists a $0 \le \beta \le 1$ such that

$$\int_0^\infty \frac{\mathrm{d}t}{t}\, \mathbf{P}(z < \xi_t \le \lambda z) \longrightarrow \beta \log(\lambda) \quad \text{as} \quad z \to +\infty \quad (\text{resp. } z \to 0+), \qquad \forall \lambda > 1.$$

The term β will be called Sinaĭ's index of ξ. Observe that if Sinaĭ's condition hold we have that

$$\int_0^\infty \frac{\mathrm{d}t}{t}\, \mathbf{P}(\lambda z < \xi_t \le z) \longrightarrow -\beta \log(\lambda) \quad \text{as} \quad z \to +\infty \quad (\text{resp. } z \to 0+), \qquad \forall \lambda \in]0,1[.$$

Example 1. A Lévy process, ξ, which satisfies Sinaĭ's condition is the strictly stable process with index $0 < \alpha \le 2$. Indeed, for every $z > 0$ and $\lambda > 1$ we have by the scaling property of ξ that

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{d}t}{t}\, \mathbf{P}\left(z < \xi_t \le \lambda z\right) &= \int_0^\infty \frac{\mathrm{d}t}{t}\, \mathbf{P}\left(z < t^{1/\alpha}\xi_1 \le \lambda z\right) \\
&= \mathbf{E}\left(1_{\{\xi_1 > 0\}} \int_0^\infty \frac{\mathrm{d}t}{t} 1_{\{(z/\xi_1)^\alpha < t \le (z\lambda/\xi_1)^\alpha\}}\right) \\
&= \mathbf{E}\left(1_{\{\xi_1 > 0\}} \log(\lambda^\alpha)\right) = \alpha\, \mathbf{P}(\xi_1 > 0) \log(\lambda).
\end{aligned}$$

Thus any stable process ξ does satisfies Sinaĭ's condition at infinity and at 0 with index $\alpha\rho$, where ρ is the positivity parameter of ξ, $\rho = \mathbf{P}(\xi_1 \ge 0)$.

We recall that a measurable function $f : [0,\infty[\to [0,\infty[$ varies regularly at infinity (resp. at 0) with index $\alpha \in \mathbb{R}$, $f \in RV_\alpha^\infty$ (resp. $\in RV_\alpha^0$), if for any $\lambda > 0$,

$$\lim \frac{f(\lambda x)}{f(x)} = \lambda^\alpha \qquad \text{at } \infty \text{ (resp. at 0)}.$$

We have all the elements to state our main result, which provides an answer to the questions above.

Theorem 2. *For $\beta \in [0,1]$, the following are equivalent*

(i) The Lévy process ξ satisfies Sinaĭ's condition at ∞ (resp. at 0) with index β.

(ii) The Laplace exponent of the ladder height subordinator H varies regularly at 0 (resp. at ∞) with index β.

The proof of this result is an easy consequence of a fluctuation identity due to Bertoin and Doney [3] and Theorem 4 below.

Proof. By the fluctuation identity of Bertoin and Doney we have that for any $z > 0, \lambda > 1$,

$$\int_0^\infty \frac{\mathrm{d}t}{t} \mathbf{P}(z < \xi_t \leq \lambda z) = \int_0^\infty \frac{\mathrm{d}t}{t} \mathbf{P}(z < H_t \leq \lambda z).$$

As a consequence, Sinaĭ's condition is satisfied by the Lévy process ξ if and only if it is satisfied by the ascending ladder height subordinator H. The result then follows from Theorem 4, which establishes that the Laplace exponent ϕ of any given subordinator, say σ, varies regularly if and only if σ satisfies Sinaĭ's condition. □

Assuming that the Lévy process ξ satisfies Sinaĭ's condition and applying known results for subordinators, when its Laplace exponent is regularly varying, we can deduce the behavior at 0 or ∞ of ξ from that of H. (See Bertoin [2] Ch. III for an account on the short and long time behavior of subordinators.) The following *spatial arc-sine law* for Lévy processes is an example of the results that can be obtained.

Corollary 1. *For $r > 0$, denote the first exit time of ξ out of $]-\infty, r]$ by $T_r = \inf\{t > 0 : \xi_t > r\}$, the undershot and overshot of the supremum of ξ by $U(r) = r - S_{T_r-}$ and $O(r) = S_{T_r} - r = \xi_{T_r} - r$. For any $\beta \in [0,1]$, the conditions (i) and (ii) in Theorem 2 are equivalent to the following conditions:*

(iii) The random variables $r^{-1}(U(r), O(r))$ converge in distribution as $r \to \infty$ (respectively, as $r \to 0$).

(iv) The random variables $r^{-1}O(r)$ converge in distribution as $r \to \infty$ (respectively, as $r \to 0$).

(v) The random variables $r^{-1}S_{T_r-}$ converge in distribution as $r \to \infty$ (respectively, as $r \to 0$).

(vi) $\lim r^{-1}\,\mathbf{E}\,(S_{T_r-}) = \beta \in [0,1]$ as $r \to \infty$ (respectively, as $r \to 0$).

In this case, the limit distribution in (iii) is determined as follows: if $\beta = 0$ (resp. $\beta = 1$), it is the Dirac mass at $(1, \infty)$ (resp. at $(0,0)$). For $\beta \in]0,1[$, it is the distribution with density

$$p_\beta(u,w) = \frac{\beta \sin \beta\pi}{\pi}(1-u)^{\beta-1}(u+w)^{-1-\beta}, \qquad 0 < u < 1, w > 0.$$

In particular, the limit law in (v) is the generalized arc-sine law of parameter β.

Proof. We recall that for every $r > 0$, the random variables $(U(r), O(r))$ are almost surely equal to the undershot and overshoot, $(U_H(r), O_H(r))$, of the ladder height subordinator H. Thus the result is a straightforward consequence of the Dynkin-Lamperti arc-sine law for subordinators, Theorem III.3.6 in [2], using the elementary relations: for every $r > 0$

$$\begin{aligned}\mathbf{P}(U_H(r) > y) &= \mathbf{P}(O_H(r-y) > y), \qquad r > y > 0,\\ \mathbf{P}(O_H(r) > x, U_H(r) > y) &= \mathbf{P}(O_H(r-y) > x+y), \qquad r > y > 0,\ x > 0.\end{aligned}$$

□

We would like to remark that in the case where Sinaĭ's condition hold for ξ with index $\beta = 1$ at ∞ (resp. at 0) we have that $r^{-1}\xi_{T_r}$ converges in law to 1 as $r \to \infty$ (resp. to 0). The almost sure convergence of this random variable was studied by Doney and Maller [7]. Precisely, Theorem 8 of Doney and Maller [7] provide necessary and sufficient conditions, on the characteristics of ξ, according to which $r^{-1}\xi_{T_r}$ converges a.s. to 1 as $r \to \infty$. Moreover, Theorem 4 in [7] establish that the latter r.v. converge a.s. to 1 as $r \to 0$ if and only if ξ creeps upward.

To summarize, in Theorem 2 we provided a necessary and sufficient condition in terms of the marginal laws of ξ which completely answers the questions posed at the beginning of this work. However, the possible drawback of this result is that in most of the cases we only know the characteristics of the Lévy process ξ, that is, its linear and Gaussian terms and Lévy measure, and so it would be suitable to have a condition in terms of the characteristics of the process. That is the purpose of the second part of this work.

One case at which Sinaĭ's condition can be verified using the characteristics of the process is the case at which the underlying Lévy process belongs to the domain of attraction at infinity (respectively, at 0) of a strictly stable law of index $0 < \alpha \leq 2$, and which does not require a centering function. That is, whenever there exists a deterministic function $b :]0, \infty[\to]0, \infty[$ such that

$$\frac{\xi_t}{b(t)} \xrightarrow{\mathcal{D}} X(1), \quad \text{as } t \to \infty \text{ (respectively, as } t \to 0), \tag{1}$$

with $X(1)$ a strictly stable random variable of parameter $0 < \alpha \leq 2$. It is well known that if such a function b exists, it is regularly varying at infinity (respectively, at 0) with index $\beta = 1/\alpha$. Plainly, the convergence in (1) can be determined in terms of the characteristic exponent Ψ of ξ, i.e. $\mathbf{E}(e^{i\lambda\xi_t}) = \exp\{t\Psi(\lambda)\}, \lambda \in \mathbb{R}$, since the latter convergence in distribution is equivalent to the validity of the limit

$$\lim t\Psi\left(\frac{\lambda}{b(t)}\right) = \Psi_\alpha(\lambda), \quad \text{as } t \to \infty, \quad \text{(respectively, as } t \to 0\ \text{) for } \lambda \in \mathbb{R}, \tag{2}$$

where Ψ_α is the characteristic exponent of a strictly stable law and is given by

$$\Psi_\alpha(\lambda) = \begin{cases} -c|\lambda|^\alpha\left(1 - i\delta\operatorname{sgn}(\lambda)\tan(\pi\alpha/2)\right) & 0 < \alpha < 1 \text{ or } 1 < \alpha < 2;\\ -c|\lambda|^\alpha\left(1 - i\delta\operatorname{sgn}(\lambda)\tan(\pi\alpha/2)\ln(|\lambda|)\right), & \alpha = 1;\\ -q^2\lambda^2/2, & \alpha = 2;\end{cases}$$

for $\lambda \in \mathbb{R}$, where $c > 0$ and the term $\delta \in [-1, 1]$ is the so called skewness parameter. We have the following theorem whose proof will be given in Section 3.

Theorem 3. *Let $0 < \alpha \leq 2$ and $\delta \in [-1, 1]$. Assume that there exits a function $b :]0, \infty[\to]0, \infty[$ such that the limit in equation (2) holds as t goes to infinity (resp. as $t \to 0$). Then the Lévy process ξ satisfies Sinai's condition at ∞ (respectively, at 0) with index $\alpha\rho$, where ρ is given by $\rho = 1/2 + (\pi\alpha)^{-1} \arctan(\delta \tan(\alpha\pi/2))$.*

With the aim of providing some other criteria in terms of the characteristics of the underlying Lévy process ξ to determine whether or not it satisfies Sinaĭ's condition, in Section 4 we will provide, under some assumptions, some estimates of the Lévy measure of H. Those estimates are more general than needed and are the Lévy processes version of the results of Veraverbeke [20] and Grübel [14] for random walks. Furthermore, they are in the same vein as those results obtained recently by Klüppelberg, Kyprianou and Maller [16], Propositions 5.3 and 5.4.

Using those estimates we will provide some criteria to determine whether or not the Lévy measure of H is regularly varying. For our ends this will be sufficient because by an Abelian-Tauberian theorem we know that this determines the regular variation of the Laplace exponent of H.

The rest of this note is organized as follows. In Section 2, we state and prove an equivalent form of the Dynkin & Lamperti's theorem for subordinators (see e.g. [2] Theorem III.6) which is interesting in itself. In Section 3, we prove Theorem 3. Finally, in Section 4 we provide some estimates of the Lévy measure of H and we use those estimates to provide some criterions in terms of the characteristics of the underlying Lévy process ξ to determine whether or not it satisfies Sinaĭ's condition. Section 4 is self contained and can be read separately from the rest of the paper.

2 A result for subordinators

Let $\sigma = (\sigma_t, t \geq 0)$ be a subordinator, possibly killed, with life time ζ, and denote by ϕ its Laplace exponent,

$$\phi(\lambda) \equiv -\log \mathbf{E}(e^{-\lambda\sigma_1}, 1 < \zeta), \qquad \lambda \geq 0.$$

It is well known that the Laplace exponent ϕ can be represented as

$$\phi(\lambda) = \kappa + \lambda d + \int_{]0,\infty[} (1 - e^{-\lambda y})\nu(\mathrm{d}y), \qquad \lambda \geq 0,$$

where $\kappa, d \geq 0$ are the killing rate and drift coefficient of σ, respectively, and ν is the Lévy measure of σ, that is, a measure on $]0, \infty[$ such that $\int_{]0,\infty[} \min\{1, y\}\nu(\mathrm{d}y) < \infty$.

The aim of this section is prove the following equivalent form of the Dynkin & Lamperti's theorem for subordinators, see e.g. [2] Theorem III.6.

Theorem 4. *For $\beta \in [0, 1]$, the following are equivalent:*

(i) The subordinator σ satisfies Sinaĭ's condition at $0+$ (resp. at $+\infty$) with index β.

(ii) The Laplace exponent ϕ is regularly varying at $+\infty$ (resp. at $0+$) with index β.

An account on necessary and sufficient conditions according to which a subordinator has a Laplace exponent such that $\phi \in RV_\beta^\infty$ or $\phi \in RV_\beta^0$ can be found in [2] page 82.

The proof of this result relies on the following elementary remark.

Remark 1. Write

$$\phi(\theta) = (1+\phi(\theta))\frac{\phi(\theta)}{1+\phi(\theta)}, \qquad \theta \geq 0.$$

The first (resp. second) factor in the right hand term of the previous equality can be used to determine the behavior at infinity (resp. at 0) of ϕ. More precisely, $\phi \in RV_\beta^\infty$, (resp. $\in RV_\beta^0$) if and only

$$1+\phi(\cdot) \in RV_\beta^\infty, \quad (\text{resp. } \frac{\phi(\cdot)}{1+\phi(\cdot)} \in RV_\beta^0).$$

The proof of Theorem 4 will be given using the previous remark and via three lemmas whose proof will be given at the end of this section. The first of them will enable us to relate the factors of the latter remark with a transformation of the type Mellin's convolution.

Lemma 1. *We have that*

(i) $1+\phi(\theta) = \exp\{\widehat{G}_1(\theta)\}$, *for* $\theta > 0$; *where the function* $\widehat{G}_1$ *is the Mellin convolution of the non-decreasing function*

$$G_1(y) = \int_0^\infty \frac{\mathrm{d}t}{t} e^{-t}\, \mathbf{P}(\sigma_t > 1/y), \qquad y > 0,$$

and the kernel $k(x) = xe^{-x}, x > 0$; *that is,*

$$\widehat{G}_1(\theta) = k^M * G_1(\theta) := \int_0^\infty \frac{\mathrm{d}x}{x} k(\theta/x) G_1(x), \qquad \theta > 0.$$

(ii) $\dfrac{\phi(\theta)}{1+\phi(\theta)} = \exp\left\{-\widehat{G}_2(\theta)\right\}$, *for* $\theta > 0$; *where* $\widehat{G}_2$ *is the Laplace transform of the measure*

$$G_2(\mathrm{d}x) = \int_0^\infty \frac{\mathrm{d}t}{t}(1-e^{-t})\, \mathbf{P}(\sigma_t \in \mathrm{d}x), \qquad x > 0;$$

which is in fact the harmonic renewal measure associated to the law $F(\mathrm{d}x) = \mathbf{P}(\sigma_\Theta \in \mathrm{d}x)$ *with* Θ *an independent random variable with exponential law of parameter* 1.

A consequence of Lemma 1 is that $1+\phi \in RV_\beta^\infty$ if and only if

$$\lim_{\theta\to\infty} \widehat{G}_1(\lambda\theta) - \widehat{G}_1(\theta) = \beta \log \lambda, \qquad \forall \lambda > 0. \tag{3}$$

Moreover, $\phi(\cdot)/(1+\phi(\cdot)) \in RV_\beta^0$ if and only if

$$\lim_{\theta\to 0} \widehat{G}_2(\lambda\theta) - \widehat{G}_2(\theta) = -\beta \log \lambda, \qquad \forall \lambda > 0. \tag{4}$$

The second of these Lemmas enable us to relate Sinai's condition with the behavior at infinity of the differences of the function G_1, and those of the function $G_2(x) \equiv G_2[0,x]$, $x > 0$.

Lemma 2. *Let $\beta \in [0,1]$.*

(i) Sinaĭ's condition holds at 0 with index β if and only if

$$\lim_{z\to\infty} G_1(\lambda z) - G_1(z) = \beta \log(\lambda), \qquad \forall \lambda > 1.$$

(ii) Let $G_2(z) := G_2[0,z], z > 0$. Sinaĭ's condition holds at infinity with index β if and only if

$$\lim_{z\to\infty} G_2(\lambda z) - G_2(z) = \beta \log(\lambda), \qquad \forall \lambda > 1.$$

The last ingredient to achieve the proof of Theorem 4 is an Abelian–Tauberian's type result relying the behavior of the differences of G_1 (resp. G_2) with those of the functions $\widehat{G}_1$ (resp. $\widehat{G}_2$).

Lemma 3. *(i) The following are equivalent*

$$\lim_{y\to\infty} G_1(\lambda y) - G_1(y) = \beta \log(\lambda), \qquad \forall \lambda > 0. \tag{5}$$

$$\lim_{\theta\to\infty} \widehat{G}_1(\lambda\theta) - \widehat{G}_1(\theta) = \beta \log(\lambda), \qquad \forall \lambda > 0. \tag{6}$$

Both imply that

$$G_1(\theta) - \widehat{G}_1(\theta) \xrightarrow[\theta\to\infty]{} \beta\gamma.$$

(ii) The following are equivalent

$$\lim_{y\to\infty} G_2(\lambda y) - G_2(y) = \beta \log(\lambda), \qquad \forall \lambda > 0. \tag{7}$$

$$\lim_{\theta\to 0} \widehat{G}_2(\lambda\theta) - \widehat{G}_2(\theta) = -\beta \log(\lambda), \qquad \forall \lambda > 0. \tag{8}$$

Both imply that

$$G_2(\theta) - \widehat{G}_2(1/\theta) \xrightarrow[\theta\to\infty]{} \beta\gamma.$$

Where γ is Euler's constant $\gamma = \int_0^\infty e^{-v}\log(v)\mathrm{d}v$.

Tacking for granted Lemmas 1,2 & 3 the proof of Theorem 4 is straightforward.

A consequence of Lemma 3 is that quantities related to Sinaĭ's condition can be used to determine whether or not σ has a finite expectation or a strictly positive drift. That is the content of the following corollary.

Corollary 2. *(i) Assume that Sinaĭ's condition holds at infinity with index $\beta = 1$ and that the lifetime of σ is infinite. Then σ has a finite mean if and only if*

$$R \equiv \lim_{\theta\to\infty} \log(\theta) - G_2(\theta) < \infty.$$

In this case $\mathbf{E}(\sigma_1) = e^{\gamma+R}$.

(ii) Assume that Sinaĭ's condition holds at 0 with index $\beta = 1$. Then σ has a strictly positive drift d if and only if

$$\widetilde{R} \equiv \lim_{\theta\to\infty} G_1(\theta) - \log(\theta) < \infty.$$

In this case $d = e^{\gamma+\widetilde{R}}$.

Remark 2. For $\beta \in]0,1[$, it is well known that $\phi \in RV_\beta^\infty$ if and only if the sequence of subordinators σ^z defined by $(\sigma_t^z = z\sigma_{t/\phi(z)}, t \geq 0)$ converge as $z \to \infty$ in the sense of finite dimensional distributions and in Skorohod's topology to a stable subordinator $\widetilde{\sigma}$ of parameter β. This is equivalent to say that for any $t > 0$

$$\mathbf{P}(1 < z\sigma_{t/\phi(z)} \leq \lambda) \xrightarrow[z\to\infty]{} \mathbf{P}(1 < \widetilde{\sigma}_t \leq \lambda), \qquad \lambda > 1, \tag{9}$$

and

$$\mathbf{P}(\lambda < z\sigma_{t/\phi(z)} \leq 1) \xrightarrow[z\to\infty]{} \mathbf{P}(\lambda < \widetilde{\sigma}_t \leq 1), \qquad 0 < \lambda < 1. \tag{10}$$

On the other hand, Theorem 4 ensures that the latter condition on ϕ holds if and only if Sinaĭ's condition holds at 0, which can be written as follows: for any $\lambda > 1$

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}s}{s}\,\mathbf{P}(1 < z^{-1}\sigma_{s/\phi(z^{-1})} \leq \lambda) &= \int_0^\infty \frac{\mathrm{d}t}{t}\,\mathbf{P}(z < \sigma_t \leq \lambda z)\\ &\xrightarrow[z\to 0]{} \beta\ln(\lambda)\\ &= \int_0^\infty \frac{\mathrm{d}s}{s}\,\mathbf{P}(1 < \widetilde{\sigma}_s \leq \lambda),\end{aligned} \tag{11}$$

where the first equality is justified by a change of variables $s = t\phi(z^{-1})$ and the last one follows from the scaling property of the stable subordinator $\widetilde{\sigma}$; and for any $0 < \lambda < 1$

$$\int_0^\infty \frac{\mathrm{d}s}{s}\,\mathbf{P}(\lambda < z^{-1}\sigma_{s/\phi(z^{-1})} \leq 1) \xrightarrow[z\to 0]{} \int_0^\infty \frac{\mathrm{d}s}{s}\,\mathbf{P}(\lambda < \widetilde{\sigma}_s \leq 1), \tag{12}$$

Putting the pieces together we get that the result in Theorem 4 can be viewed as an equivalence between the convergence of the uni-dimensional laws of σ^z in (9) & (10) and the convergence of the integrated ones in (11) & (12). An analogous fact can be deduced for the convergence of σ^z as z goes to 0 whenever Sinaĭ's condition holds at infinity.

We pass now to the proof of Lemmas 1,2 & 3.

2.1 Proof of Lemmas 1,2 & 3

Proof of (i) in Lemma 1. We have by Frullani's formula that for every $\theta > 0$

$$\begin{aligned}1 + \phi(\theta) &= \exp\left\{\int_0^\infty \frac{\mathrm{d}t}{t} e^{-t}(1 - e^{-t\phi(\theta)})\right\}\\ &= \exp\left\{\theta\int_0^\infty \mathrm{d}y e^{-\theta y}\int_0^\infty \frac{\mathrm{d}t}{t} e^{-t}\,\mathbf{P}(\sigma_t > y)\right\}\\ &= \exp\{\widehat{G}_1(\theta)\}.\end{aligned}$$

□

Proof of (ii) in Lemma 1. The equation relating ϕ and the measure G_2 can be obtained using Frullani's formula but to prove moreover that this measure is in fact is an harmonic renewal measure we proceed as follows. Let $(\mathrm{e}_k, k \geq 1)$ be a sequence of independent identically distributed random variables with exponential law of parameter 1 and independents of σ. Put $\Theta_l = \sum_{k=1}^{l} \mathrm{e}_k, l \geq 1$. It was proved by Bertoin and Doney [3] that $(\sigma_{\Theta_l}, l \geq 1)$ forms a renewal process. The harmonic renewal measure associated to $(\sigma_{\Theta_l}, l \geq 1)$ is $G_2(\mathrm{d}x)$. Indeed,

$$\begin{aligned}\sum_{l=1}^{\infty} \frac{1}{l} \mathbf{P}(\sigma_{\Theta_l} \in \mathrm{d}x) &= \sum_{l=1}^{\infty} \frac{1}{l} \int_0^{\infty} \mathrm{d}t \frac{t^{l-1}}{(l-1)!} e^{-t} \mathbf{P}(\sigma_t \in \mathrm{d}x) \\ &= \int_0^{\infty} \frac{\mathrm{d}t}{t} e^{-t}(e^t - 1) \mathbf{P}(\sigma_t \in \mathrm{d}x) = G_2(\mathrm{d}x).\end{aligned}$$

Moreover, since the l-convolution of $F(\mathrm{d}x) = \mathbf{P}(\sigma_{\Theta_1} \in \mathrm{d}x)$ is such that

$$F^{*l}(\mathrm{d}x) = \mathbf{P}(\sigma_{\Theta_l} \in \mathrm{d}x)$$

we have that the Laplace transform $\widehat{F}(\theta)$ of F is related to that of G_2 by the formula

$$1 - \widehat{F}(\theta) = \exp\{-\widehat{G}_2(\theta)\} \qquad \theta > 0.$$

Which finish the proof since $\widehat{F}(\theta) = (1 + \phi(\theta))^{-1}$, for $\theta > 0$. □

Proof of (i) in Lemma 2. We can suppose without loss of generality that $\lambda > 1$. Given that

$$\int_0^{\infty} \frac{\mathrm{d}t}{t} \mathbf{P}(z < \sigma_t \leq \lambda z) = \int_0^{\infty} \frac{\mathrm{d}t}{t} e^{-t} \mathbf{P}(z < \sigma_t \leq \lambda z) + \int_0^{\infty} \frac{\mathrm{d}t}{t}(1 - e^{-t}) \mathbf{P}(z < \sigma_t \leq \lambda z),$$

for every $z > 0$, and

$$G_1(\lambda z) - G_1(z) = \int_0^{\infty} \frac{\mathrm{d}t}{t} e^{-t} \mathbf{P}(\frac{1}{\lambda z} \leq \sigma_t < \frac{1}{z}), \qquad z > 0,$$

in order to prove (i) in Lemma 2 we only need to check that

$$\lim_{z \to 0} \int_0^{\infty} \frac{\mathrm{d}t}{t}(1 - e^{-t}) \mathbf{P}(z < \sigma_t \leq \lambda z) = 0, \qquad \forall \lambda > 1.$$

Indeed, given that for any $0 < z < \infty$

$$\int_0^1 \frac{\mathrm{d}t}{t}(1 - e^{-t}) \mathbf{P}(\sigma_t \leq \lambda z) \leq \int_0^1 \frac{\mathrm{d}t}{t}(1 - e^{-t}) < \infty,$$

we have by the monotone convergence theorem that

$$\int_0^1 \frac{\mathrm{d}t}{t}(1 - e^{-t}) \mathbf{P}(z < \sigma_t \leq \lambda z) \leq \int_0^1 \frac{\mathrm{d}t}{t}(1 - e^{-t}) \mathbf{P}(\sigma_t \leq \lambda z) \longrightarrow 0, \qquad \text{as } z \to 0, \quad \forall \lambda > 1.$$

Furthermore, Theorem IV.7 in [12] enable us to ensure that for any $0 < z < \infty$, and $1 < \lambda$,

$$\int_1^{\infty} \frac{\mathrm{d}t}{t} \mathbf{P}(\sigma_t \leq \lambda z) < \infty.$$

Thus, proceeding as in the case $\int_0^1$ we obtain that for any $\lambda > 1$,

$$\lim_{z\to 0} \int_1^\infty \frac{\mathrm{d}t}{t}(1-e^{-t})\,\mathbf{P}(z < \sigma_t \le \lambda z) = 0.$$

□

Proof of (ii) in Lemma 2. As in the proof of (i) it is enough to prove that

$$\lim_{z\to\infty} \int_0^\infty \frac{\mathrm{d}t}{t} e^{-t}\,\mathbf{P}(z < \sigma_t \le \lambda z) = 0, \qquad \forall \lambda > 1.$$

Indeed, it is straightforward that

$$\lim_{z\to\infty} \int_1^\infty \frac{\mathrm{d}t}{t} e^{-t}\,\mathbf{P}(z < \sigma_t \le \lambda z) = 0, \qquad \forall \lambda > 1.$$

To prove that

$$\lim_{z\to\infty} \int_0^1 \frac{\mathrm{d}t}{t} e^{-t}\,\mathbf{P}(z < \sigma_t \le \lambda z) = 0, \qquad \forall \lambda > 1.$$

we will use the inequality (6) in Lemma 1 of [10] which enable us to ensure that for any $u > 0$ and $z > 0$

$$\mathbf{P}(z < \sigma_t < \infty) \le \frac{t\widetilde{\phi}(u)e^{-\kappa t}}{1-e^{-uz}}, \quad \text{with } \widetilde{\phi}(u) = du + \int_{]0,\infty[} (1-e^{-ux})\pi(\mathrm{d}x).$$

Applying this inequality we get that, for any $u, z > 0$, and $\lambda > 1$,

$$\int_0^1 \frac{\mathrm{d}t}{t} e^{-t}\,\mathbf{P}(z < \sigma_t \le \lambda z) \le \frac{\widetilde{\phi}(u)}{1-e^{-uz}} \int_0^1 e^{-(1+\kappa)t}\mathrm{d}t.$$

Making, first $z \to \infty$ and then $u \to 0$ in the previous inequality, we obtain the estimate

$$\lim_{z\to\infty} \int_0^1 \frac{\mathrm{d}t}{t} e^{-nt}\,\mathbf{P}(z < \sigma_t \le \lambda z) \le \widetilde{\phi}(0) \int_0^1 e^{-(1+\kappa)t}\mathrm{d}t,$$

valid for any $\lambda > 1$. Which in fact ends the proof since $\widetilde{\phi}(0) = \phi(0) - \kappa = 0$. □

Proof of Lemma 3. The equivalence in (ii) of Lemma 3 follows from Theorem 3.9.1 in [4]. The equivalence in (i) of Lemma 3 is obtained by applying Abelian-Tauberian theorems relying the behavior of the differences of a non-decreasing function and those of their Mellin transform. Indeed, to prove that (5) implies (6) we apply an Abelian theorem that appears in [4] Section 4.11.1. To that end we just need to verify that the Mellin transform $\check{k}$ of the kernel k is finite in a set $A = \{x \in \mathbb{C} : a \le \Re(x) \le b\}$ with $a < 0 < b$. This is indeed the case since the Mellin transform of k,

$$\check{k}(x) := \int_0^\infty t^{-x}k(t)\frac{\mathrm{d}t}{t} = \int_0^\infty t^{-x}e^{-t}\mathrm{d}t, \quad x \in \mathbb{C},$$

is finite in the strip $\Re(x) < 1$. That (6) implies (5) is a direct consequence of a Tauberian theorem for differences established in [11] Theorem 2.35. □

Proof of Corollary 2. We will only prove the assertion in (i) of Corollary 2. It is well known that any subordinator has finite mean if and only if its Laplace exponent is derivable at 0. Since σ is assumed to have infinite lifetime and the following relations, which are a consequence of the Lemmas 1,2 & 3,

$$\frac{\phi(\theta)}{\theta} \sim \frac{\dfrac{\phi(\theta)}{1+\phi(\theta)}}{\theta} \sim \exp\left\{-\widehat{G}_2(\theta) - \log(\theta)\right\} \sim \exp\left\{\gamma - G_2(1/\theta) + \log(1/\theta)\right\}, \quad \text{as } \theta \to 0,$$

we have that ϕ is derivable at 0 if and only if the limit in Corollary 2 (i) holds.

The proof of the assertion in Corollary 2 (ii) uses the fact that σ has a strictly positive drift if and only if $\lim_{\theta\to\infty} \phi(\theta)/\theta > 0$ and an argument similar to that above. □

3 Proof of Theorem 3

Proof of Theorem 3. This proof is a reworking of its analogous for random walks, which was established by Rogozin [17] Theorem 9. We will prove that under the assumptions of Theorem 3, for $t \to \infty$ in equation (2), the assertion in Corollary 1 (iv) holds. (The proof of the case $t \to 0$ in equation (2) follows in a similar way and so we omit the proof.) To that end, let $(\xi^r)_{r>0}$ be the family of Lévy processes defined by, $(\xi^r(t) = \xi_{rt}/b(r), t \geq 0)$ for $r > 0$. The hypothesis of Theorem 3 is equivalent to the convergence, in the sense of finite dimensional distributions, of the sequence of Lévy processes ξ^r to a stable Lévy process X with characteristic exponent given by the formula (2). By Corollary 3.6 in Jacod–Shiryaev we have that this convergence holds also in the Skorohod topology and Theorem IV.2.3 in Gihman & Skorohod [12] enable us to ensure that there is also convergence of the first passage time above the level x and the overshoot at first passage time above the level x by ξ^r to the corresponding objects for X. That is, for any $x > 0$

$$\tau_x^r = \inf\{t > 0 : \xi^r(t) > x\}, \qquad \gamma_x^r = \xi^r(\tau_x^r) - x,$$
$$\tau_x = \inf\{t > 0 : X(t) > x\}, \qquad \gamma_x = X(\tau_x) - x,$$

we have that

$$(\tau_x^r, \gamma_x^r) \xrightarrow[r\to\infty]{\mathcal{D}} (\tau_x, \gamma_x).$$

In particular, for $x = 1$, we have that $\tau_1^r = r^{-1}T_{b(r)}$ and $\gamma_1^r = \left(\xi_{T_{b(r)}} - b(r)\right)/b(r)$, in the notation of Corollary 1, and thus that

$$\left(r^{-1}T_{b(r)}, \left(\xi_{T_{b(r)}} - b(r)\right)/b(r)\right) \xrightarrow[r\to\infty]{\mathcal{D}} (\tau_1, \gamma_1).$$

We will next prove that

$$\left(\xi_{T_r} - r\right)/r \xrightarrow[r\to\infty]{\mathcal{D}} \gamma_1,$$

which implies that the assertion (iv) in Corollary 1 holds. To that end, we introduce the generalized inverse of b, $b^{\leftarrow}(t) = \inf\{r > 0 : b(r) > t\}$ for $t > 0$. Given that b is regularly

varying at infinity it is known that $b(b^{\leftarrow}(t)) \sim t$ as $t \to \infty$, see e.g. [4] Theorem 1.5.12. Owing the following relations valid for any $\epsilon > 0$ fixed and small enough,

$$b(b^{\leftarrow}(r) - \epsilon) \leq r \leq b(b^{\leftarrow}(r)), \qquad r > 0,$$

we have that for any $x > 0$

$$\begin{aligned}\mathbf{P}\left(\frac{\xi_{T_{b(b^{\leftarrow}(r))}}}{b(b^{\leftarrow}(r))}\frac{b(b^{\leftarrow}(r))}{r} \leq x+1\right) &\leq \mathbf{P}\left(\frac{\xi_{T_r}}{r} \leq x+1\right) \\ &\leq \mathbf{P}\left(\frac{\xi_{T_{b(b^{\leftarrow}(r)-\epsilon)}}}{b(b^{\leftarrow}(r)-\epsilon)}\frac{b(b^{\leftarrow}(r)-\epsilon)}{b(b^{\leftarrow}(r))}\frac{b(b^{\leftarrow}(r))}{r} \leq x+1\right).\end{aligned}$$

Making r tend to infinity and using that $b(b^{\leftarrow}(r)-\epsilon)/b(b^{\leftarrow}(r)) \to 1$ as $r \to \infty$, we get that the left and right hand sides of the previous inequality tend to $\mathbf{P}(\gamma_1 + 1 \leq x+1)$ and so that for any $x > 0$

$$\mathbf{P}\left(\frac{\xi_{T_r} - r}{r} \leq x\right) \xrightarrow[r\to\infty]{} \mathbf{P}(\gamma_1 \leq x).$$

Furthermore, it is well known that in the case $\alpha\rho \in]0,1[$ the law of γ_1 is the generalized arc-sine law with parameter $\alpha\,\mathbf{P}(X_1 > 0) = \alpha\rho$, that is

$$\mathbf{P}(\gamma_1 \in \mathrm{d}x) = \frac{\sin(\alpha\rho\pi)}{\pi} x^{-\alpha\rho}(1+x)^{\alpha\rho-1}\mathrm{d}x, \quad x > 0.$$

In the case $\alpha\rho = 0$ the random variable γ_1 is degenerate at infinity and in the case $\alpha\rho = 1$ it is degenerate at 0. Thus, in any case the Sinai index of ξ is $\alpha\rho$. □

For shake of completeness in the following Lemma we provide necessary and sufficient conditions on the tail behavior of the Lévy measure of ξ in order that the hypotheses of Theorem 3 be satisfied. This result concerns only the case $t \to \infty$ in (i) of Theorem 3 and $\alpha \in]0,1[$. The triple (a, q^2, Π) denotes the characteristics of the Lévy process ξ, that is, its linear and Gaussian term, a, q and Lévy measure Π and are such that

$$\Psi(\lambda) = ia\lambda - \frac{\lambda^2 q^2}{2} + \int_{\mathbb{R}\setminus\{0\}} (e^{i\lambda x} - 1 - i\lambda x 1_{\{|x|<1\}})\Pi(\mathrm{d}x), \qquad \lambda \in \mathbb{R}\,.$$

By $\overline{\Pi}^+$ and $\overline{\Pi}^-$ we denote the right and left hand tails of the Lévy measure Π respectively, i.e. $\overline{\Pi}^+(x) = \Pi]x, \infty[$ and $\overline{\Pi}^-(x) = \Pi]-\infty, -x[$, for $x > 0$.

Lemma 4. *Let $\alpha \in]0,1[$ and $\delta \in [-1,1]$. The following are equivalent*

DA *There exits a function $b :]0,\infty[\to]0,\infty[$ which is regularly varying at infinity with index $\beta = 1/\alpha$ and such that the limit in equation (2) holds as $t \to \infty$.*

TB *The function $\overline{\Pi}^+(\cdot) + \overline{\Pi}^-(\cdot)$ is regularly varying at infinity with index $-\alpha$ and*

$$\frac{\overline{\Pi}^+(x)}{\overline{\Pi}^+(x) + \overline{\Pi}^-(x)} \longrightarrow p, \quad \frac{\overline{\Pi}^-(x)}{\overline{\Pi}^+(x) + \overline{\Pi}^-(x)} \longrightarrow q, \quad \text{as} \quad x \to \infty; \quad p+q = 1, p-q = \delta.$$

Remark 3. The same result holds true for $\alpha \in]0,2[$ if the Lévy process is assumed to be symmetric (the proof of Lemma 4 can be easily extended to this case). Furthermore, there is also an analogue of this result when $t \to 0$ in (i) of Theorem 3 in the cases $1 < \alpha < 2$ or $0 < \alpha < 2$ and ξ is assumed to be symmetric. Its proof is quite similar to that of Lemma 4, see e.g. the recent work of De Weert [5].

Proof of Lemma 4. It is plain, that for any $t > 0$ the function $\Psi^{(t)}(\lambda) := t\Psi\left(\frac{\lambda}{b(t)}\right)$ is the characteristic exponent of the infinitely divisible random variable $X^{(t)} := \xi_t/b(t)$, which by the hypothesis DA(α) converges to a stable law $X(1)$ whose characteristic exponent is given by equation (2). The characteristic exponent $\Psi^{(t)}$ can be written as

$$\Psi^{(t)}(\lambda) = i\lambda a^{(t)} - \lambda^2 (q^{(t)})^2/2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{i\lambda z} - 1 - ih(z)\right) \Pi^{(t)}(\mathrm{d}z),$$

where $h(z) = z1_{\{|z|\leq 1\}} + z^{-1}1_{\{|z|>1\}}$ and $(a^{(t)}, q^{(t)}, \Pi^{(t)})$ are given by

$$a^{(t)} = \frac{ta}{b(t)} + \frac{t}{b(t)} \int x1_{\{1<|x|\leq b(t)\}} \Pi(\mathrm{d}x) + tb(t) \int x^{-1}1_{\{|x|>b(t)\}} \Pi(\mathrm{d}x),$$
$$q^{(t)} = q\left(\frac{t}{b(t)^2}\right)^{1/2}, \quad \Pi^{(t)}(\mathrm{d}z) = t\Pi(b(t)\mathrm{d}z).$$

According to a well known result on the convergence of infinite divisible laws, see e.g. Sato [18], the convergence in law of $X^{(t)}$ to X as $t \to \infty$, is equivalent to the convergence of the triplet $(a^{(t)}, q^{(t)}, \Pi^{(t)})$ to $(l, 0, \Pi_S)$ as $t \to \infty$, with

$$\Pi_S(\mathrm{d}x) = \left(c_+ x^{-1-\alpha}1_{\{x>0\}} + c_-|x|^{-1-\alpha}1_{\{x<0\}}\right)\mathrm{d}x, \qquad c_+, c_- \in \mathbb{R}^+,$$

and

$$l = \frac{2(c_+ - c_-)}{1-\alpha^2},$$

and they are such that for every $\lambda \in \mathbb{R}$

$$-c|\lambda|^\alpha \left(1 - i\delta \mathrm{sgn}(\lambda)\tan(\pi\alpha/2)\right) = \int_{\mathbb{R}\setminus\{0\}} (e^{i\lambda x} - 1)\Pi_S(\mathrm{d}x)$$
$$= il + \int_{\mathbb{R}\setminus\{0\}} \left(e^{i\lambda x} - 1 - ih(x)\right) \Pi_S(\mathrm{d}x),$$

with $c > 0$ convenably chosen. The term δ in the previous equation is determined by $\delta = p - q = \frac{c_+ - c_-}{c_+ + c_-}$.

That the hypotheses on the tail behavior of Π are equivalent to the convergence of Π^t to Π_S is a quite standard fact in the theory of domains of attraction and so we refer to [4] section 8.3.2, for a proof. This implies in particular that for any $x > 0$,

$$t\overline{\Pi}^+(b(t)x) \to c_+ x^{-\alpha}, \quad \text{and} \quad t\overline{\Pi}^-(b(t)x) \to c_- x^{-\alpha}, \quad \text{as } t \to \infty.$$

The only technical detail that requires a proof is that $a^{(t)} \to l$ as $t \to \infty$. Indeed, under the conditions (ii) of Theorem 3 and $0 < p < 1$ the functions $\overline{\Pi}^+(\cdot)$ and $\overline{\Pi}^-(\cdot)$ are regularly varying at infinity with index $0 < \alpha < 1$, this implies that $at/b(t) \to 0$ as $t \to \infty$. Moreover, it is justified by making an integration by parts that

$$\begin{aligned}\int x1_{\{1<x\leq b(t)\}}\Pi(\mathrm{d}x) &\sim \overline{\Pi}^+(1) - b(t)\overline{\Pi}^+(b(t)) + \int_1^{b(t)} \overline{\Pi}^+(z)dz \\ &\sim \overline{\Pi}^+(1) - b(t)\overline{\Pi}^+(b(t)) + \frac{b(t)\overline{\Pi}^+(b(t))}{1-\alpha},\end{aligned}$$

as $t \to \infty$. Multiplying by $t/b(t)$ we get that

$$\frac{t}{b(t)}\int x1_{\{1<x\leq b(t)\}}\Pi(\mathrm{d}x) \sim -t\overline{\Pi}^+(b(t)) + \frac{t\overline{\Pi}^+(b(t))}{1-\alpha} \longrightarrow -c_+ + \frac{c_+}{1-\alpha}$$

as $t \to \infty$. Similarly, it is proved that

$$\frac{t}{b(t)}\int x1_{\{1<-x\leq b(t)\}}\Pi(\mathrm{d}x) \xrightarrow[\mathrm{t}\to\infty]{} c_- - \frac{c_-}{1-\alpha}$$

Concerning the term $\int x^{-1}1_{\{|x|>b(t)\}}\Pi(\mathrm{d}x)$, an integration by parts and Karamata's theorem yield

$$\begin{aligned}\int x^{-1}1_{\{x>b(t)\}}\Pi(\mathrm{d}x) &\sim (b(t))^{-1}\overline{\Pi}^+(b(t)) + \int_{b(t)}^{\infty} z^{-2}\overline{\Pi}^+(z)dz \\ &\sim (b(t))^{-1}\overline{\Pi}^+(b(t)) + \frac{\overline{\Pi}^+(b(t))}{b(t)(1+\alpha)},\end{aligned}$$

as $t \to \infty$ and therefore

$$tb(t)\int x^{-1}1_{\{x>b(t)\}}\Pi(\mathrm{d}x) \sim t\overline{\Pi}^+(b(t)) + \frac{t\overline{\Pi}^+(b(t))}{1+\alpha} \longrightarrow c_+ + \frac{c_+}{1+\alpha}.$$

Analogously, we prove

$$tb(t)\int x^{-1}1_{\{x<-b(t)\}}\Pi(\mathrm{d}x) \xrightarrow[\mathrm{n}\to\infty]{} -c_- - \frac{c_-}{1+\alpha}.$$

Finally, adding up these four terms it follows that

$$\lim_{t\to\infty} a^{(t)} = \frac{c_+ - c_-}{1-\alpha} + \frac{c_+ - c_-}{1+\alpha} = l.$$

The proof that $a^{(t)} \to l$ in the case $p = 1$, respectively $p = 0$, is quite similar but uses that $\overline{\Pi}^- = o(\overline{\Pi}^+)$, respectively $\overline{\Pi}^+ = o(\overline{\Pi}^-)$. □

4 Some criteria based on the characteristics of ξ

The result obtained in Theorem 2 gives a complete solution to the question posed at the introduction of this note. Nevertheless, this answer is not completely satisfactory since the criterion provided by this theorem is not given in terms of the characteristics of ξ. However, to get a result based on the characteristics of ξ one should be able to control the behavior of the dual ladder height subordinator $\widehat{H}$, that is, the ladder height subordinator of the dual Lévy process $\widehat{\xi} = -\xi$. This is due to the fact, showed by Vigon [22], that the Lévy measure of the ladder height subordinator H is determined by the Lévy measure of ξ and the potential measure of $\widehat{H}$. (See the Lemma 5 below for a precise statement.) With this in mind we will make some assumptions on the dual ladder height process, that can be verified directly from the characteristics of ξ, and provide some nasc for the regular variation, at infinity or 0, of the Laplace exponent of the ladder height subordinator H. To that end we will start by obtaining some tail estimates of the right tail of the Lévy measure of H which will provide us the necessary tools to determine whether or not the underlying Lévy process ξ satisfies Sinaĭ's condition. To tackle this task we will first introduce some notation and recall some known facts.

We will denote by (k_0, d, po) (resp. $(\widehat{k}_0, \widehat{d}, ne)$) the characteristics of the subordinator H (resp. $\widehat{H}$) that is, its killing term, drift and Lévy measure, respectively. Let $\widehat{V}$ be the potential measure of $\widehat{H}$, that is $\widehat{V}(\mathrm{d}x) = \mathbf{E}(\int_0^\infty 1_{\{\widehat{H}_t \in \mathrm{d}x\}} \mathrm{d}t)$. Furthermore, throughout this section we will denote by (a, q, Π) the characteristics of the Lévy process ξ. Finally, by the symbols $\overline{po}, \overline{ne}, \overline{\Pi}^+$, we denote the right tail of the Lévy measures of $H, \widehat{H}$ and ξ respectively, that is

$$\overline{po}(x) = po]x, \infty[, \quad \overline{ne}(x) = ne]x, \infty[, \qquad \overline{\Pi}^+(x) = \Pi]x, \infty[, \qquad x > 0,$$

and by Π^+ the restriction of Π to $]0, \infty[$, $\Pi^+ = \Pi|_{]0,\infty[}$.

We recall that the Laplace exponent, φ, of H varies regularly at 0 with index $\alpha \in]0, 1[$ if and only if the function $\overline{po}$ is regularly varying at infinity with index $-\alpha$. Analogously, φ varies regularly at infinity with index $\alpha \in]0, 1[$ if and only if the drift coefficient d is zero and the function $\overline{po}$ is regularly varying at 0 with index $-\alpha$. Owing this relations we will restrict ourselves to study the behavior of the function $\overline{po}$.

As we mentioned before, Vigon [22] established some identities "equations amicales" relying the Lévy measures po, ne and Π; these are quoted below for ease of reference.

Lemma 5 (Vigon [22], Equations amicales). *We have the following relations*

(EAI) $\overline{po}(x) = \displaystyle\int_0^\infty \widehat{V}(\mathrm{d}y)\overline{\Pi}(x+y), \qquad x > 0.$

(EA) $\overline{\Pi}^+(x) = \displaystyle\int_{]x,\infty[} po(\mathrm{d}y)\overline{ne}(y-x) + \widehat{d}\overline{p}(x) + \widehat{k}_0\overline{po}(x)$, *for any* $x > 0$; *where* $\overline{p}(x)$ *is the density of the measure* po, *which exists if* $\widehat{d} > 0$.

We will assume throughout that the underlying Lévy process is not spectrally negative, that is $\Pi]0, \infty[> 0$, since in that case the ladder height process H is simply a drift, $H_t = dt, t \geq 0$.

We will say that a measure M on $[0,\infty[$ belongs to the class $\mathcal{L}^0$ of *long tailed* measures if its tail $\overline{M}(x) = M]x,\infty[$, is such that $0 < \overline{M}(x) < \infty$ for each $x > 0$ and

$$\lim_{x\to\infty} \frac{\overline{M}(x+t)}{\overline{M}(x)} = 1, \qquad \textit{for each } t \in \mathbb{R}\,.$$

It is well known that this family includes the subexponential measures and the cases at which $\overline{M}$ is regularly varying.

In the first results of this section we relate the behavior of $\overline{\Pi}^+$ with that of $\overline{po}$ at infinity. These results are close in spirit to those obtained by Klüppelberg, Kyprianou and Maller [16]. In that work the authors assume that the Lévy process ξ drifts to $-\infty$, i.e. $\lim_{t\to\infty} \xi_t = -\infty$, $\mathbf{P}$–a.s. and obtain several asymptotic estimates of the function $\overline{po}$ in terms of the Lévy measure Π. In our setting we permit any behavior of ξ at the price of making some assumptions on the dual ladder height subordinator. The following results are the continuous time analogue of the result of Veraverbeke [20] and Grübel [14] for random walks.

Theorem 5. *(a) Assume that the dual ladder height subordinator has a finite mean* $\mu = \mathbf{E}(\widehat{H}_1) < \infty$. *The following are equivalent*

(a-1) The measure Π_I^+ *on* $[0,\infty[$ *with tail* $\overline{\Pi}_I^+(x) = \int_x^\infty \overline{\Pi}^+(z)\mathrm{d}z, x \geq 0$, *belongs to* $\mathcal{L}^0$.

(a-2) $po \in \mathcal{L}^0$.

(a-3) $\overline{po}(x) \sim \dfrac{1}{\mu}\overline{\Pi}_I^+(x)$, *as* $x \to \infty$.

(b) Assume that the dual ladder height subordinator $\widehat{H}$ *has killing term* $\widehat{k}_0 > 0$. *The following are equivalent*

(b-1) $\Pi^+ = \Pi|_{]0,\infty[} \in \mathcal{L}^0$.

(b-2) $po \in \mathcal{L}^0$,

$$\frac{\widehat{d\overline{p}}(x)}{\overline{po}(x)} \longrightarrow 0 \quad \textit{and} \quad \int_0^1 \left(\frac{\overline{po}(x) - \overline{po}(x+y)}{\overline{po}(x)}\right) ne(\mathrm{d}y) \longrightarrow 0, \quad \textit{as } x \to \infty. \tag{13}$$

(b-3) $\overline{po}(x) \sim \dfrac{1}{\widehat{k}_0}\overline{\Pi}^+(x)$, *as* $x \to \infty$.

As a corollary of the previous Theorem we have the following criterions to determine whether or not the tail of the Lévy measure of H is regularly varying.

Corollary 3. *(a) Under the assumptions of (a) in Theorem 5 and for any* $\alpha \in]0,1]$ *we have that* $\overline{\Pi}^+ \in RV_{-1-\alpha}^\infty$ *if and only if* $\overline{po} \in RV_{-\alpha}^\infty$. *Both imply that*

$$\overline{po}(x) \sim \frac{1}{\alpha\mu} x\overline{\Pi}^+(x), \qquad x \to \infty.$$

(b) Under the assumptions of (b) in Theorem 5 and for any $\alpha \in]0,1[$ we have that $\overline{\Pi}^+ \in RV^\infty_{-\alpha}$ if and only if $\overline{po} \in RV^\infty_{-\alpha}$ and

$$\int_0^1 \left(\frac{\overline{po}(x) - \overline{po}(x+y)}{\overline{po}(x)} \right) ne(\mathrm{d}y) \longrightarrow 0, \quad \text{as } x \to \infty.$$

Proof. The proof of (a) in Corollary 3 follows from the fact that under these hypotheses

$$\frac{x\overline{\Pi}^+(x)}{\int_x^\infty \overline{\Pi}^+(z)\mathrm{d}z} \longrightarrow \alpha, \qquad \text{as } x \to \infty,$$

which is a consequence of Theorem 1.5.11 in [4]. The proof of (b) is straightforward. □

The behavior at 0 of $\overline{po}$ was studied by Vigon in [21] Theorems 6.3.1 and 6.3.2. He obtained several estimations that we will use here to provide an analogue of Theorem 5 for the behavior at 0 of $\overline{po}$. (Those estimates are more general than needed, see the proof of Proposition 1.)

Proposition 1. *(a) Assume that $\widehat{H}$ has a drift $\widehat{d} > 0$. For any $\alpha \in]0,1]$ we have that $\overline{po} \in RV^0_{-\alpha}$ if and only if $\overline{\Pi}^+ \in RV^0_{-\alpha-1}$.*

(b) Assume that $\widehat{H}$ has a drift $\widehat{d} = 0$ and that the total mass of the measure ne is finite, equivalently, $\lim_{x\to 0+} \overline{ne}(x) < \infty$. Then for any $\alpha \in]0,1]$ we have that $\overline{po} \in RV^0_{-\alpha}$ if and only if $\overline{\Pi}^+ \in RV^0_{-\alpha}$. Moreover, the same assertion holds if furthermore $\alpha = 0$ and $\lim_{x\to 0+} \overline{po}(x) = \infty$.

Before passing to the proof of Theorem 5 and Proposition 1 we will make some remarks.

Remark 4. The assumptions in Theorem 5 can be verified using only the characteristics of the underlying Lévy process ξ. According to Doney and Maller [7], necessary and sufficient conditions on ξ to be such that $\mathbf{E}(\widehat{H}_1) < \infty$, are either $0 < \mathbf{E}(-\xi_1) \leq \mathbf{E}\,|\xi_1| < \infty$ or $0 = \mathbf{E}(-\xi_1) < \mathbf{E}\,|\xi_1| < \infty$ and

$$\int_{[1,\infty[} \left(\frac{x\overline{\Pi}^-(x)}{1 + \int_0^x \mathrm{d}y \int_y^\infty \overline{\Pi}^+(z)\mathrm{d}z} \right) \mathrm{d}x < \infty \quad \text{with } \overline{\Pi}^-(x) = \Pi]-\infty, -x[, \ x > 0.$$

Observe that under such assumptions the Lévy process ξ does not drift to ∞, i.e. $\liminf_{t\to\infty} \xi_t = -\infty$, $\mathbf{P}$–a.s. The case where the Lévy process ξ drift to ∞, $\lim_{t\to\infty} \xi_t = \infty$, $\mathbf{P}$–a.s. or equivalently $\widehat{k}_0 > 0$ is considered in (b). It was proved by Doney and Maller that ξ drift to ∞ if and only if

$$\int_{]-\infty,-1[} \left(\frac{|y|}{\overline{\Pi}^+(1) + \int_1^{|y|} \overline{\Pi}^+(z)\mathrm{d}z} \right) \Pi(\mathrm{d}y) < \infty = \int_1^\infty \overline{\Pi}^+(x)\mathrm{d}x \quad \text{or} \quad 0 < \mathbf{E}(\xi_1) \leq \mathbf{E}\,|\xi_1| < \infty.$$

The assumptions in Proposition 1 can be verified using the recent results of Vigon [22] and [21] Chapter 10.

Remark 5. The results in Theorem 5 concern only the case at which the underlying Lévy process does not has exponential moments and so it extends to Lévy processes the Theorem 1-(B,C) of Veraverbeke [20]. The case at which the Lévy process has exponential moments has been considered by Klüppelberg et al. [16] Proposition 5.3 under the assumption that the underlying Lévy process has positive jumps and drifts to $-\infty$, but actually the latter hypothesis is not used in their proof, and so their result is still true in this more general setting, which extends Theorem 1-A in [20].

Remark 6. The estimate of $\overline{po}$ obtained in Theorem 5-a holds whenever the function $\overline{\Pi}_I^+$ belongs to the class $\mathcal{L}^0$, but it is known that this can occurs even if $\Pi^+ \notin \mathcal{L}^0$, see e.g. Klüppelberg [15]. A question arises: Is it possible to sharpen the estimate of $\overline{po}$ provided in Theorem 5-(a) when moreover $\Pi^+ \in \mathcal{L}^0$? The following result answers this question in affirmatively.

Proposition 2. *Assume that $\mu = \mathbf{E}(\widehat{H}_1) < \infty$. The following are equivalent*

(i) $\Pi^+ \in \mathcal{L}^0$.

(ii) For any $g : \mathbb{R}^+ \to \mathbb{R}^+$ directly Riemman integrable,

$$\lim_{x\to\infty} \frac{1}{\overline{\Pi}^+(x)} \int_x^\infty po(\mathrm{d}y) g(y-x) = \frac{1}{\mu} \int_0^\infty g(z)\mathrm{d}z.$$

To our knowledge the discrete time analogue of this result, that we state below, is unknown, although it can be easily deduced from the arguments in Asmussen et al. [1] Lemma 3. We use the notation in the introduction of this work.

Proposition 3. *Assume that $m = \mathbf{E}(Z_{\widehat{N}}) < \infty$, where $\widehat{N} = \inf\{n > 0 : Z_n \leq 0\}$ and that the law of X_1 is non-lattice. The following are equivalent*

(i) The law of X_1 belongs to the class $\mathcal{L}^0$.

(ii) For any $g : \mathbb{R}^+ \to \mathbb{R}^+$ directly Riemman integrable,

$$\lim_{x\to\infty} \frac{1}{\overline{F}(x)} \int_x^\infty g(y-x)\, \mathbf{P}(Z_N \in \mathrm{d}y) = \frac{1}{m} \int_0^\infty g(z)\mathrm{d}z,$$

where $\overline{F}(x) = \mathbf{P}(X_1 > x),\ x > 0$.

The proof of this result is quite similar to that of Proposition 2 and so we will omit it.

4.1 Proofs of Theorem 5 and Propositions 1 and 2

Proof of (a) in Theorem 5. To prove that (a-1) is equivalent to (a-2) we will prove that either of this conditions implies that

$$\overline{po}(x) = \int_0^\infty \overline{\Pi}^+(x+y)\widehat{V}(\mathrm{d}y) \sim \frac{1}{\mu} \int_x^\infty \overline{\Pi}^+(z)\mathrm{d}z, \qquad \text{as } x \to \infty, \tag{14}$$

with $\mu := \mathbf{E}(\widehat{H}_1)$, from where the result follows. (Observe that the assumption that $\widehat{H}$ has a finite mean implies that $\int^{\infty} \overline{\Pi}^{+}(z)\mathrm{d}z < \infty$.)

Assume that (a-1) holds. Indeed, by the renewal theorem for subordinators we have that for any $h > 0$,

$$\lim_{t\to\infty} \widehat{V}]t, t+h] = \frac{h}{\mu}.$$

Thus, for any $h > 0$ given and any $\epsilon > 0$ there exists a $t_0(h, \epsilon) > 0$ such that

$$(1-\epsilon)\frac{h}{\mu} < \widehat{V}]t, t+h] < (1+\epsilon)\frac{h}{\mu}, \qquad \forall t > t_0,$$

and as a consequence, if N_0 is an integer such that $N_0 h > t_0$, we have the following inequalities

$$\begin{aligned}
\int_0^{\infty} \overline{\Pi}^{+}(x+y)\widehat{V}(\mathrm{d}y) &\leq \overline{\Pi}^{+}(x)\widehat{V}[0, N_0 h] + \sum_{k=N_0}^{\infty} \overline{\Pi}^{+}(kh+x)\widehat{V}]kh, kh+h] \\
&\leq \overline{\Pi}^{+}(x)\widehat{V}[0, N_0 h] + (1+\epsilon)\sum_{k=N_0}^{\infty} \overline{\Pi}^{+}(kh+x)\frac{h}{\mu} \\
&\leq \overline{\Pi}^{+}(x)\widehat{V}[0, N_0 h] + \frac{(1+\epsilon)}{\mu}\int_{(N_0-1)h}^{\infty} \overline{\Pi}^{+}(x+z)\mathrm{d}z \\
&\leq \overline{\Pi}^{+}(x)\widehat{V}[0, N_0 h] + \frac{(1+\epsilon)}{\mu}\int_{x}^{\infty} \overline{\Pi}^{+}(z)\mathrm{d}z.
\end{aligned}$$

It follows from the previous inequalities and the fact that

$$\overline{\Pi}^{+}(x)/\int_x^{\infty} \overline{\Pi}^{+}(z)\mathrm{d}z \longrightarrow 0, \qquad \text{as } x \to \infty,$$

since $\overline{\Pi}_I^{+} \in \mathcal{L}^0$ and $\overline{\Pi}^{+}$ is decreasing, that

$$\limsup_{x\to\infty} \frac{\int_0^{\infty} \overline{\Pi}^{+}(x+y)\widehat{V}(\mathrm{d}y)}{\frac{1}{\mu}\int_x^{\infty} \overline{\Pi}^{+}(z)\mathrm{d}z} \leq 1.$$

Analogously, we prove that

$$\int_0^{\infty} \overline{\Pi}^{+}(x+y)\widehat{V}(\mathrm{d}y) \geq \frac{(1-\epsilon)}{\mu}\int_x^{\infty} \overline{\Pi}^{+}(z)\mathrm{d}z - \frac{(1-\epsilon)}{\mu}\overline{\Pi}^{+}(x)(N_0+1)h, \qquad x > 0.$$

Therefore,

$$\liminf_{x\to\infty} \frac{\int_0^{\infty} \overline{\Pi}^{+}(x+y)\widehat{V}(\mathrm{d}y)}{\frac{1}{\mu}\int_x^{\infty} \overline{\Pi}^{+}(z)\mathrm{d}z} \geq 1.$$

Which ends the proof of the claim (14).

We assume now that (a-2) holds and we will prove that the estimate in (14) holds. On the one hand, we know that for every $z > 0$

$$\overline{\Pi}^{+}(z) = \int_z^{\infty} po(\mathrm{d}y)\overline{ne}(y-z) + \widehat{d\overline{p}}(z),$$

since $\widehat{k}_0 = 0$, because under our assumptions the Lévy process does not drift to ∞. Integrating this relation between x and ∞ and using Fubini's theorem we obtain that for any $x > 0$

$$\begin{aligned}\int_x^\infty \mathrm{d}z\overline{\Pi}^+(z) &= \int_x^\infty po(\mathrm{d}y)\int_0^{y-x}\mathrm{d}z\overline{ne}(z) + \widehat{d}\overline{po}(x)\\ &\leq \overline{po}(x)\left(\int_0^\infty \mathrm{d}z\overline{ne}(z) + \widehat{d}\right)\\ &= \overline{po}(x)\mu < \infty.\end{aligned}$$

Thus,

$$\limsup_{x\to\infty}\frac{\frac{1}{\mu}\int_x^\infty \mathrm{d}z\overline{\Pi}^+(z)}{\overline{po}(x)} \leq 1.$$

On the other hand, to prove that

$$\liminf_{x\to\infty}\frac{\frac{1}{\mu}\int_x^\infty \mathrm{d}z\overline{\Pi}^+(z)}{\overline{po}(x)} \geq 1,$$

we will use an argument based on some facts of renewal theory. To that end we recall that it was proved by Bertoin and Doney [3] that the potential measure of a subordinator $\widehat{H}$ is the delayed renewal measure associated to the law $F(x) = \mathbf{P}(\widehat{H}_\vartheta \leq x)$ with ϑ an exponential random variable independent of $\widehat{H}$, that is

$$\widehat{V}(\mathrm{d}y) = \sum_{n=1}^\infty F^{*n}(\mathrm{d}y).$$

We have by hypothesis that $\int_0^\infty (1-F(x))\mathrm{d}x = \mathbf{E}(\widehat{H}_1) = \mu < \infty$ and thus the measure $\widetilde{G}_F(\mathrm{d}y)$ on $]0,\infty[$, with density $G_F(z) := (1-F(z))/\mu, z > 0$ is a probability measure. By standard facts of renewal theory we know that the following equality between measures holds

$$\frac{\mathrm{d}y}{\mu} = \widetilde{G}_F(\mathrm{d}y) + \widetilde{G}_F * \widehat{V}(\mathrm{d}y), \qquad y > 0,$$

where $*$ denotes the standard convolution between measures. Using this identity and the equation (EAI) we have that for any $x > 0$,

$$\begin{aligned}\frac{1}{\mu}\int_0^\infty \mathrm{d}y\overline{\Pi}^+(x+y) &= \int_0^\infty \mathrm{d}yG_F(y)\overline{\Pi}^+(x+y) + \int_0^\infty \mathrm{d}zG_F(z)\int_0^\infty \widehat{V}(dr)\overline{\Pi}^+(x+z+r)\\ &= \int_0^\infty \mathrm{d}yG_F(y)\overline{\Pi}^+(x+y) + \int_0^\infty \mathrm{d}zG_F(z)\overline{po}(x+z),\end{aligned}$$

and by Fatou's lemma we get that

$$\liminf_{x\to\infty}\frac{\frac{1}{\mu}\int_0^\infty \mathrm{d}y\overline{\Pi}^+(x+y)}{\overline{po}(x)} \geq \int_0^\infty \mathrm{d}zG_F(z)\liminf_{x\to\infty}\frac{\overline{po}(x+z)}{\overline{po}(x)} = 1.$$

So we have proved that (a-1) and (a-2) are equivalent and imply (a-3). To finish the proof, we will prove that (a-3) implies (a-2). To that end it suffices with proving that

$$\lim_{x\to\infty}\frac{\overline{po}(x)-\overline{po}(x+y)}{\overline{po}(x)}=0,\qquad \text{for any } y>0.$$

Indeed, using the equation (EA) we have that for any $y>0$,

$$\begin{aligned}\mu-\frac{\int_0^\infty \mathrm{d}z\overline{\Pi}^+(z+x)}{\overline{po}(x)}&=\frac{\int_0^\infty \mathrm{d}z\overline{ne}(z)\overline{po}(x)+\widehat{d\overline{po}}(x)-\int_x^\infty po(\mathrm{d}y)\int_0^{y-x}\mathrm{d}zne(z)-\widehat{d\overline{po}}(x)}{\overline{po}(x)}\\&=\int_0^\infty \mathrm{d}z\overline{ne}(z)\frac{\overline{po}(x)-\overline{po}(x+z)}{\overline{po}(x)}\\&\geq\int_y^\infty \mathrm{d}z\overline{ne}(z)\frac{\overline{po}(x)-\overline{po}(x+y)}{\overline{po}(x)}\geq 0.\end{aligned}$$

and the assertion follows making $x\to\infty$ in the latter equation since by assumption its left hand term tends to 0 as $x\to\infty$. □

Proof of (b) in Theorem 5. The assumption that $\widehat{k}_0>0$, implies that the renewal measure $\widehat{V}(\mathrm{d}y)$ is a finite measure and $\widehat{V}[0,\infty[=1/\widehat{k}_0$. Thus if $\overline{\Pi}^+(x)\in\mathcal{L}^0$ we have by the equation (EA) and the dominated convergence theorem that

$$\begin{aligned}\lim_{x\to\infty}\frac{\overline{po}(x)}{\overline{\Pi}^+(x)}&=\lim_{x\to\infty}\int_0^\infty\widehat{V}(\mathrm{d}y)\frac{\overline{\Pi}^+(x+y)}{\overline{\Pi}^+(x)}\\&=\int_0^\infty\widehat{V}(\mathrm{d}y)\lim_{x\to\infty}\frac{\overline{\Pi}^+(x+y)}{\overline{\Pi}^+(x)}=\frac{1}{\widehat{k}_0}.\end{aligned}$$

Now, that (13) holds is a straightforward consequence of the following identity, for any $x>0$

$$\begin{aligned}\overline{\Pi}^+(x)&=\int_0^\infty ne(\mathrm{d}y)\left(\overline{po}(x)-\overline{po}(x+y)\right)+\widehat{k}_0\overline{po}(x)+\widehat{d\overline{p}}(x)\\&=\overline{po}(x)\int_0^1 ne(\mathrm{d}y)\left(\frac{\overline{po}(x)-\overline{po}(x+y)}{\overline{po}(x)}\right)+\int_1^\infty ne(\mathrm{d}y)\left(\overline{po}(x)-\overline{po}(x+y)\right)\\&\quad+\widehat{k}_0\overline{po}(x)+\widehat{d\overline{p}}(x),\end{aligned}\tag{15}$$

which is obtained using the equation (EAI) and Fubini's theorem. We have so proved that (b-1) implies (b-2) and (b-3). Next, to prove that (b-2) implies (b-1) and (b-3) we assume that $po\in\mathcal{L}^0$ and (13) holds. Under this assumptions we claim that

$$\overline{\Pi}^+(x)\sim\widehat{k}_0\overline{po}(x)+\widehat{d\overline{p}}(x)\qquad\text{as } x\to\infty.$$

Indeed, this can be deduced from equation (15), using that $\int_{-\infty}^0 ne(\mathrm{d}y)\min\{|y|,1\}<\infty$, that $\lim_{x\to\infty}\overline{po}(x+y)/\overline{po}(x)=1$ for any $y>0$, and the dominated convergence theorem. Furthermore, we have by hypothesis that $\widehat{d\overline{p}}(x)/\overline{po}(x)\to 0$ as $x\to\infty$, which implies that

$$\overline{\Pi}^+(x)\sim\widehat{k}_0\overline{po}(x)\qquad\text{as } x\to\infty.$$

To finish we next prove that (b-3) implies (b-2). Indeed, using the equation (EAI) and the hypothesis (b-3) we get that

$$\frac{\overline{\Pi}^+(x)}{\overline{po}(x)} - \widehat{k}_0 = \frac{\int_x^\infty po(\mathrm{d}z)\overline{ne}(z-x) + \widehat{d}\overline{p}(x)}{\overline{po}(x)}$$
$$= \int_0^\infty ne(\mathrm{d}z)\frac{(\overline{po}(x) - \overline{po}(x+z))}{\overline{po}(x)} + \frac{\widehat{d}\overline{p}(x)}{\overline{po}(x)} \longrightarrow 0 \qquad \text{as } x \to \infty.$$

We deduce therefrom that (13) holds and that $po \in \mathcal{L}^0$ since for any $y > 0$,

$$\int_0^\infty ne(\mathrm{d}z)\left(\frac{\overline{po}(x) - \overline{po}(x+z)}{\overline{po}(x)}\right) \geq \overline{ne}(y)\left(\frac{\overline{po}(x) - \overline{po}(x+y)}{\overline{po}(x)}\right) \geq 0.$$

□

Proof of (a) in Proposition 1. According to Theorem 6.3.2 in [21] under these assumptions the measure po has infinite total mass if and only if $lim_{x\to 0+}\int_x^1 \overline{\Pi}^+(z)\mathrm{d}z = \infty$ and in this case

$$\overline{po}(x) \sim \frac{1}{\widehat{d}}\int_x^1 \overline{\Pi}^+(z)\mathrm{d}z, \qquad \text{as } x \to 0+.$$

Thus the assertion in (a) Proposition 1 is a consequence of this fact and the monotone density theorem for regularly varying functions. □

Proof of (b) in Proposition 1. According to Theorem 6.3.1 in [21] under these assumptions, if we suppose $\lim_{x\to 0+}\overline{po}(x) = \infty$, then

$$\overline{po}(x) \sim \frac{1}{ne]0,\infty[+\widehat{k}_0}\overline{\Pi}^+(x), \qquad \text{as } x \to 0+.$$

The result follows. □

4.2 Proof of Proposition 2

Sketch of proof of Proposition 2. The proof of the assertion (i) implies (ii) is a reworking of the proof of Lemma 3 in Asmussen et al. [1], this can be done in our setting since the only hypothesis needed in that proof is that the dual ladder height has a finite mean.

To show that (i) implies (ii) in Proposition 2 we first prove that under the assumption $\mathbf{E}(\widehat{H}_1) = \mu < \infty$, the condition $\overline{\Pi}^+ \in \mathcal{L}^0$, implies that for any $z > 0$,

(BRT)
$$po]x, x+z[\sim \frac{z}{\mu}\overline{\Pi}^+(x), \qquad x \to \infty.$$

The latter estimate and the fact that $\Pi^+ \in \mathcal{L}^0$ implies that for any $a \geq 0$,

$$po]x+a, x+a+z[\sim \frac{z}{\mu}\overline{\Pi}^+(x), \qquad x \to \infty. \tag{16}$$

To prove that (BRT) holds, we may simply repeat the argument in the proof of Lemma 3 in Asmussen et al. [1] using instead of the equation (12) therein, the equation

$$po]x, x+z[= \int_x^\infty \Pi(dy)\widehat{V}]y-x-z, y-x[, \qquad z>0,$$

which is an elementary consequence of equation (EAI) and Fubini's theorem.

The result in (ii) in Proposition 2 follows from (BRT) in the same way that the Key renewal theorem is obtained from Blackwell's renewal theorem using the estimate in (16) and the bounds

$$\frac{po]x, x+z[}{\overline{\Pi}^+(x)} \leq \widehat{V}(z), \qquad x>0, z>0,$$

which are a simple consequence of the former equation and the fact that $\widehat{V}$ is a renewal measure and so that for any $0<z<y$, $\widehat{V}(y)-\widehat{V}(y-z) \leq \widehat{V}(z)$.

To show that (ii) implies (i) we have to verify that for any $a>0$

$$\lim_{x\to\infty} \frac{\overline{\Pi}^+(x+a)}{\overline{\Pi}^+(x)} = 1.$$

This is indeed true since using (ii) in Proposition 2 it is straightforward that for any $z>0$ the assertion in (BRT) holds and a further application of (ii) in Proposition 2 to the function $g_a(\cdot) = 1_{\{]a,a+1[\}}(\cdot)$, $a>0$, gives that for any $a>0$,

$$\lim_{x\to\infty} \frac{po]x+a, x+a+1[}{\overline{\Pi}(x)} = \frac{1}{\mu},$$

and therefore, for any $a>0$,

$$\lim_{x\to\infty} \frac{\overline{\Pi}^+(x+a)}{\overline{\Pi}^+(x)} = \lim_{x\to\infty} \frac{\overline{\Pi}^+(x+a)}{po]x+a, x+a+1[} \frac{po]x+a, x+a+1[}{\overline{\Pi}^+(x)} = 1.$$

□

Acknowledgements. It is my pleasure to thank Jean Bertoin for pointing out to me the work [13] and for insightful discussions and Ron Doney to bringing to my attention the papers [20] and [14], suggesting to me an early version of Theorem 5 and Proposition 1 and for a number of helpful comments. Part of this paper was written while I was visiting the Mathematical Sciences Institute of the Australian National University (ANU) in March 2005. I heartily thank Claudia Klüppelberg and Ross Maller for arranging this visit, their hospitality and valuable discussions. Last but not least my thanks go to Alex Lindner for his hospitality during my visit to ANU and many stimulating discussions.

References

[1] S. Asmussen, V. Kalashnikov, D. Konstantinides, C. Klüppelberg, and G. Tsitsiashvili. A local limit theorem for random walk maxima with heavy tails. *Statist. Probab. Lett.*, 56(4):399–404, 2002.

[2] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[3] J. Bertoin and R. A. Doney. Cramér's estimate for Lévy processes. *Statist. Probab. Lett.*, 21(5):363–365, 1994.

[4] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1989.

[5] F. De Weert. Attraction to stable distributions for lévy processes at zero. Technical report, University of Manchester, 2003.

[6] R. Doney. Fluctuation theory for Lévy processes. In *Lévy processes*, pages 57–66. Birkhäuser Boston, Boston, MA, 2001.

[7] R. A. Doney and R. A. Maller. Stability of the overshoot for Lévy processes. *Ann. Probab.*, 30(1):188–212, 2002.

[8] E. B. Dynkin. Some limit theorems for sums of independent random quantities with infinite mathematical expectations. *Izv. Akad. Nauk SSSR. Ser. Mat.*, 19:247–266, 1955.

[9] B. Fristedt. Sample functions of stochastic processes with stationary, independent increments. In *Advances in probability and related topics, Vol. 3*, pages 241–396. Dekker, New York, 1974.

[10] B. E. Fristedt and W. E. Pruitt. Lower functions for increasing random walks and subordinators. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 18:167–182, 1971.

[11] J. L. Geluk and L. de Haan. *Regular variation, extensions and Tauberian theorems*, volume 40 of *CWI Tract*. Stichting Mathematisch Centrum Centrum voor Wiskunde en Informatica, Amsterdam, 1987.

[12] Ĭ. Ī. Gīhman and A. V. Skorohod. *The theory of stochastic processes. II*. Springer-Verlag, New York, 1975. Translated from the Russian by Samuel Kotz, Die Grundlehren der Mathematischen Wissenschaften, Band 218.

[13] P. Greenwood, E. Omey, and J. L. Teugels. Harmonic renewal measures. *Z. Wahrsch. Verw. Gebiete*, 59(3):391–409, 1982.

[14] R. Grübel. Tail behaviour of ladder-height distributions in random walks. *J. Appl. Probab.*, 22(3):705–709, 1985.

[15] C. Klüppelberg. Subexponential distributions and integrated tails. *J. Appl. Probab.*, 25(1):132–141, 1988.

[16] C. Klüppelberg, A. E. Kyprianou, and R. A. Maller. Ruin probabilities and overshoots for general Lévy insurance risk processes. *Ann. Appl. Probab.*, 14(4):1766–1801, 2004.

[17] B. A. Rogozin. Distribution of the first laddar moment and height, and fluctuations of a random walk. *Teor. Verojatnost. i Primenen.*, 16:539–613, 1971.

[18] K.-I. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.

[19] Y. G. Sinaĭ. On the distribution of the first positive sum for a sequence of independent random variables. *Theory Prob.*, 16:122–129, 1957.

[20] N. Veraverbeke. Asymptotic behaviour of Wiener-Hopf factors of a random walk. *Stochastic Processes Appl.*, 5(1):27–37, 1977.

[21] V. Vigon. *Simplifiez vos Lévy en titillant la factorisation de Wiener-Hopf*. PhD thesis, Université Louis Pasteur, 2002.

[22] V. Vigon. Votre Lévy rampe-t-il? *J. London Math. Soc. (2)*, 65(1):243–256, 2002.