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A conjecture for q -decomposition matrices of cyclotomic v -Schur algebras

Xavier YVONNE

July 6, 2005

Abstract

The Jantzen sum formula for cyclotomic v -Schur algebras yields an identity for some q -analogues of the decomposition matrices of these algebras. We prove a similar identity for matrices of canonical bases of higher-level Fock spaces. We conjecture then that those matrices are actually identical for a suitable choice of parameters. In particular, we conjecture that decomposition matrices of cyclotomic v -Schur algebras are obtained by specializing at $q = 1$ some transition matrices between the standard basis and the canonical basis of a Fock space.

1 Introduction

In order to study representations of the Ariki-Koike algebra associated to the complex reflection group $G(l, 1, m)$, Dipper, James and Mathas introduced in 1998 the cyclotomic v -Schur algebra. This algebra depends on the two integers l and m and on some deformation parameters v, u_1, \dots, u_l . When $l = 1$, the cyclotomic v -Schur algebra coincides with the v -Schur algebra of [DJ]. It is an open problem to calculate the decomposition matrix of a cyclotomic v -Schur algebra whose parameters are powers of a given n -th root of unity. To this aim, James and Mathas proved, for cyclotomic v -Schur algebras, an important formula: the Jantzen sum formula [JM]. Given a Jantzen filtration for Weyl modules, one can define a q -analogue $D(q)$ of the decomposition matrix; the coefficients of $D(q)$ are graded decomposition numbers of the composition factors of Weyl modules (see Section 2.1). The Jantzen sum formula is equivalent to the identity $D'(1) = J^{\triangleleft} D(1)$, where J^{\triangleleft} is a matrix of \wp -adic valuations of some Gram determinants (see Lemma 1 and Theorem 2).

Let $\Delta(q)$ be the matrix of the canonical basis of the degree m homogeneous component of a Fock representation of level l of $U_q(\widehat{\mathfrak{sl}}_n)$ [U2]. Uglov provided in [U2] an algorithm for computing $\Delta(q)$.

In view of Ariki's theorem for Ariki-Koike algebras [A1], it seems natural to conjecture that for a suitable choice of parameters, one has $D(q) = \Delta(q)$. This would provide an algorithm for computing decomposition matrices of cyclotomic v -Schur algebras. Varagnolo and Vasserot [VV] proved for $l = 1$ that $D(1) = \Delta(1)$. Moreover, Ryom-Hansen showed that this conjecture (still for $l = 1$) is compatible with the Jantzen-Schaper formula [RH]. Passing to higher level $l \geq 1$ requires the introduction of an extra parameter $\mathbf{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l$, called

multi-charge; this l -tuple parametrizes the Fock space of level l introduced by Uglov. We say that \mathbf{s}_l is *dominant* (with respect to m) if for all $1 \leq d \leq l-1$, we have $s_{d+1} - s_d \geq m$. In this case, we conjecture that $D(q) = \Delta(q)$. Here, $D(q)$ comes from a Jantzen filtration of the Weyl modules of the cyclotomic v -Schur algebra $\mathcal{S}_{\mathbb{C}} = \mathcal{S}_{\mathbb{C}}(\xi; \xi^{s_1}, \dots, \xi^{s_l})$ with $\xi := \exp(\frac{2i\pi}{n})$. We do not know the existence of a prior conjecture giving accurate parameters for computing the decomposition matrix, even when $l = 2$ (type B_m).

Although $\mathcal{S}_{\mathbb{C}}$ and thus $D(1)$ do not actually depend on the multi-charge \mathbf{s}_l , but only on the sequence of the residues modulo n of its coordinates, it is necessary to impose a condition on \mathbf{s}_l for having $D(1) = \Delta(1)$. Indeed, we computed for example, for two choices of the multi-charge giving the same sequences of residues modulo n , two matrices $\Delta(q)$ that do not even have the same numbers of zero entries.

Our conjecture is motivated by the following theorem. We define in a combinatorial way a matrix J^{\prec} for any multi-charge \mathbf{s}_l ; if \mathbf{s}_l is dominant, then our matrix J^{\prec} coincides with the matrix J^{\triangleleft} of the Jantzen sum formula. We show then that for any multi-charge \mathbf{s}_l , we have $\Delta'(1) = J^{\prec} \Delta(1)$ (Theorem 3).

The proof of our theorem relies on a combinatorial expression for the derivative at $q = 1$ of the matrix $A(q)$, where $A(q)$ is the matrix of the Fock space involution used for defining $\Delta(q)$. Namely, we show that $A'(1) = 2J^{\prec}$ (Theorem 4). The coefficients of $A(q)$ are some analogues for Fock spaces of Kazhdan-Lusztig R -polynomials $R_{x,y}(q)$ for Hecke algebras. The classical computation of $R'_{x,y}(1)$ was made in [GJ], in relation with the Kazhdan-Lusztig conjecture for multiplicities of composition factors of Verma modules.

Notation. Throughout this article, we fix three integers $n \geq 1$, $l \geq 1$ and $m \geq 1$ and a multi-charge $\mathbf{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l$. We set $s := s_1 + \dots + s_l$. Let \mathbb{N} (*resp.* \mathbb{N}^*) denote the set of nonnegative (*resp.* positive) integers. Denote also by $\llbracket a, b \rrbracket$ the discrete interval $[a, b] \cap \mathbb{Z}$. Let Π be the set of partitions of any integer and Π_m^l be the set of l -multi-partitions of m . Finally, we consider the Coxeter group \mathfrak{S}_r ($r \in \mathbb{N}^*$), equipped with its generating set $\{s_i = (i, i+1), 1 \leq i \leq r-1\}$. We denote by ℓ the length function on \mathfrak{S}_r and by ω the unique element of maximal length in \mathfrak{S}_r .

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2 Statement of results

2.1 The Jantzen sum formula

Let R be a principal ideal domain, and K be the field of fractions of R . Let v be an invertible element of R and $u_1, \dots, u_l \in R$. Following [JM], let

$$\mathcal{H} = \mathcal{H}_R(v; u_1, \dots, u_l) = \mathcal{H}_{R,m}(v; u_1, \dots, u_l)$$

denote the Ariki-Koike algebra defined over R , with parameters v, u_1, \dots, u_l ; here m is the number of generators T_i , $0 \leq i \leq m-1$ involved in the presentation of \mathcal{H} . Let

$$\mathcal{S} = \mathcal{S}_R(v; u_1, \dots, u_l) = \mathcal{S}_{R,m}(v; u_1, \dots, u_l)$$

be the cyclotomic v -Schur algebra associated to \mathcal{H} . Dipper, James and Mathas (see [DJM, Theorem 6.12]) showed that \mathcal{S} is a cellular algebra in the sense of [GL]. Given $\lambda_l \in \Pi_m^l$, one defines an \mathcal{S} -module $W(\lambda_l)$ which is a free R -module of finite rank, called *Weyl module*. Since \mathcal{S} is cellular, $W(\lambda_l)$ is naturally equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$. Set

$$L(\lambda_l) := W(\lambda_l) / \text{rad } W(\lambda_l),$$

where $\text{rad } W(\lambda_l)$ is the radical of the bilinear form $\langle \cdot, \cdot \rangle$; it is also the radical (in the sense of Jacobson) of $W(\lambda_l)$. The theory of cellular algebras shows that if R is a field, then $\{L(\lambda_l), \lambda_l \in \Pi_m^l\}$ is a complete set of non-isomorphic irreducible \mathcal{S} -modules.

Let \wp be a maximal ideal in R , $F := R/\wp R$ be the residue field and ν_\wp be the \wp -adic valuation of R associated to \wp . If M is an R -module, we denote by $M_F = F \otimes_R M$ the specialized module (here, R acts on the right on F by the natural surjection $R \twoheadrightarrow R/\wp R = F$). Let M be an R -module equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$. For all $i \in \mathbb{N}$, we set

$$M(i) := \{u \in M \mid \forall v \in M, \nu_\wp(\langle u, v \rangle) \geq i\}.$$

The *Jantzen filtration* of M [AM] is the sequence

$$M_F = M_F(0) \supset M_F(1) \supset \dots,$$

where $M_F(i) := (M(i) + \wp M) / \wp M$. In particular, we have $M_F(1) = \text{rad } M_F$. If M is a free R -module of finite rank, then we have $M_F(i) = \{0\}$ for i large enough. Since the form $\langle \cdot, \cdot \rangle$ on $W(\lambda_l)$ is compatible with the action of \mathcal{S} , the $W_F(\lambda_l; i)$'s are \mathcal{S}_F -modules. Define the matrix $D(q) = (d_{\lambda_l, \mu_l}(q))_{\lambda_l, \mu_l \in \Pi_m^l}$ by

$$d_{\lambda_l, \mu_l}(q) := \sum_{i \geq 0} [W_F(\lambda_l; i) / W_F(\lambda_l; i+1) : L_F(\mu_l)] q^i \in \mathbb{N}[q] \quad (\lambda_l, \mu_l \in \Pi_m^l).$$

This matrix depends on the choice of R , on the parameters $v, u_1, \dots, u_l \in R$ and $m, l \in \mathbb{N}$, and also on the maximal ideal \wp . Notice that $d_{\lambda_l, \mu_l}(1)$ is equal to the multiplicity of $L_F(\mu_l)$ as a composition factor of $W_F(\lambda_l)$, so $D(1)$ is the decomposition matrix of \mathcal{S}_F .

The next lemma follows from an easy computation.

Lemma 1 *Let $\mathcal{R}_0(\mathcal{S}_F)$ denote the Grothendieck group of finitely-generated \mathcal{S}_F -modules. Let $J = (j_{\lambda_l, \mu_l})_{\lambda_l, \mu_l \in \Pi_m^l}$ be a matrix with integer coefficients. Then the following statements are equivalent:*

- (i) *In $\mathcal{R}_0(\mathcal{S}_F)$, we have for all $\lambda_l \in \Pi_m^l$, $\sum_{i > 0} [W_F(\lambda_l; i)] = \sum_{\nu_l \in \Pi_m^l} j_{\lambda_l, \nu_l} [W_F(\nu_l)]$,*

(ii) $D'(1) = JD(1)$.

James and Mathas proved the following result [JM, Theorem 4.3]:

Theorem 2 (the Jantzen sum formula)

Assume that the Poincaré polynomial of $\mathcal{H}(v; u_1, \dots, u_l)$ is nonzero. Then for all $\lambda_l \in \Pi_m^l$, we have

$$\sum_{i>0} [W_F(\lambda_l; i)] = \sum_{\mu_l \in \Pi_m^l} j_{\lambda_l, \mu_l}^{\triangleleft} [W_F(\mu_l)],$$

where the matrix $J^{\triangleleft} = (j_{\lambda_l, \mu_l}^{\triangleleft})_{\lambda_l, \mu_l \in \Pi_m^l}$ is given explicitly in Section 4.1.

Remark 1. It is well-known that the Poincaré polynomial of \mathcal{H}_R is nonzero if and only if \mathcal{H}_K is semisimple, or if and only if \mathcal{S}_K is semisimple.

As with the matrix $D(q)$, the matrix J^{\triangleleft} depends on the choice of the ring R , on the parameters $v, u_1, \dots, u_l \in R$ and $m, l \in \mathbb{N}$, and on the maximal ideal \wp . The coefficients of J^{\triangleleft} are \wp -adic valuations of some Gram determinants which can be described in a combinatorial way in terms of ribbons contained in diagrams of l -multi-partitions. Here \triangleleft stands for the dominance ordering on multi-partitions (see Section 3.1). We use the notation J^{\triangleleft} instead of J in order to point out that J^{\triangleleft} is lower-triangular with respect to this ordering, that is $\lambda_l \not\triangleleft \mu_l \Rightarrow j_{\lambda_l, \mu_l}^{\triangleleft} = 0$.

2.2 Statement of theorems

We first describe our choice of parameters for the cyclotomic v -Schur algebra \mathcal{S} . These parameters are similar to those used in [J] for Ariki-Koike algebras. They depend on n, l and on the l -tuple $\mathbf{s}_l \in \mathbb{Z}^l$ that we have fixed at the beginning. Let us take

$$R := \mathbb{C}[x, x^{-1}],$$

the ring of Laurent polynomials in one indeterminate over \mathbb{C} , $\zeta := \exp\left(\frac{2i\pi}{nl}\right)$ a primitive nl -th root of unity,

$$\wp := (x - \zeta), \quad v := x^l \quad \text{and} \quad u_d := \zeta^{nd} x^{ls_d - nd} \quad (1 \leq d \leq l).$$

We then have $K = \mathbb{C}(x)$, $F = \mathbb{C}$ and $\mathcal{S}_F = \mathcal{S}_{\mathbb{C}}(\xi; \xi^{s_1}, \dots, \xi^{s_l})$ with $\xi := \exp\left(\frac{2i\pi}{n}\right)$; \mathcal{S}_F depends thus only on the residues $s_d \bmod n$ ($1 \leq d \leq l$). Notice that with our choice of parameters, the algebra $\mathcal{S}_K(v; u_1, \dots, u_r)$ is semisimple. Indeed, specializing x at 1 sends $\mathcal{H}_K(v; u_1, \dots, u_r)$ on the semisimple group algebra $\mathbb{C}G(l, 1, m)$, so by the Tits deformation argument [A2], the algebra $\mathcal{H}_K(v; u_1, \dots, u_r)$ is semisimple and so is $\mathcal{S}_K(v; u_1, \dots, u_r)$.

Our choice of parameters leads thus to a Jantzen sum formula (see Theorem 2) for the

specialized cyclotomic v -Schur algebra $\mathcal{S}_F = \mathcal{S}_{\mathbb{C}}(\xi; \xi^{s_1}, \dots, \xi^{s_l})$, and therefore to the definition of a matrix J^{\triangleleft} .

Let $<$ be any ordering on Π_m^l . We can actually define a matrix $J^<$ in the same way as J^{\triangleleft} , but by replacing the dominance ordering on multi-partitions by $<$ (see Section 4.1). We consider in particular the ordering \prec ; this ordering depends on \mathbf{s}_l and is defined in Section 3.2.

We say that \mathbf{s}_l is *dominant* (with respect to m) if for all $1 \leq d \leq l-1$, we have $s_{d+1} - s_d \geq m$. We shall prove in Proposition 13 that if \mathbf{s}_l is dominant, then we have $J^{\prec} = J^{\triangleleft}$. Note that the orderings \prec and \triangleleft do not coincide in general, even if \mathbf{s}_l is dominant (see Remark 5).

Example 1. Set $n = 3$, $l = 2$, $\mathbf{s}_l = (1, 0)$ and $m = 3$. Here, \mathbf{s}_l is not dominant, and the matrices J^{\prec} and J^{\triangleleft} are different. Namely, we have on the one hand

$$J^{\prec} = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & -1 & 0 & 1 & -1 & 1 & 0 & \cdot & \cdot \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & \cdot \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} ((1, 1), (1)) \\ ((3), \emptyset) \\ (\emptyset, (3)) \\ (1), (2) \\ (\emptyset, (2, 1)) \\ (2), (1) \\ (1), (1, 1) \\ (\emptyset, (1, 1, 1)) \\ (2, 1), \emptyset \\ ((1, 1, 1), \emptyset) \end{matrix},$$

where dots over the main diagonal stand for zeroes. The l -multi-partitions of m are ordered decreasingly according to a total ordering finer than \prec and they are displayed in the column located on the right of the matrix J^{\prec} . On the other hand, if we keep the same ordering for the rows and the columns of J^{\triangleleft} , we get

$$J^{\triangleleft} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} ((1, 1), (1)) \\ ((3), \emptyset) \\ (\emptyset, (3)) \\ (1), (2) \\ (\emptyset, (2, 1)) \\ (2), (1) \\ (1), (1, 1) \\ (\emptyset, (1, 1, 1)) \\ (2, 1), \emptyset \\ ((1, 1, 1), \emptyset) \end{matrix}.$$

The latter matrix is not written in a lower-triangular way because the ordering we consider here is not compatible with \triangleleft .

Example 2. Set $n = 3$, $l = 2$, $\mathbf{s}_l = (4, -3)$ and $m = 3$. Here, \mathbf{s}_l is dominant. In this case, the orderings \prec and \triangleleft are different, but the matrices J^\prec and J^\triangleleft coincide. Namely, we have

$$J^\prec = J^\triangleleft = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -1 & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -1 & 0 & 1 & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & \cdot & \cdot \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & \cdot \end{pmatrix} \begin{matrix} ((1, 1), (1)) \\ ((3), \emptyset) \\ ((2, 1), \emptyset) \\ ((2), (1)) \\ ((1, 1, 1), \emptyset) \\ ((1), (2)) \\ ((1), (1, 1)) \\ (\emptyset, (3)) \\ (\emptyset, (2, 1)) \\ (\emptyset, (1, 1, 1)) \end{matrix}.$$

Let $\mathbf{F}_q[\mathbf{s}_l]$ be the Fock space of level l , with multi-charge \mathbf{s}_l (see Section 5.1); as a vector space, $\mathbf{F}_q[\mathbf{s}_l]$ has a natural basis $\{|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle, \boldsymbol{\lambda}_l \in \Pi^l\}$ indexed by l -multi-partitions. Let $\mathbf{F}_q[\mathbf{s}_l]_m$ be the subspace of $\mathbf{F}_q[\mathbf{s}_l]$ spanned by the $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$'s, $\boldsymbol{\lambda}_l \in \Pi_m^l$. Let $A(q)$ be the matrix of the involution of $\mathbf{F}_q[\mathbf{s}_l]_m$ with respect to the standard basis $\{|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle, \boldsymbol{\lambda}_l \in \Pi_m^l\}$, and let $\Delta(q)$ be the transition matrix between the standard basis and the canonical basis of $\mathbf{F}_q[\mathbf{s}_l]_m$ (see Sections 5.2 and 5.3). The main result of this article is the following

Theorem 3 *We have*

$$\Delta'(1) = J^\prec \Delta(1).$$

This theorem is equivalent to the following:

Theorem 4 *We have*

$$A'(1) = 2J^\prec.$$

Proof of the equivalence of Theorems 3 and 4. Since the canonical basis is invariant under the involution, we have the identity $\Delta(q) = A(q)\Delta(q^{-1})$. Taking derivatives at $q = 1$ yields $\Delta'(1) = A'(1)\Delta(1) - A(1)\Delta'(1)$. Since $A(1)$ is the identity matrix, we get $2\Delta'(1) = A'(1)\Delta(1)$. As a consequence, Theorem 4 implies Theorem 3. Since $\Delta(1)$ is unitriangular, hence invertible, the converse follows. ■

Example 1 (continued). Recall that $n = 3$, $l = 2$, $\mathbf{s}_l = (1, 0)$ and $m = 3$. If we order the rows and the columns of $A(q)$ decreasingly with respect to the same ordering used for writing J^\prec , we get

$$A(q) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & q - q^{-1} & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & q - q^{-1} & q - q^{-1} & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & q^{-2} - 1 & (q - q^{-1})^2 & 0 & q - q^{-1} & q - q^{-1} & 1 & \cdot & \cdot & \cdot \\ 0 & q^{-3}(q^2 - 1) & -(q - q^{-1}) & 0 & q^2 - 1 & q^{-2} - 1 & q - q^{-1} & 1 & \cdot & \cdot \\ 0 & q^2 - 1 & q^{-2} - 1 & 0 & 0 & q - q^{-1} & 0 & 0 & 1 & \cdot \\ 0 & -(q - q^{-1}) & -q^{-1}(q - q^{-1})^2 & 0 & q^{-2} - 1 & (q - q^{-1})^2 & q - q^{-1} & 0 & q - q^{-1} & 1 \end{pmatrix} \begin{matrix} ((1, 1), (1)) \\ ((3), \emptyset) \\ (\emptyset, (3)) \\ (1), (2) \\ (\emptyset, (2, 1)) \\ (2), (1) \\ (1), (1, 1) \\ (\emptyset, (1, 1, 1)) \\ (2, 1), \emptyset \\ ((1, 1, 1), \emptyset) \end{matrix}.$$

In this case, we do have $A'(1) = 2J^{\prec}$.

Example 2 (continued). Recall that $n = 3$, $l = 2$, $\mathbf{s}_l = (4, -3)$ and $m = 3$. If we order the rows and the columns of $A(q)$ decreasingly with respect to the same ordering used for writing J^{\prec} , we get

$$A(q) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & q - q^{-1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & q^2 - 1 & q - q^{-1} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & q^{-2} - 1 & q - q^{-1} & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & -(q - q^{-1}) & (q - q^{-1})^2 & q - q^{-1} & q - q^{-1} & 0 & 1 & \cdot & \cdot & \cdot \\ 0 & 0 & q^{-2} - 1 & q - q^{-1} & 0 & 0 & 0 & 1 & \cdot & \cdot \\ 0 & 0 & -q^{-1}(q - q^{-1})^2 & (q - q^{-1})^2 & q^{-2} - 1 & 0 & q - q^{-1} & q - q^{-1} & 1 & \cdot \\ 0 & q^{-3}(q^2 - 1) & 0 & -(q - q^{-1}) & 0 & 0 & q^2 - 1 & q^{-2} - 1 & q - q^{-1} & 1 \end{pmatrix} \begin{matrix} ((1, 1), (1)) \\ ((3), \emptyset) \\ (2, 1), \emptyset \\ (2), (1) \\ (1, 1, 1), \emptyset \\ (1), (2) \\ (1), (1, 1) \\ (\emptyset, (3)) \\ (\emptyset, (2, 1)) \\ (\emptyset, (1, 1, 1)) \end{matrix}.$$

In this case also, we do have $A'(1) = 2J^{\prec}$.

We prove Theorem 4 in the next sections. Our proof is similar to the proof of [RH] in the level one case. However the higher-level case is significantly more complicated and involves the discussion of many cases (see Section 8).

The formal analogy between Theorem 3 on one hand, and the rephrasing of the Jantzen sum formula given in Lemma 1 on the other hand leads us to set the following

Conjecture 1 *Assume that \mathbf{s}_l is dominant. Let $D(q)$ be the q -analogue of the decomposition matrix of \mathcal{S} defined in Section 2.1 with our choice of parameters given in Section 2.2. Then we have*

$$D(q) = \Delta(q).$$

Example 3. Set again $n = 3$, $l = 2$, $\mathbf{s}_l = (4, -3)$ and $m = 3$, and consider the cyclotomic v -Schur algebra

$$\mathcal{S} = \mathcal{S}_R(v; u_1, u_2)$$

defined over the ring $R = \mathbb{C}[x, x^{-1}]$ with parameters

$$v := x^2, \quad u_1 := -x^5, \quad u_2 := x^{-12}.$$

Take $\wp := \left(x - e^{\frac{i\pi}{3}}\right)$, so the specialized algebra is $\mathcal{S}_{\mathbb{C}}\left(e^{\frac{2i\pi}{3}}; e^{\frac{2i\pi}{3}}, 1\right)$. In this case, we expect the matrix $D(q)$ to be equal to the matrix

$$\Delta(q) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & q & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & q^2 & q & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & q & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & q^2 & q & q & 0 & 1 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & q & 0 & 0 & 0 & 1 & \cdot & \cdot \\ 0 & 0 & 0 & q^2 & 0 & 0 & q & q & 1 & \cdot \\ 0 & 0 & 0 & 0 & q & 0 & q^2 & 0 & q & 1 \end{pmatrix} \begin{matrix} ((1, 1), (1)) \\ ((3), \emptyset) \\ ((2, 1), \emptyset) \\ ((2), (1)) \\ ((1, 1, 1), \emptyset) \\ ((1), (2)) \\ ((1), (1, 1)) \\ (\emptyset, (3)) \\ (\emptyset, (2, 1)) \\ (\emptyset, (1, 1, 1)) \end{matrix}.$$

If we no longer assume that \mathbf{s}_l is dominant, then we expect $\Delta(q)$ to be equal to a q -analogue of the decomposition matrix of a quasi-hereditary covering (in the sense of Rouquier) of the Ariki-Koike algebra \mathcal{H} . This covering, depending on \mathbf{s}_l , could come from a rational Cherednik algebra through the Knizhnik-Zamolodchikov functor [GGOR]. It would be Morita-equivalent to the cyclotomic v -Schur algebra of [JM] if \mathbf{s}_l is dominant.

3 Combinatorics of partitions and multi-partitions

3.1 Definitions

We give here all the basic definitions about partitions and multi-partitions that we need later; our main reference is [Md]. Let $r \in \mathbb{N}$. A *partition of r* is a sequence of integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$ and $\lambda_1 + \dots + \lambda_N = r$. Each nonzero λ_i is called a *part* of λ . The sum of all the parts of λ is denoted by $|\lambda|$. We identify two partitions differing only by a tail of zeroes and we write sometimes partitions as sequences of integers with an infinite tail of zeroes. The only partition of 0 is denoted by \emptyset . An N -*multi-partition of r* is an N -tuple of partitions of integers summing up to r . Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(N)})$ be an N -multi-partition. For $1 \leq b \leq N$, write $\lambda^{(b)} = (\lambda_1^{(b)}, \lambda_2^{(b)}, \dots)$ the parts of $\lambda^{(b)}$. The *Young diagram* of $\boldsymbol{\lambda}$ is the set

$$\{(i, j, b) \in \mathbb{N}^* \times \mathbb{N}^* \times \llbracket 1, N \rrbracket \mid 1 \leq j \leq \lambda_i^{(b)}\},$$

whose elements are called *boxes* or *nodes* of $\boldsymbol{\lambda}$. If $N = 1$, namely, if $\boldsymbol{\lambda}$ is a partition, we drop the third component in the symbol (i, j, b) of a node of $\boldsymbol{\lambda}$. From now on we identify an N -multi-partition with its Young diagram. We extend the notation $|\lambda|$ in a natural way for multi-partitions, and we define the *dominance ordering* on multi-partitions as follows. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be two N -multi-partitions. We say that $\boldsymbol{\mu}$ *dominates* $\boldsymbol{\lambda}$, and we write $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$ if

$|\boldsymbol{\lambda}| = |\boldsymbol{\mu}|$ and for all $k \geq 0$, $1 \leq b \leq N$, we have

$$\sum_{i=1}^{b-1} |\lambda^{(i)}| + \sum_{j=1}^k \lambda_j^{(b)} \leq \sum_{i=1}^{b-1} |\mu^{(i)}| + \sum_{j=1}^k \mu_j^{(b)} ;$$

we write $\boldsymbol{\lambda} \triangleleft \boldsymbol{\mu}$ if $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$ and $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$.

If $\lambda, \mu \in \Pi$ are two partitions, we write $\lambda \subset \mu$ if the diagram of λ is contained in the diagram of μ , and the set-theoretic difference is called a *skew diagram*; we denote it by μ/λ . A *path* in the skew diagram θ is a sequence of boxes $(\gamma_1, \dots, \gamma_N) \in \theta^N$ such that for all $1 \leq i \leq N-1$, γ_i and γ_{i+1} have one common side. We say that θ is *connected* if given any two boxes $\gamma, \gamma' \in \theta$, there exists a path within θ connecting γ to γ' . A *ribbon* is a connected skew diagram that contains no 2×2 block of boxes. Let ρ be a ribbon. The *head* (*resp. tail*) of ρ is the node $\gamma = (i, j) \in \rho$ such that $j - i$ is minimal (*resp. maximal*); we denote this node by $\text{hd}(\rho)$ (*resp. tl*(ρ)). If $\text{hd}(\rho) = (i, j)$ and $\text{tl}(\rho) = (i', j')$, the *height* of ρ is the integer $\text{ht}(\rho) := i - i' \in \mathbb{N}$. Finally, the *length* of ρ is the number of boxes contained in ρ ; we denote it by $\ell(\rho)$.

Example 4. On Figure 2 (see Section 4.1), the set of white squares represents the partition $(4, 1)$; ρ, ρ' and ρ'' are three ribbons of respective heights 2, 1 and 0 and of respective lengths 4, 4 and 3.

A *charged N -multi-partition* is an element of $\Pi^N \times \mathbb{Z}^N$. If $(\boldsymbol{\lambda}, \mathbf{t}) \in \Pi^N \times \mathbb{Z}^N$ is a charged multi-partition and $\mathbf{t} = (t_1, \dots, t_N)$, the *content* of the node $\gamma = (i, j, b) \in \boldsymbol{\lambda}$ is the integer

$$\text{cont}(\gamma) := t_b + j - i.$$

If $M \in \mathbb{N}^*$, the *residue modulo M* of γ is

$$\text{res}_M(\gamma) := \text{cont}(\gamma) \bmod M.$$

For all $i \in \mathbb{Z}$, we set

$$N_i(\boldsymbol{\lambda}) := \#\{\gamma \in \boldsymbol{\lambda} \mid \text{res}_n(\gamma) = i \bmod n\};$$

this number depends on the multi-charge \mathbf{t} . We define in a similar way $N_i(\theta)$ if θ is a skew diagram contained in a charged partition.

3.2 The bijection τ_l

Throughout the proof of Theorem 4, we use a large amount of notation that we give here. In particular, we have to pass from l -multi-partitions (indexing the bases of the Fock space) to partitions (indexing the bases of the q -wedge space – see Section 5.1) and conversely. This passage from one indexation to the other can be described in a combinatorial way by using a bijection τ_l that we give at the end of this section. τ_l is a variant of the bijection associating to a partition its l -quotient and its l -core. We construct this bijection using abaci. This

bijection will be used in particular for defining the partial ordering \prec on Π_m^l mentioned in Section 2.2.

The Euclidean algorithm shows that any integer $k \in \mathbb{Z}$ can be written in a unique way as

$$k = c(k) + n(d(k) - 1) + nlm(k),$$

with $c(k) \in \llbracket 1, n \rrbracket$, $d(k) \in \llbracket 1, l \rrbracket$ and $m(k) \in \mathbb{Z}$. For any r -tuple $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$, we define an r -tuple

$$\mathbf{c}(\mathbf{k}) := (c(k_1), \dots, c(k_r)),$$

and in a similar way we define $\mathbf{d}(\mathbf{k})$ and $\mathbf{m}(\mathbf{k})$. The group \mathfrak{S}_r acts on the left on \mathbb{Z}^r by

$$\sigma.(k_1, \dots, k_r) = (k_{\sigma^{-1}(1)}, \dots, k_{\sigma^{-1}(r)}) \quad ((k_1, \dots, k_r) \in \mathbb{Z}^r, \sigma \in \mathfrak{S}_r),$$

and a fundamental domain for this action is

$$B := \{(b_1, \dots, b_r) \in \mathbb{Z}^r \mid b_1 \geq \dots \geq b_r\}.$$

Let $\mathbf{b}(\mathbf{k})$ denote the element of B that is conjugated to $\mathbf{d}(\mathbf{k})$ under the action of \mathfrak{S}_r , $W_{\mathbf{k}}$ the stabilizer of $\mathbf{b}(\mathbf{k})$ (this is a parabolic subgroup of \mathfrak{S}_r), and $\omega(\mathbf{k})$ the element of maximal length in $W_{\mathbf{k}}$. Let $W^{\mathbf{k}}$ be the set of minimal length representatives in the left cosets $\mathfrak{S}_r/W_{\mathbf{k}}$, and $\mathbf{v}(\mathbf{k}) \in W^{\mathbf{k}}$ be the element such that $\mathbf{d}(\mathbf{k}) = \mathbf{v}(\mathbf{k}).\mathbf{b}(\mathbf{k})$.

Example 5. Let $n = 3$, $l = 2$, $r = 4$ and $\mathbf{k} = (12, -5, 2, 17)$. Then we have $\mathbf{c}(\mathbf{k}) = (3, 1, 2, 2)$, $\mathbf{d}(\mathbf{k}) = (2, 1, 1, 2)$, $\mathbf{m}(\mathbf{k}) = (1, -1, 0, 2)$, $\mathbf{b}(\mathbf{k}) = (2, 2, 1, 1)$ and $\mathbf{v}(\mathbf{k}) = s_3s_2$.

To any charged multi-partition we can associate an abacus in the following way. By definition, a 1-runner abacus is a subset A of \mathbb{Z} such that $-k \in A$ and $k \notin A$ for all large enough $k \in \mathbb{N}$. In a less formal way, each $k \in A$ corresponds to the position of a bead on the (horizontal) abacus A which is full of beads on the left and empty on the right. Let \mathcal{A} be the set of 1-runner abaci. If $N \geq 1$, an N -runner abacus is an N -tuple of 1-runner abaci. If $\mathbf{A} = (A_1, \dots, A_N) \in \mathcal{A}^N$ is an N -runner abacus, we identify \mathbf{A} with the subset

$$\{(k, d) \mid 1 \leq d \leq N, k \in A_d\} \subset \mathbb{Z} \times \llbracket 1, N \rrbracket.$$

To $(\boldsymbol{\lambda}, \mathbf{t}) \in \Pi^N \times \mathbb{Z}^N$ we associate the N -runner abacus

$$A(\boldsymbol{\lambda}, \mathbf{t}) := \{(\lambda_i^{(d)} + t_d + 1 - i, d) \mid i \geq 1, 1 \leq d \leq N\}$$

(here the t_d 's are the components of \mathbf{t}). One checks easily that the map

$$(\boldsymbol{\lambda}, \mathbf{t}) \in \Pi^N \times \mathbb{Z}^N \mapsto A(\boldsymbol{\lambda}, \mathbf{t}) \in \mathcal{A}^N$$

is bijective. Consider now the map

$$\phi : \mathbb{Z} \rightarrow \mathbb{Z}, \quad k \mapsto c(k) + nm(k).$$

ϕ enjoys the following obvious properties, which we need later: for all $k, k' \in \mathbb{Z}$, we have

$$(P_1) \quad \phi(k) \equiv c(k) \equiv k \pmod{n},$$

$$(P_2) \quad (k < k', d(k) = d(k')) \implies \phi(k) < \phi(k'),$$

$$(P_3) \quad \phi(k) < \phi(k') \implies k < k'.$$

Notice that the map $k \in \mathbb{Z} \mapsto (\phi(k), d(k)) \in \mathbb{Z} \times \llbracket 1, l \rrbracket$ is a bijection. The bijection $\tau_l : \Pi \times \mathbb{Z} \cong \mathcal{A} \rightarrow \Pi^l \times \mathbb{Z}^l \cong \mathcal{A}^l$ is then defined in terms of abaci by the formula

$$\tau_l(A) := \{(\phi(k), d(k)), k \in A\} \in \mathcal{A}^l \quad (A \in \mathcal{A}).$$

Remark 2. Let $\lambda_l \in \Pi^l$, $\mathbf{t}_l = (t_1, \dots, t_l) \in \mathbb{Z}^l$, $\lambda \in \Pi$ and $t \in \mathbb{Z}$ be such that $(\lambda_l, \mathbf{t}_l) = \tau_l(\lambda, t)$. Then we have $t = t_1 + \dots + t_l$.

Recall that we have fixed $s = s_1 + \dots + s_l$. From now on, we write

$$\lambda_l \leftrightarrow \lambda$$

if $\lambda \in \Pi$ and $\lambda_l \in \Pi_m^l$ are related by $(\lambda_l, \mathbf{s}_l) = \tau_l(\lambda, s)$.

Example 6. Let $n = 2$, $l = 3$, $m = 5$ and $\mathbf{s}_l = (0, 0, -1)$. Then Figure 1 shows that we have

$$((1, 1), (1, 1), (1)) \leftrightarrow (4, 3, 3, 2, 1).$$

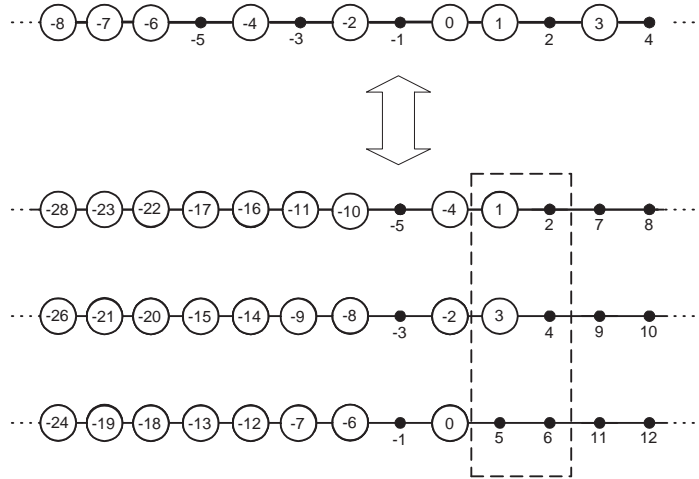


Figure 1: Computation of the bijection τ_l using abaci.

Remark 3. For another (equivalent) description of the bijection τ_l and for more examples, see [U2, Remark 4.2 (ii) & Example 4.3].

We define a partial ordering \preceq on Π_m^l as follows. Let $\lambda_l, \mu_l \in \Pi_m^l$, and $\lambda, \mu \in \Pi$ be such that $\lambda_l \leftrightarrow \lambda$ and $\mu_l \leftrightarrow \mu$. We say that λ_l precedes μ_l and we write $\lambda_l \preceq \mu_l$ if $\lambda \trianglelefteq \mu$. In particular, λ and μ must be partitions of the same integer. Notice that the ordering \preceq we defined depends on the multi-charge \mathbf{s}_l that we consider. We write $\lambda_l \prec \mu_l$ if $\lambda_l \preceq \mu_l$ and $\lambda_l \neq \mu_l$.

3.3 β -numbers and ribbons

We introduce the β -numbers associated to a charged multi-partition. Let $\lambda = (\lambda_1, \lambda_2, \dots) \in \Pi$ be a partition with at most r parts. The r -tuple

$$\beta_r(\lambda) := (\lambda_1 + s, \lambda_2 + s - 1, \dots, \lambda_r + s - r + 1) \in \mathbb{Z}^r$$

is called the r -list of β -numbers associated to (λ, s) or (with a slight abuse of notation) the list or sequence of β -numbers associated to λ . The set of integers that form $\beta_r(\lambda)$ is denoted by $B_r(\lambda)$. Notice that $\beta_r(\lambda)$ is a decreasing sequence of integers all greater than (or equal to) $s + 1 - r$. $\beta_r(\lambda)$ depends on the integer s we have fixed, but we do not mention it in our notation. A partition λ is completely determined by its sequence of β -numbers. If $r = |\lambda|$, we write more simply $\beta(\lambda) := \beta_r(\lambda)$ and $B(\lambda) := B_r(\lambda)$. If f is a function defined on \mathbb{Z}^r , it is convenient to consider f as a function (still denoted by f) defined on the set of partitions of r by the formula

$$f(\lambda) := f(\beta(\lambda)) \quad (\lambda \in \Pi, |\lambda| = r).$$

For example, we define this way for any partition λ the vectors $\mathbf{c}(\lambda)$, $\mathbf{d}(\lambda)$, etc (see Section 3.2).

In order to prove Theorem 4, we have to relate the adding/removal of a ribbon in a charged partition and the corresponding β -numbers. Let us recall a classical result on β -numbers (see e.g. [Mt, Lemma 5.26]).

Lemma 5 *Let ν and ξ be two partitions with at most r parts, and $\beta_r(\xi) = (\beta_1, \dots, \beta_r)$ be the list of β -numbers associated to ξ . Then the following statements are equivalent:*

- (i) $\nu \subset \xi$, and $\rho := \xi/\nu$ is a ribbon of length h ;
- (ii) There exist positive integers b and h such that

$$B_r(\nu) = \{\beta_1, \dots, \beta_{b-1}, \beta_b - h, \beta_{b+1}, \dots, \beta_r\}.$$

In this case, b is the row number of the tail of ρ and h is the length of ρ . Let $\sigma \in \mathfrak{S}_r$ denote the permutation obtained by arranging decreasingly the integers

$$(\beta_1, \dots, \beta_{b-1}, \beta_b - h, \beta_{b+1}, \dots, \beta_r).$$

Then we have $\ell(\sigma) = \text{ht}(\rho)$. Moreover, the content of the head of ρ is

$$\text{cont}(\text{hd}(\rho)) = \alpha_c = \beta_b - h,$$

where c is the row number of the head of ρ .

Proof. The proof of (i) \Rightarrow (ii) is easy. Conversely, assume that (ii) holds. Then we must have $\beta_b - h \geq s + 1 - r$, and there must exist $b \leq c \leq r$ such that $\beta_c > \beta_b - h > \beta_{c+1}$ (if $c = r$, we set $\beta_{c+1} := s - r$). Notice then that ν is obtained from ξ by removing a ribbon ρ , where $\rho \subset \xi$ is the ribbon whose head is located at row c of ξ and whose tail is located at row b of ξ . ρ is actually a ribbon of length h . Moreover, with the notation of the statement of this lemma, we have

$$\sigma.(\beta_1, \dots, \beta_{b-1}, \beta_b - h, \beta_{b+1}, \dots, \beta_r) = (\beta_1, \dots, \beta_{b-1}, \beta_{b+1}, \dots, \beta_c, \beta_b - h, \beta_{c+1}, \dots, \beta_r),$$

hence σ is a cycle of length $c - b = \text{ht}(\rho)$. Finally, the head of ρ has coordinates $(c, \nu_c + 1)$, so its content is equal to $\text{cont}(\text{hd}(\rho)) = s + (\nu_c + 1) - c = \alpha_c = \beta_b - h$. ■

Example 7. Let $s = 4$, $r = 5$, $\xi = (6, 5, 3, 2, 2)$ and $\nu = (6, 2, 2, 2, 2)$. Then $\rho := \xi/\nu$ is a ribbon and we have $b = 2$, $c = 3$ and $h = 4$. Moreover, we have $\beta(\xi) = (\beta_1, \dots, \beta_5) = (10, 8, 5, 3, 2)$ and $\beta(\nu) = (\beta_1, \beta_3, \beta_b - h, \beta_4, \beta_5) = (10, 5, 4, 3, 2)$. We have $\sigma = (2, 3)$, hence $\ell(\sigma) = 1 = \text{ht}(\rho)$. The head of ρ has coordinates $(3, 3)$, so its content is equal to $\text{cont}(\text{hd}(\rho)) = 4 = \beta_b - h$.

Lemma 6 *Let $\nu, \xi \in \Pi$ be such that $|\nu| = |\xi| = r$ and $\nu \neq \xi$. Let $\beta(\nu) = (\alpha_1, \dots, \alpha_r)$ and $\beta(\xi) = (\beta_1, \dots, \beta_r)$ denote the sequences of β -numbers associated to ν and ξ respectively. Set $\rho := \nu/(\nu \cap \xi)$ and $\rho' := \xi/(\nu \cap \xi)$.*

1) *Then, ρ and ρ' are two ribbons if and only if $\sharp(B(\nu) \cap B(\xi)) = r - 2$. In this case, denote by*

- . h the common length of ρ and ρ' ,
- . y the row number of the tail of ρ' ,
- . y' the row number of the head of ρ' ,
- . x' the row number of the tail of ρ , and
- . x the row number of the head of ρ .

Then we have $\{\alpha_i \mid i \neq x', y'\} = \{\beta_j \mid j \neq x, y\}$,

$$\text{cont}(\text{hd}(\rho)) = \beta_x = \alpha_{x'} - h \quad \text{and} \quad \text{cont}(\text{hd}(\rho')) = \alpha_{y'} = \beta_y - h.$$

Let $\pi \in \mathfrak{S}_r$ be the permutation obtained by arranging decreasingly the integers

$$(\beta_1, \dots, \beta_y - h, \dots, \beta_x + h, \dots, \beta_r)$$

forming $B(\nu)$. Then we have $\ell(\pi) = \text{ht}(\rho) + \text{ht}(\rho')$.

2) Assume that the conditions of 1) hold. Then we have the following equivalences, and moreover one of the two following cases occurs:

$$\begin{aligned} \text{(i)} \quad & y \leq y' < x' \leq x \iff \nu \triangleleft \xi, \\ \text{(ii)} \quad & x' \leq x < y \leq y' \iff \xi \triangleleft \nu. \end{aligned}$$

Proof. We prove 1) by applying the previous lemma to the pairs of partitions $(\nu \cap \xi, \nu)$ and $(\nu \cap \xi, \xi)$. Let us prove 2). The inequalities $y \leq y'$ and $x' \leq x$ are obvious. Since $\rho \cap \rho' = \emptyset$, one of the two following cases occurs: either $y' < x'$ and then $\nu \triangleleft \xi$, or $x < y$ and then $\xi \triangleleft \nu$. This proves both implications \Rightarrow , and since one of the two cases occurs, we get the desired equivalences. ■

4 Expressions of the matrices J^{\triangleleft} and J^{\triangleleft}

4.1 Definition of the matrix J^{\triangleleft}

Recall the notation from Section 2. Following [JM], we define a matrix $\mathcal{J} = (J_{\lambda_l, \mu_l})_{\lambda_l, \mu_l \in \Pi_m^l}$, with coefficients in R , depending on parameters $m, l \in \mathbb{N}$ and $v, u_1, \dots, u_l \in R$. Fix $\lambda_l = (\lambda^{(1)}, \dots, \lambda^{(l)})$, $\mu_l = (\mu^{(1)}, \dots, \mu^{(l)}) \in \Pi_m^l$, and consider the following cases.

- Case (J_1) . Assume that $\lambda_l \neq \mu_l$ and that there exist two integers $d, d' \in \llbracket 1, l \rrbracket$, $d \neq d'$ satisfying the following conditions: $\mu^{(d)} \subset \lambda^{(d)}$, $\lambda^{(d')} \subset \mu^{(d')}$, $\lambda^{(b)} = \mu^{(b)}$ for all $b \notin \{d, d'\}$, and $\rho := \lambda^{(d)}/\mu^{(d)}$ and $\rho' := \mu^{(d')}/\lambda^{(d')}$ are two ribbons of the same length \widehat{h} . Let $\text{hd}(\rho) = (i, j, d)$ denote the head of ρ and $\text{hd}(\rho') = (i', j', d')$ denote the head of ρ' . We set

$$\varepsilon := (-1)^{\text{ht}(\rho) + \text{ht}(\rho')} \quad \text{and} \quad J_{\lambda_l, \mu_l} := (u_d v^{j-i} - u_{d'} v^{j'-i'})^\varepsilon.$$

- Case (J_2) . Assume that $\lambda_l \neq \mu_l$ and that there exists $d \in \llbracket 1, l \rrbracket$ such that $\lambda^{(b)} = \mu^{(b)}$ for all $b \neq d$, and $\rho := \lambda^{(d)}/(\lambda^{(d)} \cap \mu^{(d)})$ and $\rho' := \mu^{(d)}/(\lambda^{(d)} \cap \mu^{(d)})$ are two ribbons of the same length \widehat{h} . By definition of ρ and ρ' , we have $\rho \cap \rho' = \emptyset$, whence we get (depending on the relative positions of ρ and ρ') that either $\lambda^{(d)} \triangleleft \mu^{(d)}$ or $\mu^{(d)} \triangleleft \lambda^{(d)}$. Assume that $\lambda^{(d)} \triangleleft \mu^{(d)}$. Let $\rho'' \subset (\lambda^{(d)} \cap \mu^{(d)})$ be the ribbon obtained by connecting the tail of ρ to the head of ρ' , excluding the two latter nodes (see Figure 2). Denote by $\text{hd}(\rho) = (i, j, d)$ (*resp.* $\text{hd}(\rho') = (i', j', d')$, *resp.* $\text{hd}(\rho'') = (i'', j'', d'')$) the head of ρ (*resp.* ρ' , *resp.* ρ''), and finally set

$$\begin{aligned} \varepsilon_1 &:= (-1)^{\text{ht}(\rho) + \text{ht}(\rho')}, & \varepsilon_2 &:= (-1)^{\text{ht}(\rho \cup \rho'') + \text{ht}(\rho'' \cup \rho')} \quad \text{and} \\ J_{\lambda_l, \mu_l} &:= (u_d (v^{j-i} - v^{j'-i'}))^{\varepsilon_1} \cdot (u_d (v^{j-i} - v^{j''-i''}))^{\varepsilon_2}. \end{aligned}$$

If $\mu^{(d)} \triangleleft \lambda^{(d)}$, we set $J_{\lambda_l, \mu_l} := J_{\mu_l, \lambda_l}$.

- Case (J_3) . In all other cases, we set

$$J_{\lambda_l, \mu_l} := 1.$$

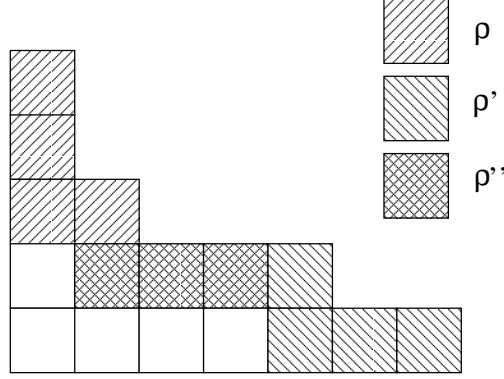


Figure 2: The ribbons ρ , ρ' and ρ'' (the nodes of $(\lambda^{(d)} \cap \mu^{(d)}) - \rho''$ are depicted in white).

Let \wp be a maximal ideal of R . We define a matrix $J = J_\wp = (j_{\lambda_l, \mu_l})_{\lambda_l, \mu_l \in \Pi_m^l}$, with integer coefficients, by the formula

$$j_{\lambda_l, \mu_l} := \nu_\wp(J_{\lambda_l, \mu_l}) \quad (\lambda_l, \mu_l \in \Pi_m^l).$$

Consider now a partial ordering \leq on Π_m^l , and write $\lambda_l < \mu_l$ if $\lambda_l \leq \mu_l$ and $\lambda_l \neq \mu_l$ ($\lambda_l, \mu_l \in \Pi_m^l$). We define a matrix $J^{<} = (j_{\lambda_l, \mu_l}^{<})_{\lambda_l, \mu_l \in \Pi_m^l}$ by the formulas

$$j_{\lambda_l, \mu_l}^{<} := \begin{cases} j_{\lambda_l, \mu_l} & \text{if } \lambda_l < \mu_l \\ 0 & \text{otherwise} \end{cases} \quad (\lambda_l, \mu_l \in \Pi_m^l).$$

Thus, the matrix $J^{<}$ depends on the same parameters as J , and also on \wp and $<$. When $< = \triangleleft$, the matrix J^{\triangleleft} described here is that given in [JM, Definition 3.36 & Remark 3.37] (up to a slight change of convention). If $<$ is the ordering \prec introduced in Section 3.2 and if R and the parameters $v, u_1, \dots, u_l \in R$ are as in Section 2.2, then we get the matrix J^{\prec} of Theorems 3 and 4.

From now on we fix the ring R and the parameters $v, u_1, \dots, u_l \in R$ as in Section 2.2, so the matrix J depends on n, l, m and s_l . The following lemma gives a simpler expression for J .

Lemma 7

1) Assume that (λ_l, μ_l) satisfies the conditions (J_1) . Then we have

$$j_{\lambda_l, \mu_l} = \begin{cases} (-1)^{\text{ht}(\rho) + \text{ht}(\rho')} & \text{if } \text{res}_n(\text{hd}(\rho)) = \text{res}_n(\text{hd}(\rho')), \\ 0 & \text{otherwise.} \end{cases}$$

2) Assume that (λ_l, μ_l) satisfies the conditions (J_2) . Then we have

$$j_{\lambda_l, \mu_l} = (-1)^{\text{ht}(\rho) + \text{ht}(\rho')} \varepsilon,$$

where

$$\varepsilon := \begin{cases} 1 & \text{if } \text{res}_n(\text{hd}(\rho)) = \text{res}_n(\text{hd}(\rho')) \quad \text{and} \quad \widehat{h} \not\equiv 0 \pmod{n}, \\ -1 & \text{if } \text{res}_n(\text{hd}(\rho)) \neq \text{res}_n(\text{hd}(\rho')) \quad \text{and} \quad \widehat{h} \equiv 0 \pmod{n}, \\ 0 & \text{otherwise,} \end{cases}$$

and \widehat{h} is the common length of ρ and ρ' .

Proof. Let us prove 1). Then we have

$$j_{\lambda_l, \mu_l} = (-1)^{\text{ht}(\rho) + \text{ht}(\rho')} \nu_{\wp}(P_{\lambda_l, \mu_l}(x)),$$

with $P_{\lambda_l, \mu_l}(x) := u_d x^{l(j-i)} - u_{d'} x^{l(j'-i')}$. Notice that

$$P_{\lambda_l, \mu_l}(x) = \zeta^{a_1} x^{a_2} - \zeta^{a_3} x^{a_4},$$

with $a_1 := dn$, $a_2 := ls_d - dn + l(j-i)$, $a_3 := d'n$ and $a_4 := ls_{d'} - d'n + l(j'-i')$. Using the fact that $\nu_{\wp}(x^N) = 0$ for all $N \in \mathbb{Z}$ and $x^N P_{\lambda_l, \mu_l}(x) \in \mathbb{C}[x]$ for a suitable $N \in \mathbb{Z}$, we get

$$\begin{aligned} \nu_{\wp}(P_{\lambda_l, \mu_l}(x)) &\geq 0, \\ \nu_{\wp}(P_{\lambda_l, \mu_l}(x)) &\geq 1 \iff P_{\lambda_l, \mu_l}(\zeta) = 0, \\ \nu_{\wp}(P_{\lambda_l, \mu_l}(x)) &\geq 2 \iff P_{\lambda_l, \mu_l}(\zeta) = P'_{\lambda_l, \mu_l}(\zeta) = 0. \end{aligned}$$

A straightforward computation shows that $\nu_{\wp}(P_{\lambda_l, \mu_l}(x)) \geq 1$ if and only if $l(s_d + j - i) \equiv l(s_{d'} + j' - i') \pmod{nl}$, that is if and only if $\text{res}_n(\text{hd}(\rho)) = \text{res}_n(\text{hd}(\rho'))$. Moreover, we have

$$\begin{aligned} \nu_{\wp}(P_{\lambda_l, \mu_l}(x)) \geq 2 &\iff \begin{cases} a_1 + a_2 \equiv a_3 + a_4 \pmod{nl} \\ a_2 \zeta^{a_1 + a_2 - 1} = a_4 \zeta^{a_3 + a_4 - 1} \end{cases} \\ &\iff \begin{cases} a_1 + a_2 \equiv a_3 + a_4 \pmod{nl} \\ a_2 = a_4 \end{cases} \\ &\iff \begin{cases} a_1 \equiv a_3 \pmod{nl} \\ a_2 = a_4 \end{cases}. \end{aligned}$$

But the condition $a_1 \equiv a_3 \pmod{nl}$ implies $d \equiv d' \pmod{l}$, which is impossible since d and d' are two distinct integers ranging from 1 to l . As a consequence, we have $\nu_{\wp}(P_{\lambda_l, \mu_l}(x)) \leq 1$, which proves 1). Let us now prove 2). With the notation of (J_2) , we have $\varepsilon_2 = -\varepsilon_1$, whence

$$j_{\lambda_l, \mu_l} = (-1)^{\text{ht}(\rho) + \text{ht}(\rho')} \nu_{\wp}(P_{\lambda_l, \mu_l}(x)), \quad \text{with}$$

$$\begin{aligned}
P_{\lambda_l, \mu_l}(x) &:= \frac{u_d(x^{l(j-i)} - x^{l(j'-i')})}{u_d(x^{l(j-i)} - x^{l(j''-i'')})} \\
&= \frac{x^{l((j'-i')-(j-i))} - 1}{x^{l((j''-i'')-(j-i))} - 1} = \frac{x^{l(\text{cont}(\text{hd}(\rho')) - \text{cont}(\text{hd}(\rho)))} - 1}{x^{\widehat{h}} - 1}.
\end{aligned}$$

In order to complete the proof, we only have to notice that for $N \in \mathbb{Z}$, we have $\nu_\varphi(x^{lN} - 1) = 1$ if nl divides lN , that is n divides N , and $\nu_\varphi(x^{lN} - 1) = 0$ otherwise. ■

4.2 A first approach for the cases (J_1) and (J_2)

We now give a characterization in terms of λ and μ of the pairs (λ_l, μ_l) that satisfy (J_1) or (J_2) .

Lemma 8 *Let $\lambda_l = (\lambda^{(1)}, \dots, \lambda^{(l)}) \in \Pi_m^l$ and $\mu_l = (\mu^{(1)}, \dots, \mu^{(l)}) \in \Pi_m^l$ be two multi-partitions, and $\lambda, \mu \in \Pi$ be such that $\lambda_l \leftrightarrow \lambda$ and $\mu_l \leftrightarrow \mu$. Then the following statements are equivalent:*

- (i) $\lambda \subset \mu$, and μ/λ is a ribbon,
- (ii) $\lambda_l \subset \mu_l$, and there exists $d \in \llbracket 1, l \rrbracket$ such that $\mu^{(d)}/\lambda^{(d)}$ is a ribbon and $\lambda^{(b)} = \mu^{(b)}$ for all $b \in \llbracket 1, l \rrbracket \setminus \{d\}$.

Proof. Let us prove (i) \Rightarrow (ii). By Lemma 5, passing from λ to μ amounts, as far as abacus diagrams are concerned, to passing from $A(\lambda, s)$ to $A(\mu, s)$ by moving a bead located at position k towards the right. As far as the l -runner abacus diagrams $A(\lambda_l, \mathbf{s}_l)$ and $A(\mu_l, \mathbf{s}_l)$ are concerned, this amounts to moving a bead located at position $\phi(k)$ on the runner $d := d(k)$ towards the right. This together with the Lemma 5 applied to $(\lambda^{(d)}, \mu^{(d)})$ proves Statement (ii). The converse is similar. ■

Applying twice the previous lemma and Lemma 6 yields the following corollary.

Corollary 9 *Let $\lambda_l, \mu_l \in \Pi_m^l$ be two distinct multi-partitions, and $\lambda, \mu \in \Pi$ be such that $\lambda_l \leftrightarrow \lambda$ and $\mu_l \leftrightarrow \mu$. Assume that $|\lambda| = |\mu| = r$. Then (λ_l, μ_l) satisfies (J_1) or (J_2) if and only if $\sharp(B(\lambda) \cap B(\mu)) = r - 2$.*

Let $\lambda_l, \mu_l \in \Pi_m^l$, and $\lambda, \mu \in \Pi$ be such that $\lambda_l \leftrightarrow \lambda$ and $\mu_l \leftrightarrow \mu$. Consider the statement

$$(\mathcal{H}) \quad : \quad \lambda_l \prec \mu_l \quad (\text{i.e. } \lambda \triangleleft \mu), \quad |\lambda| = |\mu| = r \quad \text{and} \quad \sharp(B(\lambda) \cap B(\mu)) = r - 2.$$

The previous corollary shows that if (λ_l, μ_l) does not satisfy (\mathcal{H}) , then we have $j_{\lambda_l, \mu_l}^{\prec} = 0$.

Notation. Let $\lambda_l = (\lambda^{(1)}, \dots, \lambda^{(l)}) \in \Pi_m^l$ and $\mu_l = (\mu^{(1)}, \dots, \mu^{(l)}) \in \Pi_m^l$ be two multi-partitions such that (λ_l, μ_l) satisfies (\mathcal{H}) . In this case, we use in the sequel the following notation. Denote by $\lambda, \mu \in \Pi$ the partitions such that $\lambda_l \leftrightarrow \lambda$ and $\mu_l \leftrightarrow \mu$. Let $(\alpha_1, \dots, \alpha_r)$

and $(\beta_1, \dots, \beta_r)$ denote the sequences of β -numbers associated to λ and μ respectively. By Lemma 6 applied to the pair $(\nu, \xi) = (\lambda, \mu)$, there exist positive integers y, y', x', x and h such that $\{\alpha_i \mid i \neq x', y'\} = \{\beta_j \mid j \neq x, y\}$, $\alpha_{y'} = \beta_y - h$ and $\alpha_{x'} = \beta_x + h$. Moreover, by Corollary 9, (λ_l, μ_l) satisfies (J_1) or (J_2) . Let $d, d' \in \llbracket 1, l \rrbracket$ and $\widehat{h} \in \mathbb{N}^*$ denote the integers introduced in the definition of the cases (J_1) and (J_2) (in the case (J_2) , we set $d' := d$). Finally, denote by γ , resp. $\delta \in \llbracket 0, nl - 1 \rrbracket$ the residue of $c(\beta_y) - c(\beta_x)$, resp. $n(d(\beta_y) - d(\beta_x))$ modulo nl .

Remark 4. Assume that (λ_l, μ_l) satisfies (\mathcal{H}) , and recall the above notation. Since $\lambda \triangleleft \mu$, Lemma 6 implies $y' < x'$, so we have

$$\beta_x < \beta_x + h = \alpha_{x'} < \alpha_{y'} = \beta_y - h < \beta_y .$$

These inequalities, together with $\{\alpha_i \mid i \neq x', y'\} = \{\beta_j \mid j \neq x, y\}$ and the fact that the β_i 's are pairwise distinct, imply that $\{\beta_x, \beta_y\} \cap B(\lambda) = \emptyset$.

Under assumption (\mathcal{H}) , the following lemma relates the β -numbers of $\lambda^{(d)}$, $\lambda^{(d')}$, $\mu^{(d)}$ and $\mu^{(d')}$ on the one hand to the β -numbers of λ and μ on the other hand.

Lemma 10 *Assume that (λ_l, μ_l) satisfies (\mathcal{H}) , and recall the corresponding notation.*

1) *Then, we have*

$$\{d(\beta_x), d(\beta_y)\} = \{d(\beta_x + h), d(\beta_y - h)\} = \{d, d'\}.$$

Moreover, for all $1 \leq b \leq l$, we have

$$\#\{1 \leq i \leq r \mid d(\alpha_i) = b\} = \#\{1 \leq i \leq r \mid d(\beta_i) = b\};$$

let r_b denote this common value.

2) *Assume that (λ_l, μ_l) satisfies (\mathcal{H}) and (J_1) . Then we have $d(\beta_x) = d$ and $d(\beta_y) = d'$. Let*

$$\begin{aligned} \beta_{r_d}(\lambda^{(d)}) &= (\gamma_1, \dots, \gamma_{r_d}), & \beta_{r_d}(\mu^{(d)}) &= (\delta_1, \dots, \delta_{r_d}), \\ \beta_{r_{d'}}(\lambda^{(d')}) &= (\gamma'_1, \dots, \gamma'_{r_{d'}}) & \text{and} & \quad \beta_{r_{d'}}(\mu^{(d')}) = (\delta'_1, \dots, \delta'_{r_{d'}}) \end{aligned}$$

denote the sequences of β -numbers associated to $\lambda^{(d)}$, $\mu^{(d)}$, $\lambda^{(d')}$ and $\mu^{(d')}$. Denote by b (resp. c) the row number of the tail (resp. head) of ρ , and denote by b' (resp. c') the row number of the tail (resp. head) of ρ' . The equalities $\{d(\beta_x), d(\beta_y)\} = \{d(\beta_x + h), d(\beta_y - h)\} = \{d, d'\}$ from 1) show that one of the following cases occurs.

– First case: we have $d(\beta_x) = d(\beta_x + h) = d$ and $d(\beta_y) = d(\beta_y - h) = d'$. Then we have

$$\delta_c = \phi(\beta_x), \quad \gamma_b = \phi(\beta_x + h), \quad \gamma'_{c'} = \phi(\beta_y - h), \quad \delta'_{b'} = \phi(\beta_y)$$

$$\text{and } \widehat{h} = \phi(\beta_x + h) - \phi(\beta_x) = \phi(\beta_y) - \phi(\beta_y - h).$$

– Second case: we have $d(\beta_x) = d(\beta_y - h) = d$ and $d(\beta_y) = d(\beta_x + h) = d'$. Then we have

$$\begin{aligned} \delta_c &= \phi(\beta_x), \quad \gamma_b = \phi(\beta_y - h), \quad \gamma'_{c'} = \phi(\beta_x + h), \quad \delta'_{b'} = \phi(\beta_y) \\ \text{and } \widehat{h} &= \phi(\beta_y - h) - \phi(\beta_x) = \phi(\beta_y) - \phi(\beta_x + h). \end{aligned}$$

3) Assume that $(\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l)$ satisfies (\mathcal{H}) and (J_2) . Denote by

$$\boldsymbol{\beta}_{r_d}(\lambda^{(d)}) = (\gamma_1, \dots, \gamma_{r_d}) \quad \text{and} \quad \boldsymbol{\beta}_{r_d}(\mu^{(d)}) = (\delta_1, \dots, \delta_{r_d})$$

the sequences of β -numbers associated to $\lambda^{(d)}$ and $\mu^{(d)}$. Let b (resp. c) denote the row number of the tail (resp. head) of ρ , and b' (resp. c') denote the row number of the tail (resp. head) of ρ' . Then we have $\lambda^{(d)} \triangleleft \mu^{(d)}$ and

$$\begin{aligned} \delta_c &= \phi(\beta_x), \quad \gamma_b = \phi(\beta_x + h), \quad \gamma_{c'} = \phi(\beta_y - h), \quad \delta_{b'} = \phi(\beta_y) \\ \text{and } \widehat{h} &= \phi(\beta_x + h) - \phi(\beta_x) = \phi(\beta_y) - \phi(\beta_y - h). \end{aligned}$$

Proof. We pass from $\boldsymbol{\mu}_l$ to $\boldsymbol{\lambda}_l$ by removing the ribbon ρ' and by adding the ribbon ρ . This amounts, as far as the abacus diagrams $A(\boldsymbol{\mu}_l, \mathbf{s}_l)$ and $A(\boldsymbol{\lambda}_l, \mathbf{s}_l)$ are concerned, to moving two beads (see the proof of Lemma 8). Moving these two beads amounts, as far as the abacus diagrams $A(\mu, s)$ and $A(\lambda, s)$ are concerned, to moving the beads located at positions $\{\beta_x, \beta_y\}$ towards the positions $\{\beta_x + h, \beta_y - h\}$, which proves the first two equalities of Statement 1). The last parts of Statement 1) come from this and from the equality $\{\alpha_i \mid i \neq x', y'\} = \{\beta_j \mid j \neq x, y\}$.

Let us now prove Statement 2). A careful analysis of the moves of the beads described above shows that more precisely the following properties hold:

- (i) $\{\gamma_b, \gamma'_{c'}\} = \{\phi(\beta_x + h), \phi(\beta_y - h)\}$ and $\{\delta'_{b'}, \delta_c\} = \{\phi(\beta_x), \phi(\beta_y)\}$.
- (ii) Let $k \in \{\beta_x, \beta_x + h, \beta_y - h, \beta_y\}$. Then we have $\phi(k) \in \{\gamma_b, \delta_c\}$ if and only if $d(k) = d$, and we have $\phi(k) \in \{\gamma'_{c'}, \delta'_{b'}\}$ if and only if $d(k) = d'$.

Let us first prove that $d(\beta_x) = d$. Assume that $d(\beta_x) = d'$. By (i) and (ii), we have $\phi(\beta_x) \in \{\delta'_{b'}, \delta_c\} \cap \{\gamma'_{c'}, \delta'_{b'}\}$. This implies $\phi(\beta_x) = \delta'_{b'}$. Indeed, if $\phi(\beta_x) \neq \delta'_{b'}$, we must have $\phi(\beta_x) = \delta_c = \gamma'_{c'} \in \{\phi(\beta_x + h), \phi(\beta_y - h)\}$ by (i). Let $k \in \{\beta_x + h, \beta_y - h\}$ be such that $\phi(k) = \gamma'_{c'} = \phi(\beta_x)$. By (ii), we have $d(k) = d(\beta_x)$; moreover, we have $\phi(k) = \phi(\beta_x)$, whence $k = \beta_x$. This contradicts Remark 4, so $\phi(\beta_x) = \delta'_{b'}$. Let $k \in \{\beta_x + h, \beta_y - h\}$ be such that $d(k) = d'$. By Property (P_2) and Remark 4, we have $\phi(k) > \phi(\beta_x) = \delta'_{b'}$, hence by (ii) we have $\phi(k) = \gamma'_{c'}$. As a consequence, we have $\gamma'_{c'} > \delta'_{b'}$. Moreover, by Lemma 5 applied to $(\lambda^{(d')}, \mu^{(d')})$, we have $\gamma'_{c'} = \delta'_{b'} - \widehat{h} < \delta'_{b'}$, which is absurd. By Statement 1), we thus have $d(\beta_x) = d$ and $d(\beta_y) = d'$. Let us now deal with the first case of Statement 2) (the proof for the second case is similar). By (ii) we have $\{\gamma_b, \delta_c\} = \{\phi(\beta_x), \phi(\beta_x + h)\}$. Moreover, by Lemma 5 applied to $(\lambda^{(d)}, \mu^{(d)})$, we have $\delta_c = \gamma_b - \widehat{h} < \gamma_b$. Property (P_2) then implies that

$\delta_c = \phi(\beta_x)$ and $\gamma_b = \phi(\beta_x + h)$, whence $\widehat{h} = \gamma_b - \delta_c = \phi(\beta_x + h) - \phi(\beta_x)$. By a similar argument, we get $\gamma_{c'} = \phi(\beta_y - h)$, $\delta_{b'} = \phi(\beta_y)$ and $\widehat{h} = \phi(\beta_y) - \phi(\beta_y - h)$.

Let us now prove 3). Since $\lambda \triangleleft \mu$, Remark 4 and Property (P_2) imply

$$\phi(\beta_x) < \phi(\beta_x + h) < \phi(\beta_y - h) < \phi(\beta_y).$$

Moreover, a careful analysis of the moves of the beads mentioned at the beginning of the proof shows that

$$\{\delta_{b'}, \delta_c\} = \{\phi(\beta_x), \phi(\beta_y)\} \quad \text{and} \quad \{\gamma_b, \gamma_{c'}\} = \{\phi(\beta_x + h), \phi(\beta_y - h)\}.$$

Assume that $\delta_c = \phi(\beta_y)$. Since $\phi(\beta_y) > \phi(\beta_x + h)$, $\phi(\beta_y) > \phi(\beta_y - h)$ and γ_b belongs to the set $\{\phi(\beta_x + h), \phi(\beta_y - h)\}$, we must have $\delta_c > \gamma_b$. Moreover, applying Lemma 6 to the pair $(\lambda^{(d)}, \mu^{(d)})$ yields $\gamma_b = \delta_c + \widehat{h} > \delta_c$, which is absurd. We thus have $\delta_c = \phi(\beta_x)$ and $\delta_{b'} = \phi(\beta_y)$. By Property (P_2) , we have $\delta_c < \delta_{b'}$, whence $c > b'$. Applying again Lemma 6 shows that $\lambda^{(d)} \triangleleft \mu^{(d)}$ and $b' \leq c' < b \leq c$. In particular, we have $c' < b$, whence $\gamma_b < \gamma_{c'}$. By Property (P_2) , we have $\gamma_b = \phi(\beta_x + h)$ and $\gamma_{c'} = \phi(\beta_y - h)$. Lemma 6 then implies that $\widehat{h} = \gamma_b - \delta_c = \phi(\beta_x + h) - \phi(\beta_x)$ and $\widehat{h} = \delta_{b'} - \gamma_{c'} = \phi(\beta_y) - \phi(\beta_y - h)$. ■

Lemma 11 *Let $\lambda_l, \mu_l \in \Pi_m^l$, and let $\lambda, \mu \in \Pi$ be such that $\lambda_l \leftrightarrow \lambda$ and $\mu_l \leftrightarrow \mu$. Assume that $j_{\lambda_l, \mu_l} \neq 0$. Then we have, with the notation from (J_1) , (J_2) and Section 3.1 the equalities*

$$N_i(\rho) = N_i(\rho') \quad (i \in \mathbb{Z}) \quad \text{and} \quad |\lambda| = |\mu|.$$

Proof. Since j_{λ_l, μ_l} is nonzero, Lemma 7 shows that at least one of the following cases occurs:

- First case: we have $\text{res}_n(\text{hd}(\rho)) = \text{res}_n(\text{hd}(\rho'))$. Notice that if ρ is a ribbon, then the integers $\text{cont}(\gamma), \gamma \in \rho$ are pairwise distinct, and the set formed by these numbers is exactly the interval $[\text{cont}(\text{hd}(\rho)), \text{cont}(\text{tl}(\rho))]$. Combining this with the assumption $\text{res}_n(\text{hd}(\rho)) = \text{res}_n(\text{hd}(\rho'))$ and $\widehat{h} = \ell(\rho) = \ell(\rho')$, we get that $N_i(\rho) = N_i(\rho')$ for any integer $i \in \mathbb{Z}$.
- Second case: we have $\widehat{h} \equiv 0 \pmod{n}$. Then for any $i \in \mathbb{Z}$, we have $N_i(\rho) = N_i(\rho') = \widehat{h}/n$.

Let us now show that $|\lambda| = |\mu|$. Let $\nu_l := \lambda_l \cap \mu_l$, and let $\nu \in \Pi$ be such that $\nu_l \leftrightarrow \nu$. We claim that

$$h := |\lambda| - |\nu| = ((n-1)l + 1)N_0(\rho) + (\widehat{h} - N_0(\rho)).$$

By induction on \widehat{h} , we can restrict ourselves to the case when $\widehat{h} = 1$, that is ρ contains a single node γ . Let $r \in \mathbb{N}$ be such that λ and ν have at most r parts. By Lemma 5, there exist $\alpha \in B_r(\nu)$ and $\beta \in B_r(\lambda)$ such that $B_r(\nu) \setminus \{\alpha\} = B_r(\lambda) \setminus \{\beta\}$ and $\alpha = \beta - h$. The abacus diagrams $A(\nu, s)$ and $A(\lambda, s)$ differ only by the moving of a bead; the same thing holds for the

diagrams $A(\nu_l, \mathbf{s}_l)$ and $A(\lambda_l, \mathbf{s}_l)$. By considering the initial and the final positions of these two beads, we get that

$$\phi(\beta) = \phi(\alpha) + \widehat{h} = \phi(\alpha) + 1 \quad \text{and} \quad d(\beta) = d(\alpha) = d.$$

Moreover, by Lemma 5 and Property (P_1) , we have $\text{res}_n(\gamma) = \phi(\alpha) \bmod n = \alpha \bmod n$. Let us now distinguish two cases. If $\text{res}_n(\gamma) = 0 \bmod n$ (i.e. if $N_0(\rho) = 1$), then we have $\alpha = n + n(d-1) + nlm$ with $m \in \mathbb{Z}$, whence $\phi(\beta) = \phi(\alpha) + 1 = 1 + n(m+1)$. Since $d(\beta) = d$, we get $\beta = 1 + n(d-1) + nl(m+1)$, whence $h = \beta - \alpha = (n-1)l + 1$. Similarly, if $\text{res}_n(\gamma) \neq 0 \bmod n$ (i.e. if $N_0(\rho) = 0$), then we have $h = 1$. This proves the claimed formula. In a similar way we prove that $|\mu| - |\nu| = ((n-1)l + 1)N_0(\rho') + (\widehat{h} - N_0(\rho'))$. Since $N_0(\rho) = N_0(\rho')$, we do have $|\lambda| = |\mu|$. ■

4.3 What happens if the multi-charge \mathbf{s}_l is dominant

Recall the partial ordering \preceq on Π_m^l defined in Section 3.2.

Lemma 12 *Let $\lambda_l, \mu_l \in \Pi_m^l$, and $\lambda, \mu \in \Pi$ be such that $\lambda_l \leftrightarrow \lambda$ and $\mu_l \leftrightarrow \mu$. Assume that $|\lambda| = |\mu|$. Consider the following cases:*

- 1) (λ_l, μ_l) satisfies (J_1) and \mathbf{s}_l is dominant,
- 2) (λ_l, μ_l) satisfies (J_2) .

Then in either case, we have

$$\lambda_l \prec \mu_l \iff \lambda_l \triangleleft \mu_l.$$

Proof. In either case, we can apply Corollary 9 and then Lemma 6 to get that $\lambda_l \prec \mu_l$ or $\mu_l \prec \lambda_l$. It is thus enough to prove the implication $\lambda_l \prec \mu_l \Rightarrow \lambda_l \triangleleft \mu_l$. Assume from now on that $\lambda_l \prec \mu_l$ and (λ_l, μ_l) satisfies (J_1) or (J_2) . Then by Corollary 9, (λ_l, μ_l) satisfies (\mathcal{H}) . Recall the notation from Section 4.2. If (λ_l, μ_l) satisfies (J_2) , then by Statement 3) of Lemma 10, we have $\lambda^{(d)} \triangleleft \mu^{(d)}$. Moreover, for all $b \in \llbracket 1, l \rrbracket \setminus \{d\}$ we have $\lambda^{(b)} = \mu^{(b)}$, whence $\lambda_l \triangleleft \mu_l$. Assume now that (λ_l, μ_l) satisfies (J_1) . The key point of the proof is the following. Let $\nu_l = (\nu^{(1)}, \dots, \nu^{(l)}) \in \Pi_m^l$, and $\nu \in \Pi$ be the partition such that $\nu_l \leftrightarrow \nu$. Then under the assumption that \mathbf{s}_l is dominant, we have

$$(*) \quad (d(k) < d(k'), k, k' \in B(\nu)) \Rightarrow \phi(k) > \phi(k').$$

Indeed, let $k, k' \in B(\nu)$, $b := d(k)$, $b' := d(k')$ and N (resp. N') be the number of parts of $\nu^{(b)}$ (resp. $\nu^{(b')}$). Since $\nu_l \leftrightarrow \nu$, we have $\phi(k) \in B_N(\nu^{(b)})$ and $\phi(k') \in B_{N'}(\nu^{(b')})$. As a consequence, there exist $i \in \llbracket 1, N \rrbracket$, $i' \in \llbracket 1, N' \rrbracket$ such that $\phi(k) = s_b + \nu_i^{(b)} - i + 1$ and $\phi(k') = s_{b'} + \nu_{i'}^{(b')} - i' + 1$. Since \mathbf{s}_l is dominant, we have

$$\begin{aligned} \phi(k) - \phi(k') &= (s_b - s_{b'}) + (i' + \nu_i^{(b)}) - (i + \nu_{i'}^{(b')}) \\ &> s_b - s_{b'} - (N + |\nu^{(b')}|) \quad \geq \quad s_b - s_{b'} - |\nu_l| \\ &\geq 0, \end{aligned}$$

which shows (*). Since (λ_l, μ_l) satisfies (\mathcal{H}) , Remark 4 implies $\beta_x < \beta_y$. Since \mathbf{s}_l is dominant, Statement (*) and Property (P_3) imply $d(\beta_x) \geq d(\beta_y)$. By Statement 2) of Lemma 10, we have $d(\beta_x) = d$ and $d(\beta_y) = d'$, whence $d > d'$. Moreover, we have $|\mu^{(d')}| = |\lambda^{(d')}| + \widehat{h}$, $|\mu^{(d)}| = |\lambda^{(d)}| - \widehat{h}$ and $\mu^{(b)} = \lambda^{(b)}$ for all $b \in \llbracket 1, l \rrbracket \setminus \{d, d'\}$. This and the inequality $d' < d$ imply $\lambda_l \triangleleft \mu_l$. ■

Proposition 13 *Assume that \mathbf{s}_l is dominant. Then we have $J^{\prec} = J^{\triangleleft}$.*

Proof. Let $\lambda_l, \mu_l \in \Pi_m^l$, and $\lambda, \mu \in \Pi$ be such that $\lambda_l \leftrightarrow \lambda$ and $\mu_l \leftrightarrow \mu$. It is enough to prove the equivalence $j_{\lambda_l, \mu_l}^{\prec} \neq 0 \Leftrightarrow j_{\lambda_l, \mu_l}^{\triangleleft} \neq 0$, in which case we have by definition $j_{\lambda_l, \mu_l}^{\prec} = j_{\lambda_l, \mu_l}^{\triangleleft} = j_{\lambda_l, \mu_l}$. Assume that $j_{\lambda_l, \mu_l}^{\triangleleft} \neq 0$. Then we have $\lambda_l \triangleleft \mu_l$, and by Lemma 11, we have $|\lambda| = |\mu|$. Lemma 12 then implies that $\lambda_l \prec \mu_l$, whence $j_{\lambda_l, \mu_l}^{\prec} = j_{\lambda_l, \mu_l} \neq 0$. The converse is similar. ■

Remark 5. The reader should be warned that the orderings \prec and \triangleleft do not necessarily coincide, even if \mathbf{s}_l is dominant. For example, let $n = 2$, $l = 2$, $m = 6$, $\mathbf{s}_l = (3, -3)$, $\lambda_l = ((2, 1), (1, 1, 1))$ and $\mu_l = ((3), (2, 1))$. Then we have $\lambda_l \triangleleft \mu_l$; however, the partitions λ and μ such that $\lambda_l \leftrightarrow \lambda$ and $\mu_l \leftrightarrow \mu$ are $\lambda = (9, 6, 3, 1, 1, 1, 1, 1, 1, 1)$ and $\mu = (10, 3, 3, 2, 2, 2, 1, 1, 1)$, so λ_l and μ_l are not comparable with respect to \preceq .

4.4 Expression of the coefficients of J^{\prec}

Using Lemma 7, we now derive the expressions of the $j_{\lambda_l, \mu_l}^{\prec}$'s that we need for proving Theorem 4.

Proposition 14 *Assume that (λ_l, μ_l) satisfies (J_1) and $\lambda_l \prec \mu_l$. Then with the notation from Section 4.2, we have $\delta > 0$, and*

$$j_{\lambda_l, \mu_l}^{\prec} = \begin{cases} (-1)^{\text{ht}(\rho) + \text{ht}(\rho')} & \text{if } h \equiv \gamma \pmod{nl} \quad \text{or} \quad h \equiv \delta \pmod{nl}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recall the notation from Lemma 10. The statement $\delta > 0$ comes from Statement 1) of that lemma. Applying Lemma 5 to the pairs $(\lambda^{(d)}, \mu^{(d)})$ and $(\lambda^{(d')}, \mu^{(d')})$ yields

$$\text{res}_n(\text{hd}(\rho)) = \delta_c \pmod{n} \quad \text{and} \quad \text{res}_n(\text{hd}(\rho')) = \gamma'_c \pmod{n}.$$

By Lemma 7, it is thus enough to prove the equivalence

$$\delta_c \equiv \gamma'_c \pmod{n} \iff (h \equiv \gamma \pmod{nl} \quad \text{or} \quad h \equiv \delta \pmod{nl}).$$

Let us consider again both cases from Statement 2) of Lemma 10.

- First case: we have $d(\beta_x) = d(\beta_x + h)$ and $d(\beta_y) = d(\beta_y - h)$. By Lemma 10 and Property (P_1) , we have the following equivalences, where congruences stand modulo n :

$$\delta_c \equiv \gamma'_{c'} \iff \phi(\beta_x) \equiv \phi(\beta_y - h) \iff \beta_x \equiv \beta_y - h \iff h \equiv \gamma.$$

It remains thus to prove that $\delta_c \equiv \gamma'_{c'} \pmod{n} \implies h \equiv \gamma \pmod{nl}$. Assume that $\delta_c \equiv \gamma'_{c'} \pmod{n}$; we then have $h \equiv \gamma \pmod{n}$. This and the equality $d(\beta_x) = d(\beta_x + h)$ force $d(h) = 1$, whence $h \equiv \gamma \pmod{nl}$.

- Second case: we have $d(\beta_x) = d(\beta_y - h)$ and $d(\beta_y) = d(\beta_x + h)$. By arguing as above we prove the equivalence

$$\delta_c \equiv \gamma'_{c'} \pmod{n} \iff h \equiv 0 \pmod{n}.$$

Assume that $\delta_c \equiv \gamma'_{c'} \pmod{n}$. Then we have $h \equiv 0 \pmod{n}$, whence

$$d(\beta_y) = d(\beta_x + h) \equiv d(\beta_x) + d(h) - 1 \pmod{l},$$

and then $n(d(h) - 1) \equiv n(d(\beta_y) - d(\beta_x)) \equiv \delta \pmod{nl}$. We thus do have $h \equiv \delta \pmod{nl}$. ■

Proposition 15 *Assume that (λ_l, μ_l) satisfies (J_2) and $\lambda_l \prec \mu_l$. Then with the notation from Section 4.2, we have $\delta = 0$ and*

$$j_{\lambda_l, \mu_l}^{\prec} = (-1)^{\text{ht}(\rho) + \text{ht}(\rho')} \varepsilon,$$

where

$$\varepsilon := \begin{cases} 1 & \text{if } h \equiv \gamma \pmod{nl} \text{ and } h \not\equiv 0 \pmod{nl}, \\ -1 & \text{if } h \not\equiv \gamma \pmod{nl} \text{ and } h \equiv 0 \pmod{nl}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Similar to the proof of the previous proposition. ■

5 Higher-level Fock spaces

In this section we follow [U2], to which we refer the reader for more details. The vector spaces we consider here are over $\mathbf{K} = \mathbb{C}(q)$, where q is an indeterminate over \mathbb{C} .

5.1 q -wedge products and higher-level Fock spaces

Let Λ^s denote the (semi-infinite) q -wedge space of charge s (this space is denoted by $\Lambda^{s+\frac{\infty}{2}}$ in [U2]). Λ^s is an integrable representation of level l of the quantum algebra $U_q(\widehat{\mathfrak{sl}}_n)$. As a

vector space, it has a natural basis formed by the so-called *ordered q -wedge products*. These vectors can be written as

$$u_{\mathbf{k}} = u_{k_1} \wedge u_{k_2} \wedge \cdots ,$$

where $\mathbf{k} = (k_i)_{i \geq 1}$ is a decreasing sequence of integers such that $k_i = s + 1 - i$ for $i \gg 0$. The basis formed by the ordered wedge products is called *standard*. More generally, we use the *non-ordered wedge products*; a non-ordered wedge product $u_{\mathbf{k}} = u_{k_1} \wedge u_{k_2} \wedge \cdots \in \Lambda^s$ is indexed by a sequence of integers (k_i) such that $k_i = s + 1 - i$ for $i \gg 0$, but we no longer require that (k_i) is decreasing. Any non-ordered wedge product can be written as a linear combination of ordered wedge products by using the so-called *ordering rules*, which are given in [U2, Proposition 3.16] and in a slightly different form in Section 5.4.

The vectors of the standard basis of Λ^s can also be indexed by partitions as follows. Let $u_{\mathbf{k}} = u_{k_1} \wedge u_{k_2} \wedge \cdots \in \Lambda^s$ be an ordered wedge product. For $i \geq 1$ we set $\lambda_i := k_i - s + i - 1$; then $\lambda := (\lambda_1, \lambda_2, \dots)$ is a partition, and we write $u_{\mathbf{k}} = |\lambda, s\rangle$. Notice that if λ has at most r parts, then we have $(k_1, \dots, k_r) = \beta_r(\lambda)$, which explains the definition of the β -numbers we gave in Section 3.3.

Let $\mathbf{F}_q[\mathbf{s}_l]$ be the higher-level Fock space with multi-charge \mathbf{s}_l [U2]. As a vector space, $\mathbf{F}_q[\mathbf{s}_l]$ has a natural basis $\{|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle, \boldsymbol{\lambda}_l \in \Pi^l\}$ indexed by l -multi-partitions. Recall that we have $s = s_1 + \cdots + s_l$. Then $\mathbf{F}_q[\mathbf{s}_l]$ can be identified with a subspace of Λ^s by the embedding $\mathbf{F}_q[\mathbf{s}_l] \hookrightarrow \Lambda^s, |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle \mapsto |\lambda, s\rangle$, where λ is the partition such that $\boldsymbol{\lambda}_l \leftrightarrow \lambda$ (see Section 3.2 for the notation \leftrightarrow). We make from now on this identification; in fact, Λ^s is isomorphic as a vector space to the direct sum of all the $\mathbf{F}_q[\mathbf{t}_l]$'s, where \mathbf{t}_l is any l -tuple of integers summing to s . Thus, the vectors of the standard basis of Λ^s can also be indexed by charged l -multi-partitions.

5.2 The involution $-$

In order to define the canonical basis of Λ^s , we equip this space with an involution, denoted by $-$. This map is the \mathbb{C} -vector space automorphism that maps q to q^{-1} and that acts on the standard basis of Λ^s as follows [U2, Proposition 3.23 & Remark 3.24]. Let $\lambda \in \Pi$ be a partition of r , and $\mathbf{k} = (k_i) \in \mathbb{Z}^{\mathbb{N}^*}$ be such that $u_{\mathbf{k}} = |\lambda, s\rangle$. Then we have

$$\overline{|\lambda, s\rangle} = (-1)^{\kappa(\mathbf{d}(\lambda))} q^{\kappa(\mathbf{d}(\lambda)) - \kappa(\mathbf{c}(\lambda))} (u_{k_r} \wedge \cdots \wedge u_{k_1}) \wedge u_{k_{r+1}} \wedge u_{k_{r+2}} \wedge \cdots ,$$

where for any $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r$, $\kappa(\mathbf{a})$ is the integer defined by

$$\kappa(\mathbf{a}) := \#\{(i, j) \in \mathbb{N}^2 \mid 1 \leq i < j \leq r, a_i = a_j\} .$$

We can then straighten the non-ordered wedge product in the right-hand side of the above expression in order to express it as a linear combination of ordered wedge products.

One checks that $\bar{\cdot}$ preserves the subspace

$$\mathbf{F}_q[\mathbf{s}_l]_m := \bigoplus_{\boldsymbol{\lambda}_l \in \Pi_m^l} \mathbf{K}|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle \subset \mathbf{F}_q[\mathbf{s}_l].$$

We can then define a matrix $A(q) = (a_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}(q))_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi_m^l}$ with coefficients in \mathbf{K} by

$$\overline{|\boldsymbol{\mu}_l, \mathbf{s}_l\rangle} = \sum_{\boldsymbol{\lambda}_l \in \Pi_m^l} a_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}(q) |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle \quad (\boldsymbol{\mu}_l \in \Pi_m^l);$$

the matrix $A(q)$ depends on n , l , \mathbf{s}_l and m . The ordering rules show that $A(q)$ is lower-unitriangular, that is $a_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}(q) \neq 0 \Rightarrow \boldsymbol{\mu}_l \succeq \boldsymbol{\lambda}_l$ and $a_{\boldsymbol{\lambda}_l, \boldsymbol{\lambda}_l}(q) = 1$ for all pairs of multipartitions $(\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l) \in (\Pi_m^l)^2$. The same rules also imply that $A(1)$ is the identity matrix.

5.3 Uglov's canonical basis

Since the matrix $A(q)$ of the involution of $\mathbf{F}_q[\mathbf{s}_l]_m$ is unitriangular, a classical argument can be used to prove the following result [U2].

Theorem 16 *There exists a unique basis $\{\mathcal{G}(\boldsymbol{\lambda}_l, \mathbf{s}_l), \boldsymbol{\lambda}_l \in \Pi_m^l\}$ of $\mathbf{F}_q[\mathbf{s}_l]_m$ satisfying both following conditions:*

- (i) $\overline{\mathcal{G}(\boldsymbol{\lambda}_l, \mathbf{s}_l)} = \mathcal{G}(\boldsymbol{\lambda}_l, \mathbf{s}_l)$,
- (ii) $\mathcal{G}(\boldsymbol{\lambda}_l, \mathbf{s}_l) - |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle \in \bigoplus_{\boldsymbol{\mu}_l \in \Pi_m^l} q \mathbb{C}[q] |\boldsymbol{\mu}_l, \mathbf{s}_l\rangle$.

The basis $\{\mathcal{G}(\boldsymbol{\lambda}_l, \mathbf{s}_l), \boldsymbol{\lambda}_l \in \Pi_m^l\}$ is called the *canonical basis* of $\mathbf{F}_q[\mathbf{s}_l]_m$. We define a matrix $\Delta(q) = (\Delta_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}(q))_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi_m^l}$ with coefficients in $\mathbb{C}[q]$ by

$$\mathcal{G}(\boldsymbol{\mu}_l, \mathbf{s}_l) = \sum_{\boldsymbol{\lambda}_l \in \Pi_m^l} \Delta_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}(q) |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle \quad (\boldsymbol{\mu}_l \in \Pi_m^l);$$

the matrix $\Delta(q)$ depends on n , l , \mathbf{s}_l and m . Owing to the condition (ii) written above, the matrix $\Delta(q)$ is also lower-unitriangular. By [U2, Theorem 3.26], the coefficients of $\Delta(q)$ can be expressed as Kazhdan-Lusztig polynomials related to a parabolic module of an affine Hecke algebra of type \tilde{A} , so by [KT], these coefficients belong to $\mathbb{N}[q]$.

5.4 Another basis of Λ^s . Ordering rules.

The ordering rules (R_1) - (R_4) from [U2, Proposition 3.16] do not give at $q = 1$ anticommuting relations like $u_{k_1} \wedge u_{k_2} = -u_{k_2} \wedge u_{k_1}$, because of the signs involved in Rules (R_3) and (R_4) . To fix this, we introduce another basis of Λ^s that differs from the standard basis only by

signs. The basis we consider here is actually the basis of ordered wedge products introduced in [U1]. Λ^s is graded by

$$\deg(|\lambda, s\rangle) := |\lambda| \quad (\lambda \in \Pi).$$

Let $u_{\mathbf{k}} = u_{k_1} \wedge u_{k_2} \wedge \cdots \in \Lambda^s$ be a (not necessarily ordered) wedge product of degree r . We set

$$\begin{aligned} v_{\mathbf{k}} &= v_{k_1} \wedge v_{k_2} \wedge \cdots := (-1)^{\ell(\mathbf{v}(k_1, \dots, k_r))} u_{\mathbf{k}} \\ \text{and similarly} \quad v_{k_1} \wedge \cdots \wedge v_{k_r} &:= (-1)^{\ell(\mathbf{v}(k_1, \dots, k_r))} u_{k_1} \wedge \cdots \wedge u_{k_r}, \end{aligned}$$

where $\mathbf{v}(k_1, \dots, k_r)$ is defined in Section 3.2. We say that the wedge product $v_{\mathbf{k}}$ is *ordered* if so is $u_{\mathbf{k}}$. It is straightforward to see, using the ordering rules for the $u_{\mathbf{k}}$'s given by [U2], that the ordering rules for the $v_{\mathbf{k}}$'s are given by the next proposition.

Proposition 17

(i) Let $k_1 \leq k_2$, and $\gamma \in \llbracket 0, nl - 1 \rrbracket$ (resp. $\delta \in \llbracket 0, nl - 1 \rrbracket$) denote the residue of $c(k_2) - c(k_1)$ (resp. of $n(d(k_2) - d(k_1))$) modulo nl . Then we have

$$(R_1) \quad v_{k_1} \wedge v_{k_2} = -v_{k_2} \wedge v_{k_1} \quad \text{if } \gamma = \delta = 0,$$

$$(R_2) \quad \begin{aligned} v_{k_1} \wedge v_{k_2} &= -q^{-1} v_{k_2} \wedge v_{k_1} \\ &\quad - (q^{-2} - 1) \sum_{i \geq 1} q^{-2i+1} v_{k_2-ni} \wedge v_{k_1+ni} \\ &\quad + (q^{-2} - 1) \sum_{i \geq 0} q^{-2i} v_{k_2-\gamma-ni} \wedge v_{k_1+\gamma+ni} \end{aligned} \quad \text{if } \gamma > 0, \delta = 0,$$

$$(R_3) \quad \begin{aligned} v_{k_1} \wedge v_{k_2} &= -q v_{k_2} \wedge v_{k_1} \\ &\quad - (q^2 - 1) \sum_{i \geq 1} q^{2i-1} v_{k_2-ni} \wedge v_{k_1+ni} \\ &\quad + (q^2 - 1) \sum_{i \geq 0} q^{2i} v_{k_2-\delta-ni} \wedge v_{k_1+\delta+ni} \end{aligned} \quad \text{if } \gamma = 0, \delta > 0,$$

(R₄)

$$\begin{aligned}
v_{k_1} \wedge v_{k_2} &= -v_{k_2} \wedge v_{k_1} \\
&- (q - q^{-1}) \sum_{i \geq 1} \frac{q^{2i} - q^{-2i}}{q + q^{-1}} v_{k_2 - ni} \wedge v_{k_1 + ni} \\
&- (q - q^{-1}) \sum_{i \geq 0} \frac{q^{2i+1} + q^{-2i-1}}{q + q^{-1}} v_{k_2 - \gamma - ni} \wedge v_{k_1 + \gamma + ni} \quad \text{if } \gamma > 0, \delta > 0, \\
&+ (q - q^{-1}) \sum_{i \geq 0} \frac{q^{2i+1} + q^{-2i-1}}{q + q^{-1}} v_{k_2 - \delta - ni} \wedge v_{k_1 + \delta + ni} \\
&+ (q - q^{-1}) \sum_{i \geq 0} \frac{q^{2i+2} - q^{-2i-2}}{q + q^{-1}} v_{k_2 - \gamma - \delta - ni} \wedge v_{k_1 + \gamma + \delta + ni}
\end{aligned}$$

where the sums range over the indices i such that the corresponding wedge products are ordered.

(ii) The rules from (i) are valid for any pair of adjacent factors of the wedge product $v_{\mathbf{k}} = v_{k_1} \wedge v_{k_2} \cdots$.

Let us end this section by a useful piece of notation. Let $\nu \in \Pi$ be a partition of r and $\sigma \in \mathfrak{S}_r$. We set $v_{\sigma, \nu} := v_{\sigma, \beta(\nu)}$ (it is a wedge product of r factors), and we define in a similar way $u_{\sigma, \nu}$. We say that $v_{\sigma, \nu}$ is obtained from v_{ν} by permutation.

6 Admissible sequences, good sequences

6.1 Definitions

Let us fix throughout this section an integer $r \geq 1$. Let $\mathbf{k} \in \mathbb{Z}^r$ be a (not necessarily decreasing) sequence of r integers. We examine in detail the straightening of the wedge product $v_{\mathbf{k}} = v_{k_1} \wedge \cdots \wedge v_{k_r}$. If $v_{\mathbf{k}}$ is not ordered, there are in general several ways to straighten it by applying recursively the rules (R₁)-(R₄). In the sequel, we decide to straighten at each step the *first* infraction that occurs in $v_{\mathbf{k}}$, that is, the first $v_{k_i} \wedge v_{k_{i+1}}$ with $k_i \leq k_{i+1}$. This leads to the notion of admissible sequence that we now introduce.

Let $\mathbf{k} = (k_1, \dots, k_r)$ and $\mathbf{l} = (l_1, \dots, l_r) \in \mathbb{Z}^r$. We say that the wedge products $v_{\mathbf{k}}$ and $v_{\mathbf{l}}$ are *adjacent* if there exist $1 \leq i \leq r - 1$ (i is then necessarily unique) such that:

- (i) $k_i \leq k_{i+1}$, and $k_j > k_{j+1}$ for all $1 \leq j \leq i - 1$,
- (ii) $k_j = l_j$ for all $j \in \llbracket 1, r \rrbracket \setminus \{i, i + 1\}$,

(iii) the wedge product $v_{l_i} \wedge v_{l_{i+1}}$ appears in the straightening of $v_{k_i} \wedge v_{k_{i+1}}$.

In this case, denote by $t \in \llbracket 1, 4 \rrbracket$ the index of the rule (R_t) applied for the straightening of $v_{k_i} \wedge v_{k_{i+1}}$ and by $\alpha(v_{\mathbf{k}}, v_{\mathbf{1}}) \in \mathbb{Z}[q, q^{-1}]$ the coefficient of $v_{l_i} \wedge v_{l_{i+1}}$ in the resulting linear combination. If we have $(l_i, l_{i+1}) = (k_{i+1}, k_i)$, then we write $v_{\mathbf{k}} \xrightarrow{\bullet} v_{\mathbf{1}}$ and we set $m(v_{\mathbf{k}}, v_{\mathbf{1}}) := 0$. Otherwise, we write $v_{\mathbf{k}} \xrightarrow{t} v_{\mathbf{1}}$ and we set $m(v_{\mathbf{k}}, v_{\mathbf{1}}) := 1$; notice that $t \geq 2$ in this case. In either case we sometimes write more simply $v_{\mathbf{k}} \rightarrow v_{\mathbf{1}}$.

We say that the sequence $\mathbf{V} = (v_{\mathbf{k}_i})_{0 \leq i \leq N}$ is *admissible* if each $v_{\mathbf{k}_i}$ is a wedge product of r factors and if we have

$$v_{\mathbf{k}_0} \rightarrow v_{\mathbf{k}_1} \rightarrow \cdots \rightarrow v_{\mathbf{k}_N};$$

in this case, N is called the *length* of the sequence \mathbf{V} and we set

$$\alpha_{\mathbf{V}}(q) := \prod_{i=1}^N \alpha(v_{\mathbf{k}_{i-1}}, v_{\mathbf{k}_i}) \in \mathbb{Z}[q, q^{-1}] \quad \text{and} \quad m(\mathbf{V}) := \sum_{i=1}^N m(v_{\mathbf{k}_{i-1}}, v_{\mathbf{k}_i}) \in \mathbb{N}.$$

Remark 6. The condition (i) from the definition of adjacent wedge products shows that there exists at most one admissible sequence \mathbf{V} having a given length and starting at a given wedge product such that $m(\mathbf{V}) = 0$.

Let $\lambda, \mu \in \Pi$ be two partitions of r , and recall the definition of $\omega \in \mathfrak{S}_r$. We say that the sequence of wedge products $\mathbf{V} = (v_{\mathbf{k}_i})_{0 \leq i \leq N}$ is (λ, μ) -*admissible* if it is an admissible sequence of wedge products (of r factors for each of them) such that $v_{\mathbf{k}_N} = v_{\lambda}$ and $v_{\mathbf{k}_0} = v_{\omega \cdot \mu}$.

Remark 7. It is easy to see that if $\lambda \neq \mu$, then there cannot exist any (λ, μ) -admissible sequence $\mathbf{V} = (v_{\mathbf{k}_i})_{0 \leq i \leq N}$ such that $m(\mathbf{V}) = 0$.

A (λ, μ) -admissible sequence \mathbf{V} such that $m(\mathbf{V}) = 1$ is called a *good* sequence (with respect to (λ, μ)). Such a sequence can be written as

$$v_{\omega \cdot \mu} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} \mathbf{u} \xrightarrow{t} \mathbf{v} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_{\lambda},$$

with $t \in \llbracket 2, 4 \rrbracket$.

Remark 8. If a good sequence (with respect to (λ, μ)) exists, then we have

$$\sharp(B(\lambda) \cap B(\mu)) = r - 2.$$

6.2 Reduction to the good sequences

Expressing the involution of $\mathbf{F}_q[\mathbf{s}_l]_m$ in terms of the $v_{\mathbf{k}}$'s and then using the definition of (λ, μ) -admissible sequences yields the following expression for the coefficients of $A(q)$.

Lemma 18 *Let $\lambda_l, \mu_l \in \Pi_m^l$ and $\lambda, \mu \in \Pi$ be the partitions such that $\lambda_l \leftrightarrow \lambda$ and $\mu_l \leftrightarrow \mu$. Assume that $|\lambda| = |\mu|$. Then we have*

$$a_{\lambda_l, \mu_l}(q) = \varepsilon(\lambda_l, \mu_l) q^{\kappa(\mathbf{d}(\mu)) - \kappa(\mathbf{c}(\mu))} \sum_{\mathbf{V}} \alpha_{\mathbf{V}}(q),$$

where the sum ranges over all (λ, μ) -admissible sequences \mathbf{V} , and $\varepsilon(\lambda_l, \mu_l)$ is the sign defined by

$$\varepsilon(\lambda_l, \mu_l) := (-1)^{\kappa(\mathbf{d}(\mu)) + \ell(\mathbf{v}(\omega, \mu)) + \ell(\mathbf{v}(\lambda))}.$$

Remark 9. One should be aware that in general several terms might contribute to this sum, so the statement at the beginning of [RH, Section 4] is not correct. However, we can fix the argument from [RH] by showing first that only good sequences do contribute to $a'_{\lambda_l, \mu_l}(1)$ (see next lemma), and then that there exists at most one good sequence (see Propositions 21 and 22 from next section).

Lemma 19 *Let $\lambda_l, \mu_l \in \Pi_m^l$ be two distinct multi-partitions of m , and $\lambda, \mu \in \Pi$ be the partitions such that $\lambda_l \leftrightarrow \lambda$ and $\mu_l \leftrightarrow \mu$. Assume that $|\lambda| = |\mu| = r$. Then we have*

$$a'_{\lambda_l, \mu_l}(1) = \varepsilon(\lambda_l, \mu_l) \sum_{\mathbf{V}} \alpha'_{\mathbf{V}}(1),$$

where the sum ranges over all the good sequences with respect to (λ, μ) , and $\varepsilon(\lambda_l, \mu_l)$ is the sign defined in Lemma 18.

Proof. By the previous lemma, we have

$$a_{\lambda_l, \mu_l}(q) = \varepsilon(\lambda_l, \mu_l) \sum_{\mathbf{V}} f_{\mathbf{V}}(q),$$

where the sum ranges over all the (λ, μ) -admissible sequences \mathbf{V} and $f_{\mathbf{V}}(q)$ is the Laurent polynomial defined by $f_{\mathbf{V}}(q) := q^{\kappa(\mathbf{d}(\mu)) - \kappa(\mathbf{c}(\mu))} \alpha_{\mathbf{V}}(q)$. Notice that if \mathbf{V} is an admissible sequence, then we have $m(\mathbf{V}) \geq 1$ (because $\lambda \neq \mu$), and moreover the rules (R_1) - (R_4) imply that

$$\alpha_{\mathbf{V}}(q) \in (q^2 - 1)^{m(\mathbf{V})} \mathbb{Z}[q, q^{-1}].$$

As a consequence, if \mathbf{V} is a (λ, μ) -admissible sequence such that $m(\mathbf{V}) \geq 2$, then $(q^2 - 1)^2$ divides $f_{\mathbf{V}}(q)$ in $\mathbb{Z}[q, q^{-1}]$, whence $f'_{\mathbf{V}}(1) = 0$. Moreover, if \mathbf{V} is a good sequence, then the previous discussion shows that $\alpha_{\mathbf{V}}(1) = 0$, whence $f'_{\mathbf{V}}(1) = (\kappa(\mathbf{d}(\mu)) - \kappa(\mathbf{c}(\mu))) \alpha_{\mathbf{V}}(1) + \alpha'_{\mathbf{V}}(1) = \alpha'_{\mathbf{V}}(1)$. ■

6.3 Existence and uniqueness of the good sequence

We first give (see Proposition 21) some sufficient conditions for the existence of a good sequence (with respect to a given pair (λ, μ)). In order to do this, we must study the sequence of permutations that we apply to the components of wedge products when we go through an admissible sequence

$$v_{\mathbf{k}_0} \xrightarrow{\bullet} v_{\mathbf{k}_1} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_{\mathbf{k}_N},$$

where $k_0, \dots, k_N \in \mathbb{Z}^r$ and v_{ω, \mathbf{k}_0} is ordered.

Let $\sigma, \tau \in \mathfrak{S}_r$. Write $\sigma \rightarrow \tau$ if there exists $1 \leq i \leq r-1$ such that $\sigma(i) < \sigma(i+1)$, with i minimal for this property, and such that $\tau = s_i \sigma$. The relation \rightarrow on \mathfrak{S}_r is closely related to the relation $\xrightarrow{\bullet}$ on wedge products of r factors. Recall that for $v_{\mathbf{k}} = v_{k_1} \wedge \cdots \wedge v_{k_r}$ and $\sigma \in \mathfrak{S}_r$, we have $v_{\sigma, \mathbf{k}} = v_{k_{\sigma^{-1}(1)}} \wedge \cdots \wedge v_{k_{\sigma^{-1}(r)}}$. Then by definition we have $\sigma \rightarrow \tau$ if and only if $v_{\sigma^{-1}, \mathbf{k}} \xrightarrow{\bullet} v_{\tau^{-1}, \mathbf{k}}$, where $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ is such that $v_{\omega, \mathbf{k}}$ is ordered (namely, $k_1 < \cdots < k_r$).

Consider now the following reduced expression for the longest element in \mathfrak{S}_r :

$$\omega = (s_1 s_2 \cdots s_{r-1})(s_1 s_2 \cdots s_{r-2}) \cdots (s_1 s_2)(s_1),$$

and for $0 \leq i \leq \frac{r(r-1)}{2}$ let $\omega[i]$ denote the right factor of length i in this word (by convention, we have $\omega[0] = id$). For example, for $r \geq 3$ we have $\omega[5] = s_2 s_3 s_1 s_2 s_1$. The sequence $(\omega[i])_{0 \leq i \leq \frac{r(r-1)}{2}}$ enjoys the following property: if $id = \sigma_0 \rightarrow \sigma_1 \rightarrow \cdots \rightarrow \sigma_k$ with $0 \leq k \leq \frac{r(r-1)}{2}$, then we have for all $0 \leq i \leq k$, $\sigma_i = \omega[i]$. In particular, we have $\omega[i-1] \rightarrow \omega[i]$ for all $1 \leq i \leq \frac{r(r-1)}{2}$.

Lemma 20 *Let $i, j \in \llbracket 1, r \rrbracket$ be such that $i < j$. Then there exist two integers $k \in \llbracket 1, r-1 \rrbracket$ and $e \in \llbracket 0, \frac{r(r-1)}{2} - 1 \rrbracket$, determined in a unique way by the following properties: $(\omega[e])(k) = i$, $(\omega[e])(k+1) = j$ and $\omega[e+1] = s_k \omega[e]$. Namely, we have $k = j - i$ and $e = \frac{(j-1)(j-2)}{2} + (i-1)$.*

Proof. Left to the reader. ■

Example 8. Take $r = 6$, $i = 2$ and $j = 5$. Then we have

$$\begin{pmatrix} 1,2,3,4,5,6 \\ 1,2,3,4,5,6 \end{pmatrix} \rightarrow \begin{pmatrix} 1,2,3,4,5,6 \\ 2,1,3,4,5,6 \end{pmatrix} \rightarrow \begin{pmatrix} 1,2,3,4,5,6 \\ 2,3,1,4,5,6 \end{pmatrix} \rightarrow \begin{pmatrix} 1,2,3,4,5,6 \\ 3,2,1,4,5,6 \end{pmatrix} \rightarrow \begin{pmatrix} 1,2,3,4,5,6 \\ 3,2,4,1,5,6 \end{pmatrix} \rightarrow \begin{pmatrix} 1,2,3,4,5,6 \\ 3,4,2,1,5,6 \end{pmatrix} \rightarrow \begin{pmatrix} 1,2,3,4,5,6 \\ 4,3,2,1,5,6 \end{pmatrix} = \omega[e],$$

whence $e = 7 = \frac{(j-1)(j-2)}{2} + (i-1)$ and $k = 3 = j - i$.

Proposition 21 *Assume that (λ_l, μ_l) satisfies (\mathcal{H}) . Recall the notation from Section 4.2. Assume moreover that $(\gamma, \delta) \neq (0, 0)$ and $h \equiv \eta \pmod{nl}$, with $\eta \in \{0, \gamma, \delta, \gamma + \delta\}$. Then there exists a good sequence with respect to (λ, μ) .*

Proof. Let e and $k = x - y$ be the integers given by the previous lemma applied with the integers $i = r + 1 - x$ and $j = r + 1 - y$. Then we have the sequence $\omega[0] \rightarrow \cdots \rightarrow \omega[e]$, hence the sequence $v_{\omega[0]^{-1} \cdot (\omega, \mu)} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_{\omega[e]^{-1} \cdot (\omega, \mu)}$ is admissible. Moreover, by assumption on e we have

$$\mathbf{u} := v_{\omega[e]^{-1} \cdot (\omega, \mu)} = v_{(\omega\omega[e]^{-1}) \cdot \mu} = v_{m_1} \wedge \cdots \wedge v_{m_{k-1}} \wedge v_{\beta_x} \wedge v_{\beta_y} \wedge v_{m_{k+2}} \wedge \cdots \wedge v_{m_r},$$

where the m_i 's are integers belonging to $B(\mu)$, and the next step of the straightening of \mathbf{u} consists in straightening this wedge product with respect to its k -th and $(k+1)$ -th components, namely $v_{\beta_x} \wedge v_{\beta_y}$. Due to the condition $(\gamma, \delta) \neq (0, 0)$, this elementary straightening involves Rule (R_t) with $t \in \llbracket 2, 4 \rrbracket$. We then get a linear combination of wedge products whose k -th and $(k+1)$ -th components are of the form $v_{\beta_y - nli} \wedge v_{\beta_x + nli}$, *resp.* $v_{\beta_y - \gamma - nli} \wedge v_{\beta_x + \gamma + nli}$, *resp.* $v_{\beta_y - \delta - nli} \wedge v_{\beta_x + \delta + nli}$, *resp.* $v_{\beta_y - \gamma - \delta - nli} \wedge v_{\beta_x + \gamma + \delta + nli}$ with $i \in \mathbb{N}$. Since $h \equiv \eta \pmod{nl}$, we can choose i such that $h = nli$, *resp.* $h = nli + \gamma$, *resp.* $h = nli + \delta$, *resp.* $h = nli + \gamma + \delta$. According to the expression of $\beta(\lambda)$, we see that the wedge product corresponding to this choice of i , which is actually

$$\mathbf{v} := v_{m_1} \wedge \cdots \wedge v_{m_{k-1}} \wedge v_{\beta_{y-h}} \wedge v_{\beta_{x+h}} \wedge v_{m_{k+2}} \wedge \cdots \wedge v_{m_r},$$

is obtained from v_λ by permutation. Set $\mathbf{v}_1 := \mathbf{v}$. If \mathbf{v}_1 is not ordered, then the elementary straightening of \mathbf{v}_1 gives a linear combination of wedge products, and one of them, say \mathbf{v}_2 , is obtained from \mathbf{v}_1 by permutation. If we apply this device sufficiently many times, we get eventually an ordered wedge product which is of course v_λ . As a consequence, there exists an admissible sequence $\mathbf{v} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} \mathbf{v}_z = v_\lambda$, and the sequence

$$v_{\omega, \mu} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} \mathbf{u} \xrightarrow{t} \mathbf{v} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_\lambda$$

constructed this way is a good sequence. ■

We now prove the converse of Proposition 21.

Proposition 22 *Assume that (λ_l, μ_l) satisfies (\mathcal{H}) . Recall the notation from Section 4.2. Assume moreover that there exists a good sequence*

$$v_{\omega, \mu} = v_{\mathbf{k}_0} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_{\mathbf{k}_e} \xrightarrow{t} v_{\mathbf{k}_{e+1}} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_{\mathbf{k}_N} = v_\lambda$$

with respect to (λ, μ) , with $t \in \llbracket 2, 4 \rrbracket$. Then this sequence is unique, t is also uniquely determined and moreover we have $(\gamma, \delta) \neq (0, 0)$ and $h \equiv \eta \pmod{nl}$ with $\eta \in \{0, \gamma, \delta, \gamma + \delta\}$.

Proof. By assumption, $v_{\mathbf{k}_e}$ (*resp.* $v_{\mathbf{k}_{e+1}}$) is obtained from v_μ (*resp.* v_λ) by permutation. Since $\{\beta_x, \beta_y\} \cap B(\lambda) = \emptyset$, there exist $\sigma \in \mathfrak{S}_r$ and $1 \leq k \leq r - 1$ such that

$$\begin{aligned} v_{\mathbf{k}_e} &= v_{\sigma, \mu} = v_{\beta_{\sigma^{-1}(1)}} \wedge \cdots \wedge v_{\beta_{\sigma^{-1}(k-1)}} \wedge v_{\beta_x} \wedge v_{\beta_y} \wedge v_{\beta_{\sigma^{-1}(k+2)}} \wedge \cdots \wedge v_{\beta_{\sigma^{-1}(r)}}, \\ v_{\mathbf{k}_{e+1}} &= v_{\beta_{\sigma^{-1}(1)}} \wedge \cdots \wedge v_{\beta_{\sigma^{-1}(k-1)}} \wedge v_{\beta_{y-h}} \wedge v_{\beta_{x+h}} \wedge v_{\beta_{\sigma^{-1}(k+2)}} \wedge \cdots \wedge v_{\beta_{\sigma^{-1}(r)}}, \end{aligned}$$

and $v_{\mathbf{k}_{e+1}}$ is obtained from $v_{\mathbf{k}_e}$ by straightening $v_{\beta_x} \wedge v_{\beta_y}$ with the rule (R_t) . Since $t \in \llbracket 2, 4 \rrbracket$, the last conditions of the statement of this proposition hold. Moreover, t is uniquely determined by considering whether γ and δ are zero or not. It is not hard to see that k and e satisfy the conditions of Lemma 20 with $i := r + 1 - x$ and $j := r + 1 - y$, so k and e are also determined in a unique way. This determines completely the subsequence $v_{\mathbf{k}_0} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_{\mathbf{k}_e}$. The expression of $v_{\mathbf{k}_{e+1}}$ given at the beginning of the proof shows that the wedge product $v_{\mathbf{k}_{e+1}}$ is also determined in a unique way. Let $\tau \in \mathfrak{S}_r$ be the unique permutation such that $v_{\mathbf{k}_{e+1}} = v_{\tau.\lambda}$. By Remark 6, the sequence $v_{\tau.\lambda} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_\lambda$, whose length is $\ell(\tau)$, is in turn determined in a unique way. ■

7 Computation of the length modulo 2 of the good sequence

We now deal with the technical part of the proof of Theorem 4. The next proposition will be used to show that if $a'_{\lambda_l, \mu_l}(1)$ and $j_{\lambda_l, \mu_l}^<$ are nonzero, then both numbers have the same signs. This proposition deals with the only cases that we have to consider.

Proposition 23 *Assume that (λ_l, μ_l) satisfies (\mathcal{H}) . Recall the notation from Section 4.2. Assume moreover that $(\gamma, \delta) \neq (0, 0)$ and $h \equiv \eta \pmod{nl}$ with $\eta \in \{\gamma, \delta\}$. Let then*

$$\mathbf{V} = v_{\omega.\mu} \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_{\mathbf{k}} \xrightarrow{t} v_1 \xrightarrow{\bullet} \cdots \xrightarrow{\bullet} v_\lambda$$

denote the unique good sequence with respect to (λ, μ) (see Propositions 21 and 22). Denote by N the length of this sequence. Then we have

$$(-1)^{N-1} = (-1)^{\text{ht}(\rho) + \text{ht}(\rho')} \varepsilon(\lambda_l, \mu_l) \varepsilon,$$

where $\varepsilon(\lambda_l, \mu_l)$ is the sign defined in Lemma 18 and ε is the sign defined by

$$\varepsilon := \begin{cases} 1 & \text{if } \delta > 0 \text{ and } h \equiv \delta \pmod{nl}, \\ -1 & \text{otherwise.} \end{cases}$$

Proof. Let $\sigma, \tau \in \mathfrak{S}_r$ be the permutations defined by $v_{\sigma.\omega.\mu} = v_{\mathbf{k}}$ and $v_{\tau^{-1}.\lambda} = v_1$. Then we have $N = \ell(\sigma) + 1 + \ell(\tau)$, whence $(-1)^{N-1} = \varepsilon(\sigma)\varepsilon(\tau)$. By Lemma 20, we can compute $\ell(\sigma)$ and then $\varepsilon(\sigma)$; it is however not straightforward to compute $\ell(\tau)$. We compute only $\varepsilon(\tau)$ by writing τ as a product of 7 permutations $\sigma_1, \dots, \sigma_7$ whose signs are easily computable.

* Set first

$$\sigma_1 := \sigma^{-1} \quad \text{and} \quad \mathbf{v}_1 := v_{\sigma_1.1};$$

\mathbf{v}_1 is thus the wedge product obtained from $v_{\omega.\mu}$ by replacing β_x (located at the $(r + 1 - x)$ -th component) by $\beta_y - h$, and β_y (located at the $(r + 1 - y)$ -th component) by $\beta_x + h$.

* By Lemma 10, one of the following cases occurs:

- First case: $d(\beta_x) = d(\beta_y - h)$ and $d(\beta_y) = d(\beta_x + h)$,
- Second case: $d(\beta_x) = d(\beta_x + h)$, $d(\beta_y) = d(\beta_y - h)$ and $d(\beta_x) \neq d(\beta_y)$.

It is easy to see that the first case occurs if and only if $\delta = 0$ or $h \equiv \delta \pmod{nl}$, and that the second case occurs if and only if $\delta > 0$ and $h \equiv \gamma \pmod{nl}$. Set

$$\sigma_2 := \begin{cases} id & \text{if the first case occurs} \\ (r+1-x, r+1-y) & \text{if the second case occurs} \end{cases}$$

$$\text{and } \mathbf{v}_2 := v_{\sigma_2 \sigma_1 \cdot \mathbf{1}};$$

in the second case, \mathbf{v}_2 is obtained from \mathbf{v}_1 by permuting $\beta_x + h$ and $\beta_y - h$. σ_2 is constructed in order to have $\mathbf{d}(\sigma_2 \sigma_1 \cdot \mathbf{1}) = \mathbf{d}(\omega \cdot \mu)$ and subsequently $\mathbf{v}(\sigma_2 \sigma_1 \cdot \mathbf{1}) = \mathbf{v}(\omega \cdot \mu)$.

* Set

$$\sigma_3 := \mathbf{v}(\sigma_2 \sigma_1 \cdot \mathbf{1})^{-1} = \mathbf{v}(\omega \cdot \mu)^{-1} \quad \text{and} \quad \mathbf{v}_3 := v_{\sigma_3 \sigma_2 \sigma_1 \cdot \mathbf{1}}.$$

Notice that for $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$, $\mathbf{v}(\mathbf{m})^{-1}$ is the unique permutation in \mathfrak{S}_r such that $\mathbf{m}' = (m'_1, \dots, m'_r) := \mathbf{v}(\mathbf{m})^{-1} \cdot \mathbf{m}$ satisfies:

- (i) $\forall 1 \leq i < j \leq r, d(m'_i) \geq d(m'_j)$,
- (ii) $\forall 1 \leq i, j \leq r, d(m'_i) = d(m'_j) \implies i \leq j$.

In other words, let us consider \mathbf{m} as a word formed by the letters m_i and for $1 \leq b \leq l$, denote by w_b the subword of \mathbf{m} formed by the letters m_i such that $d(m_i) = b$. Then we have $\mathbf{m}' = w_l \cdots w_1$. Applying this remark to $\mathbf{m} := \sigma_2 \sigma_1 \cdot \mathbf{1}$ shows that \mathbf{v}_3 is a wedge product that can be written as

$$\mathbf{v}_3 = v_{\mathbf{k}(l)} \wedge \cdots \wedge v_{\mathbf{k}(1)},$$

where each $v_{\mathbf{k}(b)}$, $1 \leq b \leq l$ is a wedge product such that each component v_k of $v_{\mathbf{k}(b)}$ satisfies $d(k) = b$. In this case and for the rest of the proof, we say that \mathbf{v}_3 is *block-decomposable* and call $v_{\mathbf{k}(b)}$ ($1 \leq b \leq l$) the *b-th block* of \mathbf{v}_3 .

* Set now

$$\sigma_4 := \omega(\mathbf{k}) = \omega(\mathbf{1}) \quad \text{and} \quad \mathbf{v}_4 := v_{\sigma_4 \sigma_3 \sigma_2 \sigma_1 \cdot \mathbf{1}},$$

where $\omega(\mathbf{k})$ is defined in Section 3.2 (the equality $\omega(\mathbf{k}) = \omega(\mathbf{1})$ comes from Lemma 10). For $1 \leq b \leq l$ denote by

$$r_b := \#\{1 \leq i \leq r \mid d(\alpha_i) = b\} = \#\{1 \leq i \leq r \mid d(\beta_i) = b\}$$

the number of factors of the block $v_{\mathbf{k}(b)}$ (the equality of both numbers defining r_b comes again from Lemma 10). Let $1 \leq b \leq l$. Then σ_4 acts on the b -th block of \mathbf{v}_3 as the permutation $\binom{1, \dots, r_b}{r_b, \dots, 1}$. Since v_μ is ordered, we can see that for all $b \in \llbracket 1, l \rrbracket \setminus \{d, d'\}$, the b -th block of \mathbf{v}_4 is also ordered. Here the integers d and d' are the ones defined in Lemma 10.

- * Write temporarily $\mathbf{v}_4 = v_{k_1} \wedge \cdots \wedge v_{k_r}$, and let i (*resp.* j) $\in \llbracket 1, r \rrbracket$ be such that $k_i = \beta_y - h$ (*resp.* $k_j = \beta_x + h$). Define σ_5 and \mathbf{v}_5 by

$$\sigma_5 := \begin{cases} id & \text{if } \delta > 0 \\ (i, j) & \text{if } \delta = 0 \end{cases} \quad \text{and} \quad \mathbf{v}_5 := v_{\sigma_5 \sigma_4 \sigma_3 \sigma_2 \sigma_1.1};$$

we have thus $\sigma_5 = id$ if and only if $\beta_y - h$ and $\beta_x + h$ are in the same block of \mathbf{v}_4 .

- * Let $\sigma_6 \in \mathfrak{S}_r$ be the permutation that acts separately on each block of \mathbf{v}_5 by reordering it. Let us be more precise. Set

$$\mathbf{v}_6 := v_{\sigma_6 \sigma_5 \sigma_4 \sigma_3 \sigma_2 \sigma_1.1}.$$

If $\delta = 0$, then σ_6 acts on the d -th block of \mathbf{v}_5 as the permutation π from Lemma 6, and σ_6 acts trivially on the other blocks. If $\delta > 0$, then σ_6 acts as the product of two permutations σ'_d and $\sigma'_{d'}$, where each σ'_b , $b \in \{d, d'\}$ acts on the b -th block of \mathbf{v}_5 as the permutation denoted by σ in Lemma 5 and σ'_b acts trivially on the other blocks. As a consequence, we have in both cases

$$\varepsilon(\sigma_6) = (-1)^{\text{ht}(\rho) + \text{ht}(\rho')}.$$

- * Finally, put

$$\sigma_7 := \mathbf{v}(\lambda) \quad \text{and} \quad \mathbf{v}_7 := v_{\sigma_7 \sigma_6 \sigma_5 \sigma_4 \sigma_3 \sigma_2 \sigma_1.1}.$$

Notice that v_λ is ordered, \mathbf{v}_6 is obtained from v_λ by permutation, \mathbf{v}_6 is block-decomposable and all the blocks of \mathbf{v}_6 are ordered. By the remark following the definition of σ_3 , we have $v_{\mathbf{v}(\lambda)^{-1}. \lambda} = \mathbf{v}_6$, whence $\mathbf{v}_7 = v_\lambda$.

As a consequence, we do have $\tau = \sigma_7 \cdots \sigma_1$, where the σ_i 's are defined as above, hence $(-1)^{N-1} = \varepsilon(\sigma) \prod_{i=1}^7 \varepsilon(\sigma_i)$. By considering different cases we see that $\varepsilon(\sigma_2)\varepsilon(\sigma_5) = \varepsilon$, where ε is defined as in the statement of this proposition. Moreover, we have

$$\varepsilon(\sigma_4) = \varepsilon(\omega(\mathbf{k})) = \prod_{b=1}^l (-1)^{\frac{r_b(r_b-1)}{2}} = (-1)^{\kappa(\mathbf{d}(\mu))}.$$

We then have $\varepsilon(\sigma_3)\varepsilon(\sigma_4)\varepsilon(\sigma_7) = \varepsilon(\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l)$, whence the result. \blacksquare

8 Proof of Theorem 4

Let $\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi_m^l$, and $\lambda, \mu \in \Pi$ be the partitions such that $\boldsymbol{\lambda}_l \leftrightarrow \lambda$ and $\boldsymbol{\mu}_l \leftrightarrow \mu$. We must show that $a'_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}(1) = 2j_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}^<$. If $(\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l)$ does not satisfy (\mathcal{H}) , then there cannot exist any good sequence with respect to (λ, μ) , so by Lemma 19 we have $a'_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}(1) = 0$; on the other hand, by Corollary 9 we have $j_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}^< = 0$ in this case. Assume now that $(\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l)$ satisfies (\mathcal{H}) ,

and recall the notation from Section 4.2. By Corollary 9 and Propositions 14 and 15, the case (J_1) (*resp.* (J_2)) occurs if and only if $\delta > 0$ (*resp.* $\delta = 0$), and Propositions 14 and 15 then give the expression of $j_{\lambda_l, \mu_l}^{\leftarrow}$. Moreover, Propositions 21 and 22 give necessary and sufficient conditions on γ , δ and h for the existence of a good sequence, in which case it is unique. Lemma 19 and Rules $(R_2) - (R_4)$ then give the expression of $a'_{\lambda_l, \mu_l}(1)$. In order to compare $a'_{\lambda_l, \mu_l}(1)$ and $j_{\lambda_l, \mu_l}^{\leftarrow}$, we have to consider 12 cases depending on the value of h modulo nl and on whether γ and δ are zero or not. The results are shown in Figure 3. Here N is the length of a good sequence if it exists. Theorem 4 follows by comparing the last two columns of the array and by applying Proposition 23 if the corresponding numbers are nonzero. ■

Case	Number of good sequences	$\frac{a'_{\lambda_l, \mu_l}(1)}{\varepsilon(\lambda_l, \mu_l)}$	$\frac{j_{\lambda_l, \mu_l}^{\leftarrow}}{(-1)^{\text{ht}(\rho) + \text{ht}(\rho')}}$
$\gamma = \delta = 0$	0	0	0
$\gamma > 0, \delta = 0,$ $h \not\equiv 0 \pmod{nl}, h \not\equiv \gamma \pmod{nl}$	0	0	0
$\gamma > 0, \delta = 0,$ $i := \frac{h}{nl} \in \mathbb{N}^*$	1	$(-1)^{N-1} \frac{d}{dq} \Big _{q=1} \left(-(q^{-2} - 1)q^{-2i+1} \right)$ $= 2(-1)^{N-1}$	-1
$\gamma > 0, \delta = 0,$ $i := \frac{h-\gamma}{nl} \in \mathbb{N}$	1	$(-1)^{N-1} \frac{d}{dq} \Big _{q=1} \left((q^{-2} - 1)q^{-2i} \right)$ $= -2(-1)^{N-1}$	1
$\gamma = 0, \delta > 0,$ $h \not\equiv 0 \pmod{nl}, h \not\equiv \delta \pmod{nl}$	0	0	0
$\gamma = 0, \delta > 0,$ $i := \frac{h}{nl} \in \mathbb{N}^*$	1	$(-1)^{N-1} \frac{d}{dq} \Big _{q=1} \left(-(q^2 - 1)q^{2i-1} \right)$ $= -2(-1)^{N-1}$	1
$\gamma = 0, \delta > 0,$ $i := \frac{h-\delta}{nl} \in \mathbb{N}$	1	$(-1)^{N-1} \frac{d}{dq} \Big _{q=1} \left((q^2 - 1)q^{2i} \right)$ $= 2(-1)^{N-1}$	1
$\gamma > 0, \delta > 0,$ $h \not\equiv 0 \pmod{nl}, h \not\equiv \gamma \pmod{nl},$ $h \not\equiv \delta \pmod{nl}, h \not\equiv \gamma + \delta \pmod{nl}$	0	0	0
$\gamma > 0, \delta > 0,$ $i := \frac{h}{nl} \in \mathbb{N}^*$	1	$(-1)^{N-1} \frac{d}{dq} \Big _{q=1} \left(-(q - q^{-1}) \frac{q^{2i} - q^{-2i}}{q + q^{-1}} \right)$ $= 0$	0
$\gamma > 0, \delta > 0,$ $i := \frac{h-\gamma}{nl} \in \mathbb{N}$	1	$(-1)^{N-1} \frac{d}{dq} \Big _{q=1} \left(-(q - q^{-1}) \frac{q^{2i+1} + q^{-2i-1}}{q + q^{-1}} \right)$ $= -2(-1)^{N-1}$	1
$\gamma > 0, \delta > 0,$ $i := \frac{h-\delta}{nl} \in \mathbb{N}$	1	$(-1)^{N-1} \frac{d}{dq} \Big _{q=1} \left((q - q^{-1}) \frac{q^{2i+1} + q^{-2i-1}}{q + q^{-1}} \right)$ $= 2(-1)^{N-1}$	1
$\gamma > 0, \delta > 0,$ $i := \frac{h-\gamma-\delta}{nl} \in \mathbb{N}$	1	$(-1)^{N-1} \frac{d}{dq} \Big _{q=1} \left((q - q^{-1}) \frac{q^{2i+2} - q^{-2i-2}}{q + q^{-1}} \right)$ $= 0$	0

Figure 3: List of the cases involved in the proof of Theorem 4.

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