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Physical effects of the Immirzi parameter

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The Immirzi parameter is a constant appearing in the general relativity action used as a starting point for the loop quantization of gravity. The parameter is commonly believed not to show up in the equations of motion, because it appears in front of a term in the action that vanishes on shell. We show that in the presence of fermions, instead, the Immirzi term in the action does not vanish on shell, and the Immirzi parameter does appear in the equations of motion. It determines the coupling constant of a four-fermion interaction. Therefore the Immirzi parameter leads to effects that are observable in principle, even independently from nonperturbative quantum gravity.

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Connection variables play an important role in the nonperturbative quantization of general relativity. Ashtekar realized that the gravitational field can be effectively described in terms of a selfdual $SL(2,\mathbb{C})$ Yang-Mills-like connection and its conjugate electric field, satisfying appropriate reality conditions [1], and loop quantum gravity [2] started as a canonical quantization of general relativity using these self-dual variables [3]. To circumvent difficulties related to the implementation of the reality conditions in the quantum theory and the noncompatness of $SL(2,\mathbb{C})$, the attention has later shifted to a real $SU(2)$ version of the Ashtekar connection, known as the Barbero connection [4]. An action functional that leads directly to both the Ashtekar or Barbero formalism is the following, called the Holst action [5]

$$S[e,A] = \frac{1}{16\pi G}\left(\int d^4x\; ee^a_I e^b_J\; F^{IJ}_{ab} - \frac{1}{\gamma}\int d^4x\; ee^a_I e^b_J\; {}^*F^{IJ}_{ab}\right). \tag{1}$$

Here $I, J \ldots = 0,1,2,3$ are internal Lorentz indices and $a, b \ldots = 0,1,2,3$ are spacetime indices. The field e^I_a is the tetrad field, e is its determinant and e^a_I its inverse; A^{IJ}_a is a Lorentz connection. F is the curvature of A and ${}^*F$ is its dual, defined by ${}^*F^{IJ} = \frac{1}{2}\epsilon^{IJ}{}_{KL}F^{KL}$. The coupling constant γ is the Immirzi parameter [6]. The choice $\gamma = i$ leads to the self-dual Ashtekar canonical formalism, while a real γ leads to the $SU(2)$ Barbero connection.

The first term in (1) is the familiar tetrad-Palatini action of general relativity. The second term does not affect the equations of motion, for the following reason. The equation of motion obtained varying the connection in (1) is Cartan's first structure equation

$$D_{[a}e^I_{b]} = 0, \tag{2}$$

where D_a is the covariant derivative defined by A. The solution of this equation is $A = \omega[e]$, where $\omega[e]$ is the torsion-free spin-connection of the tetrad field e. If we thus replace A by $\omega[e]$ in (1), the first term becomes the tetrad expression of the Einstein-Hilbert action, while the second term is identically zero, due to the Bianchi identity $R_{[abc]d} = 0$. Stationarity with respect to the variation of the tetrad yields then the Einstein equations. Therefore, as it is often stressed, the Immirzi parameter γ does not appear to have any effect on the equations of motion.

The parameter γ, on the other hand, plays an important role in loop quantum gravity, where the spectrum of quantum geometry operators is modulated by its value. For instance, the area of a surface and the volume of a space region are quantized in units of $\gamma\ell_p^2$ and $\gamma^{3/2}\ell_p^3$ respectively. Furthermore, the nonperturbative calculation of the entropy of a black hole appears to yield a result compatible with Hawking's semi-classical formula only for a specific value of γ. See [7] for recent evaluations and references. The role of the Immirzi parameter is often compared with the role of the Θ angle in QCD—which also appears as a constant in front of a term in the action with no effect on the equations of motion: a parameter that governs only nonperturbative quantum effects. See for instance [8]. In this letter we point out that this is in fact not the case in general.

The catch is that the second term in (1) vanishes only when equation (2) is satisfied, but equation (2) is modified by the presence of fermions (or, more in general, matter field that couple to the connection, see [9]). In the presence

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of a fermion field, (1) becomes

$$S[e,A,\psi] = S[e,A] + \frac{i}{2}\int d^4x\ e\left(\overline{\psi}\,\gamma^I e^a_I D_a\psi - \overline{D_a\psi}\,\gamma^I e^a_I\psi\right), \tag{3}$$

where γ^I are the Dirac matrices; and (2) becomes

$$D_{[a}e^I_{b]} = \textit{fermion current.} \tag{4}$$

The fermion current acts as a source for a torsion component in the connection, and the second term in (1) doesn't vanish.

In the following we compute the equations of motion of (3), namely the equations of motion of general relativity coupled with a fermionic field in the presence of a nontrivial Immirzi parameter. We solve the equations for the connection explicitly. By inserting the solution into the action we obtain an effective action that contains a four-fermion interaction. The coupling constant that determines the strength of this interaction depends explicitly on the Immirzi parameter.

Let us start by introducing the tensor

$$p_{IJ}{}^{KL} = \frac{1}{2}(\delta^K_I\delta^L_J - \delta^L_I\delta^K_J) - \frac{1}{2\gamma}\epsilon_{IJ}{}^{KL} \tag{5}$$

and its inverse

$$p^{-1}{}_{KL}{}^{IJ} = \frac{\gamma^2}{\gamma^2+1}\left(\frac{1}{2}(\delta^I_K\delta^J_L - \delta^I_L\delta^J_K) + \frac{1}{2\gamma}\epsilon_{KL}{}^{IJ}\right). \tag{6}$$

Using this, the action (3) can be written in the form

$$S[e,A,\psi] = \frac{1}{16\pi G}\int d^4x\ ee^a_I e^b_J\ p^{IJ}{}_{KL}F^{KL}_{ab} + \frac{1}{2}\int d^4x\ e\ \left(i\overline{\psi}\,\gamma^I e^a_I D_a\psi\ +\ c.c.\right), \tag{7}$$

and the equation of motion for the connection reads

$$p^{IJ}{}_{KL}D_b(e\,e^b_I e^a_J) = 8\pi G e J_{KL}{}^a, \tag{8}$$

where the fermion current $eJ_{KL}{}^a$ is the variation of the fermionic action with respect to the connection. Recalling that $D_a\psi = \partial_a\psi - 1/4\ A^{KL}_a\gamma_K\gamma_L\psi$, and using the identity $\gamma^A\gamma^{[B}\gamma^{C]} = -i\epsilon^{ABCD}\,\gamma_5\gamma_D + 2\eta^{A[B}\gamma^{C]}$, we obtain

$$J_{KL}{}^a = \frac{1}{8}(2ie^a{}_{[K}\ j_{v\,L]} + e^a_I\epsilon^I{}_{KLJ}\ j^J_a\ +\ c.c.) = \frac{1}{4}e^a_I\epsilon^I{}_{KLJ}\ j^J_a \tag{9}$$

where $j^K_v = \overline{\psi}\gamma^K\,\psi$ and $j^K_a = \overline{\psi}\,\gamma_5\gamma^K\,\psi$ are the vector and the axial fermion currents, and we have used the fact that they are real. Using the inverse tensor (6) equation (8) gives

$$D_b(e\,e^b_I e^a_J) = 8\pi G e\ p^{-1}{}_{IJ}{}^{KL}J_{KL}{}^a. \tag{10}$$

This equation can be solved for the connection. For this, we write the connection in the form $A^{IJ}_a = \omega[e]^{IJ}_a + C_a{}^{IJ}$, where $\omega[e]$ is the torsion free spin connection determined by e, namely the solution of (2), and C is the torsion. Using this, (10) gives

$$C_{b[K}{}^b e_{L]}{}^a + C_{[KL]}{}^a = 8\pi G e\ p^{-1}{}_{IJ}{}^{KL}J_{KL}{}^a. \tag{11}$$

Notice that we transform internal and spacetime indices into one another, using the tetrad field, and preserving the horizontal order of the indices. This equation can be solved by contracting the indices, and then summing terms with cyclical permutation of the indices: it is easy to verify that the solution is

$$C_a{}^{IJ} = -2\pi G\frac{\gamma}{\gamma^2+1}\left(2\ e^{[I}_a\ j^{J]}_a - \gamma\ e^K_a\epsilon_K{}^{IJ}{}_L\ j^L_a\right). \tag{12}$$

Notice that the torsion C depends on γ.

We can now obtain an equivalent action by replacing A with $\omega[e]+C$ in (3). It is easy to see that the terms linear in the fermion current are total derivatives, leaving

$$S[e,\psi] = S[e] + S_f[e,\psi] + S_{int}[e,\psi], \tag{13}$$

where the first two terms are the standard second–order tetrad action of general relativity with fermions,

$$S[e] + S_f[e,\psi] = \frac{1}{16\pi G}\int d^4x\ ee^a_I e^b_J\ F^{IJ}_{ab}[\omega[e]] \quad + \quad i\int d^4x\ e\ \overline{\psi}\,\gamma^I e^a_I D_a[\omega[e]]\psi \tag{14}$$

and the interaction term can be obtained by a tedious but straightforward calculation as

$$S_{int}[e,\psi] = -\frac{3}{2}\pi G\frac{\gamma^2}{\gamma^2+1}\int d^4x\ e\ (\overline{\psi}\gamma_5\gamma_A\psi)\ (\overline{\psi}\gamma_5\gamma^A\psi). \tag{15}$$

This term describes a four-fermion interaction mediated by a non-propagating torsion. An interaction of this form is well known: it is predicted by the Einstein-Cartan theory. The interaction is weak, because it is suppressed by one power of the Newton constant. It has never been observed, but it is compatible with all present observations and it might be observed in the future. Here we see that the coupling constant of this interaction depends on the Immirzi parameter. In the limit $\gamma\to\infty$, we recover the standard coupling of the Einstein-Cartan theory.

In summary, general relativity admits the natural formulation given by the action (3), widely used as a starting point for the nonperturbative quantization of the theory. This formulation includes a four-fermion interaction mediated by the torsion, whose strength is determined by the Immirzi parameter. The interaction is present also on a flat spacetime. The value of the Immirzi parameter is therefore observable in principle, independently from its effect on the nonperturbative quantum theory. The analogy with the Θ angle of QCD is, in this regard, misleading (see also [10]). The Immirzi parameter is a coupling constant determining the strength of a four-fermion interaction.

Note added: In the first version of this work the fermion lagrangian was taken to be $\mathcal{L}_f = ie\overline{\psi}\,\gamma^I e^a_I D_a\psi$, which yields the effective four-fermion interaction

$$S_{int}[e,\psi] = -\frac{3}{2}\pi G\frac{\gamma^2}{\gamma^2+1}\int d^4x\ e\ \left(j_a\cdot j_a + j_v\cdot j_v + \frac{2i}{\gamma}\ j_a\cdot j_v\right).$$

Such a fermion lagrangian, however, is not real. This leads to the non-unitarity which is manifest in the purely imaginary parity–violating term $j_a\cdot j_v$. We thank I.B. Khriplovich and A.A. Pomeransky, as well as L. Freidel, for pointing out to us the need of adding the complex conjugate term to the action. A parity–violating four–fermion interaction can nevertheless be still obtained without violating unitarity by starting from a non-minimally-coupled fermion field [11].

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