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HODGE METRICS AND POSITIVITY OF DIRECT IMAGES

CHRISTOPHE MOUROUGANE, SHIGEHARU TAKAYAMA

Abstract. *Building on Fujita-Griffiths method of computing metrics on Hodge bundles, we show that the direct image of an adjoint semi-ample line bundle by a projective submersion has a continuous metric with Griffiths semi-positive curvature. This shows that for every holomorphic semi-ample vector bundle E on a complex manifold, and every positive integer k, the vector bundle $S^kE \otimes \det E$ has a continuous metric with Griffiths semi-positive curvature. If E is ample on a projective manifold, the metric can be made smooth and Griffiths positive.*[1]

1. Introduction

We study the positivity properties of direct images of adjoint line bundles. We get

Theorem 1.1. *Let $\phi : Y \longrightarrow X$ be a projective submersion between two complex manifolds. Let L be a semi-ample line bundle on Y. Then the vector bundle $\phi_\star(K_{Y/X} \otimes L)$ has a continuous metric with Griffiths semi-positive curvature.*

Theorem 1.2. *Let $\phi : Y \longrightarrow X$ be a submersion between two complex projective manifolds. Let L be an ample line bundle on Y. Then the vector bundle $\phi_\star(K_{Y/X} \otimes L)$ has a smooth metric with Griffiths positive curvature.*

On grassmanian manifolds, abelian varieties and toric varieties, these results were already proved either by the Castelnuovo-Mumford criterion for global generation or by trying to mimic the Frobenius morphisms over $\mathbb{C}$ ([15], [16, Théorème 3]). These theorems should be compared with by now classical results due to Fujita, Kawamata and others in algebraic geometry setting (see for example [14, Chapter 6.3.E]). These constructions may help to find some topological properties of algebraic sub-varieties under ampleness assumptions for the normal bundle (see for example [7] and [14] part two). The interesting outcome is that metrics could be used to construct Morse functions, whose indices are computed by the curvature.

For the main application, we consider a holomorphic vector bundle E on a compact complex manifold, and intend to construct metrics on vector bundles associated to E which would reflect algebraic positivity properties of E. Just note that on the variety $\pi : \mathbb{P}(E) \longrightarrow X$ of rank one quotients of E the direct image $\pi_\star(K_{\mathbb{P}(E)/X} \otimes \mathcal{O}_E(k+r))$, where r is the rank of E, is $S^kE \otimes \det E$ to infer

Corollary 1.3. *Let E be a semi-ample vector bundle on a complex manifold. Then for all positive integer k, the vector bundles $S^kE \otimes \det E$ have continuous metrics with Griffiths semi-positive curvature.*

Corollary 1.4. *Let E be an ample vector bundle on a complex projective manifold. Then for all positive integer k, the vector bundles $S^kE \otimes \det E$ are Griffiths positive.*

The theory of resolution of the $\overline{\partial}$-equation with L^2-estimates for example shows that Griffiths positivity and ampleness are equivalent for line bundles. This implies through the curvature computation of $\mathcal{O}_E(1)$ that Griffiths positive vector bundles are ample. The converse is a problem raised by Griffiths [8, problem (0.9)], and solved positively on curves by Umemura [20] using the concept of stability (see also [3]). Our results provide a weak answer to Griffiths's problem. The appearance of the determinant line bundle has the same origin than its appearance in the vanishing theorem of Griffiths for the cohomology of ample vector bundles.

The first idea in the proof is to use natural cyclic coverings to be able to use Hodge metrics on the direct image of the relative canonical sheaf under a proper Kähler surjective map between two complex manifolds. This may be seen as a metric aspect of Ramanujam's idea to reduce vanishing theorem to topological properties [18] (see also [13]).

[1] 2000 Mathematics Subject Classification : 32L05,(14F05,32J27)

Applying Griffiths method of computing metrics on Hodge bundles, we will be able to compute the curvature of direct image of the structure sheaf for proper Kähler submersions. It turns out that the metric on the top direct image of the structure sheaf of the source manifold is semi-negatively curved.

We then have to determine the singularities of the Hodge metrics. For a proper surjective map $f : Z^{n+r} \longrightarrow X^n$, the Jacobian ideal Jac_f is the ideal of $\mathcal{O}_Z$ generated by the $n \times n$- minors of the matrix of the differential of f (computed in any chart). The discriminant locus $\Delta_f \subset X$ is defined to be the image under f of the sub-scheme of Z where the map f is not submersive, that is the sub-scheme cut out by Jac_f. Denote $X - \Delta_f$ by X^0 and $f^{-1}(X^0)$ by Z^0. The map $f^0 := f_{|Z^0} : Z^0 \longrightarrow X^0$ is then a projective submersion. With these notations, our main technical lemma generalizes Fujita's observation [6, lemma 1.12].

Lemma 1.5. *Let $f : Z \longrightarrow X$ be a proper Kähler surjective map between two complex manifolds.*

(1) *Then, the Hodge metric on $(f^0)_\star(K_{Z^0/X^0})$ extends as a metric with poles on $f_\star(K_{Z/X})$.*
(2) *The Hodge metric on $(f^0)_\star(K_{Z^0/X^0})$ extends as a smooth metric on $f_\star(Jac_f \otimes K_{Z/X})$.*

This is far simpler than general results obtained by Kawamata, Zücker, Kollár and Cattani-Kaplan-Schimd.

This paper is a revised and expanded version of our preprint [17]. Shortly before we ended the writing of our text, Bo Berndtsson gave a nice proof of the Nakano positivity of the direct image of adjoint ample line bundle in the same setting than ours [2]. We nevertheless feel worth to display our techniques for these are different from his.

Acknowledgment. The first named author warmly thanks Indranil Biswas for a collaboration in an attempt to prove similar results.

2. The tools

2.1. **Ampleness and positivity.** We refer to [11] or [14, §6] for basics about ample vector bundles and to [8] or [4, chapter VII] for basics about positive vector bundles. All vector bundles are assumed to be holomorphic.

A vector bundle E on a compact complex manifold X is said to be *semi-ample* if for some positive integer k, its symmetric power S^kE is generated by its global sections. Associated to E, we have $\pi : \mathbb{P}(E) \longrightarrow X$ the variety of rank one quotients of E together with its tautological quotient line bundle $\mathcal{O}_E(1)$. The semi-ampleness of E is rephrased that for every $x \in X$, every section s_x of $\mathcal{O}_E(k)$ over the fiber $\mathbb{P}(E_x)$ extends to a global section of $\mathcal{O}_E(k)$ over $\mathbb{P}(E)$. This in particular implies that $\mathcal{O}_E(k)$ is generated by its global sections. A vector bundle E is said to be *ample* if its associated line bundle $\mathcal{O}_E(1)$ is ample on $\mathbb{P}(E)$. This in particular implies the existence of an integer k such that for every $x \in X$, every section s_x of $\mathcal{O}_E(k)$ over the first infinitesimal neighborhood of the fiber $\mathbb{P}(E_x)$ extends to a global section of $\mathcal{O}_E(k)$ over $\mathbb{P}(E)$.

Recall the formula for the curvature of the Chern connection of the quotient metric h_q on $\mathcal{O}_E(1)$ of a chosen metric h on E. Here $a^\star$ parametrizes rank one quotients of E and $FS(h_x)$ is the Fubini-Study metric of the hermitian form h_x on E_x.

$$\begin{aligned}\Theta(\mathcal{O}_E(1), h_q) &= \sqrt{-1}\partial\overline{\partial}\log\|e_0^\star + z_1e_1^\star + \cdots z_{r-1}e_{r-1}^\star\|^2 \\ &= \Theta(\mathcal{O}_{\mathbb{P}(E_x)}(1), FS(h_x)) - \frac{\langle \pi^\star\Theta(E^\star, h^\star)a^\star, a^\star\rangle}{\langle a^\star, a^\star\rangle}.\end{aligned}$$

A vector bundle E is said to be *Griffiths positive*, if it can be endowed with a smooth hermitian metric h such that for all $x \in X$ and all non-zero decomposable tensors $v \otimes e \in TX_x \otimes E_x$, the curvature term $\langle\Theta(E,h)(v,\overline{v})e, e\rangle_h$ is positive, where $\Theta(E,h) \in \mathcal{C}^\infty_{1,1}(X, Herm(E))$ is the curvature of the Chern connection $\nabla_{E,h}$ of (E,h).

A continuous hermitian metric h on a vector bundle $b : E \longrightarrow X$ is said to be *Griffiths positive*, if there exists a smooth positive real $(1,1)$-form ω_X on X such that in the sense of currents

$$-\sqrt{-1}\partial\overline{\partial}\log h(\xi) \geq b^\star\omega_X,$$

where h is seen as a continuous quadratic function on the total space $E - X \times \{0\}$. At the points where the metric h is smooth, these two notions of Griffiths positivity coincide.

2.2. **Cyclic covers.** A reference for this part is [5, § 3]. Take a semi-ample line bundle L on a complex manifold Y and fix a positive integer k such that the tensor power L^k is generated by its global sections. Then Bertini's theorem (see for example [10, page 137]) insures that a generic section s of L^k over Y is transverse to the zero section (i.e. $ds_{|D_s} : TY_{|D_s} \longrightarrow L^k_{|D_s}$ is surjective), and defines a smooth divisor $D_s := (s = 0)$. Let

$$p = p_s : Y_s \longrightarrow Y$$

be the cyclic covering of Y obtained by taking the k-th root out of D_s. The space

$$Y_s := \{l \in L / l^k = s(p(l))\}$$

is a smooth hypersurface of the total space L, and the map p is a finite cover of degree k totally ramified along the zero locus D_s of s. The space Y_s may also be described as the spectrum $\operatorname{Spec} \mathcal{A}_s$ of the algebra

$$\mathcal{A}_s := \frac{\oplus_{i=0}^{+\infty} L^{-i}}{(l^\star - \check{s}(l^\star)\ ,\ l^\star \in L^{-k})}$$

where $\check{s}$ is the sheaf inclusion $L^{-k} \overset{\times s}{\to} \mathcal{O}_Y$. The direct image of the structure sheaf $\mathcal{O}_{Y_s}$ is hence $p_\star \mathcal{O}_{Y_s} = \mathcal{A}_s \simeq \oplus_{i=0}^{k-1} L^{-i}$.

In our setting, we consider a projective submersion $\phi : Y \longrightarrow X$ of relative dimension r between two complex manifolds and L a semi-ample line bundle on Y. A generic section of L^k gives rise to a covering of Y which we in fact regard as a family of coverings of the fibers of ϕ. Because p_s is a finite morphism, the spectral sequence of composition of direct image functors reduces to the following:

$$\begin{aligned} \mathcal{R}^r(\phi \circ p_s)_\star \mathcal{O}_{Y_s} &= \mathcal{R}^r \phi_\star \left(\mathcal{R}^0 p_\star \mathcal{O}_{Y_s}\right) \\ &= \mathcal{R}^r \phi_\star \left(\oplus_{i=0}^{k-1} L^{-i}\right) = \oplus_{i=0}^{k-1} \phi_\star \left(K_{Y/X} \otimes L^i\right)^\star . \end{aligned}$$

Here we have used Serre duality on the fibers of the smooth morphism ϕ.

2.3. **The Hodge metric.** We recall the basics on geometric variations of Hodge structures and Griffiths's computations ([9, theorem 6.2]) of the curvature the Hodge metric (see also [19, § 7] and [21, chapter 10]). Here we assume that $f : Y \longrightarrow B$ is a proper Kähler submersion of relative dimension r, in particular we regard $f : Y \longrightarrow B$ as a smooth family of compact Kähler manifolds of dimension r. Fix a non-negative integer d. The local system $\mathcal{R}^d f_\star \mathbb{C}$ can be realized as the sheaf of germs of the flat sections of the holomorphic vector bundle $\mathbf{H}^d_{\mathbb{C}}$ associated with the locally free sheaf $(\mathcal{R}^d f_\star \mathbb{C}) \otimes \mathcal{O}_B$ endowed with the flat holomorphic connection $\nabla : \mathbf{H}^d_{\mathbb{C}} \longrightarrow \Omega^1_B \otimes \mathbf{H}^d_{\mathbb{C}}$, the Gauss-Manin connection. By semi-continuity and Hodge decomposition, the vector spaces $H^{p,d-p}(Y_b, \mathbb{C})$ $(b \in B)$ have constant dimension. By elliptic theory they hence form a differentiable sub-bundle $\mathbf{H}^{p,d-p}$ of $\mathbf{H}^d_{\mathbb{C}}$. Denote by $\mathbf{F}^p$ the differentiable sub-bundle $\oplus_{i \geq p} \mathbf{H}^{i,d-i}$ of $\mathbf{H}^d_{\mathbb{C}}$. By a theorem of Griffiths, the $\mathbf{F}^p$ have natural structure of holomorphic sub-vector bundles of $\mathbf{H}^d_{\mathbb{C}}$. A relative Dolbeault theorem identifies $\mathbf{E}^p := \mathbf{F}^p / \mathbf{F}^{p+1}$ with the holomorphic vector bundle associated with the locally free sheaf $\mathcal{R}^{d-p} f_\star \Omega^p_{Y/B}$.

We now recall the construction of the Hodge metric on the primitive part of $\mathbf{E}^p$. We fix a family η_b $(b \in B)$ of polarizations given by a section of $\mathcal{R}^2 f_\star \mathbb{Z}$ (e.g. the family of Chern classes $c_1(L_{|Y_b})$ for an ample line bundle L on Y). By the hard Lefschetz theorem, the bilinear form on the fibers of $\mathbf{H}^d_{\mathbb{C}}$ given by

$$S(c_1, c_2) := (-1)^{\frac{d(d-1)}{2}} \int_{Y_b} \eta_b^{r-d} \wedge c_1 \wedge c_2$$

is non-degenerate. We define the primitive cohomology to be $\mathbf{P}^d := Ker(\eta^{r-d+1} : \mathbf{H}^d \longrightarrow \mathbf{H}^{2r-d+2})$, which is also a differentiable sub-bundle of $\mathbf{H}^d$. By the Hodge-Riemann bilinear relations, the differentiable sub-bundles $\mathbf{H}^{p,d-p}$ and $\mathbf{H}^{p',d-p'}$ are orthogonal unless $p + p' = d$ and

$$h(c) := (\sqrt{-1})^{p-q} S(c, \overline{c})$$

defines a positive definite metric on $\mathbf{H}^{p,d-p}_{prim} := \mathbf{H}^{p,d-p} \cap \mathbf{P}^d$. We set $\mathbf{F}^p_{prim} := \mathbf{F}^p \cap \mathbf{P}^d$. Those bundle also have natural holomorphic structures. We also set $\mathbf{E}^p_{prim} := \mathbf{F}^p_{prim} / \mathbf{F}^{p+1}_{prim}$. The fiber-wise isomorphism of $(\mathbf{E}^p_{prim})_b$ with $H^{p,d-p}_{prim}(Y_b, \mathbb{C}) \subset H^d(Y_b, \mathbb{C})$ enables to equip the holomorphic vector bundle $\mathbf{E}^p_{prim}$ with a smooth positive definite hermitian metric, called the Hodge metric.

We need some definitions in order to express the curvature of the corresponding Chern connection. First recall the transversality property of the Gauss-Manin connection $\nabla \mathbf{F}^p \subset \Omega^1_B \otimes \mathbf{F}^{p-1}$ which accounts for the

Cartan-Lie formula of the derivative of a family of cohomology classes (see [21, proposition 9.14]). Denote by $\overline{\nabla}^p : \mathbf{E}^p \longrightarrow \Omega^1_B \otimes \mathbf{E}^{p-1}$ the $\mathcal{O}_B$-linear map built by first lifting to $\mathbf{F}^p$ applying the Gauss-Manin connection and projecting to $\mathbf{E}^{p-1}$. The second fundamental form in $\mathcal{C}^{1,0}_\infty(B, Hom(\mathbf{F}^p, \mathbf{H}^d_{\mathbb{C}}/\mathbf{F}^p))$ of the sequence

$$0 \longrightarrow \mathbf{F}^p \longrightarrow \mathbf{H}^d_{\mathbb{C}} \longrightarrow \mathbf{H}^d_{\mathbb{C}}/\mathbf{F}^p \longrightarrow 0$$

with respect to the Gauss-Manin connection (or equivalently with the flat metric on $\mathbf{H}^d_{\mathbb{C}}$) actually induces $\overline{\nabla}^p : \mathbf{E}^p \longrightarrow \Omega^1_B \otimes \mathbf{E}^{p-1}$. Formulas for the curvature of quotient hermitian holomorphic vector bundles then lead to

Theorem 2.1. [9, theorem 5.2] *The curvature $\Theta(\mathbf{E}^p_{prim})$ of the holomorphic vector bundle $\mathbf{E}^p_{prim}$ endowed with its Hodge metric is given by*

$$\langle \Theta(\mathbf{E}^p_{prim})(V, \overline{V})\sigma, \sigma\rangle_{Hodge} = \langle \overline{\nabla}^p_V \sigma, \overline{\nabla}^p_V \sigma\rangle_{Hodge} - \langle (\overline{\nabla}^{p+1}_V)^\star \sigma, (\overline{\nabla}^{p+1}_V)^\star \sigma\rangle_{Hodge}$$

where V is a local vector field on B and σ a local section of $\mathbf{E}^p$.

We now apply this result in the case of the family of the cyclic covers $\phi \circ p_s : Y_s \longrightarrow X$ obtained by taking the k-th root of a section s of L^k transverse to the zero section as in § 2.2. We have to restrict the study over Zariski open sets outside the discriminant locus Σ_s of $\phi \circ p_s$. It is the set of $x \in X$ where $s_{|Y_x} \in \Gamma(Y_x, L^k)$ is not transverse to the zero section. We set $X^0 := X - \Sigma_s$ and $Y^0_s := (\phi \circ p_s)^{-1}(X - \Sigma_s)$ so that $\phi \circ p : Y^0_s \longrightarrow X^0$ becomes a smooth family. Then, since $\overline{\nabla}^0$ vanishes, the above theorem implies the following

Corollary 2.2. *The vector bundle $\mathbf{E}^0_{prim} = \mathbf{E}^0 = \mathcal{R}^r(\phi \circ p_s)_\star \mathcal{O}_{Y^0_s/X^0} = \oplus_{i=0}^{k-1} \mathcal{R}^r \phi_\star \left(L^{-i}\right)$ with the Hodge metric is Griffiths (actually even Nakano) semi-negative.*

2.4. **Singularities of the Hodge metric.** We now deal with the general case, namely $f : Y \longrightarrow B$ may not be smooth. We first recall the method of Fujita [6]. We therefore assume that the base B is one dimensional.

The Hodge metric on the direct image of the relative canonical sheaf is described as follows. Let $b \in B$ be a point and let (U, t) be a local coordinate centered at $b = \{t = 0\}$. A section $\omega \in \Gamma(U, f_\star K_{Y/B})$ – when regarded as a section in $\Gamma(U, f_\star Hom(f^\star K_B, K_Y))$ and applied to $f^\star dt$ – gives a section of K_Y on $f^{-1}(U)$ which we denote by $\omega \cdot dt$. If $\varphi_b \in \Gamma(Y_b, K_{Y_b})$ fulfills the relation $\omega \cdot dt = \varphi_b \wedge f^\star dt$ over Y_b (which amounts to saying that in the differentiable trivialization $Y_{|U} \simeq Y_b \times U$, the section ω is sent to φ_b), then the Hodge norm at $b \in B$ of the section ω is

$$\|\omega\|^2_{Hodge} = (\sqrt{-1})^n (-1)^{\frac{n(n-1)}{2}} \int_{Y_b} \varphi_b \wedge \overline{\varphi_b},$$

here $n = \dim Y - 1$. Fujita checked that in this setting in case $\dim B = 1$ the Hodge metric on $f_\star K_{Y/B}$ is bounded from below by a positive quantity and hence that the only possible singularities of the Hodge metric on $f_\star K_{Y/B}$ are poles (see [6, lemma 1.12]).

We just give the typical example which occurs for a local model of our cyclic covers (for some positive integer m).

$$\begin{array}{ccccc} Y_s = \{(t, z, l) \in \mathbb{C}^3 / l^k = t + z^m\} & \xrightarrow{p_s} & Y & \xrightarrow{\phi} & X = \{t \in \mathbb{C}\} \\ (t, z, l) & \mapsto & (t, z) & \mapsto & t \end{array}$$

The cotangent bundle $\Omega^1_{Y_s}$ is generated by dt, dz, dl subject to the relation $kl^{k-1}dl - dt - mz^{m-1}dz = 0$. If $\omega \cdot dt$ is written as $\eta(z, l) dz \wedge dl$ for a holomorphic function $\eta(z, l)$, then φ_0 may be chosen to be $\varphi_0 = k^{-1} l^{1-k} \eta(z, l) dz$ with a pole of order $k - 1$ on the fiber over $t = 0$ and no singularities elsewhere.

We can now go back to the proof of our main lemma.

Proof of lemma 1.5. Let $f : Z^{n+r} \longrightarrow X^n$ be a proper Kähler surjective map between two complex manifolds of relative dimension r. Let $z = (z_1, z_2, \cdots, z_{n+r})$ and $x = (x_1, x_2, \cdots, x_n)$ be local holomorphic coordinates on Z and X around z_0 and $x_0 := f(z_0)$. The map f is locally given by $(f_i(z))_{1 \leq i \leq n}$. Let $\Psi \in f_\star(K_{Z/X})$ and write

$$\begin{aligned} \Psi \cdot f^\star(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n) &= \psi dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{n+r} \\ &= \psi \det \left(\frac{\partial f_i}{\partial z_j}\right)^{-1}_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} f^\star(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n) \wedge dz_{n+1} \wedge \cdots \wedge dz_{n+r} \end{aligned}$$

the Hodge norm of Ψ is hence

$$\|\Psi\|^2_{Hodge} = \int_{Z_x} |\psi|^2 \left|\det\left(\frac{\partial f_i}{\partial z_j}\right)_{\substack{1\le i\le n\\ 1\le j\le n}}\right|^{-2} (\sqrt{-1})^r dz_{n+1}\wedge d\overline{z_{n+1}}\cdots\wedge dz_{n+r}\wedge d\overline{z_{n+r}}. \tag{2.1}$$

The Hodge metric can hence only acquire poles, located furthermore over the discriminant locus of f. If now Ψ belongs to $f_\star(Jac_f\otimes K_{Z/X})$ then ψ is a combination $\psi=\sum\psi_J\det\left(\frac{\partial f_i}{\partial z_j}\right)_{\substack{1\le i\le n\\ j\in J}}$, where the sum is taken over all multi-indexes $J\subset\{1,2,\cdots,n+r\}$ of length $|J|=n$, manely $J=\{j_1,j_2,\ldots,j_n\}$ with $1\le j_1<j_2<\ldots<j_n\le n+r$. It follows that

$$\Psi\cdot f^\star(dx_1\wedge dx_2\wedge\cdots\wedge dx_n) = f^\star(dx_1\wedge dx_2\wedge\cdots\wedge dx_n)\wedge\left(\sum\psi_J dz_{m_1}\wedge\cdots\wedge dz_{m_r}\right).$$

Here the last sum is taken over all multi-indexes $J\subset\{1,2,\cdots,n+r\}$ of $|J|=n$, and $\{m_1,\ldots,m_r\}=\{1,2,\cdots,n+r\}\setminus J$. □

In our setting, we take a global section s of L^k transverse to the zero section and consider the corresponding cyclic cover Y_s of Y. Note that if the section s is locally given by $\sigma(x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_r)$ and the map ϕ by $(x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_r)\mapsto(x_1,x_2,\cdots,x_n)$, then there is an j such that $\frac{\partial s}{\partial y_j}(p_0)\neq 0$ and the functions $(x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_{j-1},t,y_{j+1},\cdots,y_r)$ can be chosen as coordinates on Y_s near $y_0\in p_s^{-1}(y_0)$, or there is an i such that $\frac{\partial s}{\partial x_i}(p_0)\neq 0$ and the coordinates on Y_s near y_0 can be chosen to be $(x_1,x_2,\cdots x_{i-1},t,x_{i+1},\cdots,x_n,y_1,y_2,\cdots,y_r)$. In the first case, $\phi\circ p_s$ is a submersion at y_0 and its Jacobian ideal is $\mathcal{O}_{Y_s}$ at y_0. In the second case the Jacobian ideal contains

$$\frac{\partial x_i}{\partial t} = \frac{\partial s}{\partial t}\left(\frac{\partial s}{\partial x_i}\right)^{-1}\sim t^{k-1}. \tag{2.2}$$

In any case it contains $p_s^\star\mathcal{O}(-\frac{k-1}{k}D_s)$. This has to be compared with the formula $K_{Y_s/Y}=p_s^\star((k-1)L)$.

3. Proof of Theorems

Proof of Theorem 1.1. Let $\phi : Y\longrightarrow X$ be a projective submersion of relative dimension r between two complex manifolds and L be a semi-ample line bundle on Y. Note that the relative projectivity condition is only used to ensure the local freeness of $\phi_\star(K_{Y/X}\otimes L)$.

We take a positive integer k so that L^k is generated by its global sections. Take a global section s of L^k transverse to the zero section and consider the corresponding cyclic cover Y_s of Y. We have seen that the Hodge metric on $\mathcal{R}^r(\phi\circ p_s)_\star\mathcal{O}_{Y_s}$ is smooth (as a Hodge metric on a smooth family) non-degenerate and semi-negatively curved outside the discriminant locus Σ_{s_α}, is continuous on X, and may acquire zeros at the points x over which the section $s_{|\mathbb{P}(E_x)}$ is identically zero. We now explain how the semi-ampleness assumption on the line bundle L helps to remove those singularities of the Hodge metric.

When L is semi-ample, $D\in|L^k|$ moves. We choose $n+r+1$ generic sections s_α that generate L^k. The metric $h=\sum_{\alpha=1}^{n+r+1}h_\alpha$ on $\mathcal{R}^r\phi_\star(L^{-1})\subset\mathcal{R}^r(\phi\circ p_s)_\star\mathcal{O}_{Y_s}$ is continuous on X, smooth outside $\Sigma_h:=\cup_\alpha\Sigma_{s_\alpha}$ and without zeros.

Next let us discuss its curvature property. Recall the formula

$$\begin{aligned}\frac{\langle\Theta(E,h)\xi,\xi\rangle}{\|\xi\|^2} &= -\sqrt{-1}\partial\overline{\partial}\log\|\xi\|^2+\frac{\langle\nabla_{E,h}\xi,\nabla_{E,h}\xi\rangle}{\|\xi\|^2}-\frac{\sqrt{-1}\partial\|\xi\|^2\wedge\overline{\partial}\|\xi\|^2}{\|\xi\|^4}\\ &\ge -\sqrt{-1}\partial\overline{\partial}\log\|\xi\|^2\end{aligned} \tag{3.1}$$

for a nowhere zero local holomorphic section ξ of a holomorphic vector bundle E equipped with a smooth hermitian metric h. The last two terms give the norm at x of the fundamental form of the inclusion $\mathcal{O}_X\xi\subset E$. Take a point x_0 in $X-\Sigma_h$, a non-zero vector $\xi_0\in\mathcal{R}^r\phi_\star(L^{-1})^\star_{x_0}$, and a nowhere zero local holomorphic section $\xi\in\Gamma(U,\mathcal{R}^r\phi_\star(L^{-1}))$ achieving the value ξ_0 at x_0 and normal at x_0 for the metric h (i.e. $\nabla_h\xi(x_0)=0$). Then, the last two terms in the formula (3.1) vanish at x_0. The corollary 2.2 now asserts the function $\log\|\xi\|^2_{h_\alpha}$ –

whose complex Hessian (or Levi form) is the opposite of the curvature of a line sub-bundle of $\mathcal{R}^r\phi_\star(L^{-1})$ – is plurisubharmonic on U. It then follows that $\log\|\xi\|^2_h = \log(\sum_\alpha \|\xi\|^2_{h_\alpha})$ is plurisubharmonic also on U. This gives the Griffiths semi-negativity of $\mathcal{R}^r\phi_\star(L^{-1})$ on $X - \Sigma_h$. Since h is continuous and Σ_h is an analytic subset (of zero Lebesgue measure), we can conclude that the continuous metric h on $\mathcal{R}^r\phi_\star(L^{-1})$ is Griffiths semi-negative on the whole of X. □

Proof of Theorem 1.2. We now assume that L is ample. Here, the ampleness assumption ensures the vanishing of $\mathcal{R}^1\phi_\star(K_{Y/X}\otimes L)$ which in turn easily ensures the local freeness of $\phi_\star(K_{Y/X}\otimes L)$.

Let us recall Legendre-type formula applied for a metric $h = \sum_{\alpha=1}^{\ell} h_\alpha$ on $\mathcal{R}^r\phi_\star(L^{-1})$ gotten from different cyclic coverings Y_{s_α} (for a $(1,0)$-form u, $|u|^2$ denotes $\sqrt{-1}u\wedge\overline{u}$) :

$$\sqrt{-1}\partial\overline{\partial}\log(\sum_\alpha \|\xi\|^2_{h_\alpha}) = \frac{\sum_\alpha \|\xi\|^2_{h_\alpha}\sqrt{-1}\partial\overline{\partial}\log\|\xi\|^2_{h_\alpha}}{\sum_\alpha \|\xi\|^2_{h_\alpha}} + \frac{\sum_{\alpha<\beta}\left|\partial\log\|\xi\|^2_{h_\alpha} - \partial\log\|\xi\|^2_{h_\beta}\right|^2\|\xi\|^2_{h_\alpha}\|\xi\|^2_{h_\beta}}{(\sum_\alpha \|\xi\|^2_{h_\alpha})^2}.$$

Applying Griffiths curvature formula (Corollary 2.2) for individual covering and the formula (3.1) for a line sub-bundle we infer that in the right hand side, the first term is semi-positive. We need to add further Hodge metrics h_α to make the second term – hence the left hand side – strictly positive.

The explicit expression in formula (2.1) and (2.2) will help to translate the algebraic ampleness assumption on L into a negativity property for a well chosen metric on $\mathcal{R}^r\phi_\star(L^{-1})$. Hence, we have for $\Psi\in(\phi\circ p_s)_\star(K_{Y_s/Y})$

$$\begin{aligned}\partial\log h_s(\Psi) &= \partial\log\int_{(\phi\circ p_s)^{-1}(x)}|\psi|^2\left|\left(\frac{\partial s}{\partial t}\right)^{-1}\frac{\partial s}{\partial x_i}\right|^2(\sqrt{-1})^r dz_{n+1}\wedge d\overline{z_{n+1}}\cdots\wedge dz_{n+r}\wedge d\overline{z_{n+r}}\\ &= \partial\log\int_{Y_x}|\psi|^2|s|^{-2k+2}\left|\frac{1}{k}\frac{\partial s}{\partial x_i}\right|^2(\sqrt{-1})^r dz_{n+1}\wedge d\overline{z_{n+1}}\cdots\wedge dz_{n+r}\wedge d\overline{z_{n+r}}\end{aligned}$$

We now take a positive integer k so large that the map

$$H^0(Y,L^k) \longrightarrow H^0\left(Y, L^k\otimes\phi^\star(\mathcal{O}_X/\mathcal{M}_x^3)\right)$$

is surjective for every $x\in X$. By the compactness of Y, we can henceforth choose enough, but a finite number of sections $s_\alpha\in H^0(Y,L^k)$ to ensure positivity in all the directions in the Legendre formula. This gives continuous hermitian metrics on $\phi_\star(K_{Y/X}\otimes L)$ with Griffiths positive curvature. Using a regularization process as described in [16] these metrics may be smoothed keeping Griffiths positivity of the curvature. □

Remark 3.1. Griffiths [9, proposition 2.16] showed that the operator $\overline{\nabla}^p : \mathbf{E}^p \longrightarrow \Omega^1_B\otimes\mathbf{E}^{p-1}$ can be expressed as a cup product with the Kodaira-Spencer class $\rho\in\Omega^1_{B,b}\otimes H^1(Y_b,TY_b)$ of the family $f: Y\longrightarrow B$ coupled with a natural pairing. In our setting, this in turn can be related with the infinitesimal displacement of the hypersurfaces $D_{s,x}$ of Y_x given by the vanishing of the section $s_{|Y_x}$, namely (see [12, chapter 5.2 (c)])

$$\begin{array}{ccccc} TX & \longrightarrow & H^0(D_{s,x},\mathcal{O}(D_{s,x})_{|D_{s,x}}) & \left(\xrightarrow{\delta^\star}\right. & H^1(D_{s,x},TD_{s,x})\ \Big) \\ v & \mapsto & (\partial_{\widetilde{v}}s)_{|D_{s,x}} & \left(\mapsto\right. & \rho(v)\ \Big)\end{array}$$

where $\widetilde{v}$ is a holomorphic vector field lifting v on Y_x. The map $\delta^\star$ is the co-boundary map in the long exact sequence associated with the short exact sequence for the normal bundle of the divisor $D_{s,x}$

$$0\longrightarrow TD_{s,x}\longrightarrow (TY_x)_{|D_{s,x}}\longrightarrow\mathcal{O}(D_{s,x})_{|D_{s,x}}\longrightarrow 0.$$

Our computations make explicit the idea that L being ample, the sections $s_{|Y_x}$ move sufficiently to make the operator $\overline{\nabla}^p : \mathbf{E}^p\longrightarrow\Omega^1_B\otimes\mathbf{E}^{p-1}$ have non-zero contribution in the curvature formula.

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