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► **To cite this version:**

Christophe Mourougane, Shigeharu Takayama. A positivity property of ample vector bundles. 2005.
hal-00004912v1

HAL Id: hal-00004912

<https://hal.science/hal-00004912v1>

Preprint submitted on 16 May 2005 (v1), last revised 19 Nov 2005 (v2)

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A POSITIVITY PROPERTY OF AMPLE VECTOR BUNDLES

CHRISTOPHE MOURougANE, SHIGEHARU TAKAYAMA

Abstract. *Using Fujita-Griffiths method of computing metrics on Hodge bundles, we show that for every semi-ample vector bundle E on a compact complex manifold, and every positive integer k , the vector bundle $S^k E \otimes \det E$ has a continuous metric with Griffiths semi-positive curvature. If E is ample, the metric can be made smooth and Griffiths positive.*

1. INTRODUCTION

We consider a holomorphic vector bundle E on a compact complex manifold, and intend to construct metrics on vector bundles associated to E which would reflect algebraic positivity properties of E . We prove

Theorem 1.1. *Let E be a semi-ample vector bundle on a compact complex manifold. Then for all positive integer k , the vector bundles $S^k E \otimes \det E$ have continuous metrics with Griffiths semi-positive curvature.*

Theorem 1.2. *Let E be an ample vector bundle on a complex projective manifold. Then for all positive integer k , the vector bundles $S^k E \otimes \det E$ are Griffiths positive.*

Our results provide a weak answer to a question raised by Griffiths on finding an analytic characterization of ampleness for vector bundles [9, problem (0.9)]. These constructions may help to find some topological properties of algebraic sub-varieties under ampleness assumptions for the normal bundle (see for example [8] and [16] part two). The appearance of the determinant line bundle has the same origin than its appearance in the vanishing theorem of Griffiths for the cohomology of ample vector bundles. On projective spaces, abelian varieties, Grassmanian manifolds these results were already proved either by the Castelnuovo-Mumford criterion for global generation or by trying to mimic the Frobenius morphisms over \mathbb{C} ([17], [18]).

The idea to obtain the metric positivity is to construct cyclic coverings Y_s of $\mathbb{P}(E)$, the variety of rank one quotients of E , in order to relate the (sheaf of germs of the) line bundle $\mathcal{O}_E(-1)$ to the structure sheaf of Y_s . This may be seen as a metric aspect of Ramanujam's idea to reduce vanishing theorem to topological properties [19] (see also [15]). Applying Fujita-Griffiths method of computing metrics on Hodge bundles, we will be able to compute the curvature of direct image of the structure sheaf of Y_s . We next have to deal with the singularities of the gotten metric. It turns out the metric on the top direct image of \mathcal{O}_{Y_s} is semi-negatively curved and that its only singularities are zeros (i.e. may vanish on non-zero vectors). This is far simpler than general results obtained by Kawamata, Zücker, Kollár and Cattani-Kaplan-Schmid. Hence, adding metrics built from different properly chosen cyclic coverings lead to a non degenerate metric on $(S^k E \otimes \det E)^*$.

After having completed this work, we received a preprint from Bo Berndtsson where he proves similar results using the subharmonicity properties of families of Bergman kernels [1].

Acknowledgment. The first named author warmly thanks Indranil Biswas for a collaboration in an attempt to prove similar results.

2. AMPLENESS AND POSITIVITY

We refer to [12] or [16] for basics about ample vector bundles and to [9] or [4, chapter VII] for basics about positive vector bundles. All vector bundles are assumed to be holomorphic.

A vector bundle E on a compact complex manifold X is said to be *semi-ample* if for some positive integer k , its symmetric power $S^k E$ is generated by its global sections. Associated to E , we have $\pi : \mathbb{P}(E) \rightarrow X$ the variety of rank one quotients of E together with its tautological quotient line bundle $\mathcal{O}_E(1)$. The semi-amplicity of E is rephrased that for every $x \in X$, every section s_x of $\mathcal{O}_E(k)$ over the fiber $\mathbb{P}(E_x)$ extends to a global section of $\mathcal{O}_E(k)$ over $\mathbb{P}(E)$. This in particular implies that $\mathcal{O}_E(k)$ is generated by its global sections. A vector bundle E is said to be *ample* if its associated line bundle $\mathcal{O}_E(1)$ is ample on $\mathbb{P}(E)$. This in particular implies the existence of an integer k such that for every $x \in X$, every section s_x of $\mathcal{O}_E(k)$ over the first infinitesimal neighborhood of the fiber $\mathbb{P}(E_x)$ extends to a global section of $\mathcal{O}_E(k)$ over $\mathbb{P}(E)$.

A vector bundle E is said to be *Griffiths positive*, if it can be endowed with a smooth hermitian metric h such that for all $x \in X$ and all non-zero decomposable tensors $v \otimes e \in TX_x \otimes E_x$, the curvature term $\langle \Theta(E, h)(v, \bar{v})e, e \rangle_h$ is positive, where $\Theta(E, h) \in \mathcal{C}_{1,1}^\infty(X, \text{Herm}(E))$ is the curvature of the Chern connection $\nabla_{E,h}$ of (E, h) . Recall the formula

$$(2.1) \quad \frac{\langle \Theta(E, h)\xi, \xi \rangle}{\|\xi\|^2} = -\sqrt{-1}\partial\bar{\partial} \log \|\xi\|^2 + \frac{\langle \nabla_{E,h}\xi, \nabla_{E,h}\xi \rangle}{\|\xi\|^2} - \frac{\sqrt{-1}\partial\|\xi\|^2 \wedge \bar{\partial}\|\xi\|^2}{\|\xi\|^4} \\ \geq -\sqrt{-1}\partial\bar{\partial} \log \|\xi\|^2$$

for a nowhere zero local holomorphic section ξ of a holomorphic vector bundle E equipped with a smooth hermitian metric h . The last two terms give the norm at x of the fundamental form of the inclusion $\mathcal{O}_X\xi \subset E$. A continuous hermitian metric h on a vector bundle $b : E \rightarrow X$ is said to be *Griffiths positive*, if there exists a smooth positive real $(1, 1)$ -form ω_X on X such that in the sense of currents

$$-\sqrt{-1}\partial\bar{\partial} \log h(\xi) + \frac{h(\nabla\xi)}{h(\xi)} - \frac{\sqrt{-1}\partial h(\xi) \wedge \bar{\partial} h(\xi)}{h(\xi)^2} \geq b^*\omega_X,$$

where h is seen as a continuous quadratic function on the total space $E - X \times \{0\}$. At the points where the metric h is smooth, these two notions of Griffiths positivity coincide.

The theory of resolution of the $\bar{\partial}$ -equation with L^2 -estimates for example shows that Griffiths positivity and ampleness are equivalent for line bundles. This implies through the curvature computation of $\mathcal{O}_E(1)$ that Griffiths positive vector bundles are ample. The converse is a problem raised by Griffiths, and solved positively on curves by Umemura [21] using the concept of stability (see also [3]).

Notation and Assumption. In the rest of this paper, we will use the following notations. We let $E \rightarrow X$ be a holomorphic vector bundle of rank $r > 1$ on a compact connected complex manifold X , and $\pi : \mathbb{P}(E) \rightarrow X$ be the associated \mathbb{P}^{r-1} -bundle with the line bundle $\mathcal{O}_E(1)$. We assume E is semi-ample at least, and we take and fix an arbitrary positive integer k such that $S^k E$ is generated by its global sections.

3. CYCLIC COVERS

A reference for this part is [6, § 3]. By our assumption at the end of § 2, $\mathcal{O}_E(k)$ is generated by its global sections. Then Bertini's theorem (see for example [11, page 137]) insures that a generic section s of $\mathcal{O}_E(k)$ over $\mathbb{P}(E)$ is transverse to the zero section (i.e. $ds|_{D_s} : T\mathbb{P}(E)|_{D_s} \rightarrow \mathcal{O}_E(k)|_{D_s}$ is

surjective), and defines a smooth divisor $D_s := (s = 0)$. Let

$$p = p_s : Y_s \rightarrow \mathbb{P}(E)$$

be the cyclic covering of $\mathbb{P}(E)$ obtained by taking the k -th root out of D_s . We intend to study the morphism $\pi \circ p : Y_s \rightarrow X$. The space

$$Y_s := \{l \in \mathcal{O}_E(1)/l^k = s(p(l))\}$$

is a smooth hypersurface of the total space $\mathcal{O}_E(1)$, and the map p is a finite cover totally ramified along the zero locus D_s of s . The space Y_s may also be described as the spectrum $\text{Spec } \mathcal{A}_s$ of the algebra

$$\mathcal{A}_s := \frac{\bigoplus_{i=0}^{+\infty} \mathcal{O}_E(-i)}{(l^* - \check{s}(l^*), l^* \in \mathcal{O}_E(-k))}$$

where \check{s} is the sheaf inclusion $\mathcal{O}_E(-k) \xrightarrow{\times s} \mathcal{O}_E = \mathcal{O}_{\mathbb{P}(E)}$. The direct image of the structure sheaf \mathcal{O}_{Y_s} is hence $p_* \mathcal{O}_{Y_s} = \mathcal{A}_s \simeq \bigoplus_{i=0}^{k-1} \mathcal{O}_E(-i)$. Simply note that no negative power $\mathcal{O}_E(-i)$ has non-zero sections on the fibers of π to infer that the direct image $(\pi \circ p)_* \mathcal{O}_{Y_s}$ is \mathcal{O}_X . This shows that the fibers of $\pi \circ p$ are connected and that the covering space Y_s is therefore connected. One can check by a local computation that the morphism $\pi \circ p$ is smooth over the set $X - \Sigma_s$ of points $x \in X$ where $s|_{\mathbb{P}(E_x)} \in \Gamma(\mathbb{P}(E_x), \mathcal{O}_E(k))$ is transverse to the zero section. We shall call Σ_s the discriminant locus of $\pi \circ p$.

We will in fact work with the top degree direct image of the structure sheaf. Because p is a finite morphism, the spectral sequence of composition of direct image functors reduces to the following:

$$\begin{aligned} \mathcal{R}^{r-1}(\pi \circ p)_* \mathcal{O}_{Y_s} &= \mathcal{R}^{r-1} \pi_* (\mathcal{R}^0 p_* \mathcal{O}_{Y_s}) \\ &= \mathcal{R}^{r-1} \pi_* (\bigoplus_{i=0}^{k-1} \mathcal{O}_E(-i)) \\ (3.1) \quad &= \bigoplus_{i=0}^{k-1} \pi_* (\omega_{\mathbb{P}(E)/X} \otimes \mathcal{O}_E(i))^* \\ &= \bigoplus_{i=0}^{k-1} \pi_* (\mathcal{O}_E(i-r) \otimes \pi^* \det E)^* \\ &= \bigoplus_{i=r}^{k-1} (S^{i-r} E \otimes \det E)^* . \end{aligned}$$

Here we have used Serre duality on the fibers of the smooth morphism π with relative dualizing sheaf $\omega_{\mathbb{P}(E)/X} = \mathcal{O}_E(-r) \otimes \pi^* \det E$.

4. THE HODGE METRIC

We recall the basics on geometric variations of Hodge structures and Griffiths's computations ([10, theorem 6.2]) of the curvature the Hodge metric (see also [20, § 7] and [22, chapter 10]). We also recall the method of Fujita [7]. Let $f : Y \rightarrow B$ be a projective and surjective morphism of complex manifolds having connected fibers. We fix an ample line bundle on Y .

4.1. The Hodge metric. Here we assume that $f : Y \rightarrow B$ is smooth, in particular we regard $f : Y \rightarrow B$ as a smooth family of polarized complex projective manifolds of dimension n . Fix a non-negative integer d . The local system $\mathcal{R}^d f_* \mathbb{C}$ can be realized as the sheaf of germs of the flat sections of the holomorphic vector bundle $\mathbf{H}_{\mathbb{C}}^d$ associated with the locally free sheaf $(\mathcal{R}^d f_* \mathbb{C}) \otimes \mathcal{O}_B$ endowed with the flat holomorphic connection $\nabla : \mathbf{H}_{\mathbb{C}}^d \rightarrow \Omega_B^1 \otimes \mathbf{H}_{\mathbb{C}}^d$, the Gauss-Manin connection. By semi-continuity and Hodge decomposition, the vector spaces $H^{p,d-p}(Y_b, \mathbb{C})$ ($b \in B$) have constant dimension. By elliptic theory they hence form a differentiable sub-bundle $\mathbf{H}^{p,d-p}$ of $\mathbf{H}_{\mathbb{C}}^d$. Denote by \mathbf{F}^p the differentiable sub-bundle $\bigoplus_{i \geq p} \mathbf{H}^{i,d-i}$ of $\mathbf{H}_{\mathbb{C}}^d$. By a theorem of Griffiths, the \mathbf{F}^p have natural

structure of holomorphic sub-vector bundles of $\mathbf{H}_{\mathbb{C}}^d$ and they satisfy the transversality condition for the Gauss-Manin connection $\nabla \mathbf{F}^p \subset \Omega_B^1 \otimes \mathbf{F}^{p-1}$. A relative Dolbeault theorem identifies $\mathbf{E}^p := \mathbf{F}^p / \mathbf{F}^{p+1}$ with the holomorphic vector bundle associated with the locally free sheaf $\mathcal{R}^{d-p} f_* \Omega_{Y/B}^p$.

We now recall the construction of the hodge metric on the primitive part of \mathbf{E}^p . We fix a family η_b ($b \in B$) of polarizations given by a section of $\mathcal{R}^2 f_* \mathbb{Z}$. The bilinear form on the fibers of $\mathbf{H}_{\mathbb{C}}^d$ given by

$$S(c_1, c_2) := (-1)^{\frac{d(d-1)}{2}} \int_{Y_b} \eta_b^{n-d} \wedge c_1 \wedge c_2$$

is non-degenerate (hence defines a pseudo-metric) on the primitive part $\mathbf{P}^d := \text{Ker}(\eta^{n-d+1} : \mathbf{H}^d \rightarrow \mathbf{H}^{2n-d+2})$ (which is also a differentiable sub-bundle of \mathbf{H}^d). The differentiable subbundles $\mathbf{H}^{p,d-p}$ and $\mathbf{H}^{p',d-p'}$ are orthogonal unless $p + p' = d$ and

$$h(c) := (\sqrt{-1})^{p-q} S(c, \bar{c})$$

defines a positive definite metric on $\mathbf{H}_{prim}^{p,d-p} := \mathbf{H}^{p,d-p} \cap \mathbf{P}^d$. We set $\mathbf{F}_{prim}^p := \mathbf{F}^p \cap \mathbf{P}^d$. Those bundle also have natural holomorphic structures and satisfy the transversality condition for the induced Gauss-Manin connection. We also set $\mathbf{E}_{prim}^p := \mathbf{F}_{prim}^p / \mathbf{F}_{prim}^{p+1}$. The fiber-wise isomorphism of $(\mathbf{E}_{prim}^p)_b$ with $H_{prim}^{p,d-p}(Y_b, \mathbb{C}) \subset H^d(Y_b, \mathbb{C})$ enables to equip the holomorphic vector bundle \mathbf{E}_{prim}^p with a smooth positive definite hermitian metric, called the Hodge metric.

We need some definitions in order to express the curvature of the corresponding Chern connection. Denote by $\bar{\nabla}^p : \mathbf{E}^p \rightarrow \Omega_B^1 \otimes \mathbf{E}^{p-1}$ the \mathcal{O}_B -linear map built by first lifting to \mathbf{F}^p applying the Gauss-Manin connection and projecting to \mathbf{E}^{p-1} . The second fundamental form in $\mathcal{C}_{\infty}^{1,0}(B, \text{Hom}(\mathbf{F}^p, \mathbf{H}_{\mathbb{C}}^d / \mathbf{F}^p))$ of the sequence

$$0 \rightarrow \mathbf{F}^p \rightarrow \mathbf{H}_{\mathbb{C}}^d \rightarrow \mathbf{H}_{\mathbb{C}}^d / \mathbf{F}^p \rightarrow 0$$

with respect to the Gauss-Manin connection (or equivalently with the flat metric on $\mathbf{H}_{\mathbb{C}}^d$) actually induces $\bar{\nabla}^p : \mathbf{E}^p \rightarrow \Omega_B^1 \otimes \mathbf{E}^{p-1}$. Formulae for the curvature of quotient hermitian holomorphic vector bundles then lead to

Theorem 4.1. [10, theorem 5.2] *The curvature $\Theta(\mathbf{E}_{prim}^p)$ of the holomorphic vector bundle \mathbf{E}_{prim}^p endowed with its Hodge metric is given by*

$$\langle \Theta(\mathbf{E}_{prim}^p)(V, \bar{V})\sigma, \sigma \rangle_{Hodge} = \langle \bar{\nabla}_V^p \sigma, \bar{\nabla}_V^p \sigma \rangle_{Hodge} - \langle (\bar{\nabla}_V^{p+1})^* \sigma, (\bar{\nabla}_V^{p+1})^* \sigma \rangle_{Hodge}$$

where V is a local vector field on B and σ a local section of \mathbf{E}^p .

We now apply this result in the case of the family of the cyclic covers $\pi \circ p : Y_s \rightarrow X$ obtained by taking the k -th root of a section s of $\mathcal{O}_E(k)$. We have to restrict the study over Zariski open sets $Y_s^0 := (p \circ \pi)^{-1}(X - \Sigma_s)$ and $X^0 := X - \Sigma_s$ so that $\pi \circ p : Y_s^0 \rightarrow X^0$ becomes a smooth family. Then, since $\bar{\nabla}^0$ vanishes, the above theorem implies the following

Corollary 4.2. *The vector bundle $\mathbf{E}_{prim}^0 = \mathbf{E}^0 = \mathcal{R}^{r-1}(\pi \circ p)_* \mathcal{O}_{Y_s^0/X^0} = \bigoplus_{i=r}^{k-1} (S^{i-r} E \otimes \det E)_{|X^0}^*$ with the Hodge metric is Griffiths semi-negative.*

4.2. Singularities of the Hodge metric. We now deal with the general case, namely $f : Y \rightarrow B$ may not be smooth. In this section we will give rough descriptions of the singularities. Detailed results need explicit form of the isomorphisms in (3.1) and will be given in § 5.3. We therefore assume that the base B is one dimensional.

The Hodge metric on the direct image of the relative canonical sheaf is described as follows. Let $b \in B$ be a point and let (U, t) be a local coordinate centered at $b = \{t = 0\}$. A section $\omega \in \Gamma(U, f_* K_{Y/B})$

– when regarded as a section in $\Gamma(U, f_* \text{Hom}(f^* K_B, K_Y))$ and applied to $f^* dt$ – gives a section of K_Y on $f^{-1}(U)$ which we denote by $\omega \cdot dt$. If $\varphi_b \in \Gamma(Y_b, K_{Y_b})$ fulfills the relation $\omega \cdot dt = \varphi_b \wedge f^* dt$ over Y_b (which amounts to saying that in the differentiable trivialization $Y|_U \simeq Y_b \times U$, the section ω is sent to φ_b), then the Hodge norm at $b \in B$ of the section ω is

$$\|\omega\|_{Hodge}^2 = (\sqrt{-1})^n (-1)^{\frac{n(n-1)}{2}} \int_{Y_b} \varphi_b \wedge \overline{\varphi_b},$$

here $n = \dim Y - 1$. Fujita checked that in this setting in case $\dim B = 1$ the Hodge metric on $f_* K_{Y/B}$ is bounded from below by a positive quantity and hence that the only possible singularities of the Hodge metric on $f_* K_{Y/B}$ are poles (see [7, lemma 1.12]).

We just give the typical example which occurs for a local model of our cyclic covers (for some positive integer m).

$$\begin{aligned} \pi \circ p : Y_s = \{(t, z, l) \in \mathbb{C}^3 / l^k = t + z^m\} &\rightarrow X = \{t \in \mathbb{C}\} \\ (t, z, l) &\mapsto t \end{aligned}$$

The cotangent bundle $\Omega_{Y_s}^1$ is generated by dt, dz, dl subject to the relation $kl^{k-1} dl - dt - mz^{m-1} dz = 0$. If $\omega \cdot dt$ is written as $\eta(z, l) dz \wedge dl$ for a holomorphic function $\eta(z, l)$, then φ_0 may be chosen to be $\varphi_0 = k^{-1} l^{1-k} \eta(z, l) dz$ with a pole of order $k - 1$ on the fiber over $t = 0$ and no singularities elsewhere.

Now, note that over smooth fibers of f , the Serre dual of an orthonormal basis (ω_i) of $(\Gamma(Y_b, K_{Y_b}), Hodge)$ is $(\overline{\omega}_i)$, which is an orthonormal basis of $(H^{0,n}(Y_b, \mathbb{C}), Hodge)$. Hence the Hodge metric on $\mathcal{R}^{r-1}(\pi \circ p)_* \mathcal{O}_{Y_s}$ may acquire zeros at the points x over which the section $s|_{\mathbb{P}(E_x)}$ is not transverse to the zero section. This can also be inferred from the formula in § 5.3 which furthermore proves the continuity of the metric on $\mathcal{R}^{r-1}(\pi \circ p)_* \mathcal{O}_{Y_s}$.

5. EXPLICIT ISOMORPHISM

We make explicit the isomorphisms in (3.1) in terms of Dolbeault isomorphism and Serre duality for metrized vector bundles. This enables us to describe the Hodge metric. We keep the notation and assumption for $E \rightarrow X$ made at the end of § 2. We furthermore fix a reference hermitian metric g on E and then naturally on E^* and on $\mathcal{O}_E(i)$.

5.1. Calculus lemma. To make Serre duality on a projective space explicit, we pose an elementary calculus lemma. We consider a projective space \mathbb{P}^{r-1} with a homogeneous coordinates $[a] = (a_1 : \dots : a_r)$ and with the Fubini-Study Kähler form Ω . Let $z_j = a_j/a_1$ ($2 \leq j \leq r$) be a standard local coordinate. We can write the volume form as

$$\frac{\Omega^{r-1}}{(r-1)!} = \frac{(\sqrt{-1} \partial \bar{\partial} \log |a|^2)^{r-1}}{(2\pi)^{r-1} (r-1)!} = \epsilon_r \frac{dz_2 \wedge dz_3 \wedge \dots \wedge dz_r \wedge d\bar{z}_2 \wedge d\bar{z}_3 \wedge \dots \wedge d\bar{z}_r}{(1 + \|z\|^2)^r},$$

where we have set $\epsilon_r := (\sqrt{-1})^{r-1} (-1)^{\frac{(r-1)(r-2)}{2}} / (2\pi)^{r-1}$ and $\|z\|^2 = \sum_{j=2}^r |z_j|^2$.

Lemma 5.1. *Let m be a positive integer, and let $I = (i_1, i_2, \dots, i_m)$ and $J = (j_1, j_2, \dots, j_m)$ be m -tuples of integers in $\{1, 2, \dots, r\}$.*

(1) *If I and J are not equal modulo change of order, it follows from parity reasons that*

$$\int_{[a] \in \mathbb{P}^{r-1}} \frac{a_{i_1} \overline{a_{j_1}} a_{i_2} \overline{a_{j_2}} \dots a_{i_m} \overline{a_{j_m}}}{|a|^{2m}} \frac{\Omega^{r-1}}{(r-1)!} = 0.$$

(2) If I and J are equal modulo change of order, then

$$\int_{[a] \in \mathbb{P}^{r-1}} \frac{|a_1|^{2m_1} |a_2|^{2m_2} \cdots |a_r|^{2m_r}}{|a|^{2m}} \frac{\Omega^{r-1}}{(r-1)!} = \frac{\prod_{i=1}^r m_i!}{(r-1+m)!}.$$

Here the integer i appears m_i times in I (hence in J), consequently $0 \leq m_i \leq m$ and $m = m_1 + \cdots + m_r$. One uses a convention $0! = 1$.

5.2. Explicit isomorphism. Since the problem : isomorphism in (3.1) is local on X , it is enough to argue on a small open neighborhood U of a fixed point $x_0 \in X$. Choose a local frame $(e_j)_{1 \leq j \leq r}$ for E on U . A vector of E (resp. of the dual bundle E^*) will be denoted by $a \in E$ (resp. $a^* \in E^*$). Hence a point on $\mathbb{P}(E)$ will be denoted by $[a^*] \in \mathbb{P}(E)$.

We first describe the isomorphism $S^{i-r}E \otimes \det E \cong \pi_*(K_{\mathbb{P}(E)/X} \otimes \mathcal{O}_E(i))$. The isomorphism $S^{i-r}E \rightarrow \pi_*\mathcal{O}_E(i-r)$ is given over $x \in U$ with $a^* \in E_x^*$ by

$$\begin{aligned} S^{i-r}E_x &\rightarrow (\pi_*\mathcal{O}_E(i-r))_x = H^0(\mathbb{P}(E_x), \mathcal{O}_E(i-r)|_{\mathbb{P}(E_x)}) \\ f &\mapsto \left(\begin{array}{ccc} \mathbb{P}(E_x) & \rightarrow & \mathcal{O}_E(i-r)|_{\mathbb{P}(E_x)} \\ [a^*] & \mapsto & \langle f, a^{*i-r} \rangle (a^*)^{-i+r} \end{array} \right). \end{aligned}$$

Here we denote by a^{*i-r} the $(i-r)$ -fold symmetric product, and by $\langle f, a^{*i-r} \rangle$ the duality pairing. A vector of the bundle $\pi_*(K_{\mathbb{P}(E)/X} \otimes \mathcal{O}_E(i))$ is represented by a relative holomorphic $(r-1)$ -form with values in $\mathcal{O}_E(i)$ with respect to a local coordinate of the fibers. We take a point $[a_0^*] \in \mathbb{P}(E_{x_0})$ with $a_0^* = \sum_{j=1}^r a_{0j} e_j^*$, and let assume that $a_{01} \neq 0$ (this is not a special assumption). On a neighborhood of $[a_0^*] \in \mathbb{P}(E)$, we use a standard local coordinate on the fiber: $z_j = a_j/a_1$ ($2 \leq j \leq r$) for $[a^*] = [\sum_{j=1}^r a_j e_j^*] \in \mathbb{P}(E)$. There exists a canonical map $q : E^* - X \times \{0\} \rightarrow \mathbb{P}(E)$ which is expressed in local coordinates around a reference point (x_0, a_0^*) , by

$$q(x, a^*) = q(x, \sum a_j e_j^*) = (x, [a_1 : \cdots : a_r]) = (x, z_2, \dots, z_r).$$

The relative Euler sequence :

$$0 \rightarrow \mathcal{O}_E(-1) \xrightarrow{L} \pi^*E^* \xrightarrow{p} T_{\mathbb{P}(E)/X} \otimes \mathcal{O}_E(-1) \rightarrow 0$$

– which is built from the map $q : E^* - X \times \{0\} \rightarrow \mathbb{P}(E)$ – is given by

$$p \left(x, [a^*], \sum_{j \geq 1} b_j e_j^* \right) = \left(x, [a^*], \sum_{j \geq 2} \frac{b_j a_1 - b_1 a_j}{a_1^2} \frac{\partial}{\partial z_j} \otimes a^* \right).$$

Since $p(a_1 e_1^*) = \frac{\partial}{\partial z_j} \otimes a^*$, the natural determinant isomorphism for the relative Euler sequence reads

$$\begin{aligned} \det T_{\mathbb{P}(E)/X} \otimes \mathcal{O}_E(-r) &\rightarrow \pi^* \det E^* \\ \frac{\partial}{\partial z_2} \wedge \frac{\partial}{\partial z_3} \wedge \cdots \wedge \frac{\partial}{\partial z_r} \otimes (a^*)^r &\mapsto \left(\sum_{j \geq 1} a_j e_j^* \right) \wedge a_1 e_2^* \wedge a_1 e_3^* \wedge \cdots \wedge a_1 e_r^* \\ &= a_1^r e_1^* \wedge e_2^* \wedge \cdots \wedge e_r^*. \end{aligned}$$

By composition, we have an isomorphism

$$\begin{aligned} S^{i-r}E \otimes \det E &\rightarrow \pi_*(K_{\mathbb{P}(E)/X} \otimes \mathcal{O}_E(i)) \\ f \otimes e_1 \wedge e_2 \wedge \cdots \wedge e_r &\mapsto ([a^*] \mapsto \langle f, a^{*i-r} \rangle a_1^r dz_2 \wedge dz_3 \wedge \cdots \wedge dz_r \otimes (a^*)^{-i}). \end{aligned}$$

Let $(e_I)_{|I|=i-r}$ be a local frame of $S^{i-r}E$ induced from $(e_j)_{1 \leq j \leq r}$. Then by Lemma 5.1, the dual basis of $(e_I \otimes e_1 \wedge e_1 \wedge \cdots \wedge e_r)_{|I|=i-r}$ is represented through the integration along the fibers of $\pi : \mathbb{P}(E) \rightarrow X$, up to some positive constant multiple, by the following set of $\bar{\partial}$ -closed relative $(0, r-1)$ -forms with values in $\mathcal{O}_E(-i)$:

$$[a^*] \mapsto \overline{\langle e_I, a^{*i-r} \rangle a_1^r \epsilon_r} \frac{d\bar{z}_2 \wedge d\bar{z}_3 \wedge \cdots \wedge d\bar{z}_r}{(1 + \|z\|^2)^r} \otimes \frac{(a^*)^i}{\|a^*\|_{g^*}^{2i}}.$$

We secondary write the isomorphism between $\mathcal{R}^{r-1}(\pi \circ p)_* \mathcal{O}_{Y_s}$ and $\mathcal{R}^{r-1} \pi_* (\mathcal{R}^0 p_* \mathcal{O}_{Y_s})$ in Čech cohomology. There is a standard Stein covering $\mathcal{W} = \{W_j\}_{j=1}^r$ of $\pi^{-1}(U)$ with $W_j = \{[a^*] \in \mathbb{P}(E); a^* = \sum_{j=1}^r a_j e_j^* \in E_x^*, x \in U, a_j \neq 0\}$, provided U is a unit ball in a local chart. Then

$$\mathcal{R}^{r-1} \pi_* (\mathcal{R}^0 p_* \mathcal{O}_{Y_s})(U) = H^{r-1}(\pi^{-1}(U), \oplus_{i=0}^{k-1} \mathcal{O}_E(-i)) = \check{H}^{r-1}(\mathcal{W}, \oplus_{i=0}^{k-1} \mathcal{O}_E(-i)).$$

We use a Stein covering $p^{-1}\mathcal{W} = \{p^{-1}(W_j)\}_{j=1}^r$ of $(\pi \circ p)^{-1}(U)$ to compute $\mathcal{R}^{r-1}(\pi \circ p)_* \mathcal{O}_{Y_s}$. Because the higher direct images of \mathcal{O}_{Y_s} by p vanish, the Stein covering $p^{-1}\mathcal{W}$ is acyclic for the p_* -functor. Then,

$$\mathcal{R}^{r-1}(\pi \circ p)_* \mathcal{O}_{Y_s}(U) = H^{r-1}((\pi \circ p)^{-1}(U), \mathcal{O}_{Y_s}) = \check{H}^{r-1}(p^{-1}\mathcal{W}, \mathcal{O}_{Y_s}).$$

The isomorphism between $\check{H}^{r-1}(\mathcal{W}, \oplus_{i=0}^{k-1} \mathcal{O}_E(-i))$ and $\check{H}^{r-1}(p^{-1}\mathcal{W}, \mathcal{O}_{Y_s})$ is given by pull back, viewing elements of $\mathcal{O}_E(-i)$ as relative homogeneous polynomials of degree i on the total space of $\mathcal{O}_E(1)$. The correspondance between Čech and Dolbeault cohomologies can be made explicit using a partition of unity as in [2, II proposition 9.8]. We find an explicit isomorphism as follows.

Lemma 5.2. *The following map describes the isomorphism in (3.1) :*

$$\begin{aligned} \oplus_{i=r}^{k-1} S^{i-r} E^* \otimes \det E^* &\rightarrow \mathcal{R}^{r-1}(\pi \circ p)_* \mathcal{O}_{Y_s} \\ e_I^* \otimes e_1^* \wedge e_2^* \wedge \cdots \wedge e_r^* &\mapsto \left((x, [a^*], l) \mapsto \epsilon_r \langle l, (a^*)^i \rangle p^* \frac{\overline{\langle e_I, a^{*i-r} \rangle a_1^r d\bar{z}_2 \wedge d\bar{z}_3 \wedge \cdots \wedge d\bar{z}_r}}{\|a^*\|_{g^*}^{2i} (1 + \|z\|^2)^r} \right). \end{aligned}$$

5.3. Singularities of the Hodge metric (explicit formula). In our case: $\pi \circ p : Y_s \rightarrow X$, the singularities of the Hodge metric can be described more explicitly. We keep the notations of the previous paragraph.

In the local frame $(a^*)^{-k}$ for $\mathcal{O}_E(k)$, the section $s \in H^0(\mathbb{P}(E), \mathcal{O}_E(k))$ is given by a local holomorphic function $\sigma = \sigma(x, z_2, \dots, z_r)$ as $s([a^*]) = \sigma \cdot (a^*)^{-k}$. On Y_s the equality $l^k = s([a^*])$ reads $\lambda^k = \sigma$, where $\lambda = \langle l, a^* \rangle$ is the local coordinates on the fibers of $\mathcal{O}_E(1)$. Then by Lemma 5.2, after integrating along the fibers of p , the Hodge metric $h_s(e_I^* \otimes e_1^* \wedge e_2^* \wedge \cdots \wedge e_r^*)$ on a neighborhood U of $x_0 \in X$ is of the form, up to some positive constant multiple,

$$x \mapsto k \int_{[a^*] \in \mathbb{P}^{r-1}(E_x)} \frac{|\langle e_I, a^{*i-r} \rangle|^2 |a_1|^{2r} |\sigma|^{\frac{2i}{k}}}{\|a^*\|_{g^*}^{4i}} \frac{\Omega^{r-1}}{(r-1)!}$$

for $x \in U - \Sigma_s$, where Ω denotes the relative Fubini-Study Kähler form with respect to $\pi : \mathbb{P}(E) \rightarrow X$. The local expression on Y_s where σ has a k -th root may serve to check that the integrand is homogeneous in a^* and that the Hodge metric h_s is smooth on $X - \Sigma_s$. The Hodge metric h_s is initially defined on $\mathcal{R}^{r-1}(\pi \circ p)_* \mathcal{O}_{Y_s|_{X - \Sigma_s}}$ as in § 4.1. The above explicit formula on $X - \Sigma_s$ also describes the behavior of h_s around Σ_s . Thanks to this description, it is possible to extend h_s as a continuous (but maybe degenerate) hermitian metric on $\mathcal{R}^{r-1}(\pi \circ p)_* \mathcal{O}_{Y_s}$ by the same formula also at $x \in U \cap \Sigma_s$. We also call this continuous extension h_s the Hodge metric on $\mathcal{R}^{r-1}(\pi \circ p)_* \mathcal{O}_{Y_s} = \oplus_{i=r}^{k-1} S^{i-r} E^* \otimes \det E^*$.

6. PROOF OF THEOREMS ; AMPLENESS AND POSITIVITY

Proof of Theorem 1.1. Let E be a semi-ample vector bundle of rank $r > 1$ on a compact complex manifold X . We take an arbitral positive integer k so that $S^k E$ is generated by its global sections.

By the explicit expression in § 5.3, Hodge metrics on $\bigoplus_{i=r}^{k-1} (S^{i-r} E \otimes \det E)^*$ may acquire zeros at the points x over which the section $s|_{\mathbb{P}(E_x)}$ is identically zero. We now explain how the semi-ample assumption on the vector bundle E helps to remove those singularities of the Hodge metric. For every point x in X , we choose a generic section s_x of $\mathcal{O}_E(k)$ over the fiber $\mathbb{P}(E_x)$ transverse to the zero section. We extend the section s_x to a global section of $\mathcal{O}_E(k)$ over $\mathbb{P}(E)$ and take a generic section close to this extension. It will be transverse to the zero section on each fiber of points in a neighborhood of x , because this is an open condition. This shows that on the compact manifold X , adding a finite number of Hodge metrics h_α on $\bigoplus_{i=r}^{k-1} (S^{i-r} E \otimes \det E)^*$ seen as the direct image $\mathcal{R}^{r-1}(\pi \circ p_\alpha)_* \mathcal{O}_{Y_\alpha}$ for different covering maps $p_\alpha : Y_\alpha = Y_{s_\alpha} \rightarrow \mathbb{P}(E)$ of degree k , we get a continuous non-degenerate (i.e. positive on every non-zero vector) hermitian metric $h = \sum_{\alpha=1}^{\ell} h_\alpha$. The metric h_α is continuous on X , and smooth (as a Hodge metric on a smooth family) and non-degenerate outside the discriminant locus Σ_{s_α} . The set of points $x \in X$ where one of the sections $s_\alpha|_{\mathbb{P}(E_x)}$ is not transverse to the zero section is a proper Zariski closed subset of X , that we will denote by $\Sigma_h := \bigcup_{\alpha} \Sigma_{s_\alpha}$.

Next let us discuss the curvature property on each direct summand $S^{i-r} E \otimes \det E$ for $i = r, \dots, k-1$. Take a point x_0 in $X - \Sigma_h$, a non-zero vector $\xi_0 \in (S^{i-r} E \otimes \det E)_{x_0}^*$, and a nowhere zero local holomorphic section $\xi \in \Gamma(U, (S^{i-r} E \otimes \det E)^*)$ achieving the value ξ_0 at x_0 and normal at x_0 for the metric h (i.e. $\nabla_h \xi(x_0) = 0$). Then, the last two terms in the formula (2.1)

$$\frac{\langle \Theta((S^{i-r} E \otimes \det E)_{x_0}^*, h)\xi, \xi \rangle}{\|\xi\|_h^2} = -\sqrt{-1} \partial \bar{\partial} \log \|\xi\|_h^2 + \frac{\langle \nabla_h \xi, \nabla_h \xi \rangle}{\|\xi\|_h^2} - \frac{\sqrt{-1} \partial \|\xi\|_h^2 \wedge \bar{\partial} \|\xi\|_h^2}{\|\xi\|_h^4}$$

vanish at x_0 . Now, Corollary 4.2: Griffiths curvature formula for $\mathcal{R}^{r-1}(\pi \circ p_\alpha)_* \mathcal{O}_{Y_\alpha|_{X-\Sigma_h}}$ asserts that the function $\log \|\xi\|_{h_\alpha}^2$ – whose complex Hessian (or Levi form) is the opposite of the curvature of a line sub-bundle of $(S^{i-r} E \otimes \det E)^*$ – is plurisubharmonic on U . It then follows that $\log \|\xi\|_h^2 = \log(\sum_{\alpha} \|\xi\|_{h_\alpha}^2)$ is plurisubharmonic on U . This gives the Griffiths semi-negativity of $(S^{i-r} E \otimes \det E)^*$ on $X - \Sigma_h$. Since h is continuous and Σ_h is an analytic subset (of zero Lebesgue measure), we can conclude that the dual continuous metric h^* on $S^{i-r} E \otimes \det E$ is Griffiths semi-positive on the whole of X . \square

Remark 6.1. We can take the integer k large enough to make the rank of $S^k E$ exceed the dimension of X . This together with the generation by global sections ensure that a generic section s_X of $S^k E$ has no zero on X (see for example [13, II, ex 8.2]). No fiber of π is hence contained in the divisor D_s of zeros of the section $s \in H^0(\mathbb{P}(E), \mathcal{O}_E(k))$, which is associated with s_X via the isomorphism $S^k E \cong \pi_* \mathcal{O}_E(k)$. (Then we can also see, by a local computation, that every irreducible component of singular fibers of $\pi \circ p$ has multiplicity one.) In the formula in § 5.3, if s is not identically zero on \mathbb{P}^{r-1} over a point x , the right hand side integral (the Hodge norm at x) is not zero. Hence the Hodge metric associated to the covering Y_s is non-degenerate on the whole of X . The metric $h = h_s$ is continuous on X , and smooth (as a Hodge metric on a smooth family) outside Σ_s . This gives a slight simplification of the proof of Theorem 1.1.

Proof of Theorem 1.2. We now assume that E is ample. Let us recall Legendre-type formula for a metric $h = \sum_{\alpha=1}^{\ell} h_{\alpha}$ on $(S^{i-r}E \otimes \det E)^*$ gotten from different cyclic coverings $Y_{s_{\alpha}}$ (for a $(1, 0)$ -form u , $|u|^2$ denotes $\sqrt{-1}u \wedge \bar{u}$):

$$\sqrt{-1}\partial\bar{\partial} \log\left(\sum_{\alpha} \|\xi\|_{h_{\alpha}}^2\right) = \frac{\sum_{\alpha} \|\xi\|_{h_{\alpha}}^2 \sqrt{-1}\partial\bar{\partial} \log \|\xi\|_{h_{\alpha}}^2}{\sum_{\alpha} \|\xi\|_{h_{\alpha}}^2} + \frac{\sum_{\alpha < \beta} \left| \partial \log \|\xi\|_{h_{\alpha}}^2 - \partial \log \|\xi\|_{h_{\beta}}^2 \right|^2 \|\xi\|_{h_{\alpha}}^2 \|\xi\|_{h_{\beta}}^2}{\left(\sum_{\alpha} \|\xi\|_{h_{\alpha}}^2\right)^2}.$$

Applying Griffiths curvature formula (Corollary 4.2) for individual covering and the formula (2.1) for a line subbundle we infer that in the right hand side, the first term is semi-positive. We need to add further Hodge metrics h_{α} to make the second term – hence the left hand side – strictly positive.

The explicit expression in § 5.3 will help to translate the algebraic ampleness assumption on E into a positivity property for a well chosen metric on $S^{i-r}E \otimes \det E$. We may first assume that the chosen local frame $(e_j)_{1 \leq j \leq r}$ for E is normal at $x_0 \in X$ for the fixed metric g on E . Hence, denoting by π_{\star} the push forward of currents by $\pi : \mathbb{P}(E) \rightarrow X$ – in other words, the integration along the fibers of the proper submersion π –, we have

$$\partial \log h_s(e_I^* \otimes e_1^* \wedge e_2^* \wedge \cdots \wedge e_r^*) = 2i\pi_{\star} \left(\frac{|\langle e_I, a^{*i-r} \rangle|^2 |a_1|^{2r} |\sigma|^{\frac{2i}{k}-1}}{\|a^*\|_{g^*}^{4i}} \partial \sigma \wedge \frac{\Omega^{r-1}}{(r-1)!} \right).$$

We now take a positive integer k so large that the map

$$H^0(\mathbb{P}(E), \mathcal{O}_E(k)) \rightarrow H^0(\mathbb{P}(E), \mathcal{O}_E(k) \otimes \pi^*(\mathcal{O}_X/\mathcal{M}_x^2))$$

is surjective for every $x \in X$. By the compactness of X , we can henceforth choose enough, but a finite number of sections $s_{\alpha} \in H^0(\mathbb{P}(E), \mathcal{O}_E(k))$ to ensure positivity in all the directions in the Legendre formula. This gives continuous hermitian metrics on $S^k E \otimes \det E$ with Griffiths positive curvature. Using a regularization process as described in [18] these metrics may be smoothed keeping Griffiths positivity of the curvature. \square

Remark 6.2. Griffiths [10, proposition 2.16] showed that the operator $\bar{\nabla}^p : \mathbf{E}^p \rightarrow \Omega_B^1 \otimes \mathbf{E}^{p-1}$ can be expressed as a cup product with the Kodaira-Spencer class $\rho \in \Omega_{B,b}^1 \otimes H^1(Y_b, TY_b)$ of the family $f : Y \rightarrow B$ coupled with a natural pairing. In our setting, this in turn can be related with the infinitesimal displacement of the hypersurfaces $D_{s,x}$ of $\mathbb{P}(E_x)$ given by the vanishing of the section $s|_{\mathbb{P}(E_x)}$, namely (see [14, chapter 5.2 (c)])

$$\begin{array}{ccc} TX & \rightarrow & H^0(D_{s,x}, \mathcal{O}(D_{s,x})|_{D_{s,x}}) \left(\begin{array}{c} \xrightarrow{\delta^*} H^1(D_{s,x}, TD_{s,x}) \\ \mapsto \rho(v) \end{array} \right) \\ v & \mapsto & (\partial_{\tilde{v}} s)|_{D_{s,x}} \end{array}$$

where \tilde{v} is a holomorphic vector field lifting v on $\mathbb{P}(E_x)$. The map δ^* is the coboundary map in the long exact sequence associated with the short exact sequence for the normal bundle of the divisor $D_{s,x}$

$$0 \rightarrow TD_{s,x} \rightarrow T\mathbb{P}(E_x)|_{D_{s,x}} \rightarrow \mathcal{O}(D_{s,x})|_{D_{s,x}} \rightarrow 0.$$

Our computations make explicit the idea that E being ample, the sections $s|_{\mathbb{P}(E_x)}$ move sufficiently to make the operator $\bar{\nabla}^p : \mathbf{E}^p \rightarrow \Omega_B^1 \otimes \mathbf{E}^{p-1}$ have non-zero contribution in the curvature formula.

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