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# A conjecture for $q$ -decomposition matrices of cyclotomic $v$ -Schur algebras

Xavier YVONNE

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## Abstract

The Jantzen sum formula for cyclotomic  $v$ -Schur algebras yields an identity for some  $q$ -analogues of the decomposition matrices of these algebras. We prove a similar identity for matrices of canonical bases of higher-level Fock spaces. We conjecture then that those matrices are actually identical for a suitable choice of parameters. In particular, we conjecture that decomposition matrices of cyclotomic  $v$ -Schur algebras are obtained by specializing at  $q = 1$  some transition matrices between the standard basis and the canonical basis of a Fock space.

## 1 Introduction

In order to study representations of the Ariki-Koike algebra associated to the complex reflection group  $G(l, 1, m)$ , Dipper, James and Mathas introduced in 1998 the cyclotomic  $v$ -Schur algebra. This algebra depends on the two integers  $l$  and  $m$  and on some deformation parameters  $v, u_1, \dots, u_l$ . When  $l = 1$ , the cyclotomic  $v$ -Schur algebra coincides with the  $v$ -Schur algebra of [DJ]. It is an open problem to calculate the decomposition matrix of a cyclotomic  $v$ -Schur algebra whose parameters are powers of a given  $n$ -th root of unity. To this aim, James and Mathas proved, for cyclotomic  $v$ -Schur algebras, an important formula: the Jantzen sum formula [JM]. Given a Jantzen filtration for Weyl modules, one can define a  $q$ -analogue  $D(q)$  of the decomposition matrix; the coefficients of  $D(q)$  are graded decomposition numbers of the composition factors of Weyl modules (see Section 2.1). The Jantzen sum formula is equivalent to the identity  $D'(1) = J^{\triangleleft} D(1)$ , where  $J^{\triangleleft}$  is a matrix of  $q$ -adic valuations of some Gram determinants (see Lemma 1 and Theorem 2).

Let  $\Delta(q)$  be the matrix of the canonical basis of the degree  $m$  homogeneous component of a Fock representation of level  $l$  of  $U_q(\widehat{\mathfrak{sl}}_n)$  [U2]. Uglov provided in [U2] an algorithm for computing  $\Delta(q)$ .

In view of Ariki's theorem for Ariki-Koike algebras [A1], it seems natural to conjecture that for a suitable choice of parameters, one has  $D(q) = \Delta(q)$ . This would provide an algorithm for computing decomposition matrices of cyclotomic  $v$ -Schur algebras. Varagnolo and Vasserot [VV] proved for  $l = 1$  that  $D(1) = \Delta(1)$ . Moreover, Ryom-Hansen showed that this conjecture (still for  $l = 1$ ) is compatible with the Jantzen-Schaper formula [RH]. Passing to higher level  $l \geq 1$  requires the introduction of an extra parameter  $\mathbf{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l$ , called

*multi-charge*; this  $l$ -tuple parametrizes the Fock space of level  $l$  introduced by Uglov. We say that  $\mathbf{s}_l$  is *dominant* (with respect to  $m$ ) if for all  $1 \leq d \leq l-1$ , we have  $s_{d+1} - s_d \geq m$ . In this case, we conjecture that  $D(q) = \Delta(q)$ . Here,  $D(q)$  comes from a Jantzen filtration of the Weyl modules of the cyclotomic  $v$ -Schur algebra  $\mathcal{S}_{\mathbb{C}} = \mathcal{S}_{\mathbb{C}}(\xi; \xi^{s_1}, \dots, \xi^{s_l})$  with  $\xi := \exp(\frac{2i\pi}{n})$ . We do not know the existence of a prior conjecture giving accurate parameters for computing the decomposition matrix, even when  $l = 2$  (type  $B_m$ ).

Although  $\mathcal{S}_{\mathbb{C}}$  and thus  $D(1)$  do not actually depend on the multi-charge  $\mathbf{s}_l$ , but only on the sequence of the residues modulo  $n$  of its coordinates, it is necessary to impose a condition on  $\mathbf{s}_l$  for having  $D(1) = \Delta(1)$ . Indeed, we computed for example, for two choices of the multi-charge giving the same sequences of residues modulo  $n$ , two matrices  $\Delta(q)$  that do not even have the same numbers of zero entries.

Our conjecture is motivated by the following theorem. We define in a combinatorial way a matrix  $J^{\prec}$  for any multi-charge  $\mathbf{s}_l$ ; if  $\mathbf{s}_l$  is dominant, then our matrix  $J^{\prec}$  coincides with the matrix  $J^{\triangleleft}$  of the Jantzen sum formula. We show then that for any multi-charge  $\mathbf{s}_l$ , we have  $\Delta'(1) = J^{\prec} \Delta(1)$  (Theorem 3).

The proof of our theorem relies on a combinatorial expression for the derivative at  $q = 1$  of the matrix  $A(q)$ , where  $A(q)$  is the matrix of the Fock space involution used for defining  $\Delta(q)$ . Namely, we show that  $A'(1) = 2J^{\prec}$  (Theorem 4). The coefficients of  $A(q)$  are some analogues for Fock spaces of Kazhdan-Lusztig  $R$ -polynomials  $R_{x,y}(q)$  for Hecke algebras. The classical computation of  $R'_{x,y}(1)$  was made in [GJ], in relation with the Kazhdan-Lusztig conjecture for multiplicities of composition factors of Verma modules.

*Notation.* Throughout this article, we fix three integers  $n \geq 1$ ,  $l \geq 1$  and  $m \geq 1$  and a multi-charge  $\mathbf{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l$ . We set  $s := s_1 + \dots + s_l$ . Let  $\mathbb{N}$  (*resp.*  $\mathbb{N}^*$ ) denote the set of nonnegative (*resp.* positive) integers. Denote also by  $\llbracket a, b \rrbracket$  the discrete interval  $[a, b] \cap \mathbb{Z}$ . Let  $\Pi$  be the set of partitions of any integer and  $\Pi_m^l$  be the set of  $l$ -multi-partitions of  $m$ . Finally, we consider the Coxeter group  $\mathfrak{S}_r$  ( $r \in \mathbb{N}^*$ ), equipped with its generating set  $\{s_i = (i, i+1), 1 \leq i \leq r-1\}$ . We denote by  $\ell$  the length function on  $\mathfrak{S}_r$  and by  $\omega$  the unique element of maximal length in  $\mathfrak{S}_r$ .

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## 2 Statement of results

### 2.1 The Jantzen sum formula

Let  $R$  be a principal ideal domain, and  $K$  be the field of fractions of  $R$ . Let  $v$  be an invertible element of  $R$  and  $u_1, \dots, u_l \in R$ . Following [JM], let

$$\mathcal{H} = \mathcal{H}_R(v; u_1, \dots, u_l) = \mathcal{H}_{R,m}(v; u_1, \dots, u_l)$$

denote the Ariki-Koike algebra defined over  $R$ , with parameters  $v, u_1, \dots, u_l$ ; here  $m$  is the number of generators  $T_i$ ,  $0 \leq i \leq m-1$  involved in the presentation of  $\mathcal{H}$ . Let

$$\mathcal{S} = \mathcal{S}_R(v; u_1, \dots, u_l) = \mathcal{S}_{R,m}(v; u_1, \dots, u_l)$$

be the cyclotomic  $v$ -Schur algebra associated to  $\mathcal{H}$ . Dipper, James and Mathas (see [DJM, Theorem 6.12]) showed that  $\mathcal{S}$  is a cellular algebra in the sense of [GL]. Given  $\lambda_l \in \Pi_m^l$ , one defines an  $\mathcal{S}$ -module  $W(\lambda_l)$  which is a free  $R$ -module of finite rank, called *Weyl module*. Since  $\mathcal{S}$  is cellular,  $W(\lambda_l)$  is naturally equipped with a symmetric bilinear form  $\langle \cdot, \cdot \rangle$ . Set

$$L(\lambda_l) := W(\lambda_l) / \text{rad } W(\lambda_l),$$

where  $\text{rad } W(\lambda_l)$  is the radical of the bilinear form  $\langle \cdot, \cdot \rangle$ ; it is also the radical (in the sense of Jacobson) of  $W(\lambda_l)$ . The theory of cellular algebras shows that if  $R$  is a field, then  $\{L(\lambda_l), \lambda_l \in \Pi_m^l\}$  is a complete set of non-isomorphic irreducible  $\mathcal{S}$ -modules.

Let  $\wp$  be a maximal ideal in  $R$ ,  $F := R/\wp R$  be the residue field and  $\nu_\wp$  be the  $\wp$ -adic valuation of  $R$  associated to  $\wp$ . If  $M$  is an  $R$ -module, we denote by  $M_F = F \otimes_R M$  the specialized module (here,  $R$  acts on the right on  $F$  by the natural surjection  $R \twoheadrightarrow R/\wp R = F$ ). Let  $M$  be an  $R$ -module equipped with a symmetric bilinear form  $\langle \cdot, \cdot \rangle$ . For all  $i \in \mathbb{N}$ , we set

$$M(i) := \{u \in M \mid \forall v \in M, \nu_\wp(\langle u, v \rangle) \geq i\}.$$

The *Jantzen filtration* of  $M$  [AM] is the sequence

$$M_F = M_F(0) \supset M_F(1) \supset \dots,$$

where  $M_F(i) := (M(i) + \wp M) / \wp M$ . In particular, we have  $M_F(1) = \text{rad } M_F$ . If  $M$  is a free  $R$ -module of finite rank, then we have  $M_F(i) = \{0\}$  for  $i$  large enough. Since the form  $\langle \cdot, \cdot \rangle$  on  $W(\lambda_l)$  is compatible with the action of  $\mathcal{S}$ , the  $W_F(\lambda_l; i)$ 's are  $\mathcal{S}_F$ -modules. Define the matrix  $D(q) = (d_{\lambda_l, \mu_l}(q))_{\lambda_l, \mu_l \in \Pi_m^l}$  by

$$d_{\lambda_l, \mu_l}(q) := \sum_{i \geq 0} [W_F(\lambda_l; i) / W_F(\lambda_l; i+1) : L_F(\mu_l)] q^i \in \mathbb{N}[q] \quad (\lambda_l, \mu_l \in \Pi_m^l).$$

This matrix depends on the choice of  $R$ , on the parameters  $v, u_1, \dots, u_l \in R$  and  $m, l \in \mathbb{N}$ , and also on the maximal ideal  $\wp$ . Notice that  $d_{\lambda_l, \mu_l}(1)$  is equal to the multiplicity of  $L_F(\mu_l)$  as a composition factor of  $W_F(\lambda_l)$ , so  $D(1)$  is the decomposition matrix of  $\mathcal{S}_F$ .

The next lemma follows from an easy computation.

**Lemma 1** *Let  $\mathcal{R}_0(\mathcal{S}_F)$  denote the Grothendieck group of finitely-generated  $\mathcal{S}_F$ -modules. Let  $J = (j_{\lambda_l, \mu_l})_{\lambda_l, \mu_l \in \Pi_m^l}$  be a matrix with integer coefficients. Then the following statements are equivalent:*

- (i) *In  $\mathcal{R}_0(\mathcal{S}_F)$ , we have for all  $\lambda_l \in \Pi_m^l$ ,  $\sum_{i \geq 0} [W_F(\lambda_l; i)] = \sum_{\nu_l \in \Pi_m^l} j_{\lambda_l, \nu_l} [W_F(\nu_l)]$ ,*

(ii)  $D'(1) = JD(1)$ .

James and Mathas proved the following result [JM, Theorem 4.3]:

**Theorem 2 (the Jantzen sum formula)**

Assume that the Poincaré polynomial of  $\mathcal{H}(v; u_1, \dots, u_l)$  is nonzero. Then for all  $\lambda_l \in \Pi_m^l$ , we have

$$\sum_{i>0} [W_F(\lambda_l; i)] = \sum_{\mu_l \in \Pi_m^l} j_{\lambda_l, \mu_l}^{\triangleleft} [W_F(\mu_l)],$$

where the matrix  $J^{\triangleleft} = (j_{\lambda_l, \mu_l}^{\triangleleft})_{\lambda_l, \mu_l \in \Pi_m^l}$  is given explicitly in Section 4.1.

*Remark 1.* It is well-known that the Poincaré polynomial of  $\mathcal{H}_R$  is nonzero if and only if  $\mathcal{H}_K$  is semisimple, or if and only if  $\mathcal{S}_K$  is semisimple.

As with the matrix  $D(q)$ , the matrix  $J^{\triangleleft}$  depends on the choice of the ring  $R$ , on the parameters  $v, u_1, \dots, u_l \in R$  and  $m, l \in \mathbb{N}$ , and on the maximal ideal  $\wp$ . The coefficients of  $J^{\triangleleft}$  are  $\wp$ -adic valuations of some Gram determinants which can be described in a combinatorial way in terms of ribbons contained in diagrams of  $l$ -multi-partitions. Here  $\triangleleft$  stands for the dominance ordering on multi-partitions (see Section 3.1). We use the notation  $J^{\triangleleft}$  instead of  $J$  in order to point out that  $J^{\triangleleft}$  is lower-triangular with respect to this ordering, that is  $\lambda_l \not\triangleleft \mu_l \Rightarrow j_{\lambda_l, \mu_l}^{\triangleleft} = 0$ .

## 2.2 Statement of theorems

We first describe our choice of parameters for the cyclotomic  $v$ -Schur algebra  $\mathcal{S}$ . These parameters are similar to those used in [J] for Ariki-Koike algebras. They depend on  $n, l$  and on the  $l$ -tuple  $\mathbf{s}_l \in \mathbb{Z}^l$  that we have fixed at the beginning. Let us take

$$R := \mathbb{C}[x, x^{-1}],$$

the ring of Laurent polynomials in one indeterminate over  $\mathbb{C}$ ,  $\zeta := \exp\left(\frac{2i\pi}{nl}\right)$  a primitive  $nl$ -th root of unity,

$$\wp := (x - \zeta), \quad v := x^l \quad \text{and} \quad u_d := \zeta^{nd} x^{ls_d - nd} \quad (1 \leq d \leq l).$$

We then have  $K = \mathbb{C}(x)$ ,  $F = \mathbb{C}$  and  $\mathcal{S}_F = \mathcal{S}_{\mathbb{C}}(\xi; \xi^{s_1}, \dots, \xi^{s_l})$  with  $\xi := \exp\left(\frac{2i\pi}{n}\right)$ ;  $\mathcal{S}_F$  depends thus only on the residues  $s_d \bmod n$  ( $1 \leq d \leq l$ ). Notice that with our choice of parameters, the algebra  $\mathcal{S}_K(v; u_1, \dots, u_r)$  is semisimple. Indeed, specializing  $x$  at 1 sends  $\mathcal{H}_K(v; u_1, \dots, u_r)$  on the semisimple group algebra  $\mathbb{C}G(l, 1, m)$ , so by the Tits deformation argument [A2], the algebra  $\mathcal{H}_K(v; u_1, \dots, u_r)$  is semisimple and so is  $\mathcal{S}_K(v; u_1, \dots, u_r)$ .

Our choice of parameters leads thus to a Jantzen sum formula (see Theorem 2) for the

specialized cyclotomic  $v$ -Schur algebra  $\mathcal{S}_F = \mathcal{S}_{\mathbb{C}}(\xi; \xi^{s_1}, \dots, \xi^{s_l})$ , and therefore to the definition of a matrix  $J^{\triangleleft}$ .

Let  $<$  be any ordering on  $\Pi_m^l$ . We can actually define a matrix  $J^<$  in the same way as  $J^{\triangleleft}$ , but by replacing the dominance ordering on multi-partitions by  $<$  (see Section 4.1). We consider in particular the ordering  $\prec$ ; this ordering depends on  $\mathbf{s}_l$  and is defined in Section 3.2.

We say that  $\mathbf{s}_l$  is *dominant* (with respect to  $m$ ) if for all  $1 \leq d \leq l-1$ , we have  $s_{d+1} - s_d \geq m$ . We shall prove in Proposition 13 that if  $\mathbf{s}_l$  is dominant, then we have  $J^{\prec} = J^{\triangleleft}$ . Note that the orderings  $\prec$  and  $\triangleleft$  do not coincide in general, even if  $\mathbf{s}_l$  is dominant (see Remark 5).

*Example 1.* Set  $n = 3$ ,  $l = 2$ ,  $\mathbf{s}_l = (1, 0)$  and  $m = 3$ . Here,  $\mathbf{s}_l$  is not dominant, and the matrices  $J^{\prec}$  and  $J^{\triangleleft}$  are different. Namely, we have on the one hand

$$J^{\prec} = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & -1 & 0 & 1 & -1 & 1 & 0 & \cdot & \cdot \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & \cdot \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} ((1, 1), (1)) \\ ((3), \emptyset) \\ (\emptyset, (3)) \\ ((1), (2)) \\ (\emptyset, (2, 1)) \\ ((2), (1)) \\ ((1), (1, 1)) \\ (\emptyset, (1, 1, 1)) \\ ((2, 1), \emptyset) \\ ((1, 1, 1), \emptyset) \end{matrix},$$

where dots over the main diagonal stand for zeroes. The  $l$ -multi-partitions of  $m$  are ordered decreasingly according to a total ordering finer than  $\prec$  and they are displayed in the column located on the right of the matrix  $J^{\prec}$ . On the other hand, if we keep the same ordering for the rows and the columns of  $J^{\triangleleft}$ , we get

$$J^{\triangleleft} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} ((1, 1), (1)) \\ ((3), \emptyset) \\ (\emptyset, (3)) \\ ((1), (2)) \\ (\emptyset, (2, 1)) \\ ((2), (1)) \\ ((1), (1, 1)) \\ (\emptyset, (1, 1, 1)) \\ ((2, 1), \emptyset) \\ ((1, 1, 1), \emptyset) \end{matrix}.$$

The latter matrix is not written in a lower-triangular way because the ordering we consider here is not compatible with  $\triangleleft$ .

*Example 2.* Set  $n = 3$ ,  $l = 2$ ,  $\mathbf{s}_l = (4, -3)$  and  $m = 3$ . Here,  $\mathbf{s}_l$  is dominant. In this case, the orderings  $\prec$  and  $\triangleleft$  are different, but the matrices  $J^\prec$  and  $J^\triangleleft$  coincide. Namely, we have

$$J^\prec = J^\triangleleft = \begin{pmatrix} 0 & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & . & . & . & . & . \\ 0 & 1 & 0 & . & . & . & . & . & . & . & . \\ 0 & 1 & 1 & 0 & . & . & . & . & . & . & . \\ 0 & -1 & 1 & 0 & 0 & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & . & . \\ 0 & -1 & 0 & 1 & 1 & 0 & 0 & . & . & . & . \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & . & . \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & . \end{pmatrix} \begin{matrix} ((1, 1), (1)) \\ ((3), \emptyset) \\ ((2, 1), \emptyset) \\ ((2), (1)) \\ ((1, 1, 1), \emptyset) \\ ((1), (2)) \\ ((1), (1, 1)) \\ (\emptyset, (3)) \\ (\emptyset, (2, 1)) \\ (\emptyset, (1, 1, 1)) \end{matrix}.$$

Let  $\mathbf{F}_q[\mathbf{s}_l]$  be the Fock space of level  $l$ , with multi-charge  $\mathbf{s}_l$  (see Section 5.1); as a vector space,  $\mathbf{F}_q[\mathbf{s}_l]$  has a natural basis  $\{|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle, \boldsymbol{\lambda}_l \in \Pi^l\}$  indexed by  $l$ -multi-partitions. Let  $\mathbf{F}_q[\mathbf{s}_l]_m$  be the subspace of  $\mathbf{F}_q[\mathbf{s}_l]$  spanned by the  $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ 's,  $\boldsymbol{\lambda}_l \in \Pi_m^l$ . Let  $A(q)$  be the matrix of the involution of  $\mathbf{F}_q[\mathbf{s}_l]_m$  with respect to the standard basis  $\{|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle, \boldsymbol{\lambda}_l \in \Pi_m^l\}$ , and let  $\Delta(q)$  be the transition matrix between the standard basis and the canonical basis of  $\mathbf{F}_q[\mathbf{s}_l]_m$  (see Sections 5.2 and 5.3). The main result of this article is the following

**Theorem 3** *We have*

$$\Delta'(1) = J^\prec \Delta(1).$$

This theorem is equivalent to the following:

**Theorem 4** *We have*

$$A'(1) = 2J^\prec.$$

*Proof of the equivalence of Theorems 3 and 4.* Since the canonical basis is invariant under the involution, we have the identity  $\Delta(q) = A(q)\Delta(q^{-1})$ . Taking derivatives at  $q = 1$  yields  $\Delta'(1) = A'(1)\Delta(1) - A(1)\Delta'(1)$ . Since  $A(1)$  is the identity matrix, we get  $2\Delta'(1) = A'(1)\Delta(1)$ . As a consequence, Theorem 4 implies Theorem 3. Since  $\Delta(1)$  is unitriangular, hence invertible, the converse follows. ■

*Example 1 (continued).* Recall that  $n = 3$ ,  $l = 2$ ,  $\mathbf{s}_l = (1, 0)$  and  $m = 3$ . If we order the rows and the columns of  $A(q)$  decreasingly with respect to the same ordering used for writing  $J^\prec$ , we get

$$A(q) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & q - q^{-1} & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & q - q^{-1} & q - q^{-1} & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & q^{-2} - 1 & (q - q^{-1})^2 & 0 & q - q^{-1} & q - q^{-1} & 1 & \cdot & \cdot & \cdot \\ 0 & q^{-3}(q^2 - 1) & -(q - q^{-1}) & 0 & q^2 - 1 & q^{-2} - 1 & q - q^{-1} & 1 & \cdot & \cdot \\ 0 & q^2 - 1 & q^{-2} - 1 & 0 & 0 & q - q^{-1} & 0 & 0 & 1 & \cdot \\ 0 & -(q - q^{-1}) & -q^{-1}(q - q^{-1})^2 & 0 & q^{-2} - 1 & (q - q^{-1})^2 & q - q^{-1} & 0 & q - q^{-1} & 1 \end{pmatrix} \begin{matrix} ((1, 1), (1)) \\ ((3), \emptyset) \\ (\emptyset, (3)) \\ ((1), (2)) \\ (\emptyset, (2, 1)) \\ ((2), (1)) \\ ((1), (1, 1)) \\ (\emptyset, (1, 1, 1)) \\ ((2, 1), \emptyset) \\ ((1, 1, 1), \emptyset) \end{matrix}.$$

In this case, we do have  $A'(1) = 2J^{\prec}$ .

*Example 2* (continued). Recall that  $n = 3$ ,  $l = 2$ ,  $\mathbf{s}_l = (4, -3)$  and  $m = 3$ . If we order the rows and the columns of  $A(q)$  decreasingly with respect to the same ordering used for writing  $J^{\prec}$ , we get

$$A(q) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & q - q^{-1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & q^2 - 1 & q - q^{-1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & q^{-2} - 1 & q - q^{-1} & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & -(q - q^{-1}) & (q - q^{-1})^2 & q - q^{-1} & q - q^{-1} & 0 & 1 & \cdot & \cdot & \cdot \\ 0 & 0 & q^{-2} - 1 & q - q^{-1} & 0 & 0 & 0 & 1 & \cdot & \cdot \\ 0 & 0 & -q^{-1}(q - q^{-1})^2 & (q - q^{-1})^2 & q^{-2} - 1 & 0 & q - q^{-1} & q - q^{-1} & 1 & \cdot \\ 0 & q^{-3}(q^2 - 1) & 0 & -(q - q^{-1}) & 0 & 0 & q^2 - 1 & q^{-2} - 1 & q - q^{-1} & 1 \end{pmatrix} \begin{matrix} ((1, 1), (1)) \\ ((3), \emptyset) \\ ((2, 1), \emptyset) \\ ((2), (1)) \\ ((1, 1, 1), \emptyset) \\ ((1), (2)) \\ ((1), (1, 1)) \\ (\emptyset, (3)) \\ (\emptyset, (2, 1)) \\ (\emptyset, (1, 1, 1)) \end{matrix}.$$

In this case also, we do have  $A'(1) = 2J^{\prec}$ .

We prove Theorem 4 in the next sections. Our proof is similar to the proof of [RH] in the level one case. However the higher-level case is significantly more complicated and involves the discussion of many cases (see Section 8).

The formal analogy between Theorem 3 on one hand, and the rephrasing of the Jantzen sum formula given in Lemma 1 on the other hand leads us to set the following

**Conjecture 1** *Assume that  $\mathbf{s}_l$  is dominant. Let  $D(q)$  be the  $q$ -analogue of the decomposition matrix of  $\mathcal{S}$  defined in Section 2.1 with our choice of parameters given in Section 2.2. Then we have*

$$D(q) = \Delta(q).$$

*Example 3.* Set again  $n = 3$ ,  $l = 2$ ,  $\mathbf{s}_l = (4, -3)$  and  $m = 3$ , and consider the cyclotomic  $v$ -Schur algebra

$$\mathcal{S} = \mathcal{S}_R(v; u_1, u_2)$$

defined over the ring  $R = \mathbb{C}[x, x^{-1}]$  with parameters

$$v := x^2, \quad u_1 := -x^5, \quad u_2 := x^{-12}.$$



Take  $\wp := \left(x - e^{\frac{i\pi}{3}}\right)$ , so the specialized algebra is  $\mathcal{S}_{\mathbb{C}}\left(e^{\frac{2i\pi}{3}}; e^{\frac{2i\pi}{3}}, 1\right)$ . In this case, we expect the matrix  $D(q)$  to be equal to the matrix

$$\Delta(q) = \begin{pmatrix} 1 & . & . & . & . & . & . & . & . & . \\ 0 & 1 & . & . & . & . & . & . & . & . \\ 0 & q & 1 & . & . & . & . & . & . & . \\ 0 & q^2 & q & 1 & . & . & . & . & . & . \\ 0 & 0 & q & 0 & 1 & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 1 & . & . & . & . \\ 0 & 0 & q^2 & q & q & 0 & 1 & . & . & . \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 1 & . & . \\ 0 & 0 & 0 & q^2 & 0 & 0 & q & q & 1 & . \\ 0 & 0 & 0 & 0 & q & 0 & q^2 & 0 & q & 1 \end{pmatrix} \begin{matrix} ((1, 1), (1)) \\ ((3), \emptyset) \\ ((2, 1), \emptyset) \\ ((2), (1)) \\ ((1, 1, 1), \emptyset) \\ ((1), (2)) \\ ((1), (1, 1)) \\ (\emptyset, (3)) \\ (\emptyset, (2, 1)) \\ (\emptyset, (1, 1, 1)) \end{matrix}.$$

If we no longer assume that  $\mathbf{s}_l$  is dominant, then we expect  $\Delta(q)$  to be equal to a  $q$ -analogue of the decomposition matrix of a quasi-hereditary covering (in the sense of Rouquier) of the Ariki-Koike algebra  $\mathcal{H}$ . This covering, depending on  $\mathbf{s}_l$ , could come from a rational Cherednik algebra through the Knizhnik-Zamolodchikov functor [GGOR]. It would be Morita-equivalent to the cyclotomic  $v$ -Schur algebra of [JM] if  $\mathbf{s}_l$  is dominant.

### 3 Combinatorics of partitions and multi-partitions

#### 3.1 Definitions

We give here all the basic definitions about partitions and multi-partitions that we need later; our main reference is [Md]. Let  $r \in \mathbb{N}$ . A *partition of  $r$*  is a sequence of integers  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$  such that  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$  and  $\lambda_1 + \dots + \lambda_N = r$ . Each nonzero  $\lambda_i$  is called a *part* of  $\lambda$ . The sum of all the parts of  $\lambda$  is denoted by  $|\lambda|$ . We identify two partitions differing only by a tail of zeroes and we write sometimes partitions as sequences of integers with an infinite tail of zeroes. The only partition of 0 is denoted by  $\emptyset$ . An  *$N$ -multi-partition of  $r$*  is an  $N$ -tuple of partitions of integers summing up to  $r$ . Let  $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(N)})$  be an  $N$ -multi-partition. For  $1 \leq b \leq N$ , write  $\lambda^{(b)} = (\lambda_1^{(b)}, \lambda_2^{(b)}, \dots)$  the parts of  $\lambda^{(b)}$ . The *Young diagram* of  $\boldsymbol{\lambda}$  is the set

$$\{(i, j, b) \in \mathbb{N}^* \times \mathbb{N}^* \times \llbracket 1, N \rrbracket \mid 1 \leq j \leq \lambda_i^{(b)}\},$$

whose elements are called *boxes* or *nodes* of  $\boldsymbol{\lambda}$ . If  $N = 1$ , namely, if  $\boldsymbol{\lambda}$  is a partition, we drop the third component in the symbol  $(i, j, b)$  of a node of  $\boldsymbol{\lambda}$ . From now on we identify an  $N$ -multi-partition with its Young diagram. We extend the notation  $|\lambda|$  in a natural way for multi-partitions, and we define the *dominance ordering* on multi-partitions as follows. Let  $\boldsymbol{\lambda}$  and  $\boldsymbol{\mu}$  be two  $N$ -multi-partitions. We say that  $\boldsymbol{\mu}$  *dominates*  $\boldsymbol{\lambda}$ , and we write  $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$  if

$|\lambda| = |\mu|$  and for all  $k \geq 0$ ,  $1 \leq b \leq N$ , we have

$$\sum_{i=1}^{b-1} |\lambda^{(i)}| + \sum_{j=1}^k \lambda_j^{(b)} \leq \sum_{i=1}^{b-1} |\mu^{(i)}| + \sum_{j=1}^k \mu_j^{(b)} ;$$

we write  $\lambda \triangleleft \mu$  if  $\lambda \trianglelefteq \mu$  and  $\lambda \neq \mu$ .

If  $\lambda, \mu \in \Pi$  are two partitions, we write  $\lambda \subset \mu$  if the diagram of  $\lambda$  is contained in the diagram of  $\mu$ , and the set-theoretic difference is called a *skew diagram*; we denote it by  $\mu/\lambda$ . A *path* in the skew diagram  $\theta$  is a sequence of boxes  $(\gamma_1, \dots, \gamma_N) \in \theta^N$  such that for all  $1 \leq i \leq N-1$ ,  $\gamma_i$  and  $\gamma_{i+1}$  have one common side. We say that  $\theta$  is *connected* if given any two boxes  $\gamma, \gamma' \in \theta$ , there exists a path within  $\theta$  connecting  $\gamma$  to  $\gamma'$ . A *ribbon* is a connected skew diagram that contains no  $2 \times 2$  block of boxes. Let  $\rho$  be a ribbon. The *head* (*resp. tail*) of  $\rho$  is the node  $\gamma = (i, j) \in \rho$  such that  $j - i$  is minimal (*resp. maximal*); we denote this node by  $\text{hd}(\rho)$  (*resp. tl*( $\rho$ )). If  $\text{hd}(\rho) = (i, j)$  and  $\text{tl}(\rho) = (i', j')$ , the *height* of  $\rho$  is the integer  $\text{ht}(\rho) := i - i' \in \mathbb{N}$ . Finally, the *length* of  $\rho$  is the number of boxes contained in  $\rho$ ; we denote it by  $\ell(\rho)$ .

*Example 4.* On Figure 2 (see Section 4.1), the set of white squares represents the partition  $(4, 1)$ ;  $\rho$ ,  $\rho'$  and  $\rho''$  are three ribbons of respective heights 2, 1 and 0 and of respective lengths 4, 4 and 3.

A *charged  $N$ -multi-partition* is an element of  $\Pi^N \times \mathbb{Z}^N$ . If  $(\lambda, \mathbf{t}) \in \Pi^N \times \mathbb{Z}^N$  is a charged multi-partition and  $\mathbf{t} = (t_1, \dots, t_N)$ , the *content* of the node  $\gamma = (i, j, b) \in \lambda$  is the integer

$$\text{cont}(\gamma) := t_b + j - i.$$

If  $M \in \mathbb{N}^*$ , the *residue modulo  $M$*  of  $\gamma$  is

$$\text{res}_M(\gamma) := \text{cont}(\gamma) \bmod M.$$

For all  $i \in \mathbb{Z}$ , we set

$$N_i(\lambda) := \sharp\{\gamma \in \lambda \mid \text{res}_n(\gamma) = i \bmod n\};$$

this number depends on the multi-charge  $\mathbf{t}$ . We define in a similar way  $N_i(\theta)$  if  $\theta$  is a skew diagram contained in a charged partition.

### 3.2 The bijection $\tau_l$

Throughout the proof of Theorem 4, we use a large amount of notation that we give here. In particular, we have to pass from  $l$ -multi-partitions (indexing the bases of the Fock space) to partitions (indexing the bases of the  $q$ -wedge space – see Section 5.1) and conversely. This passage from one indexation to the other can be described in a combinatorial way by using a bijection  $\tau_l$  that we give at the end of this section.  $\tau_l$  is a variant of the bijection associating to a partition its  $l$ -quotient and its  $l$ -core. We construct this bijection using abaci. This

bijection will be used in particular for defining the partial ordering  $\prec$  on  $\Pi_m^l$  mentioned in Section 2.2.

The Euclidean algorithm shows that any integer  $k \in \mathbb{Z}$  can be written in a unique way as

$$k = c(k) + n(d(k) - 1) + nlm(k),$$

with  $c(k) \in \llbracket 1, n \rrbracket$ ,  $d(k) \in \llbracket 1, l \rrbracket$  and  $m(k) \in \mathbb{Z}$ . For any  $r$ -tuple  $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ , we define an  $r$ -tuple

$$\mathbf{c}(\mathbf{k}) := (c(k_1), \dots, c(k_r)),$$

and in a similar way we define  $\mathbf{d}(\mathbf{k})$  and  $\mathbf{m}(\mathbf{k})$ . The group  $\mathfrak{S}_r$  acts on the left on  $\mathbb{Z}^r$  by

$$\sigma.(k_1, \dots, k_r) = (k_{\sigma^{-1}(1)}, \dots, k_{\sigma^{-1}(r)}) \quad ((k_1, \dots, k_r) \in \mathbb{Z}^r, \sigma \in \mathfrak{S}_r),$$

and a fundamental domain for this action is

$$B := \{(b_1, \dots, b_r) \in \mathbb{Z}^r \mid b_1 \geq \dots \geq b_r\}.$$

Let  $\mathbf{b}(\mathbf{k})$  denote the element of  $B$  that is conjugated to  $\mathbf{d}(\mathbf{k})$  under the action of  $\mathfrak{S}_r$ ,  $W_{\mathbf{k}}$  the stabilizer of  $\mathbf{b}(\mathbf{k})$  (this is a parabolic subgroup of  $\mathfrak{S}_r$ ), and  $\omega(\mathbf{k})$  the element of maximal length in  $W_{\mathbf{k}}$ . Let  $W^{\mathbf{k}}$  be the set of minimal length representatives in the left cosets  $\mathfrak{S}_r/W_{\mathbf{k}}$ , and  $\mathbf{v}(\mathbf{k}) \in W^{\mathbf{k}}$  be the element such that  $\mathbf{d}(\mathbf{k}) = \mathbf{v}(\mathbf{k}).\mathbf{b}(\mathbf{k})$ .

*Example 5.* Let  $n = 3$ ,  $l = 2$ ,  $r = 4$  and  $\mathbf{k} = (12, -5, 2, 17)$ . Then we have  $\mathbf{c}(\mathbf{k}) = (3, 1, 2, 2)$ ,  $\mathbf{d}(\mathbf{k}) = (2, 1, 1, 2)$ ,  $\mathbf{m}(\mathbf{k}) = (1, -1, 0, 2)$ ,  $\mathbf{b}(\mathbf{k}) = (2, 2, 1, 1)$  and  $\mathbf{v}(\mathbf{k}) = s_3 s_2$ .

To any charged multi-partition we can associate an abacus in the following way. By definition, a 1-runner abacus is a subset  $A$  of  $\mathbb{Z}$  such that  $-k \in A$  and  $k \notin A$  for all large enough  $k \in \mathbb{N}$ . In a less formal way, each  $k \in A$  corresponds to the position of a bead on the (horizontal) abacus  $A$  which is full of beads on the left and empty on the right. Let  $\mathcal{A}$  be the set of 1-runner abaci. If  $N \geq 1$ , an  $N$ -runner abacus is an  $N$ -tuple of 1-runner abaci. If  $\mathbf{A} = (A_1, \dots, A_N) \in \mathcal{A}^N$  is an  $N$ -runner abacus, we identify  $\mathbf{A}$  with the subset

$$\{(k, d) \mid 1 \leq d \leq N, k \in A_d\} \subset \mathbb{Z} \times \llbracket 1, N \rrbracket.$$

To  $(\boldsymbol{\lambda}, \mathbf{t}) \in \Pi^N \times \mathbb{Z}^N$  we associate the  $N$ -runner abacus

$$A(\boldsymbol{\lambda}, \mathbf{t}) := \{(\lambda_i^{(d)} + t_d + 1 - i, d) \mid i \geq 1, 1 \leq d \leq N\}$$

(here the  $t_d$ 's are the components of  $\mathbf{t}$ ). One checks easily that the map

$$(\boldsymbol{\lambda}, \mathbf{t}) \in \Pi^N \times \mathbb{Z}^N \mapsto A(\boldsymbol{\lambda}, \mathbf{t}) \in \mathcal{A}^N$$

is bijective. Consider now the map

$$\phi : \mathbb{Z} \rightarrow \mathbb{Z}, \quad k \mapsto c(k) + nm(k).$$

$\phi$  enjoys the following obvious properties, which we need later: for all  $k, k' \in \mathbb{Z}$ , we have

$$(P_1) \quad \phi(k) \equiv c(k) \equiv k \pmod{n},$$

$$(P_2) \quad (k < k', d(k) = d(k')) \implies \phi(k) < \phi(k'),$$

$$(P_3) \quad \phi(k) < \phi(k') \implies k < k'.$$

Notice that the map  $k \in \mathbb{Z} \mapsto (\phi(k), d(k)) \in \mathbb{Z} \times \llbracket 1, l \rrbracket$  is a bijection. The bijection  $\tau_l : \Pi \times \mathbb{Z} \cong \mathcal{A} \rightarrow \Pi^l \times \mathbb{Z}^l \cong \mathcal{A}^l$  is then defined in terms of abaci by the formula

$$\tau_l(A) := \{(\phi(k), d(k)), k \in A\} \in \mathcal{A}^l \quad (A \in \mathcal{A}).$$

*Remark 2.* Let  $\lambda_l \in \Pi^l$ ,  $\mathbf{t}_l = (t_1, \dots, t_l) \in \mathbb{Z}^l$ ,  $\lambda \in \Pi$  and  $t \in \mathbb{Z}$  be such that  $(\lambda_l, \mathbf{t}_l) = \tau_l(\lambda, t)$ . Then we have  $t = t_1 + \dots + t_l$ .

Recall that we have fixed  $s = s_1 + \dots + s_l$ . From now on, we write

$$\lambda_l \leftrightarrow \lambda$$

if  $\lambda \in \Pi$  and  $\lambda_l \in \Pi_m^l$  are related by  $(\lambda_l, \mathbf{s}_l) = \tau_l(\lambda, s)$ .

*Example 6.* Let  $n = 2$ ,  $l = 3$ ,  $m = 5$  and  $\mathbf{s}_l = (0, 0, -1)$ . Then Figure 1 shows that we have

$$((1, 1), (1, 1), (1)) \leftrightarrow (4, 3, 3, 2, 1).$$

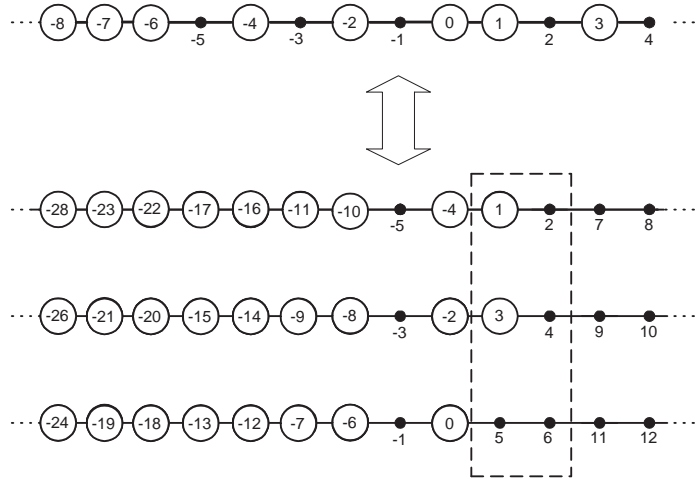


Figure 1: Computation of the bijection  $\tau_l$  using abaci.

*Remark 3.* For another (equivalent) description of the bijection  $\tau_l$  and for more examples, see [U2, Remark 4.2 (ii) & Example 4.3].

We define a partial ordering  $\preceq$  on  $\Pi_m^l$  as follows. Let  $\lambda_l, \mu_l \in \Pi_m^l$ , and  $\lambda, \mu \in \Pi$  be such that  $\lambda_l \leftrightarrow \lambda$  and  $\mu_l \leftrightarrow \mu$ . We say that  $\lambda_l$  *precedes*  $\mu_l$  and we write  $\lambda_l \preceq \mu_l$  if  $\lambda \trianglelefteq \mu$ . In particular,  $\lambda$  and  $\mu$  must be partitions of the same integer. Notice that the ordering  $\preceq$  we defined depends on the multi-charge  $\mathbf{s}_l$  that we consider. We write  $\lambda_l \prec \mu_l$  if  $\lambda_l \preceq \mu_l$  and  $\lambda_l \neq \mu_l$ .

### 3.3 $\beta$ -numbers and ribbons

We introduce the  $\beta$ -numbers associated to a charged multi-partition. Let  $\lambda = (\lambda_1, \lambda_2, \dots) \in \Pi$  be a partition with at most  $r$  parts. The  $r$ -tuple

$$\beta_r(\lambda) := (\lambda_1 + s, \lambda_2 + s - 1, \dots, \lambda_r + s - r + 1) \in \mathbb{Z}^r$$

is called the  $r$ -list of  $\beta$ -numbers associated to  $(\lambda, s)$  or (with a slight abuse of notation) the list or sequence of  $\beta$ -numbers associated to  $\lambda$ . The set of integers that form  $\beta_r(\lambda)$  is denoted by  $B_r(\lambda)$ . Notice that  $\beta_r(\lambda)$  is a decreasing sequence of integers all greater than (or equal to)  $s + 1 - r$ .  $\beta_r(\lambda)$  depends on the integer  $s$  we have fixed, but we do not mention it in our notation. A partition  $\lambda$  is completely determined by its sequence of  $\beta$ -numbers. If  $r = |\lambda|$ , we write more simply  $\beta(\lambda) := \beta_r(\lambda)$  and  $B(\lambda) := B_r(\lambda)$ . If  $f$  is a function defined on  $\mathbb{Z}^r$ , it is convenient to consider  $f$  as a function (still denoted by  $f$ ) defined on the set of partitions of  $r$  by the formula

$$f(\lambda) := f(\beta(\lambda)) \quad (\lambda \in \Pi, |\lambda| = r).$$

For example, we define this way for any partition  $\lambda$  the vectors  $\mathbf{c}(\lambda)$ ,  $\mathbf{d}(\lambda)$ , etc (see Section 3.2).

In order to prove Theorem 4, we have to relate the adding/removal of a ribbon in a charged partition and the corresponding  $\beta$ -numbers. Let us recall a classical result on  $\beta$ -numbers (see e.g. [Mt, Lemma 5.26]).

**Lemma 5** *Let  $\nu$  and  $\xi$  be two partitions with at most  $r$  parts, and  $\beta_r(\xi) = (\beta_1, \dots, \beta_r)$  be the list of  $\beta$ -numbers associated to  $\xi$ . Then the following statements are equivalent:*

- (i)  $\nu \subset \xi$ , and  $\rho := \xi/\nu$  is a ribbon of length  $h$ ;
- (ii) There exist positive integers  $b$  and  $h$  such that

$$B_r(\nu) = \{\beta_1, \dots, \beta_{b-1}, \beta_b - h, \beta_{b+1}, \dots, \beta_r\}.$$

In this case,  $b$  is the row number of the tail of  $\rho$  and  $h$  is the length of  $\rho$ . Let  $\sigma \in \mathfrak{S}_r$  denote the permutation obtained by arranging decreasingly the integers

$$(\beta_1, \dots, \beta_{b-1}, \beta_b - h, \beta_{b+1}, \dots, \beta_r).$$

Then we have  $\ell(\sigma) = \text{ht}(\rho)$ . Moreover, the content of the head of  $\rho$  is

$$\text{cont}(\text{hd}(\rho)) = \alpha_c = \beta_b - h,$$

where  $c$  is the row number of the head of  $\rho$ .

*Proof.* The proof of (i)  $\Rightarrow$  (ii) is easy. Conversely, assume that (ii) holds. Then we must have  $\beta_b - h \geq s + 1 - r$ , and there must exist  $b \leq c \leq r$  such that  $\beta_c > \beta_b - h > \beta_{c+1}$  (if  $c = r$ , we set  $\beta_{c+1} := s - r$ ). Notice then that  $\nu$  is obtained from  $\xi$  by removing a ribbon  $\rho$ , where  $\rho \subset \xi$  is the ribbon whose head is located at row  $c$  of  $\xi$  and whose tail is located at row  $b$  of  $\xi$ .  $\rho$  is actually a ribbon of length  $h$ . Moreover, with the notation of the statement of this lemma, we have

$$\sigma.(\beta_1, \dots, \beta_{b-1}, \beta_b - h, \beta_{b+1}, \dots, \beta_r) = (\beta_1, \dots, \beta_{b-1}, \beta_{b+1}, \dots, \beta_c, \beta_b - h, \beta_{c+1}, \dots, \beta_r),$$

hence  $\sigma$  is a cycle of length  $c - b = \text{ht}(\rho)$ . Finally, the head of  $\rho$  has coordinates  $(c, \nu_c + 1)$ , so its content is equal to  $\text{cont}(\text{hd}(\rho)) = s + (\nu_c + 1) - c = \alpha_c = \beta_b - h$ . ■

*Example 7.* Let  $s = 4$ ,  $r = 5$ ,  $\xi = (6, 5, 3, 2, 2)$  and  $\nu = (6, 2, 2, 2, 2)$ . Then  $\rho := \xi/\nu$  is a ribbon and we have  $b = 2$ ,  $c = 3$  and  $h = 4$ . Moreover, we have  $\beta(\xi) = (\beta_1, \dots, \beta_5) = (10, 8, 5, 3, 2)$  and  $\beta(\nu) = (\beta_1, \beta_3, \beta_b - h, \beta_4, \beta_5) = (10, 5, 4, 3, 2)$ . We have  $\sigma = (2, 3)$ , hence  $\ell(\sigma) = 1 = \text{ht}(\rho)$ . The head of  $\rho$  has coordinates  $(3, 3)$ , so its content is equal to  $\text{cont}(\text{hd}(\rho)) = 4 = \beta_b - h$ .

**Lemma 6** *Let  $\nu, \xi \in \Pi$  be such that  $|\nu| = |\xi| = r$  and  $\nu \neq \xi$ . Let  $\beta(\nu) = (\alpha_1, \dots, \alpha_r)$  and  $\beta(\xi) = (\beta_1, \dots, \beta_r)$  denote the sequences of  $\beta$ -numbers associated to  $\nu$  and  $\xi$  respectively. Set  $\rho := \nu/(\nu \cap \xi)$  and  $\rho' := \xi/(\nu \cap \xi)$ .*

1) *Then,  $\rho$  and  $\rho'$  are two ribbons if and only if  $\sharp(B(\nu) \cap B(\xi)) = r - 2$ . In this case, denote by*

- .  $h$  the common length of  $\rho$  and  $\rho'$ ,
- .  $y$  the row number of the tail of  $\rho'$ ,
- .  $y'$  the row number of the head of  $\rho'$ ,
- .  $x'$  the row number of the tail of  $\rho$ , and
- .  $x$  the row number of the head of  $\rho$ .

*Then we have  $\{\alpha_i \mid i \neq x', y'\} = \{\beta_j \mid j \neq x, y\}$ ,*

$$\text{cont}(\text{hd}(\rho)) = \beta_x = \alpha_{x'} - h \quad \text{and} \quad \text{cont}(\text{hd}(\rho')) = \alpha_{y'} = \beta_y - h.$$

*Let  $\pi \in \mathfrak{S}_r$  be the permutation obtained by arranging decreasingly the integers*

$$(\beta_1, \dots, \beta_y - h, \dots, \beta_x + h, \dots, \beta_r)$$

*forming  $B(\nu)$ . Then we have  $\ell(\pi) = \text{ht}(\rho) + \text{ht}(\rho')$ .*

2) Assume that the conditions of 1) hold. Then we have the following equivalences, and moreover one of the two following cases occurs:

$$\begin{aligned} \text{(i)} \quad & y \leq y' < x' \leq x \iff \nu \triangleleft \xi, \\ \text{(ii)} \quad & x' \leq x < y \leq y' \iff \xi \triangleleft \nu. \end{aligned}$$

*Proof.* We prove 1) by applying the previous lemma to the pairs of partitions  $(\nu \cap \xi, \nu)$  and  $(\nu \cap \xi, \xi)$ . Let us prove 2). The inequalities  $y \leq y'$  and  $x' \leq x$  are obvious. Since  $\rho \cap \rho' = \emptyset$ , one of the two following cases occurs: either  $y' < x'$  and then  $\nu \triangleleft \xi$ , or  $x < y$  and then  $\xi \triangleleft \nu$ . This proves both implications  $\Rightarrow$ , and since one of the two cases occurs, we get the desired equivalences. ■

## 4 Expressions of the matrices $J^{\triangleleft}$ and $J^{\prec}$

### 4.1 Definition of the matrix $J^{\prec}$

Recall the notation from Section 2. Following [JM], we define a matrix  $\mathcal{J} = (J_{\lambda_l, \mu_l})_{\lambda_l, \mu_l \in \Pi_m^l}$ , with coefficients in  $R$ , depending on parameters  $m, l \in \mathbb{N}$  and  $v, u_1, \dots, u_l \in R$ . Fix  $\lambda_l = (\lambda^{(1)}, \dots, \lambda^{(l)})$ ,  $\mu_l = (\mu^{(1)}, \dots, \mu^{(l)}) \in \Pi_m^l$ , and consider the following cases.

- Case  $(J_1)$ . Assume that  $\lambda_l \neq \mu_l$  and that there exist two integers  $d, d' \in \llbracket 1, l \rrbracket$ ,  $d \neq d'$  satisfying the following conditions:  $\mu^{(d)} \subset \lambda^{(d)}$ ,  $\lambda^{(d')} \subset \mu^{(d')}$ ,  $\lambda^{(b)} = \mu^{(b)}$  for all  $b \notin \{d, d'\}$ , and  $\rho := \lambda^{(d)} / \mu^{(d)}$  and  $\rho' := \mu^{(d')} / \lambda^{(d')}$  are two ribbons of the same length  $\widehat{h}$ . Let  $\text{hd}(\rho) = (i, j, d)$  denote the head of  $\rho$  and  $\text{hd}(\rho') = (i', j', d')$  denote the head of  $\rho'$ . We set

$$\varepsilon := (-1)^{\text{ht}(\rho) + \text{ht}(\rho')} \quad \text{and} \quad J_{\lambda_l, \mu_l} := (u_d v^{j-i} - u_{d'} v^{j'-i'})^\varepsilon.$$

- Case  $(J_2)$ . Assume that  $\lambda_l \neq \mu_l$  and that there exists  $d \in \llbracket 1, l \rrbracket$  such that  $\lambda^{(b)} = \mu^{(b)}$  for all  $b \neq d$ , and  $\rho := \lambda^{(d)} / (\lambda^{(d)} \cap \mu^{(d)})$  and  $\rho' := \mu^{(d)} / (\lambda^{(d)} \cap \mu^{(d)})$  are two ribbons of the same length  $\widehat{h}$ . By definition of  $\rho$  and  $\rho'$ , we have  $\rho \cap \rho' = \emptyset$ , whence we get (depending on the relative positions of  $\rho$  and  $\rho'$ ) that either  $\lambda^{(d)} \triangleleft \mu^{(d)}$  or  $\mu^{(d)} \triangleleft \lambda^{(d)}$ . Assume that  $\lambda^{(d)} \triangleleft \mu^{(d)}$ . Let  $\rho'' \subset (\lambda^{(d)} \cap \mu^{(d)})$  be the ribbon obtained by connecting the tail of  $\rho$  to the head of  $\rho'$ , excluding the two latter nodes (see Figure 2). Denote by  $\text{hd}(\rho) = (i, j, d)$  (*resp.*  $\text{hd}(\rho') = (i', j', d')$ , *resp.*  $\text{hd}(\rho'') = (i'', j'', d'')$ ) the head of  $\rho$  (*resp.*  $\rho'$ , *resp.*  $\rho''$ ), and finally set

$$\begin{aligned} \varepsilon_1 &:= (-1)^{\text{ht}(\rho) + \text{ht}(\rho')}, \quad \varepsilon_2 := (-1)^{\text{ht}(\rho \cup \rho'') + \text{ht}(\rho'' \cup \rho')} \quad \text{and} \\ J_{\lambda_l, \mu_l} &:= (u_d (v^{j-i} - v^{j'-i'}))^{\varepsilon_1} \cdot (u_d (v^{j-i} - v^{j''-i''}))^{\varepsilon_2}. \end{aligned}$$

If  $\mu^{(d)} \triangleleft \lambda^{(d)}$ , we set  $J_{\lambda_l, \mu_l} := J_{\mu_l, \lambda_l}$ .

- Case  $(J_3)$ . In all other cases, we set

$$J_{\lambda_l, \mu_l} := 1.$$

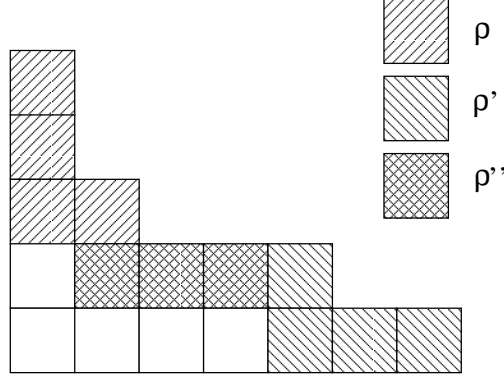


Figure 2: The ribbons  $\rho$ ,  $\rho'$  and  $\rho''$  (the nodes of  $(\lambda^{(d)} \cap \mu^{(d)}) - \rho''$  are depicted in white).

Let  $\wp$  be a maximal ideal of  $R$ . We define a matrix  $J = J_\wp = (j_{\lambda_l, \mu_l})_{\lambda_l, \mu_l \in \Pi_m^l}$ , with integer coefficients, by the formula

$$j_{\lambda_l, \mu_l} := \nu_\wp(J_{\lambda_l, \mu_l}) \quad (\lambda_l, \mu_l \in \Pi_m^l).$$

Consider now a partial ordering  $\leq$  on  $\Pi_m^l$ , and write  $\lambda_l < \mu_l$  if  $\lambda_l \leq \mu_l$  and  $\lambda_l \neq \mu_l$  ( $\lambda_l, \mu_l \in \Pi_m^l$ ). We define a matrix  $J^< = (j_{\lambda_l, \mu_l}^<)_{\lambda_l, \mu_l \in \Pi_m^l}$  by the formulas

$$j_{\lambda_l, \mu_l}^< := \begin{cases} j_{\lambda_l, \mu_l} & \text{if } \lambda_l < \mu_l \\ 0 & \text{otherwise} \end{cases} \quad (\lambda_l, \mu_l \in \Pi_m^l).$$

Thus, the matrix  $J^<$  depends on the same parameters as  $J$ , and also on  $\wp$  and  $<$ . When  $< = \triangleleft$ , the matrix  $J^{\triangleleft}$  described here is that given in [JM, Definition 3.36 & Remark 3.37] (up to a slight change of convention). If  $<$  is the ordering  $\prec$  introduced in Section 3.2 and if  $R$  and the parameters  $v, u_1, \dots, u_l \in R$  are as in Section 2.2, then we get the matrix  $J^{\prec}$  of Theorems 3 and 4.

From now on we fix the ring  $R$  and the parameters  $v, u_1, \dots, u_l \in R$  as in Section 2.2, so the matrix  $J$  depends on  $n, l, m$  and  $s_l$ . The following lemma gives a simpler expression for  $J$ .

**Lemma 7**

1) Assume that  $(\lambda_l, \mu_l)$  satisfies the conditions  $(J_1)$ . Then we have

$$j_{\lambda_l, \mu_l} = \begin{cases} (-1)^{\text{ht}(\rho) + \text{ht}(\rho')} & \text{if } \text{res}_n(\text{hd}(\rho)) = \text{res}_n(\text{hd}(\rho')), \\ 0 & \text{otherwise.} \end{cases}$$



2) Assume that  $(\lambda_l, \mu_l)$  satisfies the conditions  $(J_2)$ . Then we have

$$j_{\lambda_l, \mu_l} = (-1)^{\text{ht}(\rho) + \text{ht}(\rho')} \varepsilon,$$

where

$$\varepsilon := \begin{cases} 1 & \text{if } \text{res}_n(\text{hd}(\rho)) = \text{res}_n(\text{hd}(\rho')) \quad \text{and} \quad \widehat{h} \not\equiv 0 \pmod{n}, \\ -1 & \text{if } \text{res}_n(\text{hd}(\rho)) \neq \text{res}_n(\text{hd}(\rho')) \quad \text{and} \quad \widehat{h} \equiv 0 \pmod{n}, \\ 0 & \text{otherwise,} \end{cases}$$

and  $\widehat{h}$  is the common length of  $\rho$  and  $\rho'$ .

*Proof.* Let us prove 1). Then we have

$$j_{\lambda_l, \mu_l} = (-1)^{\text{ht}(\rho) + \text{ht}(\rho')} \nu_{\wp}(P_{\lambda_l, \mu_l}(x)),$$

with  $P_{\lambda_l, \mu_l}(x) := u_d x^{l(j-i)} - u_{d'} x^{l(j'-i')}$ . Notice that

$$P_{\lambda_l, \mu_l}(x) = \zeta^{a_1} x^{a_2} - \zeta^{a_3} x^{a_4},$$

with  $a_1 := dn$ ,  $a_2 := ls_d - dn + l(j-i)$ ,  $a_3 := d'n$  and  $a_4 := ls_{d'} - d'n + l(j'-i')$ . Using the fact that  $\nu_{\wp}(x^N) = 0$  for all  $N \in \mathbb{Z}$  and  $x^N P_{\lambda_l, \mu_l}(x) \in \mathbb{C}[x]$  for a suitable  $N \in \mathbb{Z}$ , we get

$$\begin{aligned} \nu_{\wp}(P_{\lambda_l, \mu_l}(x)) &\geq 0, \\ \nu_{\wp}(P_{\lambda_l, \mu_l}(x)) &\geq 1 \iff P_{\lambda_l, \mu_l}(\zeta) = 0, \\ \nu_{\wp}(P_{\lambda_l, \mu_l}(x)) &\geq 2 \iff P_{\lambda_l, \mu_l}(\zeta) = P'_{\lambda_l, \mu_l}(\zeta) = 0. \end{aligned}$$

A straightforward computation shows that  $\nu_{\wp}(P_{\lambda_l, \mu_l}(x)) \geq 1$  if and only if  $l(s_d + j - i) \equiv l(s_{d'} + j' - i') \pmod{nl}$ , that is if and only if  $\text{res}_n(\text{hd}(\rho)) = \text{res}_n(\text{hd}(\rho'))$ . Moreover, we have

$$\begin{aligned} \nu_{\wp}(P_{\lambda_l, \mu_l}(x)) \geq 2 &\iff \begin{cases} a_1 + a_2 \equiv a_3 + a_4 \pmod{nl} \\ a_2 \zeta^{a_1 + a_2 - 1} = a_4 \zeta^{a_3 + a_4 - 1} \end{cases} \\ &\iff \begin{cases} a_1 + a_2 \equiv a_3 + a_4 \pmod{nl} \\ a_2 = a_4 \end{cases} \\ &\iff \begin{cases} a_1 \equiv a_3 \pmod{nl} \\ a_2 = a_4 \end{cases}. \end{aligned}$$

But the condition  $a_1 \equiv a_3 \pmod{nl}$  implies  $d \equiv d' \pmod{l}$ , which is impossible since  $d$  and  $d'$  are two distinct integers ranging from 1 to  $l$ . As a consequence, we have  $\nu_{\wp}(P_{\lambda_l, \mu_l}(x)) \leq 1$ , which proves 1). Let us now prove 2). With the notation of  $(J_2)$ , we have  $\varepsilon_2 = -\varepsilon_1$ , whence

$$j_{\lambda_l, \mu_l} = (-1)^{\text{ht}(\rho) + \text{ht}(\rho')} \nu_{\wp}(P_{\lambda_l, \mu_l}(x)), \quad \text{with}$$

$$\begin{aligned}
P_{\lambda_l, \mu_l}(x) &:= \frac{u_d(x^{l(j-i)} - x^{l(j'-i')})}{u_d(x^{l(j-i)} - x^{l(j''-i'')})} \\
&= \frac{x^{l((j'-i')-(j-i))} - 1}{x^{l((j''-i'')-(j-i))} - 1} = \frac{x^{l(\text{cont}(\text{hd}(\rho')) - \text{cont}(\text{hd}(\rho)))} - 1}{x^{\widehat{h}} - 1}.
\end{aligned}$$

In order to complete the proof, we only have to notice that for  $N \in \mathbb{Z}$ , we have  $\nu_\varphi(x^{lN} - 1) = 1$  if  $nl$  divides  $lN$ , that is  $n$  divides  $N$ , and  $\nu_\varphi(x^{lN} - 1) = 0$  otherwise. ■

## 4.2 A first approach for the cases $(J_1)$ and $(J_2)$

We now give a characterization in terms of  $\lambda$  and  $\mu$  of the pairs  $(\lambda_l, \mu_l)$  that satisfy  $(J_1)$  or  $(J_2)$ .

**Lemma 8** *Let  $\lambda_l = (\lambda^{(1)}, \dots, \lambda^{(l)}) \in \Pi_m^l$  and  $\mu_l = (\mu^{(1)}, \dots, \mu^{(l)}) \in \Pi_m^l$  be two multi-partitions, and  $\lambda, \mu \in \Pi$  be such that  $\lambda_l \leftrightarrow \lambda$  and  $\mu_l \leftrightarrow \mu$ . Then the following statements are equivalent:*

- (i)  $\lambda \subset \mu$ , and  $\mu/\lambda$  is a ribbon,
- (ii)  $\lambda_l \subset \mu_l$ , and there exists  $d \in \llbracket 1, l \rrbracket$  such that  $\mu^{(d)}/\lambda^{(d)}$  is a ribbon and  $\lambda^{(b)} = \mu^{(b)}$  for all  $b \in \llbracket 1, l \rrbracket \setminus \{d\}$ .

*Proof.* Let us prove (i)  $\Rightarrow$  (ii). By Lemma 5, passing from  $\lambda$  to  $\mu$  amounts, as far as abacus diagrams are concerned, to passing from  $A(\lambda, s)$  to  $A(\mu, s)$  by moving a bead located at position  $k$  towards the right. As far as the  $l$ -runner abacus diagrams  $A(\lambda_l, \mathbf{s}_l)$  and  $A(\mu_l, \mathbf{s}_l)$  are concerned, this amounts to moving a bead located at position  $\phi(k)$  on the runner  $d := d(k)$  towards the right. This together with the Lemma 5 applied to  $(\lambda^{(d)}, \mu^{(d)})$  proves Statement (ii). The converse is similar. ■

Applying twice the previous lemma and Lemma 6 yields the following corollary.

**Corollary 9** *Let  $\lambda_l, \mu_l \in \Pi_m^l$  be two distinct multi-partitions, and  $\lambda, \mu \in \Pi$  be such that  $\lambda_l \leftrightarrow \lambda$  and  $\mu_l \leftrightarrow \mu$ . Assume that  $|\lambda| = |\mu| = r$ . Then  $(\lambda_l, \mu_l)$  satisfies  $(J_1)$  or  $(J_2)$  if and only if  $\sharp(B(\lambda) \cap B(\mu)) = r - 2$ .*

Let  $\lambda_l, \mu_l \in \Pi_m^l$ , and  $\lambda, \mu \in \Pi$  be such that  $\lambda_l \leftrightarrow \lambda$  and  $\mu_l \leftrightarrow \mu$ . Consider the statement

$$(\mathcal{H}) \quad : \quad \lambda_l \prec \mu_l \quad (\text{i.e. } \lambda \triangleleft \mu), \quad |\lambda| = |\mu| = r \quad \text{and} \quad \sharp(B(\lambda) \cap B(\mu)) = r - 2.$$

The previous corollary shows that if  $(\lambda_l, \mu_l)$  does not satisfy  $(\mathcal{H})$ , then we have  $j_{\lambda_l, \mu_l}^\prec = 0$ .

*Notation.* Let  $\lambda_l = (\lambda^{(1)}, \dots, \lambda^{(l)}) \in \Pi_m^l$  and  $\mu_l = (\mu^{(1)}, \dots, \mu^{(l)}) \in \Pi_m^l$  be two multi-partitions such that  $(\lambda_l, \mu_l)$  satisfies  $(\mathcal{H})$ . In this case, we use in the sequel the following notation. Denote by  $\lambda, \mu \in \Pi$  the partitions such that  $\lambda_l \leftrightarrow \lambda$  and  $\mu_l \leftrightarrow \mu$ . Let  $(\alpha_1, \dots, \alpha_r)$

and  $(\beta_1, \dots, \beta_r)$  denote the sequences of  $\beta$ -numbers associated to  $\lambda$  and  $\mu$  respectively. By Lemma 6 applied to the pair  $(\nu, \xi) = (\lambda, \mu)$ , there exist positive integers  $y, y', x', x$  and  $h$  such that  $\{\alpha_i \mid i \neq x', y'\} = \{\beta_j \mid j \neq x, y\}$ ,  $\alpha_{y'} = \beta_y - h$  and  $\alpha_{x'} = \beta_x + h$ . Moreover, by Corollary 9,  $(\lambda_l, \mu_l)$  satisfies  $(J_1)$  or  $(J_2)$ . Let  $d, d' \in \llbracket 1, l \rrbracket$  and  $\widehat{h} \in \mathbb{N}^*$  denote the integers introduced in the definition of the cases  $(J_1)$  and  $(J_2)$  (in the case  $(J_2)$ , we set  $d' := d$ ). Finally, denote by  $\gamma$ , resp.  $\delta \in \llbracket 0, nl - 1 \rrbracket$  the residue of  $c(\beta_y) - c(\beta_x)$ , resp.  $n(d(\beta_y) - d(\beta_x))$  modulo  $nl$ .

*Remark 4.* Assume that  $(\lambda_l, \mu_l)$  satisfies  $(\mathcal{H})$ , and recall the above notation. Since  $\lambda \triangleleft \mu$ , Lemma 6 implies  $y' < x'$ , so we have

$$\beta_x < \beta_x + h = \alpha_{x'} < \alpha_{y'} = \beta_y - h < \beta_y.$$

These inequalities, together with  $\{\alpha_i \mid i \neq x', y'\} = \{\beta_j \mid j \neq x, y\}$  and the fact that the  $\beta_i$ 's are pairwise distinct, imply that  $\{\beta_x, \beta_y\} \cap B(\lambda) = \emptyset$ .

Under assumption  $(\mathcal{H})$ , the following lemma relates the  $\beta$ -numbers of  $\lambda^{(d)}$ ,  $\lambda^{(d')}$ ,  $\mu^{(d)}$  and  $\mu^{(d')}$  on the one hand to the  $\beta$ -numbers of  $\lambda$  and  $\mu$  on the other hand.

**Lemma 10** *Assume that  $(\lambda_l, \mu_l)$  satisfies  $(\mathcal{H})$ , and recall the corresponding notation.*

1) *Then, we have*

$$\{d(\beta_x), d(\beta_y)\} = \{d(\beta_x + h), d(\beta_y - h)\} = \{d, d'\}.$$

*Moreover, for all  $1 \leq b \leq l$ , we have*

$$\#\{1 \leq i \leq r \mid d(\alpha_i) = b\} = \#\{1 \leq i \leq r \mid d(\beta_i) = b\};$$

*let  $r_b$  denote this common value.*

2) *Assume that  $(\lambda_l, \mu_l)$  satisfies  $(\mathcal{H})$  and  $(J_1)$ . Then we have  $d(\beta_x) = d$  and  $d(\beta_y) = d'$ . Let*

$$\begin{aligned} \beta_{r_d}(\lambda^{(d)}) &= (\gamma_1, \dots, \gamma_{r_d}), & \beta_{r_d}(\mu^{(d)}) &= (\delta_1, \dots, \delta_{r_d}), \\ \beta_{r_{d'}}(\lambda^{(d')}) &= (\gamma'_1, \dots, \gamma'_{r_{d'}}) & \text{and} & \quad \beta_{r_{d'}}(\mu^{(d')}) = (\delta'_1, \dots, \delta'_{r_{d'}}) \end{aligned}$$

*denote the sequences of  $\beta$ -numbers associated to  $\lambda^{(d)}$ ,  $\mu^{(d)}$ ,  $\lambda^{(d')}$  and  $\mu^{(d')}$ . Denote by  $b$  (resp.  $c$ ) the row number of the tail (resp. head) of  $\rho$ , and denote by  $b'$  (resp.  $c'$ ) the row number of the tail (resp. head) of  $\rho'$ . The equalities  $\{d(\beta_x), d(\beta_y)\} = \{d(\beta_x + h), d(\beta_y - h)\} = \{d, d'\}$  from 1) show that one of the following cases occurs.*

– First case: we have  $d(\beta_x) = d(\beta_x + h) = d$  and  $d(\beta_y) = d(\beta_y - h) = d'$ . Then we have

$$\delta_c = \phi(\beta_x), \quad \gamma_b = \phi(\beta_x + h), \quad \gamma'_{c'} = \phi(\beta_y - h), \quad \delta'_{b'} = \phi(\beta_y)$$

$$\text{and } \widehat{h} = \phi(\beta_x + h) - \phi(\beta_x) = \phi(\beta_y) - \phi(\beta_y - h).$$

– Second case: we have  $d(\beta_x) = d(\beta_y - h) = d$  and  $d(\beta_y) = d(\beta_x + h) = d'$ . Then we have

$$\begin{aligned} \delta_c &= \phi(\beta_x), \quad \gamma_b = \phi(\beta_y - h), \quad \gamma'_{c'} = \phi(\beta_x + h), \quad \delta'_{b'} = \phi(\beta_y) \\ \text{and } \widehat{h} &= \phi(\beta_y - h) - \phi(\beta_x) = \phi(\beta_y) - \phi(\beta_x + h). \end{aligned}$$

3) Assume that  $(\lambda_l, \mu_l)$  satisfies  $(\mathcal{H})$  and  $(J_2)$ . Denote by

$$\beta_{r_d}(\lambda^{(d)}) = (\gamma_1, \dots, \gamma_{r_d}) \quad \text{and} \quad \beta_{r_d}(\mu^{(d)}) = (\delta_1, \dots, \delta_{r_d})$$

the sequences of  $\beta$ -numbers associated to  $\lambda^{(d)}$  and  $\mu^{(d)}$ . Let  $b$  (resp.  $c$ ) denote the row number of the tail (resp. head) of  $\rho$ , and  $b'$  (resp.  $c'$ ) denote the row number of the tail (resp. head) of  $\rho'$ . Then we have  $\lambda^{(d)} \triangleleft \mu^{(d)}$  and

$$\begin{aligned} \delta_c &= \phi(\beta_x), \quad \gamma_b = \phi(\beta_x + h), \quad \gamma_{c'} = \phi(\beta_y - h), \quad \delta_{b'} = \phi(\beta_y) \\ \text{and } \widehat{h} &= \phi(\beta_x + h) - \phi(\beta_x) = \phi(\beta_y) - \phi(\beta_y - h). \end{aligned}$$

*Proof.* We pass from  $\mu_l$  to  $\lambda_l$  by removing the ribbon  $\rho'$  and by adding the ribbon  $\rho$ . This amounts, as far as the abacus diagrams  $A(\mu_l, \mathbf{s}_l)$  and  $A(\lambda_l, \mathbf{s}_l)$  are concerned, to moving two beads (see the proof of Lemma 8). Moving these two beads amounts, as far as the abacus diagrams  $A(\mu, s)$  and  $A(\lambda, s)$  are concerned, to moving the beads located at positions  $\{\beta_x, \beta_y\}$  towards the positions  $\{\beta_x + h, \beta_y - h\}$ , which proves the first two equalities of Statement 1). The last parts of Statement 1) come from this and from the equality  $\{\alpha_i \mid i \neq x', y'\} = \{\beta_j \mid j \neq x, y\}$ .

Let us now prove Statement 2). A careful analysis of the moves of the beads described above shows that more precisely the following properties hold:

- (i)  $\{\gamma_b, \gamma'_{c'}\} = \{\phi(\beta_x + h), \phi(\beta_y - h)\}$  and  $\{\delta'_{b'}, \delta_c\} = \{\phi(\beta_x), \phi(\beta_y)\}$ .
- (ii) Let  $k \in \{\beta_x, \beta_x + h, \beta_y - h, \beta_y\}$ . Then we have  $\phi(k) \in \{\gamma_b, \delta_c\}$  if and only if  $d(k) = d$ , and we have  $\phi(k) \in \{\gamma'_{c'}, \delta'_{b'}\}$  if and only if  $d(k) = d'$ .

Let us first prove that  $d(\beta_x) = d$ . Assume that  $d(\beta_x) = d'$ . By (i) and (ii), we have  $\phi(\beta_x) \in \{\delta'_{b'}, \delta_c\} \cap \{\gamma'_{c'}, \delta'_{b'}\}$ . This implies  $\phi(\beta_x) = \delta'_{b'}$ . Indeed, if  $\phi(\beta_x) \neq \delta'_{b'}$ , we must have  $\phi(\beta_x) = \delta_c = \gamma'_{c'} \in \{\phi(\beta_x + h), \phi(\beta_y - h)\}$  by (i). Let  $k \in \{\beta_x + h, \beta_y - h\}$  be such that  $\phi(k) = \gamma'_{c'} = \phi(\beta_x)$ . By (ii), we have  $d(k) = d(\beta_x)$ ; moreover, we have  $\phi(k) = \phi(\beta_x)$ , whence  $k = \beta_x$ . This contradicts Remark 4, so  $\phi(\beta_x) = \delta'_{b'}$ . Let  $k \in \{\beta_x + h, \beta_y - h\}$  be such that  $d(k) = d'$ . By Property  $(P_2)$  and Remark 4, we have  $\phi(k) > \phi(\beta_x) = \delta'_{b'}$ , hence by (ii) we have  $\phi(k) = \gamma'_{c'}$ . As a consequence, we have  $\gamma'_{c'} > \delta'_{b'}$ . Moreover, by Lemma 5 applied to  $(\lambda^{(d')}, \mu^{(d')})$ , we have  $\gamma'_{c'} = \delta'_{b'} - \widehat{h} < \delta'_{b'}$ , which is absurd. By Statement 1), we thus have  $d(\beta_x) = d$  and  $d(\beta_y) = d'$ . Let us now deal with the first case of Statement 2) (the proof for the second case is similar). By (ii) we have  $\{\gamma_b, \delta_c\} = \{\phi(\beta_x), \phi(\beta_x + h)\}$ . Moreover, by Lemma 5 applied to  $(\lambda^{(d)}, \mu^{(d)})$ , we have  $\delta_c = \gamma_b - \widehat{h} < \gamma_b$ . Property  $(P_2)$  then implies that

$\delta_c = \phi(\beta_x)$  and  $\gamma_b = \phi(\beta_x + h)$ , whence  $\widehat{h} = \gamma_b - \delta_c = \phi(\beta_x + h) - \phi(\beta_x)$ . By a similar argument, we get  $\gamma'_{c'} = \phi(\beta_y - h)$ ,  $\delta'_{b'} = \phi(\beta_y)$  and  $\widehat{h} = \phi(\beta_y) - \phi(\beta_y - h)$ .

Let us now prove 3). Since  $\lambda \triangleleft \mu$ , Remark 4 and Property  $(P_2)$  imply

$$\phi(\beta_x) < \phi(\beta_x + h) < \phi(\beta_y - h) < \phi(\beta_y).$$

Moreover, a careful analysis of the moves of the beads mentioned at the beginning of the proof shows that

$$\{\delta_{b'}, \delta_c\} = \{\phi(\beta_x), \phi(\beta_y)\} \quad \text{and} \quad \{\gamma_b, \gamma_{c'}\} = \{\phi(\beta_x + h), \phi(\beta_y - h)\}.$$

Assume that  $\delta_c = \phi(\beta_y)$ . Since  $\phi(\beta_y) > \phi(\beta_x + h)$ ,  $\phi(\beta_y) > \phi(\beta_y - h)$  and  $\gamma_b$  belongs to the set  $\{\phi(\beta_x + h), \phi(\beta_y - h)\}$ , we must have  $\delta_c > \gamma_b$ . Moreover, applying Lemma 6 to the pair  $(\lambda^{(d)}, \mu^{(d)})$  yields  $\gamma_b = \delta_c + \widehat{h} > \delta_c$ , which is absurd. We thus have  $\delta_c = \phi(\beta_x)$  and  $\delta_{b'} = \phi(\beta_y)$ . By Property  $(P_2)$ , we have  $\delta_c < \delta_{b'}$ , whence  $c > b'$ . Applying again Lemma 6 shows that  $\lambda^{(d)} \triangleleft \mu^{(d)}$  and  $b' \leq c' < b \leq c$ . In particular, we have  $c' < b$ , whence  $\gamma_b < \gamma_{c'}$ . By Property  $(P_2)$ , we have  $\gamma_b = \phi(\beta_x + h)$  and  $\gamma_{c'} = \phi(\beta_y - h)$ . Lemma 6 then implies that  $\widehat{h} = \gamma_b - \delta_c = \phi(\beta_x + h) - \phi(\beta_x)$  and  $\widehat{h} = \delta_{b'} - \gamma_{c'} = \phi(\beta_y) - \phi(\beta_y - h)$ . ■

**Lemma 11** *Let  $\lambda_l, \mu_l \in \Pi_m^l$ , and let  $\lambda, \mu \in \Pi$  be such that  $\lambda_l \leftrightarrow \lambda$  and  $\mu_l \leftrightarrow \mu$ . Assume that  $j_{\lambda_l, \mu_l} \neq 0$ . Then we have, with the notation from  $(J_1)$ ,  $(J_2)$  and Section 3.1 the equalities*

$$N_i(\rho) = N_i(\rho') \quad (i \in \mathbb{Z}) \quad \text{and} \quad |\lambda| = |\mu|.$$

*Proof.* Since  $j_{\lambda_l, \mu_l}$  is nonzero, Lemma 7 shows that at least one of the following cases occurs:

- First case: we have  $\text{res}_n(\text{hd}(\rho)) = \text{res}_n(\text{hd}(\rho'))$ . Notice that if  $\rho$  is a ribbon, then the integers  $\text{cont}(\gamma), \gamma \in \rho$  are pairwise distinct, and the set formed by these numbers is exactly the interval  $[\text{cont}(\text{hd}(\rho)), \text{cont}(\text{tl}(\rho))]$ . Combining this with the assumption  $\text{res}_n(\text{hd}(\rho)) = \text{res}_n(\text{hd}(\rho'))$  and  $\widehat{h} = \ell(\rho) = \ell(\rho')$ , we get that  $N_i(\rho) = N_i(\rho')$  for any integer  $i \in \mathbb{Z}$ .
- Second case: we have  $\widehat{h} \equiv 0 \pmod{n}$ . Then for any  $i \in \mathbb{Z}$ , we have  $N_i(\rho) = N_i(\rho') = \widehat{h}/n$ .

Let us now show that  $|\lambda| = |\mu|$ . Let  $\nu_l := \lambda_l \cap \mu_l$ , and let  $\nu \in \Pi$  be such that  $\nu_l \leftrightarrow \nu$ . We claim that

$$h := |\lambda| - |\nu| = ((n-1)l + 1)N_0(\rho) + (\widehat{h} - N_0(\rho)).$$

By induction on  $\widehat{h}$ , we can restrict ourselves to the case when  $\widehat{h} = 1$ , that is  $\rho$  contains a single node  $\gamma$ . Let  $r \in \mathbb{N}$  be such that  $\lambda$  and  $\nu$  have at most  $r$  parts. By Lemma 5, there exist  $\alpha \in B_r(\nu)$  and  $\beta \in B_r(\lambda)$  such that  $B_r(\nu) \setminus \{\alpha\} = B_r(\lambda) \setminus \{\beta\}$  and  $\alpha = \beta - h$ . The abacus diagrams  $A(\nu, s)$  and  $A(\lambda, s)$  differ only by the moving of a bead; the same thing holds for the

diagrams  $A(\boldsymbol{\nu}_l, \mathbf{s}_l)$  and  $A(\boldsymbol{\lambda}_l, \mathbf{s}_l)$ . By considering the initial and the final positions of these two beads, we get that

$$\phi(\beta) = \phi(\alpha) + \widehat{h} = \phi(\alpha) + 1 \quad \text{and} \quad d(\beta) = d(\alpha) = d.$$

Moreover, by Lemma 5 and Property  $(P_1)$ , we have  $\text{res}_n(\gamma) = \phi(\alpha) \bmod n = \alpha \bmod n$ . Let us now distinguish two cases. If  $\text{res}_n(\gamma) = 0 \bmod n$  (i.e. if  $N_0(\rho) = 1$ ), then we have  $\alpha = n + n(d-1) + nlm$  with  $m \in \mathbb{Z}$ , whence  $\phi(\beta) = \phi(\alpha) + 1 = 1 + n(m+1)$ . Since  $d(\beta) = d$ , we get  $\beta = 1 + n(d-1) + nl(m+1)$ , whence  $h = \beta - \alpha = (n-1)l + 1$ . Similarly, if  $\text{res}_n(\gamma) \neq 0 \bmod n$  (i.e. if  $N_0(\rho) = 0$ ), then we have  $h = 1$ . This proves the claimed formula. In a similar way we prove that  $|\mu| - |\nu| = ((n-1)l + 1)N_0(\rho') + (\widehat{h} - N_0(\rho'))$ . Since  $N_0(\rho) = N_0(\rho')$ , we do have  $|\lambda| = |\mu|$ . ■

### 4.3 What happens if the multi-charge $\mathbf{s}_l$ is dominant

Recall the partial ordering  $\preceq$  on  $\Pi_m^l$  defined in Section 3.2.

**Lemma 12** *Let  $\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi_m^l$ , and  $\lambda, \mu \in \Pi$  be such that  $\boldsymbol{\lambda}_l \leftrightarrow \lambda$  and  $\boldsymbol{\mu}_l \leftrightarrow \mu$ . Assume that  $|\lambda| = |\mu|$ . Consider the following cases:*

- 1)  $(\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l)$  satisfies  $(J_1)$  and  $\mathbf{s}_l$  is dominant,
- 2)  $(\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l)$  satisfies  $(J_2)$ .

*Then in either case, we have*

$$\boldsymbol{\lambda}_l \prec \boldsymbol{\mu}_l \iff \boldsymbol{\lambda}_l \triangleleft \boldsymbol{\mu}_l.$$

*Proof.* In either case, we can apply Corollary 9 and then Lemma 6 to get that  $\boldsymbol{\lambda}_l \prec \boldsymbol{\mu}_l$  or  $\boldsymbol{\mu}_l \prec \boldsymbol{\lambda}_l$ . It is thus enough to prove the implication  $\boldsymbol{\lambda}_l \prec \boldsymbol{\mu}_l \Rightarrow \boldsymbol{\lambda}_l \triangleleft \boldsymbol{\mu}_l$ . Assume from now on that  $\boldsymbol{\lambda}_l \prec \boldsymbol{\mu}_l$  and  $(\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l)$  satisfies  $(J_1)$  or  $(J_2)$ . Then by Corollary 9,  $(\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l)$  satisfies  $(\mathcal{H})$ . Recall the notation from Section 4.2. If  $(\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l)$  satisfies  $(J_2)$ , then by Statement 3) of Lemma 10, we have  $\lambda^{(d)} \triangleleft \mu^{(d)}$ . Moreover, for all  $b \in \llbracket 1, l \rrbracket \setminus \{d\}$  we have  $\lambda^{(b)} = \mu^{(b)}$ , whence  $\boldsymbol{\lambda}_l \triangleleft \boldsymbol{\mu}_l$ . Assume now that  $(\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l)$  satisfies  $(J_1)$ . The key point of the proof is the following. Let  $\boldsymbol{\nu}_l = (\nu^{(1)}, \dots, \nu^{(l)}) \in \Pi_m^l$ , and  $\nu \in \Pi$  be the partition such that  $\boldsymbol{\nu}_l \leftrightarrow \nu$ . Then under the assumption that  $\mathbf{s}_l$  is dominant, we have

$$(*) \quad (d(k) < d(k'), k, k' \in B(\nu)) \Rightarrow \phi(k) > \phi(k').$$

Indeed, let  $k, k' \in B(\nu)$ ,  $b := d(k)$ ,  $b' := d(k')$  and  $N$  (resp.  $N'$ ) be the number of parts of  $\nu^{(b)}$  (resp.  $\nu^{(b')}$ ). Since  $\boldsymbol{\nu}_l \leftrightarrow \nu$ , we have  $\phi(k) \in B_N(\nu^{(b)})$  and  $\phi(k') \in B_{N'}(\nu^{(b')})$ . As a consequence, there exist  $i \in \llbracket 1, N \rrbracket$ ,  $i' \in \llbracket 1, N' \rrbracket$  such that  $\phi(k) = s_b + \nu_i^{(b)} - i + 1$  and  $\phi(k') = s_{b'} + \nu_{i'}^{(b')} - i' + 1$ . Since  $\mathbf{s}_l$  is dominant, we have

$$\begin{aligned} \phi(k) - \phi(k') &= (s_b - s_{b'}) + (i' + \nu_{i'}^{(b')}) - (i + \nu_i^{(b)}) \\ &> s_b - s_{b'} - (N + |\nu^{(b')}|) \geq s_b - s_{b'} - |\boldsymbol{\nu}_l| \\ &\geq 0, \end{aligned}$$

which shows (\*). Since  $(\lambda_l, \mu_l)$  satisfies  $(\mathcal{H})$ , Remark 4 implies  $\beta_x < \beta_y$ . Since  $\mathbf{s}_l$  is dominant, Statement (\*) and Property  $(P_3)$  imply  $d(\beta_x) \geq d(\beta_y)$ . By Statement 2) of Lemma 10, we have  $d(\beta_x) = d$  and  $d(\beta_y) = d'$ , whence  $d > d'$ . Moreover, we have  $|\mu^{(d')}| = |\lambda^{(d')}| + \widehat{h}$ ,  $|\mu^{(d)}| = |\lambda^{(d)}| - \widehat{h}$  and  $\mu^{(b)} = \lambda^{(b)}$  for all  $b \in \llbracket 1, l \rrbracket \setminus \{d, d'\}$ . This and the inequality  $d' < d$  imply  $\lambda_l \triangleleft \mu_l$ . ■

**Proposition 13** *Assume that  $\mathbf{s}_l$  is dominant. Then we have  $J^< = J^{\triangleleft}$ .*

*Proof.* Let  $\lambda_l, \mu_l \in \Pi_m^l$ , and  $\lambda, \mu \in \Pi$  be such that  $\lambda_l \leftrightarrow \lambda$  and  $\mu_l \leftrightarrow \mu$ . It is enough to prove the equivalence  $j_{\lambda_l, \mu_l}^< \neq 0 \Leftrightarrow j_{\lambda_l, \mu_l}^{\triangleleft} \neq 0$ , in which case we have by definition  $j_{\lambda_l, \mu_l}^< = j_{\lambda_l, \mu_l}^{\triangleleft} = j_{\lambda_l, \mu_l}$ . Assume that  $j_{\lambda_l, \mu_l}^{\triangleleft} \neq 0$ . Then we have  $\lambda_l \triangleleft \mu_l$ , and by Lemma 11, we have  $|\lambda| = |\mu|$ . Lemma 12 then implies that  $\lambda_l \prec \mu_l$ , whence  $j_{\lambda_l, \mu_l}^< = j_{\lambda_l, \mu_l} \neq 0$ . The converse is similar. ■

*Remark 5.* The reader should be warned that the orderings  $\prec$  and  $\triangleleft$  do not necessarily coincide, even if  $\mathbf{s}_l$  is dominant. For example, let  $n = 2$ ,  $l = 2$ ,  $m = 6$ ,  $\mathbf{s}_l = (3, -3)$ ,  $\lambda_l = ((2, 1), (1, 1, 1))$  and  $\mu_l = ((3), (2, 1))$ . Then we have  $\lambda_l \triangleleft \mu_l$ ; however, the partitions  $\lambda$  and  $\mu$  such that  $\lambda_l \leftrightarrow \lambda$  and  $\mu_l \leftrightarrow \mu$  are  $\lambda = (9, 6, 3, 1, 1, 1, 1, 1, 1, 1)$  and  $\mu = (10, 3, 3, 2, 2, 2, 1, 1, 1)$ , so  $\lambda_l$  and  $\mu_l$  are not comparable with respect to  $\preceq$ .

#### 4.4 Expression of the coefficients of $J^<$

Using Lemma 7, we now derive the expressions of the  $j_{\lambda_l, \mu_l}^<$ 's that we need for proving Theorem 4.

**Proposition 14** *Assume that  $(\lambda_l, \mu_l)$  satisfies  $(J_1)$  and  $\lambda_l \prec \mu_l$ . Then with the notation from Section 4.2, we have  $\delta > 0$ , and*

$$j_{\lambda_l, \mu_l}^< = \begin{cases} (-1)^{\text{ht}(\rho) + \text{ht}(\rho')} & \text{if } h \equiv \gamma \pmod{nl} \text{ or } h \equiv \delta \pmod{nl}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Recall the notation from Lemma 10. The statement  $\delta > 0$  comes from Statement 1) of that lemma. Applying Lemma 5 to the pairs  $(\lambda^{(d)}, \mu^{(d)})$  and  $(\lambda^{(d')}, \mu^{(d')})$  yields

$$\text{res}_n(\text{hd}(\rho)) = \delta_c \pmod{n} \quad \text{and} \quad \text{res}_n(\text{hd}(\rho')) = \gamma'_{c'} \pmod{n}.$$

By Lemma 7, it is thus enough to prove the equivalence

$$\delta_c \equiv \gamma'_{c'} \pmod{n} \iff (h \equiv \gamma \pmod{nl} \text{ or } h \equiv \delta \pmod{nl}).$$

Let us consider again both cases from Statement 2) of Lemma 10.

- First case: we have  $d(\beta_x) = d(\beta_x + h)$  and  $d(\beta_y) = d(\beta_y - h)$ . By Lemma 10 and Property  $(P_1)$ , we have the following equivalences, where congruences stand modulo  $n$ :

$$\delta_c \equiv \gamma'_{c'} \iff \phi(\beta_x) \equiv \phi(\beta_y - h) \iff \beta_x \equiv \beta_y - h \iff h \equiv \gamma.$$

It remains thus to prove that  $\delta_c \equiv \gamma'_{c'} \pmod{n} \implies h \equiv \gamma \pmod{nl}$ . Assume that  $\delta_c \equiv \gamma'_{c'} \pmod{n}$ ; we then have  $h \equiv \gamma \pmod{n}$ . This and the equality  $d(\beta_x) = d(\beta_x + h)$  force  $d(h) = 1$ , whence  $h \equiv \gamma \pmod{nl}$ .

- Second case: we have  $d(\beta_x) = d(\beta_y - h)$  and  $d(\beta_y) = d(\beta_x + h)$ . By arguing as above we prove the equivalence

$$\delta_c \equiv \gamma'_{c'} \pmod{n} \iff h \equiv 0 \pmod{n}.$$

Assume that  $\delta_c \equiv \gamma'_{c'} \pmod{n}$ . Then we have  $h \equiv 0 \pmod{n}$ , whence

$$d(\beta_y) = d(\beta_x + h) \equiv d(\beta_x) + d(h) - 1 \pmod{l},$$

and then  $n(d(h) - 1) \equiv n(d(\beta_y) - d(\beta_x)) \equiv \delta \pmod{nl}$ . We thus do have  $h \equiv \delta \pmod{nl}$ . ■

**Proposition 15** *Assume that  $(\lambda_l, \mu_l)$  satisfies  $(J_2)$  and  $\lambda_l \prec \mu_l$ . Then with the notation from Section 4.2, we have  $\delta = 0$  and*

$$j_{\lambda_l, \mu_l}^{\prec} = (-1)^{\text{ht}(\rho) + \text{ht}(\rho')} \varepsilon,$$

where

$$\varepsilon := \begin{cases} 1 & \text{if } h \equiv \gamma \pmod{nl} \text{ and } h \not\equiv 0 \pmod{nl}, \\ -1 & \text{if } h \not\equiv \gamma \pmod{nl} \text{ and } h \equiv 0 \pmod{nl}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Similar to the proof of the previous proposition. ■

## 5 Higher-level Fock spaces

In this section we follow [U2], to which we refer the reader for more details. The vector spaces we consider here are over  $\mathbf{K} = \mathbb{C}(q)$ , where  $q$  is an indeterminate over  $\mathbb{C}$ .

### 5.1 $q$ -wedge products and higher-level Fock spaces

Let  $\Lambda^s$  denote the (semi-infinite)  $q$ -wedge space of charge  $s$  (this space is denoted by  $\Lambda^{s+\frac{\infty}{2}}$  in [U2]).  $\Lambda^s$  is an integrable representation of level  $l$  of the quantum algebra  $U_q(\widehat{\mathfrak{sl}}_n)$ . As a



vector space, it has a natural basis formed by the so-called *ordered  $q$ -wedge products*. These vectors can be written as

$$u_{\mathbf{k}} = u_{k_1} \wedge u_{k_2} \wedge \cdots ,$$

where  $\mathbf{k} = (k_i)_{i \geq 1}$  is a decreasing sequence of integers such that  $k_i = s + 1 - i$  for  $i \gg 0$ . The basis formed by the ordered wedge products is called *standard*. More generally, we use the *non-ordered wedge products*; a non-ordered wedge product  $u_{\mathbf{k}} = u_{k_1} \wedge u_{k_2} \wedge \cdots \in \Lambda^s$  is indexed by a sequence of integers  $(k_i)$  such that  $k_i = s + 1 - i$  for  $i \gg 0$ , but we no longer require that  $(k_i)$  is decreasing. Any non-ordered wedge product can be written as a linear combination of ordered wedge products by using the so-called *ordering rules*, which are given in [U2, Proposition 3.16] and in a slightly different form in Section 5.4.

The vectors of the standard basis of  $\Lambda^s$  can also be indexed by partitions as follows. Let  $u_{\mathbf{k}} = u_{k_1} \wedge u_{k_2} \wedge \cdots \in \Lambda^s$  be an ordered wedge product. For  $i \geq 1$  we set  $\lambda_i := k_i - s + i - 1$ ; then  $\lambda := (\lambda_1, \lambda_2, \dots)$  is a partition, and we write  $u_{\mathbf{k}} = |\lambda, s\rangle$ . Notice that if  $\lambda$  has at most  $r$  parts, then we have  $(k_1, \dots, k_r) = \beta_r(\lambda)$ , which explains the definition of the  $\beta$ -numbers we gave in Section 3.3.

Let  $\mathbf{F}_q[\mathbf{s}_l]$  be the higher-level Fock space with multi-charge  $\mathbf{s}_l$  [U2]. As a vector space,  $\mathbf{F}_q[\mathbf{s}_l]$  has a natural basis  $\{|\lambda_l, \mathbf{s}_l\rangle, \lambda_l \in \Pi^l\}$  indexed by  $l$ -multi-partitions. Recall that we have  $s = s_1 + \cdots + s_l$ . Then  $\mathbf{F}_q[\mathbf{s}_l]$  can be identified with a subspace of  $\Lambda^s$  by the embedding  $\mathbf{F}_q[\mathbf{s}_l] \hookrightarrow \Lambda^s, |\lambda_l, \mathbf{s}_l\rangle \mapsto |\lambda, s\rangle$ , where  $\lambda$  is the partition such that  $\lambda_l \leftrightarrow \lambda$  (see Section 3.2 for the notation  $\leftrightarrow$ ). We make from now on this identification; in fact,  $\Lambda^s$  is isomorphic as a vector space to the direct sum of all the  $\mathbf{F}_q[\mathbf{t}_l]$ 's, where  $\mathbf{t}_l$  is any  $l$ -tuple of integers summing to  $s$ . Thus, the vectors of the standard basis of  $\Lambda^s$  can also be indexed by charged  $l$ -multi-partitions.

## 5.2 The involution $-$

In order to define the canonical basis of  $\Lambda^s$ , we equip this space with an involution, denoted by  $-$ . This map is the  $\mathbb{C}$ -vector space automorphism that maps  $q$  to  $q^{-1}$  and that acts on the standard basis of  $\Lambda^s$  as follows [U2, Proposition 3.23 & Remark 3.24]. Let  $\lambda \in \Pi$  be a partition of  $r$ , and  $\mathbf{k} = (k_i) \in \mathbb{Z}^{\mathbb{N}^*}$  be such that  $u_{\mathbf{k}} = |\lambda, s\rangle$ . Then we have

$$\overline{|\lambda, s\rangle} = (-1)^{\kappa(\mathbf{d}(\lambda))} q^{\kappa(\mathbf{d}(\lambda)) - \kappa(\mathbf{c}(\lambda))} (u_{k_r} \wedge \cdots \wedge u_{k_1}) \wedge u_{k_{r+1}} \wedge u_{k_{r+2}} \wedge \cdots ,$$

where for any  $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r$ ,  $\kappa(\mathbf{a})$  is the integer defined by

$$\kappa(\mathbf{a}) := \#\{(i, j) \in \mathbb{N}^2 \mid 1 \leq i < j \leq r, a_i = a_j\} .$$

We can then straighten the non-ordered wedge product in the right-hand side of the above expression in order to express it as a linear combination of ordered wedge products.

One checks that  $-$  preserves the subspace

$$\mathbf{F}_q[\mathbf{s}_l]_m := \bigoplus_{\lambda_l \in \Pi_m^l} \mathbf{K} |\lambda_l, \mathbf{s}_l\rangle \subset \mathbf{F}_q[\mathbf{s}_l].$$

We can then define a matrix  $A(q) = (a_{\lambda_l, \mu_l}(q))_{\lambda_l, \mu_l \in \Pi_m^l}$  with coefficients in  $\mathbf{K}$  by

$$\overline{|\mu_l, \mathbf{s}_l\rangle} = \sum_{\lambda_l \in \Pi_m^l} a_{\lambda_l, \mu_l}(q) |\lambda_l, \mathbf{s}_l\rangle \quad (\mu_l \in \Pi_m^l);$$

the matrix  $A(q)$  depends on  $n, l, \mathbf{s}_l$  and  $m$ . The ordering rules show that  $A(q)$  is lower-unitriangular, that is  $a_{\lambda_l, \mu_l}(q) \neq 0 \Rightarrow \mu_l \succeq \lambda_l$  and  $a_{\lambda_l, \lambda_l}(q) = 1$  for all pairs of multi-partitions  $(\lambda_l, \mu_l) \in (\Pi_m^l)^2$ . The same rules also imply that  $A(1)$  is the identity matrix.

### 5.3 Uglov's canonical basis

Since the matrix  $A(q)$  of the involution of  $\mathbf{F}_q[\mathbf{s}_l]_m$  is unitriangular, a classical argument can be used to prove the following result [U2].

**Theorem 16** *There exists a unique basis  $\{\mathcal{G}(\lambda_l, \mathbf{s}_l), \lambda_l \in \Pi_m^l\}$  of  $\mathbf{F}_q[\mathbf{s}_l]_m$  satisfying both following conditions:*

- (i)  $\overline{\mathcal{G}(\lambda_l, \mathbf{s}_l)} = \mathcal{G}(\lambda_l, \mathbf{s}_l)$ ,
- (ii)  $\mathcal{G}(\lambda_l, \mathbf{s}_l) - |\lambda_l, \mathbf{s}_l\rangle \in \bigoplus_{\mu_l \in \Pi_m^l} q \mathbb{C}[q] |\mu_l, \mathbf{s}_l\rangle$ .

The basis  $\{\mathcal{G}(\lambda_l, \mathbf{s}_l), \lambda_l \in \Pi_m^l\}$  is called the *canonical basis* of  $\mathbf{F}_q[\mathbf{s}_l]_m$ . We define a matrix  $\Delta(q) = (\Delta_{\lambda_l, \mu_l}(q))_{\lambda_l, \mu_l \in \Pi_m^l}$  with coefficients in  $\mathbb{C}[q]$  by

$$\mathcal{G}(\mu_l, \mathbf{s}_l) = \sum_{\lambda_l \in \Pi_m^l} \Delta_{\lambda_l, \mu_l}(q) |\lambda_l, \mathbf{s}_l\rangle \quad (\mu_l \in \Pi_m^l);$$

the matrix  $\Delta(q)$  depends on  $n, l, \mathbf{s}_l$  and  $m$ . Owing to the condition (ii) written above, the matrix  $\Delta(q)$  is also lower-unitriangular. By [U2, Theorem 3.26], the coefficients of  $\Delta(q)$  can be expressed as Kazhdan-Lusztig polynomials related to a parabolic module of an affine Hecke algebra of type  $\tilde{A}$ , so by [KT], these coefficients belong to  $\mathbb{N}[q]$ .

### 5.4 Another basis of $\Lambda^s$ . Ordering rules.

The ordering rules  $(R_1)$ – $(R_4)$  from [U2, Proposition 3.16] do not give at  $q = 1$  anticommuting relations like  $u_{k_1} \wedge u_{k_2} = -u_{k_2} \wedge u_{k_1}$ , because of the signs involved in Rules  $(R_3)$  and  $(R_4)$ . To fix this, we introduce another basis of  $\Lambda^s$  that differs from the standard basis only by

signs. The basis we consider here is actually the basis of ordered wedge products introduced in [U1].  $\Lambda^s$  is graded by

$$\deg(|\lambda, s\rangle) := |\lambda| \quad (\lambda \in \Pi).$$

Let  $u_{\mathbf{k}} = u_{k_1} \wedge u_{k_2} \wedge \cdots \in \Lambda^s$  be a (not necessarily ordered) wedge product of degree  $r$ . We set

$$\begin{aligned} v_{\mathbf{k}} &= v_{k_1} \wedge v_{k_2} \wedge \cdots := (-1)^{\ell(\mathbf{v}(k_1, \dots, k_r))} u_{\mathbf{k}} \\ \text{and similarly} \quad v_{k_1} \wedge \cdots \wedge v_{k_r} &:= (-1)^{\ell(\mathbf{v}(k_1, \dots, k_r))} u_{k_1} \wedge \cdots \wedge u_{k_r}, \end{aligned}$$

where  $\mathbf{v}(k_1, \dots, k_r)$  is defined in Section 3.2. We say that the wedge product  $v_{\mathbf{k}}$  is *ordered* if so is  $u_{\mathbf{k}}$ . It is straightforward to see, using the ordering rules for the  $u_{\mathbf{k}}$ 's given by [U2], that the ordering rules for the  $v_{\mathbf{k}}$ 's are given by the next proposition.

**Proposition 17**

(i) Let  $k_1 \leq k_2$ , and  $\gamma \in \llbracket 0, nl - 1 \rrbracket$  (resp.  $\delta \in \llbracket 0, nl - 1 \rrbracket$ ) denote the residue of  $c(k_2) - c(k_1)$  (resp. of  $n(d(k_2) - d(k_1))$ ) modulo  $nl$ . Then we have

$$(R_1) \quad v_{k_1} \wedge v_{k_2} = -v_{k_2} \wedge v_{k_1} \quad \text{if } \gamma = \delta = 0,$$

$$\begin{aligned} v_{k_1} \wedge v_{k_2} &= -q^{-1} v_{k_2} \wedge v_{k_1} \\ (R_2) \quad &-(q^{-2} - 1) \sum_{i \geq 1} q^{-2i+1} v_{k_2-nli} \wedge v_{k_1+nli} \quad \text{if } \gamma > 0, \delta = 0, \\ &+(q^{-2} - 1) \sum_{i \geq 0} q^{-2i} v_{k_2-\gamma-nli} \wedge v_{k_1+\gamma+nli} \end{aligned}$$

$$\begin{aligned} v_{k_1} \wedge v_{k_2} &= -q v_{k_2} \wedge v_{k_1} \\ (R_3) \quad &-(q^2 - 1) \sum_{i \geq 1} q^{2i-1} v_{k_2-nli} \wedge v_{k_1+nli} \quad \text{if } \gamma = 0, \delta > 0, \\ &+(q^2 - 1) \sum_{i \geq 0} q^{2i} v_{k_2-\delta-nli} \wedge v_{k_1+\delta+nli} \end{aligned}$$

(R<sub>4</sub>)

$$\begin{aligned}
v_{k_1} \wedge v_{k_2} &= -v_{k_2} \wedge v_{k_1} \\
&- (q - q^{-1}) \sum_{i \geq 1} \frac{q^{2i} - q^{-2i}}{q + q^{-1}} v_{k_2 - nli} \wedge v_{k_1 + nli} \\
&- (q - q^{-1}) \sum_{i \geq 0} \frac{q^{2i+1} + q^{-2i-1}}{q + q^{-1}} v_{k_2 - \gamma - nli} \wedge v_{k_1 + \gamma + nli} \\
&+ (q - q^{-1}) \sum_{i \geq 0} \frac{q^{2i+1} + q^{-2i-1}}{q + q^{-1}} v_{k_2 - \delta - nli} \wedge v_{k_1 + \delta + nli} \\
&+ (q - q^{-1}) \sum_{i \geq 0} \frac{q^{2i+2} - q^{-2i-2}}{q + q^{-1}} v_{k_2 - \gamma - \delta - nli} \wedge v_{k_1 + \gamma + \delta + nli}
\end{aligned}$$

if  $\gamma > 0, \delta > 0$ ,

where the sums range over the indices  $i$  such that the corresponding wedge products are ordered.

(ii) The rules from (i) are valid for any pair of adjacent factors of the wedge product  $v_{\mathbf{k}} = v_{k_1} \wedge v_{k_2} \cdots$ .

Let us end this section by a useful piece of notation. Let  $\nu \in \Pi$  be a partition of  $r$  and  $\sigma \in \mathfrak{S}_r$ . We set  $v_{\sigma, \nu} := v_{\sigma, \beta(\nu)}$  (it is a wedge product of  $r$  factors), and we define in a similar way  $u_{\sigma, \nu}$ . We say that  $v_{\sigma, \nu}$  is obtained from  $v_\nu$  by permutation.

## 6 Admissible sequences, good sequences

### 6.1 Definitions

Let us fix throughout this section an integer  $r \geq 1$ . Let  $\mathbf{k} \in \mathbb{Z}^r$  be a (not necessarily decreasing) sequence of  $r$  integers. We examine in detail the straightening of the wedge product  $v_{\mathbf{k}} = v_{k_1} \wedge \cdots \wedge v_{k_r}$ . If  $v_{\mathbf{k}}$  is not ordered, there are in general several ways to straighten it by applying recursively the rules (R<sub>1</sub>)-(R<sub>4</sub>). In the sequel, we decide to straighten at each step the *first* infraction that occurs in  $v_{\mathbf{k}}$ , that is, the first  $v_{k_i} \wedge v_{k_{i+1}}$  with  $k_i \leq k_{i+1}$ . This leads to the notion of admissible sequence that we now introduce.

Let  $\mathbf{k} = (k_1, \dots, k_r)$  and  $\mathbf{l} = (l_1, \dots, l_r) \in \mathbb{Z}^r$ . We say that the wedge products  $v_{\mathbf{k}}$  and  $v_{\mathbf{l}}$  are *adjacent* if there exist  $1 \leq i \leq r-1$  ( $i$  is then necessarily unique) such that:

- (i)  $k_i \leq k_{i+1}$ , and  $k_j > k_{j+1}$  for all  $1 \leq j \leq i-1$ ,
- (ii)  $k_j = l_j$  for all  $j \in \llbracket 1, r \rrbracket \setminus \{i, i+1\}$ ,

(iii) the wedge product  $v_{l_i} \wedge v_{l_{i+1}}$  appears in the straightening of  $v_{k_i} \wedge v_{k_{i+1}}$ .

In this case, denote by  $t \in \llbracket 1, 4 \rrbracket$  the index of the rule  $(R_t)$  applied for the straightening of  $v_{k_i} \wedge v_{k_{i+1}}$  and by  $\alpha(v_{\mathbf{k}}, v_1) \in \mathbb{Z}[q, q^{-1}]$  the coefficient of  $v_{l_i} \wedge v_{l_{i+1}}$  in the resulting linear combination. If we have  $(l_i, l_{i+1}) = (k_{i+1}, k_i)$ , then we write  $v_{\mathbf{k}} \xrightarrow{\bullet} v_1$  and we set  $m(v_{\mathbf{k}}, v_1) := 0$ . Otherwise, we write  $v_{\mathbf{k}} \xrightarrow{t} v_1$  and we set  $m(v_{\mathbf{k}}, v_1) := 1$ ; notice that  $t \geq 2$  in this case. In either case we sometimes write more simply  $v_{\mathbf{k}} \rightarrow v_1$ .

We say that the sequence  $\mathbf{V} = (v_{\mathbf{k}_i})_{0 \leq i \leq N}$  is *admissible* if each  $v_{\mathbf{k}_i}$  is a wedge product of  $r$  factors and if we have

$$v_{\mathbf{k}_0} \rightarrow v_{\mathbf{k}_1} \rightarrow \cdots \rightarrow v_{\mathbf{k}_N} ;$$

in this case,  $N$  is called the *length* of the sequence  $\mathbf{V}$  and we set

$$\alpha_{\mathbf{V}}(q) := \prod_{i=1}^N \alpha(v_{\mathbf{k}_{i-1}}, v_{\mathbf{k}_i}) \in \mathbb{Z}[q, q^{-1}] \quad \text{and} \quad m(\mathbf{V}) := \sum_{i=1}^N m(v_{\mathbf{k}_{i-1}}, v_{\mathbf{k}_i}) \in \mathbb{N} .$$

*Remark 6.* The condition (i) from the definition of adjacent wedge products shows that there exists at most one admissible sequence  $\mathbf{V}$  having a given length and starting at a given wedge product such that  $m(\mathbf{V}) = 0$ .

Let  $\lambda, \mu \in \Pi$  be two partitions of  $r$ , and recall the definition of  $\omega \in \mathfrak{S}_r$ . We say that the sequence of wedge products  $\mathbf{V} = (v_{\mathbf{k}_i})_{0 \leq i \leq N}$  is  $(\lambda, \mu)$ -*admissible* if it is an admissible sequence of wedge products (of  $r$  factors for each of them) such that  $v_{\mathbf{k}_N} = v_{\lambda}$  and  $v_{\mathbf{k}_0} = v_{\omega \cdot \mu}$ .

*Remark 7.* It is easy to see that if  $\lambda \neq \mu$ , then there cannot exist any  $(\lambda, \mu)$ -admissible sequence  $\mathbf{V} = (v_{\mathbf{k}_i})_{0 \leq i \leq N}$  such that  $m(\mathbf{V}) = 0$ .

A  $(\lambda, \mu)$ -admissible sequence  $\mathbf{V}$  such that  $m(\mathbf{V}) = 1$  is called a *good* sequence (with respect to  $(\lambda, \mu)$ ). Such a sequence can be written as

$$v_{\omega \cdot \mu} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} \mathbf{u} \xrightarrow{t} \mathbf{v} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_{\lambda},$$

with  $t \in \llbracket 2, 4 \rrbracket$ .

*Remark 8.* If a good sequence (with respect to  $(\lambda, \mu)$ ) exists, then we have

$$\sharp(B(\lambda) \cap B(\mu)) = r - 2.$$

## 6.2 Reduction to the good sequences

Expressing the involution of  $\mathbf{F}_q[\mathbf{s}_l]_m$  in terms of the  $v_{\mathbf{k}}$ 's and then using the definition of  $(\lambda, \mu)$ -admissible sequences yields the following expression for the coefficients of  $A(q)$ .

**Lemma 18** *Let  $\lambda_l, \mu_l \in \Pi_m^l$  and  $\lambda, \mu \in \Pi$  be the partitions such that  $\lambda_l \leftrightarrow \lambda$  and  $\mu_l \leftrightarrow \mu$ . Assume that  $|\lambda| = |\mu|$ . Then we have*

$$a_{\lambda_l, \mu_l}(q) = \varepsilon(\lambda_l, \mu_l) q^{\kappa(\mathbf{d}(\mu)) - \kappa(\mathbf{c}(\mu))} \sum_{\mathbf{V}} \alpha_{\mathbf{V}}(q),$$

where the sum ranges over all  $(\lambda, \mu)$ -admissible sequences  $\mathbf{V}$ , and  $\varepsilon(\lambda_l, \mu_l)$  is the sign defined by

$$\varepsilon(\lambda_l, \mu_l) := (-1)^{\kappa(\mathbf{d}(\mu)) + \ell(\mathbf{v}(\omega, \mu)) + \ell(\mathbf{v}(\lambda))}.$$

*Remark 9.* One should be aware that in general several terms might contribute to this sum, so the statement at the beginning of [RH, Section 4] is not correct. However, we can fix the argument from [RH] by showing first that only good sequences do contribute to  $a'_{\lambda_l, \mu_l}(1)$  (see next lemma), and then that there exists at most one good sequence (see Propositions 21 and 22 from next section).

**Lemma 19** *Let  $\lambda_l, \mu_l \in \Pi_m^l$  be two distinct multi-partitions of  $m$ , and  $\lambda, \mu \in \Pi$  be the partitions such that  $\lambda_l \leftrightarrow \lambda$  and  $\mu_l \leftrightarrow \mu$ . Assume that  $|\lambda| = |\mu| = r$ . Then we have*

$$a'_{\lambda_l, \mu_l}(1) = \varepsilon(\lambda_l, \mu_l) \sum_{\mathbf{V}} \alpha'_{\mathbf{V}}(1),$$

where the sum ranges over all the good sequences with respect to  $(\lambda, \mu)$ , and  $\varepsilon(\lambda_l, \mu_l)$  is the sign defined in Lemma 18.

*Proof.* By the previous lemma, we have

$$a_{\lambda_l, \mu_l}(q) = \varepsilon(\lambda_l, \mu_l) \sum_{\mathbf{V}} f_{\mathbf{V}}(q),$$

where the sum ranges over all the  $(\lambda, \mu)$ -admissible sequences  $\mathbf{V}$  and  $f_{\mathbf{V}}(q)$  is the Laurent polynomial defined by  $f_{\mathbf{V}}(q) := q^{\kappa(\mathbf{d}(\mu)) - \kappa(\mathbf{c}(\mu))} \alpha_{\mathbf{V}}(q)$ . Notice that if  $\mathbf{V}$  is an admissible sequence, then we have  $m(\mathbf{V}) \geq 1$  (because  $\lambda \neq \mu$ ), and moreover the rules  $(R_1)$ – $(R_4)$  imply that

$$\alpha_{\mathbf{V}}(q) \in (q^2 - 1)^{m(\mathbf{V})} \mathbb{Z}[q, q^{-1}].$$

As a consequence, if  $\mathbf{V}$  is a  $(\lambda, \mu)$ -admissible sequence such that  $m(\mathbf{V}) \geq 2$ , then  $(q^2 - 1)^2$  divides  $f_{\mathbf{V}}(q)$  in  $\mathbb{Z}[q, q^{-1}]$ , whence  $f'_{\mathbf{V}}(1) = 0$ . Moreover, if  $\mathbf{V}$  is a good sequence, then the previous discussion shows that  $\alpha_{\mathbf{V}}(1) = 0$ , whence  $f'_{\mathbf{V}}(1) = (\kappa(\mathbf{d}(\mu)) - \kappa(\mathbf{c}(\mu))) \alpha_{\mathbf{V}}(1) + \alpha'_{\mathbf{V}}(1) = \alpha'_{\mathbf{V}}(1)$ . ■

### 6.3 Existence and uniqueness of the good sequence

We first give (see Proposition 21) some sufficient conditions for the existence of a good sequence (with respect to a given pair  $(\lambda, \mu)$ ). In order to do this, we must study the sequence of permutations that we apply to the components of wedge products when we go through an admissible sequence

$$v_{\mathbf{k}_0} \xrightarrow{\bullet} v_{\mathbf{k}_1} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_{\mathbf{k}_N},$$

where  $k_0, \dots, k_N \in \mathbb{Z}^r$  and  $v_{\omega, \mathbf{k}_0}$  is ordered.

Let  $\sigma, \tau \in \mathfrak{S}_r$ . Write  $\sigma \rightarrow \tau$  if there exists  $1 \leq i \leq r-1$  such that  $\sigma(i) < \sigma(i+1)$ , with  $i$  minimal for this property, and such that  $\tau = s_i \sigma$ . The relation  $\rightarrow$  on  $\mathfrak{S}_r$  is closely related to the relation  $\xrightarrow{\bullet}$  on wedge products of  $r$  factors. Recall that for  $v_{\mathbf{k}} = v_{k_1} \wedge \cdots \wedge v_{k_r}$  and  $\sigma \in \mathfrak{S}_r$ , we have  $v_{\sigma, \mathbf{k}} = v_{k_{\sigma^{-1}(1)}} \wedge \cdots \wedge v_{k_{\sigma^{-1}(r)}}$ . Then by definition we have  $\sigma \rightarrow \tau$  if and only if  $v_{\sigma^{-1}, \mathbf{k}} \xrightarrow{\bullet} v_{\tau^{-1}, \mathbf{k}}$ , where  $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$  is such that  $v_{\omega, \mathbf{k}}$  is ordered (namely,  $k_1 < \cdots < k_r$ ).

Consider now the following reduced expression for the longest element in  $\mathfrak{S}_r$  :

$$\omega = (s_1 s_2 \cdots s_{r-1})(s_1 s_2 \cdots s_{r-2}) \cdots (s_1 s_2)(s_1),$$

and for  $0 \leq i \leq \frac{r(r-1)}{2}$  let  $\omega[i]$  denote the right factor of length  $i$  in this word (by convention, we have  $\omega[0] = id$ ). For example, for  $r \geq 3$  we have  $\omega[5] = s_2 s_3 s_1 s_2 s_1$ . The sequence  $(\omega[i])_{0 \leq i \leq \frac{r(r-1)}{2}}$  enjoys the following property: if  $id = \sigma_0 \rightarrow \sigma_1 \rightarrow \cdots \rightarrow \sigma_k$  with  $0 \leq k \leq \frac{r(r-1)}{2}$ , then we have for all  $0 \leq i \leq k$ ,  $\sigma_i = \omega[i]$ . In particular, we have  $\omega[i-1] \rightarrow \omega[i]$  for all  $1 \leq i \leq \frac{r(r-1)}{2}$ .

**Lemma 20** *Let  $i, j \in \llbracket 1, r \rrbracket$  be such that  $i < j$ . Then there exist two integers  $k \in \llbracket 1, r-1 \rrbracket$  and  $e \in \llbracket 0, \frac{r(r-1)}{2} - 1 \rrbracket$ , determined in a unique way by the following properties:  $(\omega[e])(k) = i$ ,  $(\omega[e])(k+1) = j$  and  $\omega[e+1] = s_k \omega[e]$ . Namely, we have  $k = j-i$  and  $e = \frac{(j-1)(j-2)}{2} + (i-1)$ .*

*Proof.* Left to the reader. ■

*Example 8.* Take  $r = 6$ ,  $i = 2$  and  $j = 5$ . Then we have

$$\begin{pmatrix} 1,2,3,4,5,6 \\ 1,2,3,4,5,6 \end{pmatrix} \rightarrow \begin{pmatrix} 1,2,3,4,5,6 \\ 2,1,3,4,5,6 \end{pmatrix} \rightarrow \begin{pmatrix} 1,2,3,4,5,6 \\ 2,3,1,4,5,6 \end{pmatrix} \rightarrow \begin{pmatrix} 1,2,3,4,5,6 \\ 3,2,1,4,5,6 \end{pmatrix} \rightarrow \begin{pmatrix} 1,2,3,4,5,6 \\ 3,2,4,1,5,6 \end{pmatrix} \rightarrow \begin{pmatrix} 1,2,3,4,5,6 \\ 3,4,2,1,5,6 \end{pmatrix} \rightarrow \begin{pmatrix} 1,2,3,4,5,6 \\ 4,3,2,1,5,6 \end{pmatrix} = \omega[e],$$

whence  $e = 7 = \frac{(j-1)(j-2)}{2} + (i-1)$  and  $k = 3 = j - i$ .

**Proposition 21** *Assume that  $(\lambda_l, \mu_l)$  satisfies  $(\mathcal{H})$ . Recall the notation from Section 4.2. Assume moreover that  $(\gamma, \delta) \neq (0, 0)$  and  $h \equiv \eta \pmod{nl}$ , with  $\eta \in \{0, \gamma, \delta, \gamma + \delta\}$ . Then there exists a good sequence with respect to  $(\lambda, \mu)$ .*

*Proof.* Let  $e$  and  $k = x - y$  be the integers given by the previous lemma applied with the integers  $i = r + 1 - x$  and  $j = r + 1 - y$ . Then we have the sequence  $\omega[0] \rightarrow \cdots \rightarrow \omega[e]$ , hence the sequence  $v_{\omega[0]^{-1} \cdot (\omega, \mu)} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_{\omega[e]^{-1} \cdot (\omega, \mu)}$  is admissible. Moreover, by assumption on  $e$  we have

$$\mathbf{u} := v_{\omega[e]^{-1} \cdot (\omega, \mu)} = v_{(\omega\omega[e]^{-1}) \cdot \mu} = v_{m_1} \wedge \cdots \wedge v_{m_{k-1}} \wedge v_{\beta_x} \wedge v_{\beta_y} \wedge v_{m_{k+2}} \wedge \cdots \wedge v_{m_r},$$

where the  $m_i$ 's are integers belonging to  $B(\mu)$ , and the next step of the straightening of  $\mathbf{u}$  consists in straightening this wedge product with respect to its  $k$ -th and  $(k+1)$ -th components, namely  $v_{\beta_x} \wedge v_{\beta_y}$ . Due to the condition  $(\gamma, \delta) \neq (0, 0)$ , this elementary straightening involves Rule  $(R_t)$  with  $t \in \llbracket 2, 4 \rrbracket$ . We then get a linear combination of wedge products whose  $k$ -th and  $(k+1)$ -th components are of the form  $v_{\beta_y - nli} \wedge v_{\beta_x + nli}$ , *resp.*  $v_{\beta_y - \gamma - nli} \wedge v_{\beta_x + \gamma + nli}$ , *resp.*  $v_{\beta_y - \delta - nli} \wedge v_{\beta_x + \delta + nli}$ , *resp.*  $v_{\beta_y - \gamma - \delta - nli} \wedge v_{\beta_x + \gamma + \delta + nli}$  with  $i \in \mathbb{N}$ . Since  $h \equiv \eta \pmod{nl}$ , we can choose  $i$  such that  $h = nli$ , *resp.*  $h = nli + \gamma$ , *resp.*  $h = nli + \delta$ , *resp.*  $h = nli + \gamma + \delta$ . According to the expression of  $\beta(\lambda)$ , we see that the wedge product corresponding to this choice of  $i$ , which is actually

$$\mathbf{v} := v_{m_1} \wedge \cdots \wedge v_{m_{k-1}} \wedge v_{\beta_y - h} \wedge v_{\beta_x + h} \wedge v_{m_{k+2}} \wedge \cdots \wedge v_{m_r},$$

is obtained from  $v_\lambda$  by permutation. Set  $\mathbf{v}_1 := \mathbf{v}$ . If  $\mathbf{v}_1$  is not ordered, then the elementary straightening of  $\mathbf{v}_1$  gives a linear combination of wedge products, and one of them, say  $\mathbf{v}_2$ , is obtained from  $\mathbf{v}_1$  by permutation. If we apply this device sufficiently many times, we get eventually an ordered wedge product which is of course  $v_\lambda$ . As a consequence, there exists an admissible sequence  $\mathbf{v} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} \mathbf{v}_z = v_\lambda$ , and the sequence

$$v_{\omega, \mu} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} \mathbf{u} \xrightarrow{t} \mathbf{v} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_\lambda$$

constructed this way is a good sequence. ■

We now prove the converse of Proposition 21.

**Proposition 22** *Assume that  $(\lambda_l, \mu_l)$  satisfies  $(\mathcal{H})$ . Recall the notation from Section 4.2. Assume moreover that there exists a good sequence*

$$v_{\omega, \mu} = v_{\mathbf{k}_0} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_{\mathbf{k}_e} \xrightarrow{t} v_{\mathbf{k}_{e+1}} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_{\mathbf{k}_N} = v_\lambda$$

*with respect to  $(\lambda, \mu)$ , with  $t \in \llbracket 2, 4 \rrbracket$ . Then this sequence is unique,  $t$  is also uniquely determined and moreover we have  $(\gamma, \delta) \neq (0, 0)$  and  $h \equiv \eta \pmod{nl}$  with  $\eta \in \{0, \gamma, \delta, \gamma + \delta\}$ .*

*Proof.* By assumption,  $v_{\mathbf{k}_e}$  (*resp.*  $v_{\mathbf{k}_{e+1}}$ ) is obtained from  $v_\mu$  (*resp.*  $v_\lambda$ ) by permutation. Since  $\{\beta_x, \beta_y\} \cap B(\lambda) = \emptyset$ , there exist  $\sigma \in \mathfrak{S}_r$  and  $1 \leq k \leq r - 1$  such that

$$v_{\mathbf{k}_e} = v_{\sigma, \mu} = v_{\beta_{\sigma^{-1}(1)}} \wedge \cdots \wedge v_{\beta_{\sigma^{-1}(k-1)}} \wedge v_{\beta_x} \wedge v_{\beta_y} \wedge v_{\beta_{\sigma^{-1}(k+2)}} \wedge \cdots \wedge v_{\beta_{\sigma^{-1}(r)}},$$

$$v_{\mathbf{k}_{e+1}} = v_{\beta_{\sigma^{-1}(1)}} \wedge \cdots \wedge v_{\beta_{\sigma^{-1}(k-1)}} \wedge v_{\beta_y - h} \wedge v_{\beta_x + h} \wedge v_{\beta_{\sigma^{-1}(k+2)}} \wedge \cdots \wedge v_{\beta_{\sigma^{-1}(r)}},$$



and  $v_{\mathbf{k}_{e+1}}$  is obtained from  $v_{\mathbf{k}_e}$  by straightening  $v_{\beta_x} \wedge v_{\beta_y}$  with the rule  $(R_t)$ . Since  $t \in \llbracket 2, 4 \rrbracket$ , the last conditions of the statement of this proposition hold. Moreover,  $t$  is uniquely determined by considering whether  $\gamma$  and  $\delta$  are zero or not. It is not hard to see that  $k$  and  $e$  satisfy the conditions of Lemma 20 with  $i := r + 1 - x$  and  $j := r + 1 - y$ , so  $k$  and  $e$  are also determined in a unique way. This determines completely the subsequence  $v_{\mathbf{k}_0} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_{\mathbf{k}_e}$ . The expression of  $v_{\mathbf{k}_{e+1}}$  given at the beginning of the proof shows that the wedge product  $v_{\mathbf{k}_{e+1}}$  is also determined in a unique way. Let  $\tau \in \mathfrak{S}_r$  be the unique permutation such that  $v_{\mathbf{k}_{e+1}} = v_{\tau, \lambda}$ . By Remark 6, the sequence  $v_{\tau, \lambda} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_\lambda$ , whose length is  $\ell(\tau)$ , is in turn determined in a unique way. ■

## 7 Computation of the length modulo 2 of the good sequence

We now deal with the technical part of the proof of Theorem 4. The next proposition will be used to show that if  $a'_{\lambda_l, \mu_l}(1)$  and  $j_{\lambda_l, \mu_l}^<$  are nonzero, then both numbers have the same signs. This proposition deals with the only cases that we have to consider.

**Proposition 23** *Assume that  $(\lambda_l, \mu_l)$  satisfies  $(\mathcal{H})$ . Recall the notation from Section 4.2. Assume moreover that  $(\gamma, \delta) \neq (0, 0)$  and  $h \equiv \eta \pmod{nl}$  with  $\eta \in \{\gamma, \delta\}$ . Let then*

$$\mathbf{V} = v_{\omega, \mu} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_{\mathbf{k}} \xrightarrow{t} v_1 \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_\lambda$$

*denote the unique good sequence with respect to  $(\lambda, \mu)$  (see Propositions 21 and 22). Denote by  $N$  the length of this sequence. Then we have*

$$(-1)^{N-1} = (-1)^{\text{ht}(\rho) + \text{ht}(\rho')} \varepsilon(\lambda_l, \mu_l) \varepsilon,$$

*where  $\varepsilon(\lambda_l, \mu_l)$  is the sign defined in Lemma 18 and  $\varepsilon$  is the sign defined by*

$$\varepsilon := \begin{cases} 1 & \text{if } \delta > 0 \text{ and } h \equiv \delta \pmod{nl}, \\ -1 & \text{otherwise.} \end{cases}$$

*Proof.* Let  $\sigma, \tau \in \mathfrak{S}_r$  be the permutations defined by  $v_{\sigma\omega, \mu} = v_{\mathbf{k}}$  and  $v_{\tau^{-1}, \lambda} = v_1$ . Then we have  $N = \ell(\sigma) + 1 + \ell(\tau)$ , whence  $(-1)^{N-1} = \varepsilon(\sigma)\varepsilon(\tau)$ . By Lemma 20, we can compute  $\ell(\sigma)$  and then  $\varepsilon(\sigma)$ ; it is however not straightforward to compute  $\ell(\tau)$ . We compute only  $\varepsilon(\tau)$  by writing  $\tau$  as a product of 7 permutations  $\sigma_1, \dots, \sigma_7$  whose signs are easily computable.

\* Set first

$$\sigma_1 := \sigma^{-1} \quad \text{and} \quad \mathbf{v}_1 := v_{\sigma_1, 1};$$

$\mathbf{v}_1$  is thus the wedge product obtained from  $v_{\omega, \mu}$  by replacing  $\beta_x$  (located at the  $(r + 1 - x)$ -th component) by  $\beta_y - h$ , and  $\beta_y$  (located at the  $(r + 1 - y)$ -th component) by  $\beta_x + h$ .

\* By Lemma 10, one of the following cases occurs:

- First case:  $d(\beta_x) = d(\beta_y - h)$  and  $d(\beta_y) = d(\beta_x + h)$ ,
- Second case:  $d(\beta_x) = d(\beta_x + h)$ ,  $d(\beta_y) = d(\beta_y - h)$  and  $d(\beta_x) \neq d(\beta_y)$ .

It is easy to see that the first case occurs if and only if  $\delta = 0$  or  $h \equiv \delta \pmod{nl}$ , and that the second case occurs if and only if  $\delta > 0$  and  $h \equiv \gamma \pmod{nl}$ . Set

$$\sigma_2 := \begin{cases} id & \text{if the first case occurs} \\ (r+1-x, r+1-y) & \text{if the second case occurs} \end{cases}$$

$$\text{and } \mathbf{v}_2 := v_{\sigma_2 \sigma_1.1};$$

in the second case,  $\mathbf{v}_2$  is obtained from  $\mathbf{v}_1$  by permuting  $\beta_x + h$  and  $\beta_y - h$ .  $\sigma_2$  is constructed in order to have  $\mathbf{d}(\sigma_2 \sigma_1.1) = \mathbf{d}(\omega.\mu)$  and subsequently  $\mathbf{v}(\sigma_2 \sigma_1.1) = \mathbf{v}(\omega.\mu)$ .

\* Set

$$\sigma_3 := \mathbf{v}(\sigma_2 \sigma_1.1)^{-1} = \mathbf{v}(\omega.\mu)^{-1} \quad \text{and} \quad \mathbf{v}_3 := v_{\sigma_3 \sigma_2 \sigma_1.1}.$$

Notice that for  $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$ ,  $\mathbf{v}(\mathbf{m})^{-1}$  is the unique permutation in  $\mathfrak{S}_r$  such that  $\mathbf{m}' = (m'_1, \dots, m'_r) := \mathbf{v}(\mathbf{m})^{-1}.\mathbf{m}$  satisfies:

- (i)  $\forall 1 \leq i < j \leq r, d(m'_i) \geq d(m'_j)$ ,
- (ii)  $\forall 1 \leq i, j \leq r, d(m'_i) = d(m'_j) \implies i \leq j$ .

In other words, let us consider  $\mathbf{m}$  as a word formed by the letters  $m_i$  and for  $1 \leq b \leq l$ , denote by  $w_b$  the subword of  $\mathbf{m}$  formed by the letters  $m_i$  such that  $d(m_i) = b$ . Then we have  $\mathbf{m}' = w_l \cdots w_1$ . Applying this remark to  $\mathbf{m} := \sigma_2 \sigma_1.1$  shows that  $\mathbf{v}_3$  is a wedge product that can be written as

$$\mathbf{v}_3 = v_{\mathbf{k}(l)} \wedge \cdots \wedge v_{\mathbf{k}(1)},$$

where each  $v_{\mathbf{k}(b)}$ ,  $1 \leq b \leq l$  is a wedge product such that each component  $v_k$  of  $v_{\mathbf{k}(b)}$  satisfies  $d(k) = b$ . In this case and for the rest of the proof, we say that  $\mathbf{v}_3$  is *block-decomposable* and call  $v_{\mathbf{k}(b)}$  ( $1 \leq b \leq l$ ) the *b-th block* of  $\mathbf{v}_3$ .

\* Set now

$$\sigma_4 := \omega(\mathbf{k}) = \omega(1) \quad \text{and} \quad \mathbf{v}_4 := v_{\sigma_4 \sigma_3 \sigma_2 \sigma_1.1},$$

where  $\omega(\mathbf{k})$  is defined in Section 3.2 (the equality  $\omega(\mathbf{k}) = \omega(1)$  comes from Lemma 10). For  $1 \leq b \leq l$  denote by

$$r_b := \sharp\{1 \leq i \leq r \mid d(\alpha_i) = b\} = \sharp\{1 \leq i \leq r \mid d(\beta_i) = b\}$$

the number of factors of the block  $v_{\mathbf{k}(b)}$  (the equality of both numbers defining  $r_b$  comes again from Lemma 10). Let  $1 \leq b \leq l$ . Then  $\sigma_4$  acts on the *b-th block* of  $\mathbf{v}_3$  as the permutation  $\binom{1, \dots, r_b}{r_b, \dots, 1}$ . Since  $v_\mu$  is ordered, we can see that for all  $b \in \llbracket 1, l \rrbracket \setminus \{d, d'\}$ , the *b-th block* of  $\mathbf{v}_4$  is also ordered. Here the integers  $d$  and  $d'$  are the ones defined in Lemma 10.

- \* Write temporarily  $\mathbf{v}_4 = v_{k_1} \wedge \dots \wedge v_{k_r}$ , and let  $i$  (*resp.*  $j$ )  $\in \llbracket 1, r \rrbracket$  be such that  $k_i = \beta_y - h$  (*resp.*  $k_j = \beta_x + h$ ). Define  $\sigma_5$  and  $\mathbf{v}_5$  by

$$\sigma_5 := \begin{cases} id & \text{if } \delta > 0 \\ (i, j) & \text{if } \delta = 0 \end{cases} \quad \text{and} \quad \mathbf{v}_5 := v_{\sigma_5 \sigma_4 \sigma_3 \sigma_2 \sigma_1.1} ;$$

we have thus  $\sigma_5 = id$  if and only if  $\beta_y - h$  and  $\beta_x + h$  are in the same block of  $\mathbf{v}_4$ .

- \* Let  $\sigma_6 \in \mathfrak{S}_r$  be the permutation that acts separately on each block of  $\mathbf{v}_5$  by reordering it. Let us be more precise. Set

$$\mathbf{v}_6 := v_{\sigma_6 \sigma_5 \sigma_4 \sigma_3 \sigma_2 \sigma_1.1} .$$

If  $\delta = 0$ , then  $\sigma_6$  acts on the  $d$ -th block of  $\mathbf{v}_5$  as the permutation  $\pi$  from Lemma 6, and  $\sigma_6$  acts trivially on the other blocks. If  $\delta > 0$ , then  $\sigma_6$  acts as the product of two permutations  $\sigma'_d$  and  $\sigma'_{d'}$ , where each  $\sigma'_b$ ,  $b \in \{d, d'\}$  acts on the  $b$ -th block of  $\mathbf{v}_5$  as the permutation denoted by  $\sigma$  in Lemma 5 and  $\sigma'_b$  acts trivially on the other blocks. As a consequence, we have in both cases

$$\varepsilon(\sigma_6) = (-1)^{\text{ht}(\rho) + \text{ht}(\rho')} .$$

- \* Finally, put

$$\sigma_7 := \mathbf{v}(\lambda) \quad \text{and} \quad \mathbf{v}_7 := v_{\sigma_7 \sigma_6 \sigma_5 \sigma_4 \sigma_3 \sigma_2 \sigma_1.1} .$$

Notice that  $v_\lambda$  is ordered,  $\mathbf{v}_6$  is obtained from  $v_\lambda$  by permutation,  $\mathbf{v}_6$  is block-decomposable and all the blocks of  $\mathbf{v}_6$  are ordered. By the remark following the definition of  $\sigma_3$ , we have  $v_{\mathbf{v}(\lambda)^{-1}. \lambda} = \mathbf{v}_6$ , whence  $\mathbf{v}_7 = v_\lambda$ .

As a consequence, we do have  $\tau = \sigma_7 \dots \sigma_1$ , where the  $\sigma_i$ 's are defined as above, hence  $(-1)^{N-1} = \varepsilon(\sigma) \prod_{i=1}^7 \varepsilon(\sigma_i)$ . By considering different cases we see that  $\varepsilon(\sigma_2) \varepsilon(\sigma_5) = \varepsilon$ , where  $\varepsilon$  is defined as in the statement of this proposition. Moreover, we have

$$\varepsilon(\sigma_4) = \varepsilon(\omega(\mathbf{k})) = \prod_{b=1}^l (-1)^{\frac{r_b(r_b-1)}{2}} = (-1)^{\kappa(\mathbf{d}(\mu))} .$$

We then have  $\varepsilon(\sigma_3) \varepsilon(\sigma_4) \varepsilon(\sigma_7) = \varepsilon(\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l)$ , whence the result. ■

## 8 Proof of Theorem 4

Let  $\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi_m^l$ , and  $\lambda, \mu \in \Pi$  be the partitions such that  $\boldsymbol{\lambda}_l \leftrightarrow \lambda$  and  $\boldsymbol{\mu}_l \leftrightarrow \mu$ . We must show that  $a'_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}(1) = 2j_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}^<$ . If  $(\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l)$  does not satisfy  $(\mathcal{H})$ , then there cannot exist any good sequence with respect to  $(\lambda, \mu)$ , so by Lemma 19 we have  $a'_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}(1) = 0$ ; on the other hand, by Corollary 9 we have  $j_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}^< = 0$  in this case. Assume now that  $(\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l)$  satisfies  $(\mathcal{H})$ ,

and recall the notation from Section 4.2. By Corollary 9 and Propositions 14 and 15, the case  $(J_1)$  (*resp.*  $(J_2)$ ) occurs if and only if  $\delta > 0$  (*resp.*  $\delta = 0$ ), and Propositions 14 and 15 then give the expression of  $j_{\lambda_l, \mu_l}^{\prec}$ . Moreover, Propositions 21 and 22 give necessary and sufficient conditions on  $\gamma$ ,  $\delta$  and  $h$  for the existence of a good sequence, in which case it is unique. Lemma 19 and Rules  $(R_2) - (R_4)$  then give the expression of  $a'_{\lambda_l, \mu_l}(1)$ . In order to compare  $a'_{\lambda_l, \mu_l}(1)$  and  $j_{\lambda_l, \mu_l}^{\prec}$ , we have to consider 12 cases depending on the value of  $h$  modulo  $nl$  and on whether  $\gamma$  and  $\delta$  are zero or not. The results are shown in Figure 3. Here  $N$  is the length of a good sequence if it exists. Theorem 4 follows by comparing the last two columns of the array and by applying Proposition 23 if the corresponding numbers are nonzero. ■

Case	Number of good sequences	$\frac{a'_{\lambda_l, \mu_l}(1)}{\varepsilon(\lambda_l, \mu_l)}$	$\frac{j_{\lambda_l, \mu_l}^{\prec}}{(-1)^{\text{ht}(\rho) + \text{ht}(\rho')}}$
$\gamma = \delta = 0$	0	0	0
$\gamma > 0, \delta = 0,$ $h \not\equiv 0 \pmod{nl}, h \not\equiv \gamma \pmod{nl}$	0	0	0
$\gamma > 0, \delta = 0,$ $i := \frac{h}{nl} \in \mathbb{N}^*$	1	$(-1)^{N-1} \frac{d}{dq} \Big _{q=1} \left( -(q^{-2} - 1)q^{-2i+1} \right)$ $= 2(-1)^{N-1}$	-1
$\gamma > 0, \delta = 0,$ $i := \frac{h-\gamma}{nl} \in \mathbb{N}$	1	$(-1)^{N-1} \frac{d}{dq} \Big _{q=1} \left( (q^{-2} - 1)q^{-2i} \right)$ $= -2(-1)^{N-1}$	1
$\gamma = 0, \delta > 0,$ $h \not\equiv 0 \pmod{nl}, h \not\equiv \delta \pmod{nl}$	0	0	0
$\gamma = 0, \delta > 0,$ $i := \frac{h}{nl} \in \mathbb{N}^*$	1	$(-1)^{N-1} \frac{d}{dq} \Big _{q=1} \left( -(q^2 - 1)q^{2i-1} \right)$ $= -2(-1)^{N-1}$	1
$\gamma = 0, \delta > 0,$ $i := \frac{h-\delta}{nl} \in \mathbb{N}$	1	$(-1)^{N-1} \frac{d}{dq} \Big _{q=1} \left( (q^2 - 1)q^{2i} \right)$ $= 2(-1)^{N-1}$	1
$\gamma > 0, \delta > 0,$ $h \not\equiv 0 \pmod{nl}, h \not\equiv \gamma \pmod{nl},$ $h \not\equiv \delta \pmod{nl}, h \not\equiv \gamma + \delta \pmod{nl}$	0	0	0
$\gamma > 0, \delta > 0,$ $i := \frac{h}{nl} \in \mathbb{N}^*$	1	$(-1)^{N-1} \frac{d}{dq} \Big _{q=1} \left( -(q - q^{-1}) \frac{q^{2i} - q^{-2i}}{q + q^{-1}} \right)$ $= 0$	0
$\gamma > 0, \delta > 0,$ $i := \frac{h-\gamma}{nl} \in \mathbb{N}$	1	$(-1)^{N-1} \frac{d}{dq} \Big _{q=1} \left( -(q - q^{-1}) \frac{q^{2i+1} + q^{-2i-1}}{q + q^{-1}} \right)$ $= -2(-1)^{N-1}$	1
$\gamma > 0, \delta > 0,$ $i := \frac{h-\delta}{nl} \in \mathbb{N}$	1	$(-1)^{N-1} \frac{d}{dq} \Big _{q=1} \left( (q - q^{-1}) \frac{q^{2i+1} + q^{-2i-1}}{q + q^{-1}} \right)$ $= 2(-1)^{N-1}$	1
$\gamma > 0, \delta > 0,$ $i := \frac{h-\gamma-\delta}{nl} \in \mathbb{N}$	1	$(-1)^{N-1} \frac{d}{dq} \Big _{q=1} \left( (q - q^{-1}) \frac{q^{2i+2} - q^{-2i-2}}{q + q^{-1}} \right)$ $= 0$	0

Figure 3: List of the cases involved in the proof of Theorem 4.

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LABORATOIRE DE MATHÉMATIQUES NICOLAS ORESME, UNIVERSITÉ DE CAEN, BP 5186, 14032 CAEN CEDEX, FRANCE.

*E-mail address:* xyvonne@math.unicaen.fr