On the explicit reconstruction of a Riemann surface from its Dirichlet-Neumann operator

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ON THE EXPLICIT RECONSTRUCTION OF A RIEMANN SURFACE FROM ITS DIRICHLET-NEUMANN OPERATOR

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Abstract. This article gives a complex analysis lighting on the problem which consists in restoring a bordered connected riemaniann surface from its boundary and its Dirichlet-Neumann operator. The three aspects of this problem, unicity, reconstruction and characterization are approached.

Contents

1. Statements of the main results

Let $\mathcal{X}$ be an open bordered riemannian real surface (i.e. the interior of an oriented riemannian two dimensional real manifold whose all components have non trivial one dimensional smooth boundary) and g its metric. Using the boundary control method, Belishev and Kurylev [4][6] have started the study of the inverse problem consisting in recovering $(\mathcal{X}, g)$ from the operators $N_\lambda : C^\infty(b\mathcal{X}) \ni u \mapsto (\partial\widetilde{u}_\lambda/\partial\nu)_{b\mathcal{X}}$ where $b\mathcal{X}$ is the topological boundary of $\mathcal{X}$, ν is the normal exterior unit to $b\mathcal{X}$ and $\widetilde{u}_\lambda$ is the unique solution of $\Delta_g U = \lambda U$ such that $U|_\gamma = u$. The principal result of [6] implies that the

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knowledge of $\lambda \mapsto N_\lambda$ on an non empty open set of $\mathbb{R}_+$ determines $(\mathcal{X}, g)$ up to isometry. The important question to know if $(\mathcal{X}, g)$ is uniquely determined by only one operator N_{λ_*} with $\lambda_* \neq 0$ remains open. This article mainly deals with the case of the Dirichlet-Neumann operator $N_{\mathcal{X}} := N_0$. Section 2 gives an intrinsic interpretation Electrical Impedance Tomography on manifolds, EIT for short, in terms of Inverse-Dirichlet-Neumann problem for twisted Laplacian. In dimension two, this clearly underline how the complex structure of Riemannian surfaces is involved.

Two surfaces in the same conform class which have the same oriented boundary and whose metrics coincides there, need to have the same Dirichlet-Neumann operator. Conversely, Lassas and Uhlmann [21] have proved for a connected $\mathcal{X}$ that the conform class and so the complex structure of $(\mathcal{X}, g)$ is determined by $N_{\mathcal{X}}$. Hence, it is relevant to consider $\mathcal{X}$ as a Riemann surface. In [5], also the full knowledge of $N_{\mathcal{X}}$, Belishev gives another proof of the above unicity by recovering abstractly $\mathcal{X}$ as the spectre of the algebra of boundary values of functions holomorphic on $\mathcal{X}$ and continuous on $\overline{\mathcal{X}} = \mathcal{X} \cup b\mathcal{X}$. It turns out in our theorems 1 and 2 that only three generic functions on the boundary and their images by $N_{\mathcal{X}}$ are sufficient for unicity to hold and to reconstruct $\mathcal{X}$ by integral Cauchy type formulas. Theorems 3a, 3b and 3c deal with characterizations of datas of the type $(b\mathcal{X}, N_{\mathcal{X}})$ where $\mathcal{X}$ is a Riemann surface.

While the frame of bordered manifolds is sufficient for real analytic boundaries, characterization statements lead to consider a wider class of manifolds. In this article, $\left(\overline{\mathcal{X}}, \gamma\right)$ is a *Riemann surface with almost smooth boundary* if the following holds[1] : $\overline{\mathcal{X}}$ is a compact metrizable topological manifold which is the closure of $\mathcal{X} = \overline{\mathcal{X}} \backslash \gamma$, $\mathcal{X}$ is a Riemann surface and[2] $h^2\left(\overline{\mathcal{X}}\right) < \infty$, γ is a smooth real curve[3] and the set $\sigma = \overline{\mathcal{X}}_{\text{sing}}$ ($\overline{\mathcal{X}} \backslash \overline{\mathcal{X}}_{\text{sing}}$ is denoted by $\overline{\mathcal{X}}_{\text{reg}}$) of points of γ where $\left(\overline{\mathcal{X}}, \gamma\right)$ is not a manifold with boundary satisfies $h^1(\sigma) = 0$.

If $\left(\overline{\mathcal{X}}, \gamma\right)$ is a Riemann surface with almost smooth boundary, classical results contained in [1] implies a Riemann's existence theorem : a real valued function u of class C^1 on γ has a unique continuous extension $\widetilde{u}$ to $\overline{\mathcal{X}}$ which is harmonic on $\mathcal{X}$, smooth on $\overline{\mathcal{X}}_{\text{reg}}$ and satisfy $\int_{\mathcal{X}} i\,\partial\widetilde{u} \wedge \overline{\partial}\widetilde{u} < +\infty$. Moreover, $N_{\mathcal{X}} u$ still make sense as the element of the dual space of $C^1(\gamma)$ which equals $\partial\widetilde{u}/\partial\nu$ on $\gamma \backslash \overline{\mathcal{X}}_{\text{sing}}$ (see proposition 12).

In the sequel, γ is a smooth compact oriented real curve without component reduced to a point, N is an operator from $C^1(\gamma)$ to the space of currents on γ of degree 0 and order 1, τ is a smooth generating section of $T\gamma$ and ν is another vector field along γ such that the bundle $\mathcal{T}$ generated by (ν_x, τ_x), $x \in \gamma$, has rank 2 ; γ is assumed to be oriented by τ and $\mathcal{T}$ by (ν, τ).

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[1] A Stokes formula holds automatically for such manifolds (see lemma 11 in section 3).

[2] h^d is the d-dimensionnial Hausdorff measure.

[3] It could have been possible to allow singularities on γ itself but we have avoid it for the sake of simplicity of statements. For the same reason, we consider only smooth DN-datas in the sequel.

The inverse Dirichlet-Neumann problem for $(\gamma, N, \mathcal{T})$ is to find, when it exists, an open riemaniann surface $(\mathcal{X}, g)$ with almost smooth boundary γ such that for all $x \in \gamma \cap \overline{\mathcal{X}}_{\mathrm{reg}}$, (ν_x, τ_x) is a positively oriented orthonormal basis of $T_x\overline{\mathcal{X}}$ and for all $u \in C^1(\gamma)$, $Nu = N_{\mathcal{X}}u$ in the sense of currents. As these conditions do not distinguished between metrics g in a same conformal class, we look after $\mathcal{X}$ as a Riemann surface. The connection between real and complex analysis in the IDN-problem is realized through the operators L and θ defined for $u \in C^1(\gamma)$ by

$$Lu = \frac{1}{2}(Nu + iTu) \quad \& \quad \theta u = (Lu)(\nu^* + i\tau^*) \tag{1.1}$$

where T is the tangential derivation by τ and (ν_x^*, τ_x^*) is the dual basis of (ν_x, τ_x) for every $x \in \gamma$. Note that in the sense of currents, the equality $Nu = N_{\mathcal{X}}u$ is equivalent to the identity $\partial\widetilde{u} = \theta u$, the tilde denoting, as through all this article, continuous harmonic extension to $\mathcal{X}$

If an open Riemann surface $\mathcal{X}$ has almost smooth boundary γ, g is a hermitian metric on $\mathcal{X}$ for which (τ_x, ν_x) is a positively oriented orthonormal basis of $T_x^*\mathcal{X}$ when $x \in \gamma$ is outside $\sigma = \overline{\mathcal{X}}_{\mathrm{sing}}$ and if ρ is a continuous defining function of γ in $\overline{\mathcal{X}}$ smooth on $\overline{\mathcal{X}}\backslash\sigma$, then $(\nu^*, \tau^*) = \frac{1}{|d\rho|_g}(d\rho, d^c\rho)$ on $\gamma\backslash\sigma$ where $d^c = i(\overline{\partial} - \partial)$ and $\partial\widetilde{u} = (Lu)\,|\partial\rho|_g^{-1}\,\partial\rho = \theta u$ on $\gamma\backslash\sigma$ for all $u \in C^1(\gamma)$.

Main hypothesis. In addition to the assumptions on γ, we consider in all this paper $u_0, u_1, u_2 \in C^\infty(\gamma)$ three real valued functions only ruled by the hypothesis that

$$f = (f_1, f_2) = \left((Lu_\ell)\,/\,(Lu_0)\right)_{\ell=1,2} = \left((\theta u_\ell)\,/(\theta u_0)\right)_{\ell=1,2} \tag{1.2}$$

is an embedding of γ in $\mathbb{C}^2$ considered as the complement of the hyperplane $\{w_0 = 0\}$ of the complex projective plane $\mathbb{CP}_2$ with homogeneous coordinates $(w_0 : w_1 : w_2)$. This is somehow generic :

Proposition 0. *Assume γ is real analytic. Fix $u_1, u_0 \in C^\omega(\gamma)$ so that $f_1 = (Lu_1)\,/\,(Lu_0)$ is non constant on every connected component of γ. For any function $u_2 \in C^\omega(\gamma)$, one can construct $v_2 \in C^\omega(\gamma)$, arbitrarily close to u_2 in C^2 norm, such that $(f_1, (Lv_2)\,/\,(Lu_0))$ is an embedding of γ into $\mathbb{C}^2$.*

If $u = (u_\ell)_{0\leqslant\ell\leqslant2}$ is a triplet of smooth functions on γ, we set $\theta u = (\theta u_\ell)_{0\leqslant\ell\leqslant2}$ and call $(\gamma, u, \theta u)$ a *restricted DN-data* for an open Riemann surface $\mathcal{X}$ if $\mathcal{X}$ has almost smooth boundary γ, $\partial\widetilde{u_\ell} = \theta u_\ell$ for $0 \leqslant \ell \leqslant 2$ and the well defined meromorphic quotient F_ℓ of (1,0)-forms $(\partial\widetilde{u_\ell})\,/\,(\partial\widetilde{u_0})$ extends f_ℓ to $\mathcal{X}$ in the sense that for every $x_0 \in \gamma$, $\lim\limits_{x\to x_0,\ x\in\mathcal{X}} F(x)$ exists and equals $f(x_0)$. If γ is real analytic, this last property holds automatically.

We define an isomorphism between two Riemann surfaces with almost smooth boundary $\left(\overline{\mathcal{X}}, \gamma\right)$ and $\left(\overline{\mathcal{X}'}, \gamma'\right)$ as a map from $\overline{\mathcal{X}}$ to $\overline{\mathcal{X}'}$ which realizes a complex analytic isomorphism between $\mathcal{X}$ and $\mathcal{X}'$. As the definition of a Riemann surface with almost smooth boundary implies that its boundary is locally a Jordan curve in its double which is the compact Riemann surface obtained by gluing along its boundary its conjugate (see [1]), a theorem of Caratherory implies that if $\Phi : \mathcal{X} \to \mathcal{X}'$ is a complex analytic isomorphism, Φ and Φ^{-1} extend continuously to γ and γ' so that Φ becomes a homeomorphism from $\overline{\mathcal{X}}$ to $\overline{\mathcal{X}'}$. As a consequence, Φ realizes also a diffeomorphism between manifolds with boundary from $\overline{\mathcal{X}}_{\mathrm{reg}} \cap \Phi^{-1}\left(\overline{\mathcal{X}'}_{\mathrm{reg}}\right)$ to $\overline{\mathcal{X}'}_{\mathrm{reg}} \cap \Phi\left(\overline{\mathcal{X}}_{\mathrm{reg}}\right)$.

The first theorem of this article is a significative improvement of results in [6][21] on how unique $\mathcal{X}$ can be when a restricted DN-data is specified.

Theorem 1. *Assume that the main hypothesis is valid, that $\mathcal{X}$ and $\mathcal{X}'$ are open Riemann surfaces with almost smooth boundary γ and restricted DN-data $(\gamma, u, \theta u)$. Then, there is an isomorphism of Riemann surfaces with almost smooth boundary between $\mathcal{X} \cup \gamma$ and $\mathcal{X}' \cup \gamma$ whose restriction on γ is the identity.*

Remarks. 1. If E is any subset of γ which has positive 1-dimensional Hausdorff measure on every connected component of γ, the conclusions of theorem 1 hold when $N_{\mathcal{X}'}u_\ell = N_{\mathcal{X}}u_\ell$ is ensured only on E and the meromorphic functions $(\partial\widetilde{u}_\ell)/(\partial\widetilde{u}_0)$ are continuous near γ. This phenomena is a simple consequence of the fact that a meromorphic function is uniquely determined as soon as it is known on E. This includes [21, th. 1.1.i] which is stated for a connected $\mathcal{X}$.

2. The proof of theorem 1 also contains the fact that two connected compact Riemann surfaces $\mathcal{Z}$ and $\mathcal{Z}'$ are isomorphic when they share a same real smooth curve γ which can be embedded into $\mathbb{C}^2$ by a map which extends meromorphically both to $\mathcal{Z}$ and $\mathcal{Z}'$ and are continuous near γ.

The assumption on u_0, u_1 and u_2 is used only to ensure that the map f defined by (1.2) is an embedding of γ into $\mathbb{C}^2$ extending meromorphically to $\mathcal{X}$ into $F = (\partial\widetilde{u}_\ell/\partial\widetilde{u}_0)_{\ell=1,2}$. Moreover, theorem 10 in section 3 implies that if $\mathcal{X}$ has almost smooth boundary and solves the the IDN-problem, the map F enable to see $\overline{\mathcal{X}}$ as a kind of normalization of the closure of a complex curve[(4)] of $\mathbb{CP}_2 \backslash f(\gamma)$ uniquely determined by γ. This generalization shows that in each characterization theorem 3a, 3b, 3c, the constructed Riemann surface is, up to isomorphism, the only one which has a chance to solve the IDN-problem.

Our next result explains how to recover $F(\mathcal{X})$ and $\partial\widetilde{u}_\ell$ from θu_ℓ and the intersection of $F(\mathcal{X})$ with the lines $\{z_2 := w_2/w_0 = \xi\}$, $\xi \in \mathbb{C}$. Desingularization arguments enable then the reconstruction of $\mathcal{X}$ from $F(\mathcal{X})$.

Theorem 2. *Assume that the main hypothesis is valid and that $\left(\overline{\mathcal{X}}, \gamma\right)$ is a Riemann surface with almost smooth boundary almost smooth boundary γ restricted DN-data $(\gamma, u, \theta u)$. Then, the following holds :*

1) The map f defined by (1.2) has a meromorphic extension F to $\mathcal{X}$ and if $\mathcal{Y} = F(\mathcal{X}) \backslash f(\gamma)$, there are discrete sets $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{X}$ and $\mathcal{Y}$ respectively such that $F : \mathcal{X}\backslash\mathcal{A} \to \mathcal{Y}\backslash\mathcal{B}$ is one to one.

2) Almost all $\xi_* \in \mathbb{C}$ has a neighborhood W_{ξ_*} such that for all ξ in W_{ξ_*}, $\mathcal{Y}_\xi = \mathcal{Y} \cap \{z_2 := w_2/w_0 = \xi\} = \underset{1\leqslant j\leqslant p}{\cup}\{(h_j(\xi), \xi)\}$ where $h_1, ..., h_p$ are p

[4]A complex curve of an open set U of $\mathbb{CP}_2$ is a one pure dimensionnal complex analytic subset of U.

mutually distinct holomorphic functions on W_{ξ_*} *whose symmetric functions* $S_{h,m} = \sum\limits_{1\leqslant j\leqslant p} h_j^m$ *are recovered by the Cauchy type integral formulas*

$$(E_{m,\xi}) \qquad \frac{1}{2\pi i}\int_\gamma \frac{f_1^m}{f_2-\xi}df_2 = S_{h,m}(\xi) + P_m(\xi)\,,\ m\in\mathbb{N},$$

where P_m *is a polynomial of degree at most* m. *More precisely, the system* $E_\xi = (E_{m,\xi_\nu})_{\substack{0\leqslant m\leqslant B-1\\ o\leqslant \nu\leqslant A-1}}$ *enables the explicit computation of* $h_j(\xi_\nu)$ *and* P_m *if* $A\geqslant B$ *and* $B\geqslant 2p+1$ *and* $\xi_0,...,\xi_A$ *are mutually distinct points.*

3) For almost all $\xi_*\in\mathbb{C}$, W_{ξ_*} *can be chosen such that* $\mathcal{Y}_\xi\cap\mathcal{B}=\varnothing$ *when* $\xi\in W_{\xi_*}$ *and the forms* $\partial\widetilde{u}_\ell$, $0\leqslant\ell\leqslant 2$, *can be reconstructed in* $F^{-1}(\bigcup\limits_{\xi\in W_{\xi_*}}\mathcal{Y}_\xi)$ *from the well defined meromorphic quotient of (1,0)-forms* $(\partial\widetilde{u}_\ell)/(\partial F_2)$ *thanks to the Cauchy type formulas :*

$(T_{m,\xi})$

$$\frac{1}{2\pi i}\int_\gamma \frac{f_1^m}{f_2-\xi}\theta u_\ell = \sum_{1\leqslant j\leqslant p} h_j(\xi)^m\frac{\partial\widetilde{u}_\ell}{\partial F_2}\left(F^{-1}(h_j(\xi),\xi)\right),\ m\in\mathbb{N},\ \xi\in W_{\xi_*}.$$

Remark. The number α of connected components of $\mathcal{X}$ can be computed by the following algorithm : let γ_1 be a connected component of γ and let λ_1 be a function which is zero on γ_j for $j\neq 0$ and non constant on γ_1 ; then if $\mathcal{X}_1$ is the component of $\mathcal{X}$ whose boundary contains γ_1, $N\lambda_1\neq 0$ on each component $\gamma_1,...,\gamma_k$ of γ which with γ_1 are the components of $b\mathcal{X}_1$. Iterating this with components γ different from $\gamma_1,...,\gamma_k$, yields a process with α steps.

The numerical resolution of (E_ξ) and the study of its stability requires an estimate of the number I_{Δ_ξ} of intersection points, multiplicities taken in account, of $\mathcal{Y}$ with $\Delta_\xi=\{z_2:=w_2/w_0=\xi\}$. To get this information, it is sufficient to estimates the number I_Δ of intersection points of $\mathcal{Y}$ with a $\mathbb{CP}_2$-line Δ generic in the sense where Δ does not contained the germ of a component of $\mathcal{Y}$ near γ. Indeed, if L (resp. L_ξ) denotes a linear homogeneous form defining Δ (resp. Δ_ξ),

$$I_{\Delta_\xi}-I_\Delta = \frac{1}{2\pi i}\int_\gamma (L_\xi/L)^{-1}d(L_\xi/L)\,.$$

This underline the usefulness of knowing an a priori an upper bound of I_Δ for any particular line Δ.

This open problem is related, because of the Ahlfors theorem on covering surface, to the computation of the genius $g_\mathcal{X}$ of $\mathcal{X}$ from some DN-datas when $\mathcal{X}$ is connected. Belishev [5] has shown that $2g_\mathcal{X}$ is the rank of $Id+\left(N_\mathcal{X}T^{-1}\right)^2$ acting on the space of smooth functions on γ admitting a smooth primitive, T^{-1} being a primitive operator. Yet, a formula for $g_\mathcal{X}$ not using the full knowledge of $N_\mathcal{X}$ but only three functions satisfying the main hypothesis has to be found.

The third aspect of the IDN-problem, characterization of what can and should be a DN-data, is more delicate. As mentioned before, we have been led to consider the class of manifolds where while $\mathcal{X}$ is an open Riemann surface, $\left(\overline{\mathcal{X}},\gamma\right)$ has only almost smooth boundary. Theorem 3a below explicitly characterizes the only right candidate for $\mathcal{X}$ and while its part C gives a test which discriminates which $(\gamma, u, \theta u)$ are DN-datas and which are not.

Then, a *Green function for* $\overline{\mathcal{X}}$ relatively to a domain $\mathcal{D}$ of $\mathcal{Z}$ containing $\overline{\mathcal{X}}$, is a smooth symmetric function g defined on $\mathcal{D}\times\mathcal{D}$ without its diagonal such that each $g(.,z)$ is harmonic on $\mathcal{D}\backslash\{z\}$ and has singularity $\frac{1}{2\pi}\ln\operatorname{dist}(.,z)$ at z, the distance being computed in any hermitian metric on $\mathcal{Z}$.

Theorem 3a. *Assume that the main hypothesis is valid and consider*

$$G:\mathbb{C}^2\ni(\xi_0,\xi_1)\mapsto\frac{1}{2\pi i}\int_\gamma f_1\frac{d(\xi_0+\xi_1f_1+f_2)}{\xi_0+\xi_1f_1+f_2}. \tag{1.3}$$

A. *Assume that $\mathcal{X}$ is an open Riemann surface with almost smooth boundary γ and restricted DN-data $(\gamma, u, \theta u)$. Then almost all point $(\xi_{0*}:\xi_{1*}:1)$ of $\mathbb{CP}_2$ has a neighborhood where one can find mutually distinct holomorphic functions $h_1,...,h_p$ such that*

$$0=\frac{\partial^2}{\partial\xi_0^2}(G-\sum_{1\leqslant j\leqslant p}h_j) \tag{1.4}$$

$$h_j\frac{\partial h_j}{\partial\xi_0}=\frac{\partial h_j}{\partial\xi_1}, 1\leqslant j\leqslant p. \tag{1.5}$$

B. *Conversely, assume γ is connected and that the conclusion of A is satisfied in a connected neighborhood W_{ξ_*} of one point (ξ_{0*},ξ_{1*}). Then, if $\left(\partial^2G/\partial\xi_0\right)_{|W_{\xi_*}}\neq0$, there is an open Riemann surface $\mathcal{X}$ with almost smooth boundary*[(5)] *γ where f extends meromorphically. If $\left(\partial^2G/\partial\xi_0\right)_{|W_{\xi_*}}=0$, the same conclusion holds for a suitable orientation of γ.*

C. *Assume that $\left(\overline{\mathcal{X}},\gamma\right)$ is a Riemann surface with almost smooth boundary. Let $\mathcal{Z}$ be the double of $\mathcal{X}$, $\mathcal{D}$ a smooth domain of $\mathcal{Z}$ containing $\overline{\mathcal{X}}$ and g a Green function for $\overline{\mathcal{X}}$ relatively to $\mathcal{D}$. Then, $(\gamma,u,\theta u)$ is actually a restricted DN-data if and only if*

$$\int_\gamma u_\ell(\zeta)\,\partial_\zeta g(\zeta,z)+g(\zeta,z)\,\overline{\theta u_\ell(\zeta)}=0,\ 0\leqslant\ell\leqslant2, \tag{1.6}$$

for any $z\in\mathcal{D}\backslash\overline{\mathcal{X}}$.

Remarks. 1. The connectness assumption on γ is essentially used to ensure that any possible solution to the IDN-problem has to be connected. Taking in account the remark following theorem 2, one may weaken the connectness assumption on γ into the requirement that the given DN-data ensures that possible solutions are connected. Then, the conclusions of theorem 3a.B are still true (see the proof).

2. The proof includes that if γ is real analytic, $\left(\overline{\mathcal{X}},\gamma\right)$ is a manifold with boundary in the classical sense.

3. Emphasizing on f_2 instead of f_1, one can consider

$$G_2:\xi\mapsto\frac{1}{2\pi i}\int_\gamma f_2\frac{d(\xi_0+\xi_1f_1+f_2)}{\xi_0+\xi_1f_1+f_2}.$$

[5]With [18, example 10.5], one can construct a smooth curve γ and smooth restricted DN-datas for which the solution of the IDN-problem is a manifold with only almost smooth boundary.

Then, if we link $h_1, ..., h_p$ to $h_{1,2}, ..., h_{p,2}$ by the relations $0 = \xi_0 + \xi_1 h_j + h_{j,2}$, $(h_1, ..., h_p)$ satisfy (1.5) and (1.4) if and only $\frac{\partial^2}{\partial\xi_0^2}(G_2 - \sum\limits_{1\leqslant j\leqslant p} h_{j,2}) = 0$ and $h_j \frac{\partial h_{j,2}}{\partial\xi_0} = \frac{\partial h_{j,2}}{\partial\xi_1}$, $1 \leqslant j \leqslant p$.

4. Select $\mathcal{H} = \{h_1, ..., h_p\}$ satisfying (1.5) and minimal for (1.4). Then, section 5.2 and proposition 14 shows that there is $\tau_2 \subset \delta = f(\gamma)$ such that $h^1(\tau_2) = 0$ with the following properties. If $\mathcal{H} = \varnothing$, $\mathcal{X}$ is a normalization of $\mathcal{Y} \cup \tau_2$ where $\mathcal{Y}$ is the polynomial hull of δ in the affine complex plane

$$\mathbb{C}^2_{\xi_*} = \left\{ w \in \mathbb{CP}^2 \ ; \ \xi_* w = \xi_{0*} w_0 + \xi_{1*} w_1 + w_2 \neq 0 \right\}.$$

When $\mathcal{H}$ is not empty, $\mathcal{X}$ is a normalization of $\mathcal{Y} \cup \tau_2$ where $\mathcal{Y}$ is the analytic extension in $\mathbb{CP}_2 \backslash \delta$ of the union of the graphs of the functions $(h_j, -\xi_0 - \xi_1 h_j)$, $1 \leqslant j \leqslant p$. In both case, τ_2 is the set of points z of δ such that $\mathcal{Y} \cup \{z\}$ is analytic near z in $\{z\} \cup (\mathbb{CP}_2 \backslash \delta)$. Hence, when $\mathcal{H}$ is minimal, decomposition (1.4) of G is unique up to order and Card $\mathcal{H}$ is the minimal number p for which such a decomposition exists. Moreover, theorem 10 implies that the only Riemann surfaces $\mathcal{X}$ which has a chance to solve the IDN-problem are normalizations of $\mathcal{Y}$.

The vanishing of $\partial^2 G / \partial\xi_0$ in a connected neighborhood W_{ξ_*} of $\xi_* \in \mathbb{C}^2$ is known to be equivalent to the fact that $\delta = f(\gamma)$ satisfy the classical Wermer-Harvey-Lawson moment condition in[6] $\mathbb{C}^2_{\xi_*}$: for all $k_1, k_2 \in \mathbb{N}$, $\int_\delta z_1^{k_1} z_2^{k_2} dz_2 = 0$ where $z_j = w_j / \xi_* w$, $j = 1, 2$ (see [12, cor. 1.6.2]). It is proved in [29] for the real analytic case and in [7][18] for the smooth case that for a suitable orientation of γ, this moment condition guarantees the existence in $\mathbb{C}^2_{\xi_*} \backslash \delta$ of a unique complex curve $\mathcal{Y}$ with finite mass and boundary $\pm\delta$ in the sense of currents. In [2], Alexander and Wermer have improved this Wermer-Bishop-Harvey-Lawson statement by showing that a closed oriented smooth connected real curve δ of $\mathbb{C}^2$ is, with its given orientation, the boundary, in the sense of currents, of a complex curve of finite mass in $\mathbb{C}^2_{\xi_*} \backslash \delta$ if and only if $\frac{1}{2\pi i} \int_\delta \frac{dA}{A} \geqslant 0$ for any polynomial A which does not vanish on γ. Hence, in case $\left(\partial^2 G / \partial\xi_0\right)_{|W_{\xi_*}} = 0$ it is sufficient to find one polynomial A such that $\int_{f(\gamma)} \frac{dA}{A} \neq 0$ to determine the correct orientation of γ.

Note that the case $\left(\partial^2 G / \partial\xi_0\right)_{|W_{\xi_*}} = 0$ occurs only for very special DN-datas since it implies that for $\ell = 1, 2$, f_ℓ admits a $\mathbb{C}^2_{\xi_*}$-valued holomorphic extension to $\mathcal{X}$. Proposition 20 proposes another result of this kind for some other special DN-datas when they are available.

To palliate the fact that Green functions are difficult to compute, the theorem below proposes another way to achieve the same goals : selection of the right candidate for $\mathcal{X}$ and extension of θu_ℓ, then checking it is actually solutions.

[6] When ξ_* belongs to the connected component of infinity of $\left\{ \xi \in \mathbb{C}^2 \ ; \ \forall w \in \delta, \ \xi w \neq 0 \right\}$, this moment condition is equivalent to the moment condition in $\mathbb{C}^2_{(1,0)}$ and also to the vanishing of G on this component.

Theorem 3b. *Assume that the main hypothesis is valid. Let G be the function defined by (1.3) and let be $\widetilde{G}$ the form which in $\mathbb{CP}_2$ with homogenous coordinates $\eta = (\eta_0 : \eta_1 : \eta_2)$ is given by*

$$\widetilde{G} = \sum_{0\leqslant \ell\leqslant 2} \frac{1}{2\pi i}\left(\int_\gamma \frac{\theta u_\ell}{\eta_0 + \eta_1 f_1 + \eta_2 f_2}\right) d\eta_\ell = \sum_{0\leqslant \ell\leqslant 2} \widetilde{G}_\ell \, d\eta_\ell \tag{1.7}$$

A. *Assume that $\mathcal{X}$ is an open Riemann surface such that $\overline{\mathcal{X}} = \mathcal{X}\cup\gamma$ is an manifold with almost smooth boundary and restricted DN-data $(\gamma, u, \theta u)$. Then,*
a1) *Almost all points $\eta_* = (\xi_{*0} : \xi_{*1} : 1)$ of $\mathbb{CP}_2$ has a neighborhood where $\widetilde{G}$ can be written as the sum of p holomorphic closed forms $g_j = \sum_{0\leqslant \ell\leqslant 2} g_{j,\ell}\, d\eta_\ell$ such that $(h_j) = (g_{j,1}/g_{j,0})_{1\leqslant j\leqslant p}$ satisfy (1.5).*
a2) *The form $\Theta_\ell = \partial \widetilde{u_\ell}$, $0 \leqslant \ell \leqslant 2$, satisfies*

$$\int_c \operatorname{Re}\Theta_\ell = 0 \tag{1.8}$$

for all c in the first homology group $H_1(\mathcal{X})$ of $\mathcal{X}$.

B. b1) *Assume γ is connected and that there is $\eta_* = (\xi_{*0} : \xi_{*1} : 1)$ and a connected neighborhood W_{ξ_*} of $\xi_* = (\xi_{*0}, \xi_{*1})$ such that (a1) is true for all $\eta \in W_{\eta_*} = \{(\xi_0 : \xi_1 : 1)\ ;\ (\xi_0, \xi_1) \in W_{\xi_*}\}$. Then, there exists an open Riemann surface $\mathcal{X}$, topologically bordered by γ, where f extends meromorphically and each θu_ℓ extends weakly[7] into a meromorphic (1,0)-form Θ_ℓ outside a set Σ of zero length[8].*
b2) *In addition to (a1), assume that Θ_ℓ satisfies $\int i\,\Theta_\ell\wedge\overline{\Theta_\ell} < +\infty$ and (1.8), Then if $\left(\partial^2 G/\partial\xi_0\right)_{|W_{\xi_*}} \neq 0$, then $\left(\overline{\mathcal{X}}, \gamma\right)$ is a manifold with almost smooth boundary ; the same conclusion holds when $\widetilde{G}\,\big|_{W_{\eta_*}} = 0$ if γ has a suitable orientation. If $\left(\partial^2 G/\partial\xi_0\right)_{|W_{\xi_*}} = 0$ but $\widetilde{G}\,\big|_{W_{\eta_*}} \neq 0$, then either $\mathcal{X}$ is a domain with boundary γ in a normalization of an algebraic curve of $\mathbb{CP}_2$, either $\overline{\mathcal{X}}$ is a compact Riemann surface where γ is a slit[9]. In all cases, there is a continuous extension $\widetilde{u_\ell}$ of u_ℓ to $\overline{\mathcal{X}}$ which is harmonic in $\mathcal{X}$ and such that $\Theta_\ell = \partial\widetilde{u_\ell}$ which means that Nu_ℓ is actually the DN-data of $\mathcal{X}$ for u_ℓ.*

Remarks. 1. When (a1) holds, $h_{j,2} = \frac{g_{j,2}}{g_{j,0}}$ verify $h_j \frac{\partial h_{j,2}}{\partial \xi_0} = \frac{\partial h_{j,2}}{\partial \xi_1}$, $1 \leqslant j \leqslant p$.

2. The formulas $(E_{m,\xi})$ and $(T_{m,\xi})$ enable direct reconstruction of a projective presentation of $\mathcal{X}$ and forms Θ_ℓ.

Theorem 3b is obtained by a normalization of a singular version of the IDN-problem which is more explicit. When $\mathcal{X}$ is smooth, the harmonicity of a distribution U is equivalent to the fact that ∂U is holomorphic. For the

[7]It means that $\int_\gamma \varphi\theta u_\ell = \int_{\mathcal{X}} d(\varphi\Theta_\ell) = \int_{\mathcal{X}} \left(\overline{\partial}\varphi\right)\wedge\Theta_\ell$ holds for any Lipschitz function φ on $\overline{\mathcal{X}}$ which is a holomorphic function of f near points of Σ and singular points of $\left(\overline{\mathcal{X}}, \gamma\right)$; if $\left(\overline{\mathcal{X}}, \gamma\right)$ is a manifold with boundary, this definition means that $\Theta_\ell\,|_\gamma = \theta u_\ell$ in the usual sense.

[8]Basing on [11, example 1], one can construct examples where (a1) is satisfied while the weak extension Θ_ℓ has essential singularities on some zero length set Σ.

[9]This means that $\overline{\mathcal{X}}\backslash\gamma$ is connected.

case where $\mathcal{X}$ is a complex curve of an open set in $\mathbb{CP}_2$, we need two of the several non equivalent definitions of holomorphic (1,0)-forms.

At first, we use the *weakly holomorphic* forms introduced by Rosenlicht [25] which can be defined as meromorphic (1,0)-forms ψ such that $\psi \wedge [\mathcal{X}]$ is $\overline{\partial}$-closed current of $\mathbb{CP}_2$. Such forms ψ are also characterized by the fact that $p_*\psi$ is a usual holomorphic $(1,0)$-form for any holomorphic proper function $p : \mathcal{X} \to \mathbb{C}$. A distribution U is defined as *weakly harmonic* if ∂U is weakly holomorphic.

Assume now $\mathcal{X}$ lies in $\mathbb{CP}_2$ and that $\mathcal{X}$ is bounded in the sense of currents by γ. A distribution U on $\mathcal{X}$ is said almost smooth up to the boundary if it is the case near each $p \in \gamma$ where $\left(\overline{\mathcal{X}}, \gamma\right)$ is a manifold with boundary and if U has a restriction on γ in the sense of currents. When u is a smooth function on γ, a weakly harmonic extension of u to $\mathcal{X}$ is a weakly harmonic distribution U almost smooth up to boundary whose restriction on γ is u. Since two weakly harmonic extension U_1 and U_2 of u to $\mathcal{X}$ are equal when $\partial U_1 = \partial U_2$ on γ in the sense of currents, we consider a *weak Cauchy-Dirichlet-problem* : a data for this problem is a smooth function u on γ and a smooth section λ of $T^*_\gamma \mathcal{X}$; a solution to this problem is a weakly harmonic function U almost smooth up to γ such that $u = U\,|_\gamma$ and $\lambda = (\partial U)_\gamma$ in the sense of currents ; when it exists, such an U is unique and is denoted $\widetilde{u}$ as any harmonic extension in this article. In connection with this notion, we define a *weak restricted data* as a triplet $(\gamma, u, \theta u)$ where $u = (u_\ell)_{0 \leqslant \ell \leqslant 2}$ (resp. $\theta u = (\theta u_\ell)_{0 \leqslant \ell \leqslant 2}$) is a triplet of smooth functions (resp. (1,0)-forms) on γ such that $\theta u_\ell = (\partial \widetilde{u_\ell})_\gamma$ in the sense of currents.

The weak CD-problem has its own interest and arise naturally in the proof of theorem 3b. However, the original IDN-problem requires a more restrictive notion of harmonicity. According to Griffiths [15], *holomorphic forms* (resp. *harmonic functions*) are by definition push forward of holomorphic forms (resp. harmonic functions) on a normalization of $\mathcal{X}$. Equivalently, a real function U on $\mathcal{X}$ is harmonic if and only if U is harmonic in the regular part $\mathcal{X}_{\mathrm{reg}}$ of $\mathcal{X}$ and $\int_{\mathcal{X}_{\mathrm{reg}}} i\,\partial U \wedge \overline{\partial} U < +\infty$. This notion is closed in spirit to a Riemann characterization of the harmonic function with a given boundary value u as the smooth function with restriction u on the boundary and minimizing the preceding integral. We can now state a singular version of theorem 3b.

Theorem 3c. *Consider in* $\mathbb{CP}_2 \backslash \{w_0 = 0\}$ *a smooth oriented real curve* γ*, three functions* u_0*,* u_1*,* u_2 *in* $C^\infty(\gamma)$ *and* θ_0*,* θ_1*,* θ_2 *three smooth sections of* $\Lambda^{1,0} T^*_\gamma \mathbb{CP}_2$ *such that* $du_\ell = 2\,\mathrm{Re}\,\theta_\ell$*,* $0 \leqslant \ell \leqslant 2$ *and linked by the relations* $\theta_1 = z_1 \theta_0$*,* $\theta_2 = z_2 \theta_0$*. Let* G *and* $\widetilde{G}$ *be the form given by (1.3) and (1.7) but with* $(f_1, f_2) = (z_1, z_2)$*.*

A. *Assume* γ *bounds, in the sense of currents, a complex curve* $\mathcal{X}$ *of* $\mathbb{CP}_2 \backslash \gamma$ *which has finite volume and weak restricted DN-data* $(\gamma, u, \theta u)$*. Then,*
a1) *The conclusions of theorem 3b.A.a1 are valid.*
a2) *The form* $\Theta_\ell = \partial \widetilde{u_\ell}$ *satisfies (1.8) for all* c *in* $H_1(\mathcal{X}_{\mathrm{reg}})$*.*

B. b1) *Conversely, assume that* γ *is connected and that (a1) is valid for one point* $\eta_* = (\xi_{0*}, \xi_{1*}, 1)$*. Then, there is in* $\mathbb{CP}_2 \backslash \gamma$ *a complex curve* $\mathcal{X}$ *of finite mass where each* θ_ℓ *extends weakly on* $\mathcal{X}$ *into a weakly holomorphic*

$(1,0)$-form Θ_ℓ. Moreover, if $\left(\partial^2 G/\partial\xi_0\right)_{|W_{\xi_*}} \neq 0$, then $\mathcal{X}$ has boundary γ in the sense of currents ; the same conclusion holds if $\widetilde{G}\,|_{W_{\eta_*}} = 0$ but for a suitable orientation of γ. If $\left(\partial^2 G/\partial\xi_0\right)_{|W_{\xi_*}} = 0$ but $\widetilde{G}\,|_{W_{\eta_*}} \neq 0$, either $\mathcal{X}$ is a domain in an algebraic curve of $\mathbb{CP}_2$ and has boundary γ in the sense of currents, either $\overline{\mathcal{X}}$ is itself an algebraic curve of $\mathbb{CP}_2$ where γ is a slit.

***b2)** If in addition (1.8) is satisfied by Θ_ℓ for all $c \in H_1(\mathcal{X}_{\text{reg}})$, then u_ℓ has a (unique) weakly harmonic extension $\widetilde{u_\ell}$ and $\Theta_\ell = \partial\widetilde{u_\ell}$. If Θ_ℓ also satisfy $\int_{\mathcal{X}_{\text{reg}}} i\,\Theta_\ell \wedge \overline{\Theta_\ell} < +\infty$, then $\widetilde{u_\ell}$ is harmonic.*

Remark. It is possible that $\mathcal{X}$ has zero boundary in the sense of currents. This occurs only in the exceptional case where $\overline{\mathcal{X}}$ is a compact complex curve of $\mathbb{CP}_2$ and (so is algebraic) where γ is a slit. In the other cases, $\mathcal{X}$ has boundary $\pm\gamma$ in the sense of currents and a result of Chirka [10] gives that outside a zero one Hausdorff dimensional subset, $\left(\overline{\mathcal{X}}, \pm\gamma\right)$ is locally a manifold with boundary.

The proofs of the preceding theorems are given in sections three to five. They use the results on the complex Plateau problem started in [29][7], developed in [18][17][11] for $\mathbb{C}^n$ and in [20][12][19] for $\mathbb{CP}_n$.

The non constructive existence criterions of theorems 3a, 3b and 3c may incite to seek a less general but more effective characterization. It has been already mentioned after theorem 3a that in the special case $p = 0$, the condition $\left(\partial^2 G/\partial\xi_0\right)_{|W_{\xi_*}} = 0$ together with the Alexander-Wermer moment criterion gives an effective tool but only when special DN-datas are at hand.

For $p > 0$, the main result of [12] is that conditions of type (1.4) and (1.5) characterize the fact that a given closed smooth and orientable real chain γ of $\mathbb{CP}_2$ is, with adequate orientation, the boundary of some holomorphic chain of $\mathbb{CP}_2\backslash\gamma$. These conditions have been qualified as mysterious in [19] because the functions satisfying these relations are produced "deus ex machina". The following criterion, which completes for a closed connected curve γ the one of [12], is obtained in [19] : Suppose that the second coordinate f_2 of $\mathbb{C}^2$ does not vanish on γ, then there exists in $\mathbb{CP}_2\backslash\gamma$ a connected complex curve $\mathcal{X}$ with boundary $\pm\gamma$ in the sense of currents if and only if there exist $p \in \mathbb{N}$ and A_d in the space $\mathcal{O}(d)$ of holomorphic homogeneous polynomials of order d, $1 \leqslant d \leqslant p$, such that for ξ_0 in some neighborhood of 0, $C_m(\xi_0) = \frac{1}{2\pi i}\int_\gamma \frac{f_1^m}{f_2+\xi_0} df_2$ satisfies

$$\begin{aligned} C_k &= Q_{k,p}(C_1, \ldots, C_p) \ \mathit{mod}\ \mathcal{O}(k)\ ,\ k > p \\ C_d(\xi_0) &= \sum_{k>d} \frac{(-\xi_0)^k}{2\pi i} \int_\gamma \frac{f_1^d}{f_2^{k+1}} df_2 + A_d(\xi_0)\,,\ 1 \leqslant d \leqslant p \end{aligned}$$

where $Q_{k,p}$ are universal homogeneous polynomials.

The tools of section 6.1 enable us, for $p > 0$, to develop theorem 3a into theorem 4 below which gives a more effective criterion for the Plateau problem in $\mathbb{CP}_2$ and also for the IDN-problem. This new criterion follows from considerations on sums of shock wave functions modulo affine functions in ξ_0. Even if decompositions in sum of shock wave functions are studied for

ξ_0-affine functions, theorem 4 do not consider the case where G is of that type since it corresponds to a plain case of (1.4).

Theorem 6.1 uses the following notation : if H and u are holomorphic functions in a simply connected domain D, we set $\mathcal{D}_H = \frac{\partial}{\partial \xi_1} - H_{\xi_0}$ and denote by $\mathcal{L}_H u$ the unique function $v \in \mathcal{O}(D)$ such that $\partial v / \partial \xi_0 = \mathcal{D}_H u$ and $v(0,.) = 0$.

Theorem 4. *Let f be defined by (1.2) and consider the function G defined by (1.3). We assume that γ is connected and that f_2 does not vanish on γ so that G, which is assumed to be not affine in ξ_0, is defined in a simply connected neighborhood D of 0 in $\mathbb{C}^2$; we denote by Δ the image of D by the projection $(\xi_0, \xi_1) \mapsto \xi_0$.*

A/ *If there exits an open Riemann surface $\mathcal{X}$ such that $\overline{\mathcal{X}} = \mathcal{X} \cup \gamma$ is a manifold with almost smooth boundary and such that f extends meromorphically to $\mathcal{X}$, then the following assertions hold for G*

(1) *There is $p \in \mathbb{N}^*$ and holomorphic functions $a, b, \lambda_1, ..., \lambda_{p-1}$ on Δ such that the integro-differential equation*

$$-\mathcal{D}_{G+L}\mathcal{L}_{G+L}^{p-1}(G+L) + \sum_{1 \leqslant j \leqslant p-1} \mathcal{D}_{G+L}\mathcal{L}_{G+L}^{p-1-j}\widetilde{\lambda}_j = 0, \tag{1.9}$$

is valid with $L(\xi_0, \xi_1) = a(\xi_1)\xi_0 + b(\xi_1)$ and $\widetilde{\lambda}_k(\xi_0, \xi_1) = \lambda_j(\xi_1), 1 \leqslant j \leqslant p$.

(2) *For $s_k = -\mathcal{L}_{G+L}^{k-1}G + \mathcal{L}_{G+L}^{k-2}\widetilde{\lambda}_1 + \cdots + \mathcal{L}_{G+L}^{0}\widetilde{\lambda}_{k-1}$, $1 \leqslant k \leqslant p$, the discriminant of $T_{\xi_0,\xi_1} = X^p + \sum\limits_{1 \leqslant k \leqslant p} s_k(\xi_0, \xi_1) X^{p-k}$ does not vanish identically in D.*

(3) *$G = (-s_1) - L$*

(4) *There is $q \in \mathbb{N}$, $\alpha, \beta \in \mathbb{C}_q[\xi_1]$ such that $\alpha(0) = 0$, $\deg \beta < q$ and $a = \frac{\alpha'}{1-\alpha}$, $b = \frac{\beta}{1-\alpha}$*

Moreover, if p is the least integer such that (1-2-3) assertions holds, (γ, f) uniquely determines $(a, b, \lambda_1, ..., \lambda_{p-1})$.

B/ *Assume (1-2-3) holds for some $p \in \mathbb{N}^*$. Then, there exists an open Riemann surface $\mathcal{X}$ such that $\overline{\mathcal{X}} = \mathcal{X} \cup \gamma$ is a manifold with almost smooth boundary where f extends meromorphically. Moreover, (4) holds.*

Remarks. 1. Non unicity of $(a, b, \lambda_1, ..., \lambda_{p-1})$ solving (1-2-3) means that $\mathcal{X}$ exists but p is not minimal.

2. It is possible that γ, regardless of its orientation, is the boundary of an open Riemann surface $\mathcal{X}$ with almost smooth boundary where f extends meromorphically. It is the case when γ divides a compact Riemann surface $\mathcal{Z}$ into two smooth domains and f is the restriction to γ of an analytic map from $\mathcal{Z}$ to $\mathbb{CP}_2$.

2. Intrinsic EIT on Riemann surfaces

The Inverse Dirichlet-Neuman problem, which goes back to Calderon [9] and which is called now Electrical-Impedance-Tomography problem, can be sketch like this : suppose that a bounded domain $\mathcal{X}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is an ohmic conductor which means that the density of current j it may have is proportional (in isotropic cases) to the electrical field $e = \nabla U$ where U is an electrical potential. The scalar function σ such that $j = \sigma e$ is then called the conductivity of $\mathcal{X}$; $\rho = 1/\sigma$ is called the resistivity. When there is no time dependence and no source or sink of current, the equation $\operatorname{div} j = \operatorname{div}(\sigma e)$ must hold and Calderon's problem is then to recover σ on the whole of $\mathcal{X}$ from the operator $C^\infty(\gamma) \ni u \mapsto (\sigma \nabla \widetilde{u})_\gamma$, $\widetilde{u}$ being the unique solution of $\operatorname{div}(\sigma \nabla \widetilde{u}) = 0$ with boundary value u.

In what follows, linking the Calderon problem to the Belishev problem mentioned in the introduction, we formulate the EIT-problem for a more general setting than the case of domains in $\mathbb{R}^n$. The second part of this section, despite the fact it is also quite elementary, seems to be new and clearly underline how complex structure is involved in the dimension two case.

General dimension. Assume that $\mathcal{X}$ is an open oriented bordered manifold of dimension n with boundary γ and that $\mathcal{X}$ comes with a volume form μ and a conductivity σ which is modelled as a tensor from $T^*\mathcal{X}$ to $\Lambda^{n-1}T^*\mathcal{X}$ (see [27]). The gradient associated to σ relatively to μ is the differential operator which to any $f \in C^1(\mathcal{X})$ associates the tangent vector field $\nabla_{\mu,\sigma} f$ characterized by

$$(\nabla_{\mu,\sigma} f) \lrcorner \mu = \sigma(df)$$

where $\lrcorner$ is the interior product. When $U \in C^1(\mathcal{X})$ is some given potential, the density of physical current J is by definition[(10)]

$$J = \nabla_{\mu,\sigma} U.$$

If $\mathcal{X}$ has no source or sink of currents and if U has no time dependence, the flux of current through the boundary of any domain is zero. Using Stokes formula, this can be modeled by the simplified Maxwell equation

$$0 = \operatorname{div}_\mu J = \operatorname{div}_\mu(\nabla_{\mu,\sigma} U) \tag{2.1}$$

where div_μ is[(11)] the divergence with respect to the volume form μ. Going back to the definition of gradient and divergence, we see that (2.1) is equivalent to the intrinsic equation formulated in [27] for domains in $\mathbb{R}^n$

$$d\,\sigma(du) = 0 \tag{2.2}$$

Since μ is no longer involved, the usual DN-operator has to be replaced by the operator Θ which to $u \in C^1(\gamma)$ associates $\sigma(du)_\gamma$ which is a section of $\Lambda^{n-1}T^*_\gamma\mathcal{X}$. The IDN-problem, or Electrical Impedance Tomography problem, is then to reconstruct $(\mathcal{X}, \sigma)$ from its DN-map Θ. Of course, the two other aspects of this problem, unicity and characterization, also has to be studied.

The problem in such a generality is still widely open ; almost all publications are about domains in $\mathbb{R}^3$. In such a case, (2.1) is generally written in euclidean global coordinates. However, when $\mathcal{X}$ is a manifold, (2.2) yields the same equation in any chart (W, x) ; setting $dx_j = \sum\limits_{1 \leqslant k \leqslant n} \sigma_{kj} (-1)^k dx_{\widehat{k}}$

[10]That $\nabla_{\mu,\sigma} U$ truly models the density of current is the assumption of Ohm's law.

[11]If t is a differentiable vector field, $\operatorname{div}_\mu t$ is defined by $d(t \lrcorner \mu) = (\operatorname{div}_\mu t)\,\mu$.

with $dx_{\widehat{k}} = \underset{j\neq k}{\wedge} dx_j$, (2.2) becomes

$$\sum_{1\leqslant k\leqslant n}\sum_{1\leqslant j\leqslant n}\frac{\partial}{\partial x_k}\left(\sigma_{k,j}\frac{\partial U}{\partial x_j}\right)=0 \tag{2.3}$$

When the conductivity σ is symmetric and invertible, it is possible to design a natural metric $g_{\mu,\rho}$ associated to the resistivity map $\rho = \sigma^{-1}$ by the well defined by quotient of n-forms :

$$g_{\mu,\sigma^{-1}}(t) = \frac{\sigma^{-1}(t \lrcorner \mu)\wedge(t \lrcorner \mu)}{\mu},\ t\in T\mathcal{X}. \tag{2.4}$$

If (W, x) is any coordinates chart for $\mathcal{X}$, a direct calculus in x-coordinates shows that for $t = \Sigma t_k \partial/\partial x_k$ (2.4) becomes

$$g_{\mu,\rho}(t) = \sum_{k,\ell} t_\ell\, t_k\, \lambda\rho_{k,\ell} \tag{2.5}$$

where $(\rho_{k,\ell})$ is the matrix of the resistivity $\rho = \sigma^{-1}$ when at any given point z the chosen basis for $\Lambda^{n-1}T_z^*\mathcal{X}$ and $T_z^*\mathcal{X}$ are $(\,(-1)^k\, dx_{\widehat{k}}\,)$ and (dx_k) respectively. When $(\sigma_{j,k})$ is positive definite, $g_{\mu,\rho}$ is a metric on $\mathcal{X}$.

When $n \geqslant 3$, there is a specially adequate choice of metric and volume.

Proposition 5. *Assume $n \geqslant 3$. Then one can correctly design a global volume form μ by letting it be defined by $\mu = \left[\det\left(\rho_{k,\ell}\right)\right]^{\frac{-1}{n-2}} dx_1\wedge\cdots\wedge dx_n$ in any coordinates chart (W, x) for $\mathcal{X}$. For this specific volume form, σ is the Hodge star operator of $g_{\mu,\rho}$ and μ is the riemannian volume form of $g_{\mu,\rho}$.*

This statement, already pointed out by Bossavit and Lee-Uhlmann (see [3] and [21]) for domains in affine spaces, follows from straightforward calculus in coordinates.

The interest of proposition 5 is to state the strict equivalence between the IDN-problem for riemannian manifolds and the EIT-problem when $n \geqslant 3$. When $\dim\mathcal{X} \geqslant 3$ and $\overline{\mathcal{X}}$ is a riemannian real analytic manifold with boundary, Lassas and Uhlmann have proved in [21] that the DN-operator uniquely determine $\mathcal{X}$ and its metric.

The two dimensional case. We now assume $n = 2$ and $\sigma = \rho^{-1}$ is symmetric and positive so that $(\mathcal{X}, g_{\mu,\rho})$ becomes a riemannian manifold whose volume form is thereafter denoted by $V_{\mu,\rho}$. Let us emphasize the complex structure associated to the conformal class of $(\mathcal{X}, g_{\mu,\rho})$ by choosing isothermal coordinates charts for it, that is holomorphic charts (see e.g. [28]). In such a chart (W, z), $(\mathrm{Re}\, z = x,\ y = \mathrm{Im}\, z)$ the metric $g_{\mu,\rho}$ has the expression

$$g_{\mu,\rho} = \kappa_{\mu,\rho}\,(dx\otimes dx + dy\otimes dy) = \mathrm{Re}\,(\kappa_{\mu,\rho}dz\otimes d\overline{z})$$

where $\kappa_{\mu,\rho}\in C^1\left(W, \mathbb{R}_+^*\right)$. Hence, in these coordinates, $(\sigma_{k,\ell}) = s\,\mathrm{diag}\,(1,1)$ with $\kappa_{\mu,\rho} = \lambda/s$ and $\lambda \in C^1\left(W, \mathbb{R}_+^*\right)$ is defined by $\mu = \lambda\, dx\wedge dy = \lambda\,\frac{i}{2} dz\wedge d\overline{z}$. Note that s is a global positive function on $\mathcal{X}$ since it is a well defined quotient of volume forms :

$$s = \frac{\mu}{V_{\mu,\rho}}.$$

Note also that s does not depend on μ and that (2.2) finally evolutes into

$$d\left(sd^{c}U\right)=0 \tag{2.6}$$

where $d^{c}=i\left(\overline{\partial}-\partial\right)$, $\overline{\partial}$ and ∂ being the usual global differential operators associated to the complex structure of the conformal class of $(\mathcal{X},g_{\mu,\rho})$. Hence, we have proved the following which generalizes a result written by Sylvester [27] for domains in $\mathbb{R}^{2}$.

Proposition 6. *Let $\mathcal{X}$ a real two dimensional manifold equipped with a symmetric and positive tensor $\sigma:T^{*}\mathcal{X}\to T^{*}\mathcal{X}$. Then, there is a complex structure on $\mathcal{X}$ and $s\in C^{1}\left(\mathcal{X},\mathbb{R}_{+}^{*}\right)$, called scalar conductivity, such that (2.2) is equivalent to (2.6).*

The beginning of this paper has shown that the data $\partial U/\partial\nu$ is equivalent to the data $(\partial U)_{\gamma}$ which don't involve any metric. Since the knowledge of $(\partial U)_{\gamma}$ is equivalent to the knowledge of $(sd^{c}U)_{\gamma}$, we consider $(sd^{c}U)_{\gamma}$ as the DN-data. We can now state an intrinsic IDN-problem for two dimensional ohmic conductors ; for the sake of simplicity, we limit ourselves to manifolds with boundary and smooth datas.

A *two dimensional ohmic conductor* is a couple $(\mathcal{X},\rho)$ where $\mathcal{X}$ is an open oriented bordered two dimensional real surface (with boundary γ),the conductivity $\sigma=\rho^{-1}$ is a positive definite tensor from $T^{*}\mathcal{X}$ to $T^{*}\mathcal{X}$ and $\mathcal{X}$ is equipped with the complex structure associated to the riemannian metric $g_{\mu,\rho}$ defined by (2.4) where μ is any volume form of $\mathcal{X}$. In this setting, the scalar conductivity is the function $s=\mu/V_{\mu,\rho}$ where $V_{\mu,\rho}$ is the volume associated to $g_{\mu,\rho}$. The DN-operator is the operator $\theta_{\mathcal{X},\rho}$ defined by

$$\theta_{\mathcal{X},\rho}:C^{1}\left(\gamma\right)\ni u\mapsto\left(sd^{c}\widetilde{u}\right)_{\gamma}\in T_{\gamma}^{*}\mathcal{X}$$

where $\widetilde{u}$ is the unique solution of the following Dirichlet problem

$$U\left|_{\gamma}=u\quad\&\quad d\left(s\,d^{c}U\right)=0.\right. \tag{2.7}$$

The IDN-problem associated to this setting is threefold :

Unicity. Assume that two dimensional ohmic conductors $(\mathcal{X},\rho)$ and $(\mathcal{X}',\rho')$ share the same boundary γ and the same DN-operator θ. Is it true that there is a diffeomorphism $\varphi:\overline{\mathcal{X}}\to\overline{\mathcal{X}'}$ between manifolds with boundaries such that $\varphi:\mathcal{X}\to\mathcal{X}'$ is analytic and $s=s'\circ\varphi$ where s and s' are scalar conductivities of $\mathcal{X}$ and $\mathcal{X}'$?

Reconstruction. Assume that $(\mathcal{X},\rho)$ is a two dimensional ohmic conductor. How from its DN-operator one can reconstruct a two dimensional ohmic conductor $(\mathcal{X}',\rho')$ which is isomorphic (in the above sense) to $(\mathcal{X},\rho)$.

Characterization. Let γ be a smooth abstract real curve, L a complex line bundle along γ and $\theta:C^{1}\left(\gamma\right)\to L$ an operator. Find a non trivial necessary and sufficient condition on (γ,L,θ) which ensures that there exists a two dimensional ohmic conductor $(\mathcal{X},\rho)$ such that $L=\Lambda^{1,0}T_{\gamma}^{*}\mathcal{X}$ and $\theta=\theta_{\mathcal{X},\rho}$.

All these problems are open. In the particular case where the resistivity ρ is such that its scalar conductivity σ is constant, the Dirichlet problem (2.7) becomes

$$U\left|_{\gamma}=u\ \&\ dd^{c}U=0\right.$$

where $dd^c = i\partial\overline{\partial}$ is the usual Laplacian. Hence, our article gives with theorems 1 to 3c a rather complete answer to the EIT-problem with constant scalar conductivity.

Concerning the main results given in the literature about unicity, reconstruction and stability for the important case where $\mathcal{X}$ is a domain in $\mathbb{R}^2$ but the scalar conductivity is not constant, see [8][22] and references therein. Note that the exact method of reconstruction for this case goes back to [24].

3. Unicity under existence assumption

The notations and hypothesis are taken from theorem and section 1 but $\mathcal{X}$ is now an open Riemann surface equipped with a hermitian metric[12] g. Let $\overline{\mathcal{X}} = \mathcal{X} \cup \gamma$ be a manifold with almost smooth boundary. Hence, there is a compact subset σ of γ such that $h^1(\sigma) = 0$ and $\left(\overline{\mathcal{X}}, \gamma\right)$ is a manifold with boundary near each point of $\gamma \backslash \sigma$.

When $\left(\overline{\mathcal{X}}, \gamma\right)$ has only almost smooth boundary and u is a smooth function on γ, proposition 12 implies that u has a continuous harmonic extension $\widetilde{u}$ with finite Dirichlet integral on $\mathcal{X}$. An elementary calculus gives then that for a fixed continuous defining function ρ of γ, smooth on $\overline{\mathcal{X}}\backslash\sigma$, the operator L defined by (1.1) determines for all $u \in C^\infty(\gamma)$ the trace on γ of the holomorphic $(1,0)$-form $\partial\widetilde{u}$: $\partial\widetilde{u} = (Lu)\,|\partial\rho|_g^{-1}\,\partial\rho$ on $\gamma\backslash\sigma$. With (1.2), this implies that f is the restriction to γ of a function $F = (F_1, F_2)$ meromorphic on $\mathcal{X}$, smooth on $\gamma\backslash\sigma$. Since $(\gamma, u, \theta u)$ is assumed to be a DN-data for $\mathcal{X}$, F is continuous in a neighborhood of γ in $\overline{\mathcal{X}}$.

The proof of theorem 1 relies on the following lemmas which enable to see $\overline{\mathcal{X}}$ as a normalization of $\overline{\mathcal{Y}}$.

Lemma 7. *Set $\delta = f(\gamma)$. Then $\mathcal{Y} = F(\mathcal{X})\backslash\delta$ is a complex curve of $\mathbb{CP}_2\backslash\delta$ without compact component, which has finite mass and satisfies $d[\mathcal{Y}] = [\delta]$. Moreover, each regular point of $\overline{\mathcal{X}}$ has in $\overline{\mathcal{X}}$ a neighborhood V such that $F : V \to F(V)$ is diffeomorphism between manifolds with smooth boundary.*

Proof. Since F is continuous in a neighborhood of γ in $\overline{\mathcal{X}}$, $\mathcal{Y}$ is a closed set of $\mathbb{CP}_2\backslash\delta$. As $\mathcal{Y}$ is also locally the image of a Riemann surface by an analytic map, $\mathcal{Y}$ is a complex curve of $\mathbb{CP}_2\backslash\delta$. Since $F_*[\mathcal{X}]$ is a locally flat current, the Federer support theorem (see [17, p. 316][14, 4.1.15 & 4.1.20]) produces a locally integrable function λ on $\mathcal{Y}$ such that $F_*[\mathcal{X}] = \lambda[\mathcal{Y}]$ on the regular part $\mathcal{Y}_{\mathrm{reg}}$ of $\mathcal{Y}$; since $d^2 = 0$, λ is locally constant. Since f embeds γ into $\mathbb{C}^2$, each point x in γ which is a regular boundary point of $\overline{\mathcal{X}}$ has a neighborhood V of x in $\overline{\mathcal{X}}$ such that $F : V_x \to F(V_x)$ is a diffeomorphism between classical manifolds with boundary. Let F_W be the restriction of F to the Riemann surfaces $W = \cup V_x$ and $W' = F(W)$. The degree of F_W is at most 1 otherwise almost all points of W' would have at least two different preimages which would imply that df is zero at almost all points of γ. So this degree is 1 and $F_*[\mathcal{X}] = [\mathcal{Y}]$ on W'. Hence, $\lambda = 1$ on each connected component of $\mathcal{Y}$ and $d[\mathcal{Y}] = [\delta]$.

If $\mathcal{Y}$ contains a compact complex curve $\mathcal{Z}$, $F^{-1}(\mathcal{Z})$ is a complex curve in $\mathcal{X}$ without boundary which is impossible unless $\mathcal{Z} = \varnothing$. The fact that $\mathcal{Y}$

[12] Harmonicity does not depend of the chosen hermitian metric.

has a finite mass follows from a theorem of Wirtinger (see [17, Lemma 1.5 p. 315]). ∎

As δ is smooth, the conclusion of lemma 7 implies, thanks to [18], that δ contains a compact set τ such that $h^1(\tau) = 0$ and $\left(\overline{\mathcal{Y}}, \delta\right)$ is manifold with boundary near points of $\delta \backslash \tau$. The lemma below described how $\mathcal{Y}$ is near a point y of τ.

Lemma 8. *Assume $\mathcal{Y}$ is a complex curve of $\mathbb{CP}_2 \backslash \delta$ with finite mass satisfying $d[\mathcal{Y}] = \pm[\delta]$. Let y be a point of σ and U a domain containing y. Then, among the components of $\mathcal{Y} \cap U$, $\mathcal{C}_{y,1}^U$, ..., $\mathcal{C}_{y,m_U}^U$, one, says $\mathcal{C}_{y,1}^U$, satisfies $d\left[\mathcal{C}_{y,1}^U\right] = \pm[\delta]|_{U_y}$ whereas for $j \geqslant 2$, $\overline{\mathcal{C}_{y,j}^U} \cap U$ is a complex curve of U.*

Proof. [18, th. 4.7] implies that for each j there is $n_j \in \mathbb{Z}$ such that $d\left[\mathcal{C}_{y,j}^U\right] = n_j[\delta]$ on U. As $d[\mathcal{Y}] = \pm[\delta]$, $\Sigma d\left[\mathcal{C}_{y,j}^U\right] = \pm 1$ and at least one $\mathcal{C}_{y,j}^U$, says $\mathcal{C}_{y,1}^U$, is such that $n_j \neq 0$. Because $h^1(\sigma) = 0$, $\delta \cap U$ contains a point q not in σ. Then, if V is a sufficiently small ball centered at q, $\mathcal{Y} \cap V$ is submanifold of V with boundary $\delta \cap V$ and $\mathcal{Y} \cap V$ has only one connected component which can be nothing else than $\mathcal{C}_{y,1}^U \cap V$. Hence $n_1 = \pm 1$. Since two different bordered Riemann surfaces of some open set of $\mathbb{CP}_2$ meet at most in a set of zero one dimensional Hausdorff measure, this implies that $n_j = 0$ for $j \neq 1$. Thus, if $j \geqslant 2$, $d\left[\mathcal{C}_{y,j}^U\right] = 0$ and with [17, th. 2.1 p. 337] we conclude that $\overline{\mathcal{C}_{y,j}^U} \cap U$ is a complex curve of U_y. ∎

If $y \in \delta$, we denote by m_y the limit of m_U (see lemma 8) when the diameter of U goes to 0, U neighborhood of y ; if $m_y \geqslant 2$, then $y \in \tau$. We says that $\overline{\mathcal{Y}}$ has a *weak singularity* at $y \in \delta$ if $m_y \geqslant 2$ but y is regular point of $\mathcal{C}_{y,1}^U$; $y \in \delta$ is called a *strong singularity* of $\overline{\mathcal{Y}}$ if y is not a regular point of $\mathcal{C}_{y,1}^U$.

We denote by τ_1 (resp. τ_2) the sets of points where $\overline{\mathcal{Y}}$ has weak (resp. strong) singularity. Then $\tau = \tau_1 \cup \tau_2$ and $\tau_1 \cap \tau_2 = \varnothing$. Note that both τ_1 and τ_2 may contains points y where $m_y \geqslant 2$.

We denote by $\overline{\mathcal{Y}}_{\text{sing}} = \mathcal{Y}_{\text{sing}} \cup \tau$ the singular locus of $\overline{\mathcal{Y}}$, that is the set of points of $\overline{\mathcal{Y}}$ where $\overline{\mathcal{Y}}$ is not a smooth manifold with boundary and we set $\mathcal{B} = f(\sigma) \cup \overline{\mathcal{Y}}_{\text{sing}}$, $\mathcal{A} = F^{-1}(\mathcal{B})$ and $\mathcal{X}_\circ = \mathcal{X} \backslash F^{-1}(\delta) = \mathcal{X} \backslash F^{-1}(\tau_2)$.

Lemma 9. *The map $F : \overline{\mathcal{X}} \to \overline{\mathcal{Y}}$ is a normalization in the following sense : $F : \mathcal{X}_\circ \to \mathcal{Y}$ is a (usual) normalization and $F : \overline{\mathcal{X}} \backslash \mathcal{A} \to \overline{\mathcal{Y}} \backslash \mathcal{B}$ is a diffeomorphism between manifolds with boundary.*

Proof. Since $\mathcal{X}_\circ = \mathcal{X} \backslash F^{-1}(\delta)$, the properness of $F|_{\mathcal{X}_\circ}$ and the finiteness of its fibers are elementary. For each connected component $\mathcal{C}$ of $\mathcal{X}_\circ \backslash \mathcal{A}$, the degree $m_{\mathcal{C}}$ of $F : \mathcal{C} \to F(\mathcal{C})$ as a Riemann surfaces morphism is finite and $F_*[\mathcal{C}] = \delta_{\mathcal{C}}[F(\mathcal{C})]$. Reasoning as in lemma 7's proof, we get $m_{\mathcal{C}} = 1$. As $\mathcal{Y}_{\text{sing}}$ contains all the points of $\mathcal{Y}$ which has more than one preimage by F, $F : \mathcal{C} \to F(\mathcal{C})$ is thus an isomorphism. Let $\mathcal{C}'$ be another connected component of $\mathcal{X}_\circ \backslash \mathcal{A}$ and assume that $F(\mathcal{C})$ and $F(\mathcal{C}')$ meet at q. Since $q \notin \mathcal{B}$, the germs of $F(\mathcal{C})$ and $F(\mathcal{C}')$ at q are equal. This leads to $F(\mathcal{C}) = F(\mathcal{C}')$ which yields the contradiction $d[\mathcal{Y}] = 2[\delta]$ near regular boundary points of

δ in $bF(\mathcal{C})$. Hence, $F : \mathcal{X}_\circ \backslash \mathcal{A} \to \mathcal{Y} \backslash \mathcal{B} = \mathcal{Y}_{\text{reg}}$ is an isomorphism of complex manifolds. As $\mathcal{X}_\circ \cap \mathcal{A} = F^{-1}(\mathcal{Y}_{\text{sing}})$ has empty interior, $F : \mathcal{X}_\circ \to \mathcal{Y}$ is a usual normalization.

Set $\widetilde{\mathcal{X}} = \overline{\mathcal{X}} \backslash \mathcal{A}$, $\widetilde{\mathcal{Y}} = \overline{\mathcal{Y}} \backslash \mathcal{B}$, $\widetilde{\tau} = \tau \cup f(\sigma)$ and $\widetilde{\delta} = \delta \backslash \widetilde{\tau}$; by definition of $\mathcal{B}$, $\widetilde{\mathcal{Y}}$ is a manifold with smooth boundary $\widetilde{\delta} = \delta \backslash \widetilde{\tau}$ and $\widetilde{\mathcal{X}}$ has smooth boundary $\gamma \backslash \widetilde{\sigma}$ where $\widetilde{\sigma} = f^{-1}(\tau) = \sigma \cup f^{-1}(\tau)$. The map $F : \widetilde{\mathcal{X}} \to \widetilde{\mathcal{Y}}$ is onto by construction. It is injective because the maps $F : \mathcal{X}_\circ \to \mathcal{Y}$ and $f : \gamma \to \delta$ are so and because if $x_1 \in \mathcal{X}$ and $x_2 \in \gamma$ have the same image y by F, then $m_y \geqslant 2$, $y \in \tau$ and $x_1, x_2 \in \mathcal{A}$. Since $\widetilde{\mathcal{Y}} \backslash \widetilde{\tau} = \mathcal{Y}_{\text{reg}}$ and $F : \mathcal{X}_\circ \backslash \mathcal{A} \to \mathcal{Y}_{\text{reg}}$ is a diffeomorphism, the fact that $F : \widetilde{\mathcal{X}} \to \widetilde{\mathcal{Y}}$ is a diffeomorphism between manifolds with boundary has only to be check locally near boundary points. If $x \in \gamma \backslash \widetilde{\sigma}$, then $y = f(x) \notin \tau$ and the last conclusion of lemma 7 implies that there are open neighborhoods V and W of x and y in $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$ such that $F : V \to W$ is diffeomorphism between manifolds with boundary. ∎

3.1. **Proof of theorem 1.** Let L' be the operator defined by (1.1) when N is changed for N', let us denote F' the meromorphic extension of f to $\mathcal{X}'$ and let $\mathcal{Y}' = F'(\mathcal{X}') \backslash \delta$ where $\delta = f(\gamma)$. By lemma 7, the sets $\mathcal{Y}'$ and $\mathcal{Y}$ are two complex curves of $\mathbb{CP}_2 \backslash \delta$ which has no compact component and both are bordered by $[\delta]$ in the sense of currents. Hence they are identical by a consequence of a Harvey-Shiffman theorem (see [12, prop. 1.4.1]).

Taking in account lemma 9 and the fact that $\mathcal{B} \cap \mathcal{Y} = \mathcal{Y}_{\text{sing}} = \mathcal{B}' \cap \mathcal{Y}$, this implies that $\Phi = F^{-1} \circ F'$ is an analytic isomorphism between $\mathcal{X}' \backslash F'^{-1}(\mathcal{Y}_{\text{sing}})$ and $\mathcal{X}' \backslash F^{-1}(\mathcal{Y}_{\text{sing}})$. Using the properness of $F : \mathcal{X}_\circ \to \mathcal{Y}$ and $F' : \mathcal{X}'_\circ \to \mathcal{Y}$, we conclude that Φ extends holomorphically to $\mathcal{X}'$. Likewise, $\Psi = F'^{-1} \circ F$ extends holomorphically to $\mathcal{X}$. As $\Phi(\Psi(x')) = x'$ and $\Psi(\Phi(x)) = x$ for almost all $x' \in \mathcal{X}'$ and $x \in \mathcal{X}$, the extension of Φ is an isomorphism from $\mathcal{X}$ to $\mathcal{X}'$.

As F and F' extend f to $\mathcal{X}$ and $\mathcal{X}'$, Φ extend continuously to γ by the identity map on γ . Set $\sigma = \overline{\mathcal{X}}_{\text{sing}}$ and $\sigma' = \overline{\mathcal{X}'}_{\text{sing}}$ and let x' be in $\gamma \backslash (\sigma \cup \sigma')$. Then if $y = f(x) \notin \tau = \delta \cap \overline{\mathcal{Y}}_{\text{sing}}$, Φ is a diffeomorphism between neighborhoods of x in $\overline{\mathcal{X}}$ and $\overline{\mathcal{X}'}$ because F (resp. F') is a diffeomorphism between manifold with boundary from a neighborhood of x in $\overline{\mathcal{X}}$ (resp. in $\overline{\mathcal{X}'}$) to a neighborhood of y in $\overline{\mathcal{Y}}$. If $y \in \tau$, then the last conclusion of lemma 7 implies that $y \in \tau_1$ so that there is a open neighborhood U of y, a component $\mathcal{C}^U_{y,1}$ of $\mathcal{Y} \cap U$ and open neighborhoods V and V' of x in $\overline{\mathcal{X}}$ and $\overline{\mathcal{X}'}$ such that that $F : V \to \mathcal{C}^U_{y,1}$ and $F' : V' \to \mathcal{C}^U_{y,1}$ are diffeomorphism between manifolds with smooth boundary. Hence, $\Phi : V \to V'$ is diffeomorphism between manifolds with smooth boundary. Finally, Φ realizes a diffeomorphism between manifolds with smooth boundary from $\mathcal{X}' \backslash (\sigma \cup \sigma')$ to $\mathcal{X} \backslash (\sigma \cup \sigma')$ and the proof is complete. ∎

The proof contains the following variation of theorem 1.

Theorem 10. *Assume that $\mathcal{X}$ and $\mathcal{X}'$ are open Riemann surfaces with almost smooth boundary γ such that the map f defined by (1.2) is an embedding of γ into $\mathbb{CP}_2$ and has a meromorphic extension F to $\mathcal{X}$ and F' to $\mathcal{X}'$ which are continuous near γ. Then $F(\mathcal{X}) \backslash f(\gamma) = F'(\mathcal{X}') / f(\gamma) \overset{def}{=} \mathcal{Y}$ is a complex curve of $\mathbb{CP}_2 \backslash \delta$ without compact component, which has finite mass* *and satisfies $d[\mathcal{Y}] = [\delta]$. Moreover, $\overline{\mathcal{X}}$ and $\overline{\mathcal{X}'}$ are normalizations of $\overline{\mathcal{Y}}$ in the sense of lemma 9.*

This result shows that the Riemann surfaces constructed in the converse part of theorems 3a, 3b and 3c are the only possible candidates for a solution to the IDN-problem.

4. Existence and reconstruction, proof of theorem 2

We first prove that the Stokes formula holds in manifolds with almost smooth boundary ; one can see also [14].

Lemma 11. *Let $\left(\overline{\mathcal{X}}, \gamma\right)$ be a Riemann surface with almost smooth boundary. Then for any 1-form φ which is continuous on $\overline{\mathcal{X}}$ such that $d\varphi$ exists as an integrable differential on $\mathcal{X}$, we have*

$$\int_{\mathcal{X}} d\varphi = \int_{\gamma} \varphi \tag{4.1}$$

Proof. Set $\sigma = \overline{\mathcal{X}}_{\text{sing}}$. Since $h^2\left(\overline{\mathcal{X}}\right) < \infty$ and $h^1(\sigma) = 0$, there is an increasing sequence $(\mathcal{X}_k)$ of smooth open sets of $\mathcal{X}$ such that $\left(h^1(b\mathcal{X}_k)\right)$ and $\left(h^2\left(\overline{\mathcal{X}}\backslash\mathcal{X}_k\right)\right)$ both have limit zero and $\overline{\mathcal{X}}\backslash\overline{\mathcal{X}_k}$ is contained in a 2^{-k}-neighborhood of σ. Let φ be as above. Since $d\varphi$ is integrable and $\lim h^2(\mathcal{X}_k) = 0$, $\left(\int_{\mathcal{X}_k} d\varphi\right)$ has limit $\int_{\mathcal{X}} d\varphi$. As $b\mathcal{X}_k = \left(\gamma \cap \overline{\mathcal{X}_k}\right) \cup [(b\mathcal{X}_k)\backslash\gamma]$ and $\lim h^1(b\mathcal{X}_k) = 0$, $\left(\int_{(b\mathcal{X}_k)\backslash\gamma} \varphi\right)$ converges to 0. Hence, $\lim \int_{\gamma\cap\overline{\mathcal{X}_k}} \varphi = \int_{\gamma} \varphi$ and the classical Stokes formula for φ and $\mathcal{X}_k$ yields (4.1). ∎

We prove now a variation of the Riemann's existence theorem.

Proposition 12. *Let $\left(\overline{\mathcal{X}}, \gamma\right)$ be a Riemann surface with almost smooth boundary and u a real valued lipschitzian function on γ. Then u has a unique continuous harmonic extension $\widetilde{u}$ of u to $\mathcal{X}$ and $\widetilde{u}$ has finite Dirichlet integral $\int i\,\partial\widetilde{u} \wedge \overline{\partial}\widetilde{u}$. Moreover, $N_{\mathcal{X}}u$ defined as $\partial\widetilde{u}/\partial\nu$ on $\gamma\backslash\overline{\mathcal{X}}_{\text{sing}}$ admits an extension on γ as a current of order 1 on γ.*

Proof. Following the lines of Riemann's method for harmonic extension of smooth functions, we first construct an adequate space $W^1(\mathcal{X})$.

Since $\left(\overline{\mathcal{X}}, \gamma\right)$ is at least a topological bordered manifold, for every fixed point x in γ we can choose in $\mathcal{X}$ an open set Δ_x whose closure in $\overline{\mathcal{X}}$ is a neighborhood of x and which is mapped by a complex coordinate φ_x into the closure of the unit disk $\mathbb{D}$ of $\mathbb{C}$, φ_x being a homeomorphism from $\overline{\Delta_x}$ to $\overline{\mathbb{D}}$. Note that if $x' \in \gamma \cap \overline{\Delta_x}$ is a regular point of $\overline{\mathcal{X}}$, φ_x has to be diffeomorphism between manifolds with boundary from a neighborhood of x' to a neighborhood of $\varphi_x(x')$ in $\overline{\mathbb{D}}$. If $x \in \mathcal{X}$, we choose a conformal open disk $\varphi_x : \Delta_x \to \mathbb{D}$ of $\mathcal{X}$ centered at x. With the help of a continuous partition of unity, we can now construct a continuous hermitian metric h on $\mathcal{X}$ by gluing together the local metrics $(\varphi_x)_* dz \wedge d\overline{z}$ where z is the standard coordinate of $\mathbb{C}$. We then denote by $W^1(\mathcal{X})$ the Sobolev space of functions in $L^2(\mathcal{X}, h)$ with finite Dirichlet integral.

By construction, any function A in $W^1(\mathcal{X})$ is such that for each $x \in \gamma$, $B_x = (\varphi_x)_* A|_{\Delta_x}$ is a square integrable function for the standard metric of $\mathbb{D}$. Since the values of Dirichlet integrals are conformal invariants, it follows that B_x is in the standard Sobolev space $W^1(\mathbb{D})$ and hence admits a boundary value b_x on $\mathbb{T} = b\mathbb{D}$ which is in $W^{1/2}(\mathbb{T})$. As b_x is punctually defined almost everywhere, $a_x = b_x \circ \varphi$ is defined almost everywhere in $\gamma \cap \overline{\Delta_x}$. The constructions made for each $x \in \gamma$ glue together to form a function defined almost everywhere in γ which we call the boundary value of A.

We consider now the subset F of $W^1(\mathcal{X})$ with boundary value u. It is closed and non empty since by a result of McShane [23], u admits a Lipschitz extension to $\mathcal{X}$. It follows now from classical arguments that the Dirichlet integral can be minimized in F at some function $\widetilde{u}$ which has to be harmonic in $\mathcal{X}$. It remains only to show that $\widetilde{u}$ is continuous on $\overline{\mathcal{X}}$. If $x \in \gamma$, what precedes implies that $v_x = \left(u_x \circ \varphi_x^{-1}\right)|_{\mathbb{T}}$ is in $W^{1/2}(\mathbb{T})$, continuous near $\varphi_x(x)$ and is the boundary value of $\widetilde{v}_x = \widetilde{u} \circ \varphi_x^{-1}$. Hence, classical Poisson formula for the disc implies that near $\varphi_x(x)$ in $\overline{\mathbb{D}}$, $\widetilde{v}_x$ is continuous up to $\mathbb{T}$ with restriction v_x on $\mathbb{T}$. Since φ_x is an homeomorphism, we get that $\widetilde{u}$ is continuous at x with value $u(x)$.

Let θu be the form defined by 1.1. The Stokes formula (4.1) implies that if $\varphi \in C^1(\gamma)$ and Φ is a Lipschitz extension of φ on $\mathcal{X}$,

$$\int_\gamma \varphi\theta u = -\int_{\mathcal{X}} \partial\widetilde{u} \wedge \overline{\partial}\Phi.$$

As the last integral does not depend of the choice of the Lipschitz extension of φ, this means that θu and hence $N_{\mathcal{X}} u$, are well defined currents of order 1. ∎

Assume now that the hypothesis of theorem 2 are true. Lemma 7 points out that F projects γ on a smooth curve δ of $\mathbb{C}^2$ which bounds in the sense of currents a complex curve $\mathcal{Y}$ of $\mathbb{CP}_2 \backslash \delta$ which has finite mass and no compact component and theorem 10 implies that for some subset $\mathcal{X}_\circ$ of $\mathcal{X}$ with discrete complement in $\mathcal{X}$, $F : \mathcal{X}_\circ \to \mathcal{Y}$ is usual normalization. Hence, if $\mathcal{B} = \mathcal{Y}_{\text{sing}}$ and $\mathcal{A} = F^{-1}(\mathcal{B})$, $F : \mathcal{X}\backslash\mathcal{A} \to \mathcal{Y}\backslash\mathcal{B}$ is one to one. This is part 1 of theorem 2.

Before proving the second claim of theorem 2, we recall that $\mathbb{CP}_2$ is equipped with homogenous coordinates w and $\mathbb{C}^2$ identified with $\{w_0 \neq 0\}$ has affine coordinates $z_1 = w_1/w_0$ and $z_2 = w_2/w_0$. Set $\Delta_\infty = \{w_0 = 0\}$, $\mathcal{Y}_\infty = \mathcal{Y} \cap \Delta_\infty$ and if $\xi \in \mathbb{C}$, we set $\Delta_\xi = \{w_2 = \xi w_0\}$ and $\mathcal{Y}_\xi = \mathcal{Y} \cap \Delta_\xi$. Set

$$\Omega_\xi^m = \frac{z_1^m}{z_2 - \xi} dz_2 = \frac{w_1^m}{w_0^m} \frac{dw_2}{w_2 - \xi w_0} - \frac{w_1^m}{w_0^{m+1}} \frac{w_2 dw_0}{w_2 - \xi w_0}.$$

Applying the Stokes formula either for $\mathcal{Y}$ or $\mathcal{X}$, it turns out that $\frac{1}{2\pi i}\int_\gamma \frac{f_1^m}{f_2 - \xi} df_2$ equals $S_m(\xi) + P_m(\xi)$ where

$$S_m(\xi) = \sum_{z \in \mathcal{Y}_\xi} \operatorname{Res}\left(\eta^*\Omega_\xi^m, z\right) \; ; \; P_m(\xi) = \sum_{z \in \mathcal{Y}_\infty} \operatorname{Res}\left(\eta^*\Omega_\xi^m, z\right)$$

and $\eta : \mathcal{Y} \to \mathbb{CP}_2$ is the canonical injection.

For almost all ξ_* in $\mathbb{C}$, $\mathcal{Y}$ meets transversely Δ_{ξ_*} only in $\mathbb{C}^2 \cap \mathcal{Y}_{\text{reg}}$; for such a fixed ξ_*, set $p = \operatorname{Card} \mathcal{Y}_{\xi_*}$ and $\mathcal{Y}_{\xi_*} = \{z_{1*}, ..., z_{p*}\}$. For ξ in a sufficiently small connected neighborhood W_{ξ_*} of ξ_*, $\mathcal{Y}_\xi$ lies then in $\mathbb{C}^2 \cap \mathcal{Y}_{\mathrm{reg}}$ and can be written $\{z_1(\xi), ..., z_p(\xi)\}$ with $z_j(\xi) = (h_j(\xi), \xi)$ where h_j is holomorphic in W_{ξ_*} and has value z_{j*} at ξ_*, $1 \leqslant j \leqslant p$. Direct calculation shows (see [12]) that the poles of Ω_ξ^m in $\mathbb{C}^2$ are $z_1(\xi), ..., z_p(\xi)$ with residue $h_1(\xi)^m, ..., h_p(\xi)^m$. Hence, $S_m = S_{h,m}$ in V.

Reasoning as in lemma 13 in next section, we can assume without loss of generality that $\mathcal{Y}$ meets transversely Δ_∞ and that $\mathcal{Y}_\infty \subset \mathcal{Y}_{\mathrm{reg}}$. In this situation, a direct calculus (see [12]) gives that at $z \in \mathcal{Y}_\infty$, Ω_ξ^m has a pole of order $m+1$ with a residue which is a polynomial in ξ of degree at most m. Hence, P_m is is a polynomial in ξ of degree at most m ; formula $(E_{m,\xi})$ is proved.

If $A \geqslant B$ and $\xi_0, ..., \xi_{A-1}$ are mutually distinct, the Vandermonde matrix $(\xi_\nu^\mu)_{0 \leqslant \nu, \mu \leqslant B-1}$ is invertible and hence, the system $(E_{m,\xi_\nu})_{0 \leqslant \nu \leqslant B-1}$ enable to write the coefficients of P_m as a linear combination of the $S_{h,m}(\xi_\nu)$, $0 \leqslant \nu \leqslant B-1$. Introducing this result in $(E_\xi) = (E_{m,\xi_\nu})_{\substack{0 \leqslant m \leqslant B-1 \\ 0 \leqslant \nu \leqslant B-1}}$ we get a linear system which, since $AB - \frac{1}{2}B(B+1) \geqslant pA$ when $B \geqslant 2p+1$, enable to compute for a generic ξ the unknowns $S_{h,m}(\xi_\nu)$ and, thanks to the Newton-Girard formulas, the elementary symmetric functions of $h_1(\xi_\nu), ..., h_p(\xi_\nu)$; finally we get the intersection points $(h_j(\xi_\nu), \xi_\nu)$ of $\mathcal{Y}$ with Δ_{ξ_ν}.

We prove the third assertion of theorem 2. Almost all ξ_* in $\mathbb{C}^2$ has a connected neighborhood W_{ξ_*} such that there is a compact of $\mathbb{CP}_2 \backslash (\delta \cup \mathcal{Y}_{\mathrm{sing}})$ containing all $\mathcal{Y}_\xi$ when $\xi \in W_{\xi_*}$. When ξ_* is such and $\xi \in W_{\xi_*}$ the form $\Phi_\xi^{m,\ell} = \frac{F_1^m}{F_2 - \xi} \partial \widetilde{u}_\ell$ has pole only in a compact of $\mathcal{X}$. Since $\widetilde{u}_\ell$ is the continuous harmonic extension of u_ℓ on $\mathcal{X}$, $\int_{\mathcal{X}} i\, \partial \widetilde{u}_\ell \wedge \overline{\partial} \widetilde{u}_\ell < +\infty$ by proposition 12 and we can apply the Stokes formula (4.1) to it on $\mathcal{X}$. After a residue calculus, it comes that

$$\frac{1}{2\pi i} \int_\gamma \frac{f_1^m}{f_2 - \xi} \theta u_\ell = \sum_{x \in \mathcal{X}} \mathrm{Res}\left(\Phi_\xi^{m,\ell}, x\right) = \sum_{1 \leqslant j \leqslant p} h_j(\xi)^m \frac{\partial \widetilde{u}_\ell}{\partial F_2}\left(F^{-1}(h_j(\xi), \xi)\right)$$

for all $\xi \in W_{\xi_*}$.$\square$

Remark. The $\frac{1}{2}B(B+1)$ coefficients of the polynomials P_k come from the intersection points of $\mathcal{Y}$ with $\{w_0 = 0\}$. In the generic case, $\mathcal{Y}$ is given near theses points as the graph of holomorphic functions $\psi_1, ..., \psi_q$ of the variable w_0/w_2, it appears that the coefficients of P_k are ruled by the derivatives of order at most k at 0 of the ψ_ℓ. The reconstruction of $\mathcal{X}$ is thus possible with a non linear system with only $pA + q(B+1)$ unknowns.

5. Proofs of characterizations theorem 3a, 3b and 3c

The proofs of theorems 3a, 3b and 3c follow a similar schema. The function f defined by (1.2) embeds γ into a smooth real curve $\delta = f(\gamma)$ of $\mathbb{C}^2$. The necessary conditions for the existence of a solution to the IDN-problem for γ are drawn from the fact this existence implies that δ bounds a "concrete" Riemann surface in $\mathbb{CP}_2$ or $\mathbb{C}^2$. The sufficient part of theorem 3c reconstructs the concrete but singular solution to the IDN-problem ; a normalization gives then the sufficient part of 3b. The proof of theorem 3a follows a similar scheme.

5.1. **Proof of theorem 3a.A.** Assume that $\mathcal{X}$ is an open bordered riemannian surface of finite volume with restricted DN-data $(\gamma, u, \theta u)$. Then the functions F_j $(j = 1,2)$ which are the well defined quotients of forms $(\partial\widetilde{u_j})/(\partial\widetilde{u}_0)$ are meromorphic and letting $F = (F_1, F_2)$, lemma 7 implies that $\mathcal{Y} = F(\mathcal{X})\backslash\delta$, $\delta = f(\gamma)$, is a complex curve of finite volume, without compact component and bordered by $[\delta]$ in the sense of currents.

Moreover, the function G has the expression

$$G(\xi_0, \xi_1) = \frac{1}{2\pi i}\int_\delta \Omega_\xi \ , \ \Omega_\xi = \frac{w_1}{w_0}\frac{d\Lambda_\xi(w)}{\Lambda_\xi(w)} - \frac{w_1}{w_0^2}dw_0 \ ,$$

where $\delta = f(\gamma)$, $(w_0 : w_1 : w_2)$ are homogenous coordinates for $\mathbb{CP}_2$ and $\Lambda_\xi(w) = \xi_0 w_0 + \xi_1 w_1 + w_2$.

For almost all $\xi_* = (\xi_{0*}, \xi_{1*})$ and for all ξ in a sufficiently small connected neighborhood W_{ξ_*} of ξ_*, $\mathcal{Y}$ meets $\Delta_\xi = \{\Lambda_\xi = 0\}$ transversely, $\mathcal{Y}_\xi = \mathcal{Y} \cap \Delta_\xi \subset \mathbb{C}^2 \cap \mathcal{Y}_{\mathrm{reg}}$ so that there exists $p = \mathrm{Card}\,\mathcal{Y}_{\xi_*}$ holomorphic functions $H_j = (1 : h_j : h_{j,2}) : W_{\xi_*} \to \mathbb{CP}_2$ such that $\mathcal{Y}_\xi = \{H_j(\xi)\,,\ 1 \leqslant j \leqslant p\}$ and $(h_j)_{1\leqslant j\leqslant p}$ are mutually distinct. Direct calculations[(13)] shows that these functions satisfy the shock wave equation (1.5).

Let $\eta : \mathcal{Y} \to \mathbb{CP}_2$ the canonical injection. Since $\eta^*\Omega_\xi$ may only have poles in $\mathcal{Y}_\xi \cup \mathcal{Y}_\infty$, the Stokes formula gives that near ξ_*, $G = H + L$ where

$$H(\xi) = \sum_{z\in\mathcal{Y}_\xi} \mathrm{Res}\,(\eta^*\Omega_\xi, z) \qquad , \qquad L(\xi) = \sum_{z\in\mathcal{Y}_\infty} \mathrm{Res}\,(\eta^*\Omega_\xi, z)\,.$$

By construction, $\eta^*\Omega_\xi$ has residue $h_j(\xi)$ at $z = H_j(\xi) \in \mathcal{Y}_\xi$ and it remains only to know that L is affine in ξ_0 to prove theorem 3a. The second part of the lemma below is needed in the proof of theorem 4.

Lemma 13. *If W_{ξ_*} is small enough, $L = \Sigma h_j - G$ is affine in ξ_0. In addition, there is an integer q such that L is the limit in $\mathcal{O}\left(W_{\xi^*}\right)$ of a continuous one parameter family of ξ_0-affine functions which are sum of q mutually distinct shock wave functions.*

Proof. With no loss of generality, we assume $\xi_* = 0$ for the proof. For small complex parameters ε, we consider the homogeneous coordinates $w^\varepsilon = (w_0 + \varepsilon w_1 : w_1 : w_2 + \varepsilon w_1)$. For ε in a sufficiently small neighborhood of 0 the intersection of $\mathcal{Y}$ with the zero set of $\Lambda_\xi : w \mapsto \xi_0 + \xi_1 w_1^\varepsilon + w_2^\varepsilon$ is still generic in the sense that it is transverse and lies in $\{w_2^\varepsilon \neq 0\} \cap \mathcal{Y}_{\mathrm{reg}}$. Hence, setting $\Omega_\xi^\varepsilon = \frac{w_1}{w_0^\varepsilon}\frac{d\Lambda_\xi^\varepsilon(w)}{\Lambda_\xi^\varepsilon(w)} - \frac{w_1}{\left(w_0^\varepsilon\right)^2}dw_0^\varepsilon$, the function $G^\varepsilon : \xi \mapsto \frac{1}{2\pi i}\int_\delta \Omega_\xi^\varepsilon$ is on W_{ξ_*} the sum of p mutually distinct shock wave functions $h_1^\varepsilon, ..., h_p^\varepsilon$. When ε is generic, $\mathcal{Y}$ meets also transversely $\Delta_\infty^\varepsilon = \{w_0^\varepsilon = 0\}$ and $\mathcal{Y}_\infty^\varepsilon = \mathcal{Y} \cap \{w_0 = 0\}$ lies in $\mathcal{Y}_{\mathrm{reg}} \cap \{w_2^\varepsilon \neq 0\}$. Hence we know thanks to [12, Lemme 2.3.1] that $L^\varepsilon = \Sigma h_j^\varepsilon - G^\varepsilon$ is affine in ξ_0. The dependence of G^ε is clearly holomorphic in ε. The same holds for each h_j^ε since what precedes have shown that $h_j^\varepsilon(\xi) = \mathrm{Res}\left(\eta^*\Omega_\xi^\varepsilon, H_j(\xi)\right) = \frac{1}{2\pi i}\int_{\mathcal{Y}\cap\partial U_j} \eta^*\Omega_\xi^\varepsilon$ where U_j is any sufficiently small neighborhood of $H_j(\xi)$ in $\mathbb{CP}_2$ whose boundary is smooth and transverse to $\mathcal{Y}$. Hence L^ε is holomorphic in ε and has to be affine in ξ_0 when $\varepsilon = 0$.

[13]This lemma which goes back to Darboux is proved in [12, lemma 2.4].

Let q be the number of points in $\mathcal{Y}_\infty$ counted with their multiplicities ; when ξ is generic, q is either defined by

$$p-q=\frac{1}{2\pi i}\int_\gamma \frac{d\left(\xi_0+\xi_1 f_1+f_2\right)}{\xi_0+\xi_1 f_1+f_2}. \tag{5.1}$$

For sufficiently small generic ε, [12, Lemme 2.3.1] gives more precisely that $L^\varepsilon=\sum\limits_{1\leqslant j\leqslant q} h_j^{\varepsilon,\infty}$ with

$$h_j^{\varepsilon,\infty}=-\operatorname{Res}\left(\eta^*\Omega_\xi, z_j^\varepsilon\right)=\frac{-\xi_0\psi_j^\varepsilon(0)+\psi_j^{\varepsilon\prime}(0)}{1+\xi_1\psi_j^\varepsilon(0)}$$

where $\mathcal{Y}_\infty^\varepsilon=\left\{z_1^\varepsilon,...,z_q^\varepsilon\right\}$ and $\psi_j^\varepsilon\in\mathcal{O}\left(U^\varepsilon\right)$, U^ε open neighborhood of 0 in $\mathbb{C}$, enable to give in the affine coordinates $\zeta^\varepsilon=(w_j^\varepsilon/w_2^\varepsilon)_{j=0,1}$ the set $\mathcal{Y}$ as a graph above U^ε : $\mathcal{Y}\cap V_j^\varepsilon=\left\{(\zeta_0^\varepsilon:\psi_j^\varepsilon(\zeta_0^\varepsilon):1)\ ;\ \zeta_0^\varepsilon\in U^\varepsilon\right\}$. Each $h_j^{\varepsilon,\infty}$ is clearly a shock wave function, that is a solution to $h_{\xi_1}=h_{\xi_0}h$. ∎

Remark. When ε goes to a non generic value, the fact that L is a sum of q shock wave functions may not be preserved as section 6.1 shows.

5.2. **Proof of theorem 3a.B.** Assume that γ satisfies (1.4) in a connected neighborhood W_{ξ_*} of one point $(\xi_{0*}:\xi_{1*}:1)$ of $\mathbb{CP}_2$. If $\left(\partial^2G/\partial\xi_0\right)_{|W_{\xi_*}}=0$, then γ satisfies the classical Wermer-Harvey-Lawson moment condition in $\mathbb{C}_{\xi_*}^2=\mathbb{CP}_2\backslash\left\{\xi_{0*}w_0+\xi_{1*}w_1+w_2=0\right\}$ (see [12, cor. 1.6.2]) and [29][18] implies that if δ is suitably oriented, the polynomial hull of δ in $\mathbb{C}_{\xi_*}^2$ is the unique complex curve $\mathcal{Y}$ of finite mass of $\mathbb{C}_{\xi_*}^2\backslash\delta$ such that $d\left[\mathcal{Y}\right]=\left[\delta\right]$.

Assume now $\left(\partial^2G/\partial\xi_0\right)_{|W_{\xi_*}}\neq 0$. Then we can choose a minimal $\mathcal{H}=\left\{h_1,...,h_p\right\}$ in the sense that no proper subset of $\mathcal{H}$ satisfy (1.4). Although it is not explicitly mentioned by their authors, the heart of the arguments of [12, th. II p. 390] is that $\pm\left[\delta\right]=d\left[\mathcal{Y}\right]$ where $\mathcal{Y}$ is the analytic extension $\mathcal{Y}$ in $\mathbb{CP}_2\backslash\delta$ of the union Γ of the graphs Γ_j of the functions

$$H_j:\xi\mapsto\left(1:h_j(\xi):-\xi_0-\xi_1h_j(\xi)\right), 1\leqslant j\leqslant p.$$

This fact, not totally explicit in [20, p. 264], can be recovered *a posteriori* by a kind of trick which has been used Poly in [13] and is developed later in the proof of theorem 3c : for the curve $\widetilde{\gamma}$ which is the union of γ with the boundaries of Γ_j negatively oriented, one goes back to the $\mathbb{C}_{\xi_*}^2$-case where $\left(\partial^2G/\partial\xi_0\right)_{|W_{\xi_*}}=0$. If $d\left[\mathcal{Y}\right]$ is $-\left[\delta\right]$ and not $\left[\delta\right]$, then the same arguments which has proved theorem 3a.A would give that the functions h_j, geometrically defined as the first coordinates of the intersection points of $\mathcal{Y}$ with generic lines Λ_ξ, should satisfy not only the shock wave equation $h_{\xi_1}=h_{\xi_0}h$ but also the "negative" shock wave equation $h_{\xi_1}=-h_{\xi_0}h$. As this is impossible, $d\left[\mathcal{Y}\right]=\left[\delta\right]$.

In both cases, we have found (up to a change of orientation if $\partial^2G/\partial\xi_0$ vanish on W_{ξ_*}) a complex curve $\mathcal{Y}$ of finite mass of $\mathbb{CP}_2\backslash\delta$ such that $d\left[\mathcal{Y}\right]=\left[\delta\right]$.

As δ is smooth, we know from [18] that there is in δ a compact set τ such that $h^1(\tau)=0$ and for which each point of $y\in\delta\backslash\tau$ has a neighborhood U_y where $\overline{\mathcal{Y}}\cap U_y$ is a closed bordered submanifold of U_y with boundary $\delta\cap U_y$. Lemma 8 in section 3 describes how $\mathcal{Y}$ is near points of τ. Using the notations and definitions introduced after its proof, we denote by τ_2 the set of points of δ where $\overline{\mathcal{Y}}$ has a strong singularity, set $\tau_1 = \tau \backslash \tau_2$ and finally by τ' the set of points y of δ where $m_y \geqslant 2$.

We denote by $\widetilde{\mathcal{Y}}$ the abstract complex curve which is the union of $\mathcal{Y}$ and of τ_2 and we consider a normalization $\pi : \mathcal{X} \to \widetilde{\mathcal{Y}}$. Lemma 8 implies that π is an open mapping. Let $\mathcal{Z}$ be the disjoint and abstract union $\mathcal{X} \cup \gamma$. If $x \in \gamma$, we define a neighborhood of x in $\mathcal{Z}$ as a subset of $\mathcal{Z}$ which contains a set of the kind $\pi^{-1}\left(C_{y,1}^U\right)$ where $y = f(x)$ and U is a neighborhood of y in $\mathbb{CP}_2$. Then $(\mathcal{Z}, \gamma)$ is a compact metrizable topological manifold with boundary which has finite 2-dimensionnal Hausdorff measure and smooth boundary at every point of $\sigma = f^{-1}(\tau \backslash \tau_2)$. Since f is an embedding, $h^1(\sigma) = 0$ and $(\mathcal{Z}, \gamma)$ is a manifold with almost smooth boundary. Moreover, it follows by construction that the meromorphic extension $F : \mathcal{Z} \to \overline{\mathcal{Y}}$ of f to $\mathcal{X}$ defined by $F\,|_{\mathcal{X}} = \pi$ is a normalization of $\overline{\mathcal{Y}}$ in the sense of lemma 9.$\square$

Remark. When γ is real analytic, [18, th. II] implies that $C_{p,1}$ is a manifold with boundary in the classical sense. So, in that case, $(\mathcal{Z}, \gamma)$ is a classical manifold with boundary.

The following proposition which clarifies some results of [12] justifies the fourth remark after theorem 3a.

Proposition 14. *Assume δ is connected. Then if $\left(\partial^2 G/\partial\xi_0\right)_{|W_{\xi_*}} = 0$, the polynomial hull of γ in $\mathbb{C}^2 = \mathbb{CP}_2 \backslash \{\xi_{0*} w_0 + \xi_{1*} w_1 + w_2 = 0\}$ has boundary $\pm[\gamma]$. If $\left(\partial^2 G/\partial\xi_0\right)_{|W_{\xi_*}} \neq 0$ and no proper subset of $\mathcal{H} = \{h_1, ..., h_p\}$ satisfies (1.4), then the analytic extension $\mathcal{Y}$ in $\mathbb{CP}_2 \backslash \delta$ of the union Γ of the graphs Γ_j of the functions $H_j : \xi \mapsto (1 : h_j(\xi) : -\xi_0 - \xi_1 h_j(\xi))$, $1 \leqslant j \leqslant p$, is the complex curve which has minimal volume among complex curves $\mathcal{Z}$ such that $d[\mathcal{Z}] = [\delta]$.*

Proof. Let $Y_0 = \Sigma n_j [\mathcal{Z}_j]$ a holomorphic 1-chain of finite volume in $\mathbb{CP}_2 \backslash \delta$ such that $dY_0 = [\delta]$; each $n_j \in \mathbb{Z}$ and each $\mathcal{Z}_j$ is an irreducible complex curve of $\mathbb{CP}_2 \backslash \delta$. Removing each $\mathcal{Z}_j$ which is compact, we still get a holomorphic 1-chain Y such that $dY = [\delta]$. As δ is connected and smooth, $Y = \left[\widetilde{\mathcal{Y}}\right]$ where $\widetilde{\mathcal{Y}}$ is an irreducible complex curve. To verify this, we can follow [17] :

Let $\widetilde{\mathcal{Y}}$ be one of the component of the support of Y. As $\widetilde{\mathcal{Y}}$ has finite volume, the natural extension T of $\left[\widetilde{\mathcal{Y}}\right]$ to $\mathbb{CP}_2$ is a locally rectifiable current. Hence, T and dT are locally flat currents and dT has to be of the form $h[\delta]$ where h is a locally integrable function ; h is locally constant since $ddT = 0$. As there is at least one point where $(\widetilde{\mathcal{Y}}, \delta)$ is a smooth manifold with boundary, we get $h = \pm 1$ from the fact δ is connected. Hence, for some ε' in $\{-1, 1\}$, $dY = \varepsilon' d\left[\widetilde{\mathcal{Y}}\right]$. From Harvey-Shiffman theorem (see [12, prop. 1.4.1]) we get then that $Y - \varepsilon'\left[\widetilde{\mathcal{Y}}\right]$ is a holomorphic chain of $\mathbb{CP}_2$, that is a finite sum $\Sigma n_j [\mathcal{S}_j]$ where each $n_j \in \mathbb{Z}$ and each $\mathcal{S}_j$ is an irreducible complex curve of $\mathbb{CP}_2$. So $Y = \varepsilon'\left[\widetilde{\mathcal{Y}}\right] + \Sigma n_j [\mathcal{S}_j]$ and from the definition of Y, we get $Y = \varepsilon'\left[\widetilde{\mathcal{Y}}\right]$ and $\varepsilon' = 1$.

Since $\mathcal{Y}$ is one of the complex curves of $\mathbb{CP}_2 \backslash \delta$ with boundary $[\delta]$, the minimality of the volume of $\widetilde{\mathcal{Y}}$ implies that $\mathcal{Y} = \widetilde{\mathcal{Y}} \cup \mathcal{Z}$ where $\mathcal{Z}$ is a union of compact complex curves of $\mathbb{CP}_2$. But the intersection of any compact curve with a line Λ_ξ is described, for ξ in a neighborhood of a generic ξ_*, as a finite union of graphs of function $(1 : g_j : g_{j,2})$ whose second homogeneous coordinate satisfy the shock wave equation and such that Σg_j is affine in ξ_0 (see [20, section 2]). Since $\mathcal{H}$ is minimal, no such g_j belongs to $\mathcal{H}$ and it appears that Γ has to be contained in $\widetilde{\mathcal{Y}}$. Hence, $\mathcal{Y} \subset \widetilde{\mathcal{Y}}$ and, finally, $\mathcal{Y} = \widetilde{\mathcal{Y}}$. ∎

5.3. **Proof of theorem 3a.C.** ¿From theorem 1 and its following remark, the above $\mathcal{X}$ is, up to isomorphism, the only possible candidate to a solution to the IDN-problem for γ. Let $\mathcal{Z}$, $\mathcal{D}$ and g be as in the statement. Let $u \in C^1(\gamma)$ and for $z \in \mathcal{D} \backslash \gamma$ set

$$\Omega_z = u \partial_\zeta g_z + g_z \overline{\theta u}.$$

If $\mathcal{Y}$ is a smooth domain in $\mathcal{Z}$, the Stokes formula (4.1) implies that value of $\mathbf{1}_{\mathcal{Y}}(z) \widehat{u}(z) + \int_{\mathcal{Y}} \overline{\partial} \widehat{u} \wedge \partial_\zeta g_z - \int_{(\partial \mathcal{Y}) \backslash \gamma} u \partial_\zeta g_z$ does not depend of the Lipschitz extension $\widehat{u}$ of u to $\mathcal{Z}$ and equals $\int_{\gamma \cap \mathcal{Y}} u \partial_\zeta g_z$ when $\mathcal{X}_{\text{sing}} = \varnothing$. Hence we can take it as a definition for $\int_{\gamma \cap \mathcal{Y}} u \partial_\zeta g_z$ in the general case. Let then F be the function defined for $z \in \mathcal{D} \backslash \gamma$ by

$$F(z) = \frac{2}{i} \int_{\zeta \in \gamma} \Omega_z(\zeta) \ , \ \Omega_z = u \partial_\zeta g_z + g_z \overline{\theta u},$$

where $g_z = g(., z)$.

Since $\widetilde{u}$, u and θu are continuous, the conclusion of part C follows from the lemma bellow which gives $\widetilde{u} = F|_{\mathcal{X}}$ and $\theta u = \partial \widetilde{u}$ on $\gamma \backslash \sigma$ if $F\Big|_{\mathcal{D} \backslash \overline{\mathcal{X}}} = 0$.

Lemma 15. *$F_+ = F|_{\mathcal{X}}$ and $F_- = F\Big|_{\mathcal{D} \backslash \overline{\mathcal{X}}}$ are real valued harmonic functions such that*

$$u = F_+ - F_- \ \& \ \overline{\theta u} = \overline{\partial} F_+ - \overline{\partial} F_- \text{ on } \gamma \backslash \sigma. \tag{5.2}$$

Proof. The harmonicity of F is a simple consequence of the properties of g. Let ω be the metric on $\overline{\mathcal{X}}$ and $L_{\mathcal{X}}$ the DN-operator of $\mathcal{X}$.

Fix now p in $\gamma \backslash \sigma$ and in neighborhood $\mathcal{U}$ of p in $\mathcal{D}$, a holomorphic chart $\mathcal{U} \to U$ centered at p ; a hat "$\widehat{\ }$" denotes thereafter the coordinate expression of a function, a form or a set. For $z \in U \backslash \gamma$ let us define $\varphi(z)$ and $R_1(z)$ by the formulas

$$\varphi(z) = -2i \int_{\gamma \cap \mathcal{U}} \Omega_z \ \& \ R_1(z) = -2i \int_{\gamma \backslash \mathcal{U}} \Omega_z.$$

Then R_1 is smooth on $\mathcal{U}$ and the proof of (5.2) at p reduces to the same but for $\varphi_+ = \varphi|_{\mathcal{X} \cap \mathcal{U}}$ and $\varphi_- = \varphi|_{\mathcal{U} \backslash \overline{\mathcal{X}}}$ instead of F_+ and F_-.

If y and x are the coordinates of $\zeta \in \gamma \cap \mathcal{U}$ and $z \in \mathcal{D} \backslash \gamma$, $\widehat{g}(y, x) = g(\zeta, z)$ can be written in the form $\widehat{g}(y, x) = \frac{1}{2\pi} \ln|y - x| + h(y, x)$ where h is a smooth function on $U \times U$, harmonic in each its variable. Hence,

$$\widehat{\varphi}(x) = \frac{1}{\pi i} \int_{\widehat{\gamma} \cap U} \left[\frac{\widehat{u}(y)}{y - x} + 2\overline{\lambda(y)} \ln|y - x| \right] dy + R_2(x) \tag{5.3}$$

where λ is defined by $\widehat{\theta u}(y) = \lambda(y)\,dy$ and $R_2 \in C^0(U)$. Since $y \mapsto \ln|y|$ is integrable on γ, we obtain

$$\widehat{\varphi}(x) = \frac{1}{\pi i}\int_{\widehat{\gamma}\cap U}\frac{\widehat{u}(y)}{y-x}dy + R_3(x)$$

where $R_3 \in C^0(U)$. From the classical Sohotsky-Plemelj formula, we know that the above integral has jump $i\pi\,\widehat{u}$ across $\widehat{\gamma}$ in the sense of distribution and pointwise near each regular boundary point. So the jump of F through γ in $\mathcal{U}$ is u. Since

$$\overline{\partial}\widehat{\varphi}(x) = \frac{1}{\pi i}d\overline{x}\int_{\widehat{\gamma}\cap U}\frac{\overline{\lambda(y)}}{\overline{y}-\overline{x}}d\overline{y} + \overline{\partial}R_2(x),$$

the jump of $\overline{\partial}\widehat{\varphi}$ that through $\widehat{\gamma}$ is likewise $\overline{\lambda dx}$, that is $\overline{\widehat{\theta u}}(x)$.

In order to check that the values of F are real, we choose a smooth function ρ in $\mathcal{Z}$ such that $\gamma = \{\rho = 0\}$, $\mathcal{X} = \{\rho < 0\}$ and $|d\rho|_\omega = 1$; hence $(\nu^*, \tau^*) = (d\rho, d^c\rho)$. Let $z \in \mathcal{D}\backslash\gamma$. Since

$$\partial g_z = (L_{\mathcal{X}}g_z)(\nu^* + i\tau^*) \;\;\&\;\; \theta u = (Lu)(\nu^* + i\tau^*)$$

and ν^* vanishes on $T\gamma$,we get

$$F(z) = \int_\gamma (u\,N_{\mathcal{X}}g_z - g_z Nu)\,d^c\rho + i\int_\gamma (u\,\tau g_z + g_z\tau u)\,\tau^*$$

As $(u\,\tau g_z + g_z\tau u)\,\tau^* = ud(g_z|_\gamma) + g_z du = d(ug_z|_\gamma)$, the second integral is zero. The proof is complete. ∎

5.4. **Proof of theorem 3c.A.** We assume that γ is in the sense of currents the boundary of a complex curve $\mathcal{X}$ of $\mathbb{CP}_2\backslash\gamma$ for which $(\gamma, u, \theta u)$ is a restricted DN-data. Then as in the proof of theorem 3a.A, for almost all $(\xi_{*0} : \xi_{*1} : 1)$ in $\mathbb{CP}_2$ there exists a neighborhood W_{ξ_*} of $\xi_* = (\xi_{*0}, \xi_{*1})$ such that for every $\xi \in W_{\xi_*}$, $\mathcal{X} \cap \Delta_\xi$ lies in $\mathbb{C}^2$ and equals $\Gamma \cap \Delta_\xi$ where Γ is the union of the graphs Γ_j of $H_j = (1 : h_j : h_{j,2}) : W_{\xi_*} \to \mathbb{CP}_2$, $1 \leqslant j \leqslant p$ where H_j is holomorphic in W_{ξ_*}. Since we are concerned only by generic ξ_*, we can suppose that $\Gamma_j = \left\{\left(\varphi_j(z_2), z_2\right)\ ;\ z_2 \in U_j\right\}$ where U_j is a neighborhood of $z^*_{j,2} = h_{j,2}(\xi_*)$ and $\varphi_j \in \mathcal{O}\{U_j\}$. The decomposition sought for $\widetilde{G}$ in (a1) can be then found in [20]. However, this residues calculus is needed in part B and we include it here

When $\xi \in W_{\xi_*}$, $\widetilde{G}_\ell(\xi_0 : \xi_1 : 1)$ is the sum of the residues of

$$\Lambda_\ell = \frac{z_\ell}{\xi_0 + \xi_1 z_1 + z_2}\Theta_0$$

(for convenience $z_\ell = 1$ if $\ell = 0$) in $\mathcal{X}$. Set $\Theta_0 = A_j(z_2)\,dz_2$ in each Γ_j and let us abbreviate $H_j(\xi)$ in z_j. Then $z_{j,2}$ is the only pole of Λ_ℓ in Γ_j. It is a simple one and its residue is

$$g_{j,\ell} = \frac{z_{j,\ell}\;A_j(z_{j,2})}{\xi_1\varphi'_j(z_{j,2}) + 1} = z_{j,\ell}\;g_{j,0} \tag{5.4}$$

where $z_{j,\ell} = 1$ if $\ell = 0$. As Γ_j is also parametrized by $\xi \mapsto H_j(\xi)$, we can set

$$g_j = \sum_{0\leqslant\ell\leqslant 2} g_{j,\ell}\,d\eta_\ell \text{ on } \Gamma_j$$

and get that $g_{j,1}/g_{j,0} = z_{j,1} = h_j$ satisfy (1.5) ; $g_{j,2}/g_{j,0} = z_{j,2} = h_{j,2}$ satisfy then $h_j \frac{\partial h_{j,2}}{\partial \xi_0} = \frac{\partial h_{j,2}}{\partial \xi_1}$ because $h_{j,2} = -\xi_0 - \xi_1 h_j$. Note that the dependence in ξ of g can be made more clear if the identity $h_j - \varphi_j(-\xi_0 - \xi_1 h_j) = 0$ is used. Indeed, this relation implies

$$\begin{cases} \left(1+\xi_1\varphi_j'\right)\partial_{\xi_0} h_j + \varphi_j' = 0 \\ \left(1+\xi_1\varphi_j'\right)\partial_{\xi_1} h_j + h_j\varphi_j' = 0 \end{cases}.$$

Hence $\varphi_j' = -\left(\partial_{\xi_0} h_j\right)/\left(1+\xi_1\partial_{\xi_0} h_j\right)$ and

$$\frac{1}{1+\xi_1\varphi_j'} = \frac{\partial_{\xi_1} h_j}{\partial_{\xi_0} h_j}\frac{1+\xi_1\partial_{\xi_0} h_j}{h_j} = 1+\xi_1\partial_{\xi_0} h_j = \partial_{\xi_0} h_{j,2}$$

since h_j satisfies the shock wave equation $h_j\partial_{\xi_0} h_j = \partial_{\xi_1} h_j$. So we have now instead of (5.4)

$$g_{j,\ell} = A_j\left(H_j\right) h_{j,\ell}\frac{\partial h_{j,2}}{\partial \xi_0},\ 1 \leqslant j \leqslant p,\ 0 \leqslant \ell \leqslant 2. \tag{5.5}$$

where $h_{j,0} = 1$ for convenience.

¿From the definition we get that when expressed in the affine coordinates ξ, g_j is given by the integral formula

$$2\pi i\ g_j = \left(\int_{\partial\Gamma_j} \frac{\Theta_0}{\xi_0+\xi_1 z_1+z_2}\right) d\xi_0 + \left(\int_{\partial\Gamma_j} \frac{z_1\Theta_0}{\xi_0+\xi_1 z_1+z_2}\right) d\xi_1$$

from which it is clear that g_j is closed.

To achieve the proof of part A, it is enough to remark that (a2) is a direct consequence of the fact $\operatorname{Re}\Theta_\ell = d\widetilde{u_\ell}$ is exact.

5.5. **Proof of theorem 3c.B.** Assume that the hypothesis of (b1) is true and γ is connected.

Case $\widetilde{G}\big|_{W_{\eta_*}} \neq 0$. This mean $\mathcal{H} \neq \varnothing$ when $\mathcal{H}$ is minimal in the sense that no proper subset of $\mathcal{H}$ gives a decomposition of $\widetilde{G}$ with the same properties. Let Γ be the union of the graphs Γ_j of the functions $H_j = (1 : h_j : h_{j,2})$, $1 \leqslant j \leqslant p$, where $h_{j,2} = -\xi_0 - \xi_1 h_j$. If needed, we can choose another ξ_* in order that Γ does not meet Λ_{ξ_*} in $\{w_0 = 0\}$. Then for any ξ in a neighborhood Ω of ξ_*, the $H_j(\xi)$ are mutually distinct and are the points of $\Gamma \cap L_\xi$. Finally, we assume, which it is not a restriction, that Γ has a smooth oriented boundary $\partial\Gamma$.

Let $\widetilde{\gamma}$ be the union of $\partial\Gamma$ with opposite orientation and γ and let φ_η be the linear function $z \mapsto \eta_0+\eta_1 z_1+\eta_2 z_2$. From the hypothesis we get directly that

$$\frac{1}{2\pi i}\int_\gamma \varphi_\eta^{-1}\theta_\ell = \sum_{1\leqslant j\leqslant p} g_{j,\ell} = \sum_{1\leqslant j\leqslant p} g_{j,\ell}h_{j,\ell}$$

where $h_{j,\ell} = 1$ if $\ell = 0$. On the other hand, if we set

$$\Theta_0 = \left(\partial_{\xi_0} h_{j,2}\right)^{-1} g_{j,0}\, dh_{j,2},\ \text{on } \overline{\Gamma_j},\ 1 \leqslant j \leqslant p \tag{5.6}$$

and $z_0 = 1$, the residues calculus made in the proof of part A implies that

$$\int_{\partial\Gamma} z_\ell \varphi_\eta^{-1} \theta_0 = \sum_{1 \leqslant j \leqslant p} g_{j,0} h_{j,\ell}\ ,\ 1 \leqslant j \leqslant p.$$

Hence, $\int_{\widetilde{\gamma}} \varphi_\eta \theta_\ell = 0$. As $\widetilde{\gamma}$ is contained in the affine space $E_{\xi_*} = \mathbb{CP}_2 \backslash \Lambda_{\xi_*}$, we can apply [20, cor. 4.2 p. 265] and [11, prop. 1] and get in $E_{\xi_*} \backslash \widetilde{\gamma}$ a complex curve $\widetilde{\mathcal{X}}$ of finite volume where θ_0 extends weakly in a weakly holomorphic form Θ_0 satisfying[(14)]

$$\int_{\mathcal{X}} \left(\overline{\partial}\varphi\right) \wedge \Theta_0 = \int_{\mathcal{X}} d\left(\varphi\Theta_0\right) = \int_\gamma \varphi\theta_0 \tag{5.7}$$

holds for any function φ smooth in a neighborhood of $\overline{\mathcal{X}}$ and analytic in a neighborhood of σ.

By construction, $\mathcal{X} = \widetilde{\mathcal{X}} \cup \overline{\Gamma}$ is a complex curve of $\mathbb{CP}_2 \backslash \gamma$ where θ_0 extends as a weakly holomorphic form Θ_0, this extension coinciding in Γ with the form defined by (5.6). If $\ell = 1, 2$, the form $\Theta_\ell = z_\ell \Theta_0$ is a weakly holomorphic extension of θ_ℓ to $\mathcal{X}$.

Let $\varepsilon > 0$ and let $\mathcal{W}_\varepsilon$ an ε-neighborhood of Λ_{ξ_*}. As $\mathcal{X}_\varepsilon = \mathcal{X} \backslash \overline{\mathcal{W}_\varepsilon}$ lies in the affine space $E_{\xi_*} \backslash \mathcal{W}_\varepsilon$, [18, th. 4.7] implies that

$$d\left[\mathcal{X}_\varepsilon\right] = n_\varepsilon \left[\gamma\right] + \Sigma_{1 \leqslant j \leqslant p} n_{\varepsilon,j} \left[\gamma_{\varepsilon,j}\right] \tag{5.8}$$

where for $1 \leqslant j \leqslant p$, $n_\varepsilon, n_{\varepsilon,j} \in \mathbb{Z}$ and $\gamma_{\varepsilon,j}$ is the (smooth) boundary of the smooth (manifold) $\mathcal{W}_\varepsilon \cap \Gamma_j$. If $0 < \varepsilon' < \varepsilon$,

$$\begin{aligned} \sum_{1 \leqslant j \leqslant p} d\left[\Gamma_j \cap \left(\mathcal{W}_\varepsilon \backslash \mathcal{W}_{\varepsilon'}\right)\right] &= -d\left[\mathcal{X}_\varepsilon\right] + d\left[\mathcal{X}_{\varepsilon'}\right] \\ &= \left(-n_\varepsilon + n_{\varepsilon'}\right)\left[\gamma\right] + \Sigma_{1 \leqslant j \leqslant p} \left(-n_{\varepsilon,j} + n_{\varepsilon',j}\right)\left[\gamma_{\varepsilon,j}\right] \end{aligned}$$

Hence, $n_\varepsilon = n_{\varepsilon'} \overset{def}{=} n$ and as each $\Gamma_j \cap \mathcal{W}_\varepsilon \backslash \mathcal{W}_{\varepsilon'}$ is a smooth manifold with boundary, $n_{\varepsilon,j} = n_{\varepsilon',j} = -1$. Taking now limits in (5.8) when ε goes to zero, we get $d\left[\mathcal{X}\right] = n\left[\gamma\right]$. We suppress from $\mathcal{X}$ any compact component it may have and still denote the result by $\mathcal{X}$; note that $\mathcal{X}$ has now to be connected. Since $\mathcal{X}$ is a complex curve of $\mathbb{CP}_2 \backslash \gamma$, [10] implies that if $n \neq 0$, there is in γ a compact set σ such that $h^1(\sigma) = 0$ and $\left(\overline{\mathcal{X}}, \pm\gamma\right)$ is manifold with boundary near points of $\gamma \backslash \sigma$; as $\mathcal{X}$ is connected, this implies $n = \pm 1$. When $n = 0$, the structure theorems of Harvey-Shiffman [17] implies that $\mathcal{Z} = \overline{\mathcal{X}}$ is then a complex compact curve of $\mathbb{CP}_2$; since γ is smooth, γ is locally a Jordan curve of $\mathcal{Z}_{\mathrm{reg}}$ and the points where γ may meet the finite set $\mathcal{Z}_{\mathrm{sing}}$ are only self-intersection points of $\mathcal{Z}$.

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[14]In [20][11], (5.7) is in fact obtained only for φ smooth on $\mathcal{X}$ and holomorphic in a neighborhood of γ but (5.7) follows from this together with (5.5) and the residue relations (very close in spirit to the relations $T_{m,\xi}$)

$$\frac{1}{2\pi i} \int_\gamma z_1^m \frac{\theta_0}{\xi_0 + \xi_1 z_1 + z_2} = \Sigma_{1 \leqslant j \leqslant p} h_j^m(\xi)\, g_{j,0}(\xi)$$

where $h_j(\xi)$, $h_{j,2}(\xi)$ and $g_{j,0}(\xi)$ are as above ; indeed these relations enable for generic ξ the computation of $g_{j,0}(\xi)$ by a kramerian system and hence imply the smoothness of Θ_0 near points of $\gamma \backslash \sigma$. However, this precision is used not essentialy in the sequel.

Since $\mathcal{X}\cap\Lambda_\xi=\{(1:h_j(\xi):h_{j,2}(\xi)),\ 1\leqslant j\leqslant p\}$, the Stokes's formula gives $G=\Sigma h_j$ (see proof of theorem 3a.A).

Assume that $\left(\partial^2 G/\partial\xi_0\right)_{|W_{\xi_*}}\neq 0$. Then $n\neq 0$ because otherwise for ξ closed to ξ_*, the intersection of $\mathcal{X}$ with the line Λ_ξ should have to be the intersection with Λ_ξ of a compact Riemann surface, namely $\overline{\mathcal{X}}$, which by a theorem of Reiss would force Σh_j to be affine in ξ_0 (see [16, ch. 5.2] or [20, section 2]). Reasoning like in the proof of 3a.B, we also eliminate the possibility $d[\mathcal{X}]=-[\gamma]$ because it would imply that if $\mathcal{H}=\{h_1,...,h_p\}$ is minimal in the sense that no proper subset of $\mathcal{H}$ satisfy (a1), each h_j also satisfies $h_{j,y}=-h_{j,x}h_j$. Hence $n=1$.

When $\left(\partial^2 G/\partial\xi_0\right)_{|W_{\xi_*}}=0$, that is when Σh_j is affine in ξ_0, and when $\overline{\mathcal{X}}$ is not an algebraic curve where γ is a slit, n has to be non null and hence is ±1. Since $\widetilde{G}\left|_{W_{\eta_*}}\right.\neq 0$, the minimal $\mathcal{H}$ in the above sense is not empty and reasoning likewise, we get $d[\mathcal{X}]=[\gamma]$.

Case $\widetilde{G}\left|_{W_{\eta_*}}\right.=0$. This means that the minimal $\mathcal{H}$ is empty. Then, we can apply [20, th. 4.2 p. 264] in the affine case (see [11] for a generalization and a detailed proof in this case) to get in $\mathbb{C}^2\backslash\gamma$ (here $\mathbb{C}^2$ is the complement of $\Lambda_{\xi_*}=\{\xi_{*0}w_0+\xi_{1*}w_1+w_2=0\}$ in $\mathbb{CP}_2$) a complex curve $\mathcal{X}$ of finite volume where θ_0 extends weakly in a weakly holomorphic form Θ_0 with the property (5.7).

Since θ_0 yields a non zero measure on γ which, because $\widetilde{G}\left|_{W_{\eta_*}}\right.=0$, is orthogonal to all polynomials of $\mathbb{C}^2\sim\mathbb{CP}_2\backslash\Lambda_{\xi_*}$, we can apply Bishop [7] and [26] (Wermer originates and solves this problem [29] for the real analytic case) to get that $\mathcal{X}$ is the polynomial hull $\widetilde{\gamma}$ of γ in $\mathbb{C}^2$ and $d(\pm[\mathcal{X}])=[\gamma]$. [18] (see also [10]) implies then that γ contains a compact set σ such that $h^1(\sigma)=0$ and $\left(\overline{\mathcal{X}},\gamma\right)$ is manifold with boundary near points of $\gamma\backslash\sigma$.

To prove (b2), we go back to the assumption that (a1) is true and we assume in addition that Θ_ℓ is holomorphic and satisfies (1.8) for all $c\in H_1(\mathcal{X}_{\mathrm{reg}})$. Then, there exists $U_\ell\in C^\infty(\mathcal{X}_{\mathrm{reg}})$ such that $\Theta_\ell=dV_\ell$ on $\mathcal{X}_{\mathrm{reg}}$; since Θ_ℓ is a (1,0)-form, $\partial V_\ell=dV_\ell=\Theta_\ell$. We know from the preceding point that $d[\mathcal{X}]=n[\gamma]$ where $n\in\{0,1\}$, up to a change of orientation of γ when $\widetilde{G}\left|_{W_{\eta_*}}\right.=0$. Assume at first that $n=1$. As d is elliptic up to the boundary, V_ℓ has to be smooth up to the boundary in the classical sense near points of γ outside σ and the preceding equality (5.7) yields $dv_\ell=du_\ell$ where $v_\ell=V_\ell\left|_\gamma\right.$. Hence, z_0 is a point of γ where $\left(\overline{\mathcal{X}},\gamma\right)$ is a manifold with boundary, there is a constant c such that $v_\ell=u_\ell+c$ near z_0 in γ. Since $h^1(\sigma)=0$ and $dv_\ell=du_\ell$ is smooth we have $v_\ell(z)-v_\ell(z_0)=\int_{\gamma_{z_0,z}}du_\ell$ where $\gamma_{z_0,z}$ is the positively oriented path of γ starting at z_0 and ending at z. Hence, $v_\ell(z)-u_\ell(z_0)-c=u_\ell(z)-u_\ell(z_0)$ and $v_\ell(z)=u_\ell(z)+c$. This implies that U_ℓ is a weakly harmonic extension of u_ℓ. When $n=0$, $\overline{\mathcal{X}}$ is two sided locally near points of γ and each local side has the same boundary regularity as in the case $n=1$. Hence, we can reason as in this case and get that there is a weakly harmonic extension of u_ℓ to $\mathcal{X}$.

When we assume also that $\int_{\mathcal{X}_{\mathrm{reg}}}\Theta_\ell\wedge\overline{\Theta_\ell}<+\infty$, Θ_ℓ is holomorphic in the sense that its pullback to any normalization $\pi:\mathcal{Z}\to\mathcal{X}$ of $\mathcal{X}$ is holomorphic and not only meromorphic. The isolated singularities that the pullback V_ℓ of

U_ℓ may have in $\mathcal{Z}$ are removable because dV_ℓ is smooth. So U_ℓ is harmonic on $\mathcal{X}$ and U_ℓ is the harmonic extension of u_ℓ to $\mathcal{X}$. The proof is complete.

5.6. **Proof of theorem 3b.** The map f enable us to embed the abstract IDN-problem of theorem 3b in the projective but concrete frame of theorem 3c. So theorem 3b.A is a direct consequence of 3c.A.

For the converse part B.b1, we apply 3c.B to $\delta = f(\gamma)$ and $\theta_\ell = \theta u_\ell$, $\ell = 0, 1, 2$. We get in $\mathbb{CP}_2 \backslash \delta$ an irreducible complex curve $\mathcal{Y}$ such that $d[\mathcal{Y}] = n[\delta]$ where $n \in \{0, 1\}$ and up to a change of orientation when $\widetilde{G}\,\big|_{W_{\eta_*}} = 0$; in addition, each θ_ℓ extends weakly to $\mathcal{Y}$ into a weakly holomorphic (1,0)-form $\Theta_\ell^{\mathcal{Y}}$.

When $n = 1$, the boundary regularity of $\mathcal{Y}$ mentioned in the proof of theorem 3c.B enable to apply readily the construction made in the proof of theorem 3a : adding to $\mathcal{Y}$ a subset σ_2 of γ of zero one dimensional Hausdorff measure, we get an abstract complex curve $\widetilde{\mathcal{Y}}$ which can be normalized in the classical sense into an abstract Riemann surface $\mathcal{X}$; γ can then be topologically glued to $\mathcal{X}$ so that $\left(\overline{\mathcal{X}}, \gamma\right)$ becomes a manifold with almost smooth boundary where the pullback F to $\mathcal{X}$ of the meromorphic map $\mathbb{CP}_2 \ni z \mapsto (z_1, z_2)$ gives a meromorphic extension of f to $\mathcal{X}$. Since the forms $\Theta_\ell^{\mathcal{Y}}$ are meromorphic on $\mathcal{Y}$, $\Theta_\ell = F^* \Theta_\ell^{\mathcal{Y}}$ is well defined and meromorphic outside $\mathcal{X} \backslash F^{-1}(\sigma_2)$ which has zero length.

When $n = 0$, $\mathcal{Z} = \overline{\mathcal{X}}$ is an algebraic curve and one can use a standard normalization of $\mathcal{Z}$ to get the same kind of conclusions.

The supplementary hypothesis of part B.b2 force each Θ_ℓ to have only removable singularities. Reasoning like in the proof of 3c.B.b2, (1.8) implies that $\Theta_\ell = dV_\ell = \partial V_\ell$ for some harmonic function V_ℓ smooth up to regular boundary points of $\overline{\mathcal{X}}$ (when $n = 0$, γ cuts locally $\mathcal{X}$ into two domains and this means that each restriction of V_ℓ to these domains is smooth up to γ) and that there is a constant c such that $U_\ell + c$ agrees with u_ℓ on γ.$\square$

6. Characterizations, effective or affine

6.1. **Explicit integro-differential characterization.** In this section where theorem 4 is proved, (ξ_0, ξ_1) is replaced by the simpler (x, y) and we reason in a neighborhood of $(0, 0)$; the set of one variable holomorphic functions near 0 is denoted by $\mathcal{O}_0$. The notation $\mathcal{O}_0 \otimes \mathbb{C}_d[Z]$ stands for the set of polynomials of degree at most d with indeterminacy Z and coefficients in $\mathcal{O}_0$. An element of $\mathcal{O}_0 \otimes \mathbb{C}_d[X]$ (resp. $\mathcal{O}_0 \otimes \mathbb{C}_d[Y]$) should be think as a function of the type $(x, y) \mapsto \sum_{0 \leqslant j \leqslant d} \lambda_j(x)\, y^j$ (resp. $(x, y) \mapsto \sum_{0 \leqslant j \leqslant d} \lambda_j(y)\, x^j$) where each $\lambda_j \in \mathcal{O}_0$.

If h is differentiable, the derivative of h with respect to one of its variable u is denoted h_u. If U is an open set in $\mathbb{C}^n$, $\mathcal{O}(U)$ is the space of holomorphic functions in U ; if $h \in \mathcal{O}(U)$ and $0 \in U$, we set $h_{x^{-0}, y^0} = h$ and in any simply connected neighborhood of 0 in U, we denote by $h_{x^{-\alpha-1}, y^\beta}$ $(\alpha, \beta \in \mathbb{N})$ the function which vanishes at 0 and satisfies $\partial\left(h_{x^{-\alpha-1}, y^\beta}\right)/\partial x = h_{x^{-\alpha}, y^\beta}$; $h_{x,^\alpha y^{-\beta-1}}$ is defined similarly.

A two variable function h is called a *shock wave function* on a domain D of $\mathbb{C}^2$ if it is holomorphic and satisfies $h_y = h_x h$ on D.

A *p-algebroïde* function on D is a p-uple $h=(h_1,...,h_m)$ of functions from D to $\mathbb{C}$ for which one can find p holomorphic functions $a_0,,...,a_{p-1}$ in D such that for $z\in D$, $h_1(z),...,h_p(z)$ are the roots, with multiplicities, of the polynomial $T^h=X^p+a_{p-1}(z)X^{p-1}+\cdots+a_1(z)X+a_0(z)$.

A *p-multivaluate shock wave function* on D is a p-algebroïde function h on D such that $\Sigma=\left\{\operatorname{Discr}T^h=0\right\}$ is a hypersurface of D and for any $z_*\in D\backslash\Sigma$, the holomorphic functions $h_1^*,..,h_p^*$ which near z_* describe the roots of T_z^h are non null shock wave functions ; the first symmetric function of T^h, that is the sum of the roots of T^h, is called the *trace* of T^h of h. Functions which are trace of p-multivaluate shock wave function are called *p-shock wave functions.*

Lemma 16. *Consider p mutually distinct functions $h_1,...,h_p$ holomorphic in a domain D of $\mathbb{C}^2$. Then, each h_j is a shock wave function if and only the functions $\sigma_k=(-1)^k\sum\limits_{1\leqslant j_1<\cdots<j_k\leqslant p}h_{j_1}...h_{j_k}$ satisfy the following system*

$$\sigma_p\sigma_{1,x}+\sigma_{p,y}=0\quad\&\quad\sigma_k\sigma_{1,x}+\sigma_{k,y}=\sigma_{k+1,x}\ ,\ 1\leqslant k\leqslant p-1. \tag{6.1}$$

Proof. Set $T=X^p+\sum\limits_{1\leqslant k\leqslant p}\sigma_kX^{p-k}$, $T'=\partial T/\partial X$ and for $u=x,y$, $T_u=\sum\limits_{1\leqslant k\leqslant p}\sigma_{k,u}X^{p-k}$. When $h\in\{h_1,...,h_p\}$ and $u=x,y$, $0=(Th)_u=(T'h)h_u+T_uh$ so that

$$(T'h)(h_y-hh_x)=\sigma_{1,x}h^p+\sum_{2\leqslant k\leqslant p}\sigma_{k,x}h^{p-k+1}-\sum_{1\leqslant k\leqslant p}\sigma_{k,y}h^{p-k}. \tag{6.2}$$

Since $h^p=-\sum\limits_{1\leqslant k\leqslant p}\sigma_kh^{p-k}$, this yields $(T'h)(h_y-hh_x)=Sh$ where

$$S=\sum_{1\leqslant k\leqslant p-1}\left[\sigma_{k+1,x}-\sigma_k\sigma_{1,x}-\sigma_{k,y}\right]X^{p-k}-(\sigma_p\sigma_{1,x}+\sigma_{p,y})$$

Since $\deg S\leqslant p-1$, the fact that each h_j is a shock wave function implies then that the coefficients of S vanish in a non empty open set and thus in the domain D. If $S=0$ at every point of D, then (6.2) proves that each h_j verifies $h_y-h_xh=0$ in the domain D because $T'h_j$ cannot be zero everywhere. ∎

Proposition 17. *Let D be a simply connected domain of $\mathbb{C}^2$ containing 0, Δ its image by the projection $(x,y)\mapsto y$ and and $H\in\mathcal{O}(D)$. When u is differentiable, we set*

$$\mathcal{D}_Hu=e^{H_{x,y^{-1}}}\frac{\partial}{\partial y}ue^{-H_{x,y^{-1}}}\quad\&\quad\mathcal{L}_Hu=(\mathcal{D}_Hu)_{x^{-1}}$$

The following two assertions are equivalent

1. H is a p-shock wave function in D.

2. There exists $\lambda_1,...,\lambda_{p-1}\in\mathcal{O}(\Delta)$ such that for $\widetilde{\lambda}_j(x,y)=\lambda_j(y)$, $1\leqslant j\leqslant p-1$,

$$\mathcal{D}_H\mathcal{L}_H^{p-1}H=\mathcal{D}_H\mathcal{L}_H^{p-2}\widetilde{\lambda}_1+\cdots+\mathcal{D}_H\mathcal{L}_H^0\widetilde{\lambda}_{p-1}. \tag{6.3}$$

$$\operatorname{Discr}T_z\not\equiv0 \tag{6.4}$$

where $T_z=X^p+\sum\limits_{1\leqslant k\leqslant p}s_k(z)X^{p-k}$ with

$$s_k=-\mathcal{L}_H^{k-1}H+\mathcal{L}_H^{k-2}\widetilde{\lambda}_1+\cdots+\mathcal{L}_H^0\widetilde{\lambda}_{k-1},\ 1\leqslant k\leqslant p. \tag{6.5}$$

More precisely, in case (2) is true, T determines a p-multivaluate shock wave function with trace H. Conversely, if H is the trace of a p-multivaluate shock wave function T, the p holomorphic functions which near a point z_* in $\{\operatorname{Discr} T \neq 0\}$ describes the roots of T_z have symmetric functions $(-1)^k s_k, 1 \leqslant k \leqslant p$ which satisfy (6.3) and (6.5).

Proof. 1) Assume H is a p-shock wave function in D. Then, H is the first symmetric function of a some $T \in \mathcal{O}(D) \otimes \mathbb{C}_p[Z]$ such that $\Sigma = \{\operatorname{Discr} T = 0\} \neq D$ and for any $z_* \in D \backslash \Sigma$, $\deg T_z = p$ and the holomorphic functions $h_1^*, .., h_p^*$ which near z_* describe the roots of T_z are non null shock wave functions. Consider $z_* = (x_*, y_*) \in D$ outside Σ. In a sufficient small convex neighborhood $W = U \times V$ of z_*, the roots h_1, ..., h_p of T are non null shock wave functions in W with no common value and $H = \sum\limits_{1 \leqslant j \leqslant p} h_j$ on W. For $k \in \{1, ..., p\}$ and $z \in W$, set $\rho_k(z) = \sigma_k(z) e^{-H_{x,y^{-1}}(z)}$ where σ_k defined in lemma 16. Then (6.1) implies $\rho_{p,y} = 0$ and that for $k \in \{1, ..., p-1\}$,

$$\left(\rho_{k+1} e^{H_{x,y^{-1}}}\right)_x = -H_x \sigma_k + \sigma_{k,y} = e^{H_{x,y^{-1}}} \frac{\partial}{\partial y} \sigma_k e^{-H_{x,y^{-1}}} = e^{H_{x,y^{-1}}} \rho_{k,y}$$

which implies that there exists $\lambda_k \in \mathcal{O}(V)$ such that

$$\rho_{k+1}(x, y) e^{H_{x,y^{-1}}} = \left[e^{H_{x,y^{-1}}} \rho_{k,y}\right]_{x^{-1}} + \lambda_k(y).$$

Since $\sigma_1 = -H$, we get $e^{-H_{x,y^{-1}}} \left[e^{H_{x,y^{-1}}} \rho_{1,y}\right]_{x^{-1}} = -e^{-H_{x,y^{-1}}} \mathcal{L}_H H$. Setting $\widetilde{\lambda}_0 = -H$, we obtain $\rho_2 = e^{-H_{x,y^{-1}}} (\mathcal{L}_H \lambda_0 + \widetilde{\lambda}_1)$ and a straightforward finite recurrence gives

$$\sigma_k = \rho_k e^{H_{x,y^{-1}}} = \mathcal{L}_H^{k-1} \widetilde{\lambda}_0 + \cdots + \mathcal{L}_H^0 \widetilde{\lambda}_{k-1} \tag{6.6}$$

for $1 \leqslant k \leqslant p$. In particular, $k = p$ yields (6.3), $1 \leqslant j \leqslant p-1$ because $\rho_{p,y} = 0$; (6.3) follows from the fact that the discriminant of T_z don't vanish in W since $h_1, ..., h_p$ have no common value. Since (6.6) also reads $\sigma_{k+1} = \mathcal{L}_H \sigma_k + \widetilde{\lambda}_k$ we obtain that $\lambda_k = \sigma_{k+1}(0, .)$ on V. Hence, $\lambda_1, ..., \lambda_p$ do not depend on z_* so that they are well defined holomorphic function on Δ.

2) Assume now (2) is true. We only have to check that $T = X^p - \sum\limits_{1 \leqslant k \leqslant p} s_k X^{p-k}$ is actually a p-multivaluate shock wave function. Formulas (6.5) also read $s_{k+1} = \mathcal{L}_H s_k + \widetilde{\lambda}_k$ for $1 \leqslant k \leqslant p-1$ and (6.3) means that $\mathcal{D}_H s_p = 0$. Hence $0 = s_{p,y} - H_x s_p = s_{p,y} - s_{1,x} s_p$ and $s_{k+1,x} = \mathcal{D}_H s_k = s_{k,y} - H_x s_k = s_{k,y} + s_{1,x} s_k$. So, if $z_* \in D$ is outside $\Sigma = \{\operatorname{Discr} T = 0\}$, lemma 16 implies that the holomorphic functions $h_1, ..., h_p$ which near z_* describe the roots of T_z are mutually distinct shock wave function. ∎

The following describes p-shock wave functions which are affine in x.

Proposition 18. *Let D be a simply connected domain of $\mathbb{C}^2$ containing 0 and $H : (x, y) \mapsto a(y) x + b(y)$ where a, b are holomorphic. Then, H is a p-shock wave function, if and only if there exists $Q_0, Q_1 \in \mathbb{C}_{p-1}[Y]$ such* *that*

$$a=\frac{Q_1}{1-Q_{1,y^{-1}}}\ \&\ b=\frac{Q_0}{1-Q_{1,y^{-1}}} \tag{6.7}$$

$$\operatorname{Discr}\left(1-Q_{1,y^{-1}}\right)\neq 0, \tag{6.8}$$

In addition, when (6.7) and (6.8) are satisfied, the decomposition of H in elementary fractions gives a decomposition of H in a sum of shock wave functions.

Proof. Assume $H=\sum\limits_{1\leqslant j\leqslant p}h_j$ where h_1, ...,h_p are mutually distinct shock wave functions ; with proposition 17's notations, $(-1)^1 s_1,...,(-1)^p s_p$ are the symmetric functions of h_1, ...,h_p and satisfy the relations $\mathcal{D}_H s_p=0$ and $s_{k+1}=\mathcal{L}_H s_k+\widetilde{\lambda}_k$, $1\leqslant k\leqslant p-1$.

Set $H_{x,y^{-1}}(x,y)=A(y)$ where $A=a_{y^{-1}}$; $\mathcal{D}_H=\frac{\partial}{\partial y}-a$ and if $y\mapsto u_0(y)$ is holomorphic

$$\mathcal{L}_H^k\frac{x^m}{m!}\otimes u_0=\frac{x^{k+m}}{(k+m)!}\otimes u_k$$

where (u_k) is defined by $u_{k+1}=u_k'-au_k$, $k\in\mathbb{N}$. If it happens that $u_{r+1}=0$ for some $r\in\mathbb{N}$, u_r satisfies the differential equation $u_r'-au_r=0$ and a straightforward induction gives the existence of $U\in\mathbb{C}_r[Y]$ such that $u_s=U^{(s)}e^{-A}$ for all s in $\{0,..,r\}$. Hence, one can find sequences $(\lambda_{j,k})_{1\leqslant j\leqslant p-1,\,k\in\mathbb{N}}$ and $(a_k)_{k\in\mathbb{N}}$ of holomorphic functions such that

$$\mathcal{L}_H^k\widetilde{\lambda}_j=\frac{x^k}{k!}\otimes\lambda_{j,k}\ ,\qquad \mathcal{L}_H^kH=\frac{x^{k+1}}{(k+1)!}\otimes a_k+\frac{x^k}{k!}\otimes b_k\ ,\ k\in\mathbb{N}$$

which all are solutions of the recurrence $u_{k+1}=u_k'-au_k$, $k\in\mathbb{N}$. Since $\mathcal{D}_H\mathcal{L}_H^k u=\frac{x^k}{k!}\otimes u_{k+1}$ for any k, (6.3) yields the vanishing of the x-polynomial

$$\frac{x^p}{p!}\otimes a_p+\frac{x^{p-1}}{(p-1)!}\otimes b_p-\sum_{1\leqslant j\leqslant p-1}\frac{x^{p-1-j}}{(p-1-j)!}\otimes\lambda_{j,p-j}$$

and hence ensures $a_p=b_p=\lambda_{1,p-1}=\lambda_{2,p-2}=\cdots=\lambda_{p-1,1}=0$. So, one can find $Q_1,Q_0\in\mathbb{C}_{p-1}[Y]$ and $\Lambda_j\in\mathbb{C}_{p-j-1}[Y]$, $1\leqslant j\leqslant p-1$, such that

$$a_k=Q_1^{(k)}e^A\ ,\ b_k=Q_0^{(k)}e^A,\ \lambda_{j,k}=\Lambda_j^{(k)}e^A,\ k\in\mathbb{N}.$$

This already guarantees (6.7) since $a=a_0=Q_1e^A$ implies $1-e^{-A}=Q_{1,y^{-1}}$.

Conversely, assume that (6.7) and (6.8) are true for some $Q_0,Q_1\in\mathbb{C}_{p-1}[Y]$. Then, the decomposition in elementary fraction of H is Σh_j where $h_j(x,y)=\frac{q_jx+c_j}{1-q_jy}$, $1\leqslant j\leqslant p$ and $q_1^{-1},...,q_p^{-1}$ are the roots of $1-Q_{1y^{-1}}$. It is quite evident that $h_1,...,h_p$ are mutually distinct shock wave functions. The proof is complete. ∎

Remark 1. It is worth noting that if $x\otimes a+1\otimes b$ is the sum of p-mutually distinct shock wave functions h_1, ...,h_p each h_j needs to be algebraic since for $1\leqslant k\leqslant p$ (6.5) yields

$$s_k(x,y)=\frac{S_k(x,y)}{1-Q_{1y^{-1}}(y)}$$

where S_k is the polynomial defined by

$$S_k = -\frac{x^k \otimes Q_1^{(k-1)}}{k!} - \frac{x^{k-1} \otimes Q_0^{(k-1)}}{(k-1)!} + \sum_{1 \leqslant j \leqslant k-1} \frac{x^{k-1-j} \otimes \Lambda_j^{(k-1-j)}}{(k-1-j)!}.$$

Remark 2. When Q_1 is allowed to be non generic, $1 - Q_{1y^{-1}}$ can only be written in the form

$$1 - Q_{1,y^{-1}} = \Pi_{1 \leqslant j \leqslant m} (1 - q_j y)^{\alpha_j}$$

with $\alpha_1, ..., \alpha_p \in \mathbb{N}^*$. Hence, if $Q_0 \in \mathbb{C}_{p-1}[Y]$ and (a, b) is defined by (6.7), there is constants $c_{j,\ell}$ such that $H = x \otimes a + 1 \otimes b$ admits the following decomposition :

$$H = \sum_{1 \leqslant j \leqslant m} \sum_{1 \leqslant \ell \leqslant \alpha_j} h_{j,\ell}$$

where $h_{j,\ell} = \frac{q_j x}{1 - q_j y} + \frac{c_{j,\ell}}{(1-q_j y)^{\ell}}$, $1 \leqslant j \leqslant m$, $1 \leqslant \ell \leqslant \alpha_j$. Each $h_{j,\ell}$ is now a *generalized shock wave function* in the sense it is a solution of the equation

$$h_y - h_x h = (\ell - 1) \kappa h_x^{\ell+1}$$

where $\kappa \in \mathbb{C}$ is equal to $c_{j,\ell}/q_j^{\ell}$.

Generalized shock wave functions arise when an affine function $H = x \otimes a + b$ has coefficient a and b given by (6.7) not constraint to (6.8). In that case, H is clearly a limit of p-shock wave functions. The lemma below prove the converse and so, shows that generalized shock wave functions occur naturally.

Lemma 19. *Let $\left(H^t\right)_{t \in T} = \left(x \otimes a^t + 1 \otimes b^t\right)_{t \in T}$ be a continuous family of holomorphic affine functions in a simply connected domain D such that the set T_{reg} of parameters t for which H_t is a p-shock wave function is dense in T, then there exists in $\mathbb{C}_{p-1}[Y]$ continuous family of holomorphic polynomials $\left(Q_1^t\right)$ and $\left(Q_0^t\right)$ such that for any $t \in T$, $a^t = Q_1^t \left(1 - Q_{1,y^{-1}}^t\right)^{-1}$ and $b^t = Q_{1,0}^t \left(1 - Q_{1,y^{-1}}^t\right)^{-1}$. In particular, H^t is a p-shock wave function when* Discr $\left(1 - Q_{1,y^{-1}}^t\right) \neq 0$ *and a p-generalized shock wave function when* Discr $\left(1 - Q_{1,y^{-1}}\right) = 0$.

Proof. For each $t \in T_{\mathrm{reg}}$, one can find $Q_1^t, Q_0^t \in \mathbb{C}_{p-1}[Y]$ such that $a^t = Q_1^t \left(1 - Q_{1,y^{-1}}^t\right)^{-1}$ and $b^t = Q_{1,0}^t \left(1 - Q_{1,y^{-1}}\right)^{-1}$. The formula $Q_1^t = a^t e^{-a^t_{y^{-1}}}$ implies that if $t \to t_* \in T$, $t \mapsto Q_1^t$ converges in $\mathcal{O}(D)$ and so in $\mathbb{C}_{p-1}[Y]$ to some polynomial Q_1^* ; hence $Q_0^t = H^t(1 - Q_{1,y^{-1}}^t) - x \otimes a^t$ also converges toward some $Q_0^* \in \mathbb{C}_{p-1}[Y]$ when $t \to t_*$. Hence, $a^{t_*} = Q_1^{t_*}(1 - Q_{1,y^{-1}}^{t_*})^{-1}$ and $b^{t_*} = Q_{1,0}^{t_*}(1 - Q_{1,y^{-1}})^{-1}$. ∎

6.1.1. *Proof of theorem 4.* **1)** Assume that there is an open Riemann surface $\mathcal{X}$ such that $\overline{\mathcal{X}} = \mathcal{X} \cup \gamma$ is a manifold with almost smooth boundary. Then, theorem 3a gives that almost all point $(\xi_{0*}, \xi_{1*}, 1)$ of $\mathbb{CP}_2$ has a neighborhood W_{ξ_*} for which one can find an integer p and p mutually distinct shock wave functions $h_1, ..., h_p$ on W_{ξ_*} such that $L = \Sigma h_j - G$ is affine in ξ_0.

Set $s_k = (-1)^k \sum_{1\leqslant j_1<\cdots<j_k\leqslant p} h_{j_1}...h_{j_k}$ and $T = X^p + \sum_{1\leqslant k\leqslant p} s_k X^{p-k}$. Then $\operatorname{Discr} T \not\equiv 0$ and proposition 17 implies that (1.9) with unknowns L and $\lambda_1, ..., \lambda_p$ have a solution.

Assume now that $\widetilde{p}$ is be the least integer q such that there is a q-multivaluate shock wave function whose trace differs from $G\Big|_{W_{\xi_*}}$ only by a ξ_0-affine function. Let $\widetilde{T}$ be a $\widetilde{p}$-multivaluate shock wave function with trace $\widetilde{H}$ such that $\widetilde{L} = \widetilde{H} - G\Big|_{W_{\xi_*}}$ is affine in ξ_0. Let $\widetilde{h}_1, ..., \widetilde{h}_{\widetilde{p}}$ be the holomorphic function on W_{ξ_*} which describes the roots of $\widetilde{T}$. Then $\left\{\widetilde{h}_1, ..., \widetilde{h}_{\widetilde{p}}\right\}$ is minimal in the sense of the fourth remark below theorem 3a.

When $\widetilde{p} \geqslant 1$, this remark says that $\mathcal{X}$ is a normalization of the analytic extension $\mathcal{Y}$ in $\mathbb{CP}_2 \backslash f(\gamma)$ of the union of the graphs of the functions $\left(1 : \widetilde{h}_j : -\xi_0 - \xi_1 \widetilde{h}_j\right)$, $1 \leqslant j \leqslant \widetilde{p}$, and for any $\xi \in W_{\xi_*}$, the intersection of $\mathcal{Y}$ with the projective lines $\xi_0 w_0 + \xi_1 w_1 + w_2 = 0$ is

$$\left\{\left(1 : \widetilde{h}_j(\xi) : \xi_0 - \xi_1 \widetilde{h}_j(\xi)\right) \ ; \ , 1 \leqslant j \leqslant \widetilde{p}\right\}.$$

Since by prop 10, $\mathcal{Y}$ is uniquely determined by (γ, f) each $\widetilde{h}_j$ and so each symmetric function $\widetilde{s}_k$ of $\widetilde{h}_1, ..., \widetilde{h}_k$ is also uniquely determined by (γ, f). If $\lambda_1, ..., \lambda_{\widetilde{p}}$ are any one variable holomorphic function such that assertion (2) of prop. 17 holds, with $\widetilde{H}$ instead of H, then $\lambda_{k-1}(\xi_1) = \widetilde{s}_k(0, \xi_1)$, $1 \leqslant k \leqslant p-1$. Hence, $\lambda_1, ..., \lambda_{\widetilde{p}}$ are uniquely determined by (γ, f).

Thanks to lemma 13, we know that L is affine in ξ_0 and is the limit of a continuous one parameter family of holomorphic ξ_0-affine functions which are all q-shock wave functions for the same integer q. Thanks to lemma 19, this implies that $L(\xi_0, \xi_1) = \xi_0 \frac{\alpha'(\xi_1)}{1-\alpha(\xi_1)} + \frac{\beta(\xi_1)}{1-\alpha(\xi_1)}$ where $\alpha \in \mathbb{C}_q[\xi_1]$ vanish at zero and $b \in \mathbb{C}_{q-1}[\xi_1]$.

2) Assume now that (1.9) with unknowns L and $\lambda_1, ..., \lambda_p$ have a solution. Set $T = X^p + \sum_{1\leqslant k\leqslant p} s_k X^{p-k}$ where now is defined by (6.5). Proposition 17 implies then that T is determines a p-multivaluate shock wave function whose trace is $G + L$. Since L is affine in ξ_0, G satisfies the hypothesis of theorem 3a.B. As G is not affine in ξ_0 by hypothesis, γ, with its given orientation, is the boundary of an open Riemann surface $\mathcal{X}$ with the sought properties. Hence, (4) has to be satisfied from the direct part of theorem 4.

6.1.2. *Theorem 4 when* $p = 2$. We write here a simpler form of the first condition of theorem 4 when the minimal number p which appears therein is equal to 2. In this case, the function G defined by (1.3) is expected to not be a shock wave function but the sum of two such functions. Using the variables (x, y) instead of (ξ_0, ξ_1) in order to simplify notations, the first condition of theorem 4 is equivalent to the existence in a neighborhood of $(0, 0)$ of holomorphic functions $L(x, y) = a(y)x + b(y)$ and $\widetilde{\lambda}(x, y) = \lambda(y)$ such that

$$\mathcal{D}_{G+L}\mathcal{L}_{G+L}(G + L) = \mathcal{D}_{G+L}\widetilde{\lambda} = \lambda' - (G_x + a)\lambda \tag{6.9}$$

When $x=0$, the left term of (6.9) becomes zero so that λ has to be of the form $c\,\mathrm{e}^{g_{x,y^{-1}}+a_{y^{-1}}}$ where $c\in\mathbb{C}$ and $g_{x,y^{-1}}=G_{x,y^{-1}}(0,.)$. Hence, $\lambda'-(G_x+a)\lambda=(g_x-G_x)\lambda$ where $g_x=G_x(0,.)$. Even if the condition (4) of theorem 4 is a consequence of conditions (1-2-3), it is convenient to use this a priori information. Hence, there is $\alpha_1,\alpha_2,\beta_0,\beta_1\in\mathbb{C}$ such that

$$a=\frac{\alpha'}{1-\alpha}\ \ \&\ \ b=\frac{\beta}{1-\alpha}.$$

where $\alpha=\alpha_1 y-\alpha_2 y^2$ and $\beta=\beta_0+\beta_1 y$. After elementary calculus, it appears that (6.9) is

$$\mathcal{D}_G\mathcal{L}_G G=cC+\alpha_1A_1+\alpha_2A_2+\beta_0B_0+\beta_1B_1 \tag{6.10}$$

where

$$\begin{aligned} C &= (G_x-g_x)\,\mathrm{e}^{g_{x,y^{-1}}} \\ A_1 &= y\mathcal{D}_G\mathcal{L}_G G+(x\mathcal{L}_G G)_x\ ,\ A_2=y^2\mathcal{D}_G\mathcal{L}_G G+\left(2xy\mathcal{L}_G G+x^2G\right)_x \\ B_0 &= \mathcal{D}_G G-\mathcal{D}_G g\ ,\ B_1=yB_0+g-G \end{aligned}$$

For minimal p equal to 2, theorem 4 says that γ bounds almost smoothly an open Riemann surface $\mathcal{X}$ where f extends meromorphically and G is, near $(0,0)$, if and only if (6.10) is valid for some constants $c,\alpha_1,\alpha_2,\beta_0,\beta_1$.

6.2. **Affine characterization.** The existence part of theorem 3a essentially describes a projective presentation of $\mathcal{X}$ since it recovers at first its image $\mathcal{Y}$ by the meromorphic extension F of f to $\mathcal{X}$. A characterization for an affine presentation is possible even if it is less efficient since some very special data have to be selected.

Assume γ bounds an open riemannian surface $\mathcal{X}$ of finite volume and denotes by j the canonical injection of γ into $\overline{\mathcal{X}}$. Note first that if $H\in\mathcal{O}(\mathcal{X})$, then $d(V\circ j)=j^*d^cU$ where $U=\operatorname{Re}H$ and $V=\operatorname{Im}H$. For $u=U\,|_\gamma$, we get $U=\widetilde{u}$ and a straightforward computation gives that $j^*d^cU=\frac{i}{2}(Nu)\ j^*\tau^*$. Since $\tau^*\in T^*\gamma$, we identify τ^* and $j^*\tau^*$, so u satisfies the obvious

$$\int_\gamma (Nu)\ \tau^*=0. \tag{6.11}$$

When (6.11) is true, $\frac{i}{2}(Nu)\,\tau^*$ has a primitive v on γ and the holomorphic extension to $\mathcal{X}$ of $h=u+iv$ is equivalent to the moments condition

$$\forall\Psi\in H^{1,0}(\overline{\mathcal{X}}),\ \int_\gamma h\Psi=0. \tag{6.12}$$

If now $H=(H_0,...,H_N)\in\mathcal{O}(\overline{\mathcal{X}})^{N+1}$ is such that $h=H\,|_\gamma$ embeds γ in $\mathbb{C}^{N+1}$, $\mathcal{Y}=H(\mathcal{X})\backslash h(\gamma)$ is a complex curve of $\mathbb{C}^{N+1}\backslash h(\gamma)$ which has finite volume and has boundary $h(\gamma)$. So γ has to satisfy the Harvey-Lawson-moment conditions

$$\forall k_0,...,k_N\in\mathbb{N},\ \int_\gamma h_0^{k_0}...h_N^{k_N}\,dh_0=0 \tag{6.13}$$

which by the way contains also (6.11).

Proposition 20. *Let $u_0, ..., u_N \in C^\infty(\gamma)$ be functions satisfying (6.11). Let $v_\ell \in C^\infty(\gamma)$ be a primitive of $\frac{i}{2}(Nu_\ell)\tau^*$, $0 \leqslant \ell \leqslant N$. Assume that $h = (u_\ell + iv_\ell)_{0 \leqslant \ell \leqslant N}$ is an embedding of γ in $\mathbb{C}^{N+1}$. Then (6.13) is a necessary condition to the existence of an open Riemann surface $\mathcal{X}$ such that $\overline{\mathcal{X}} = \mathcal{X} \cup \gamma$ is a manifold with almost smooth boundary such that each u_ℓ extends to $\mathcal{X}$ in a harmonic function with harmonic conjugate function. The converse is true when γ is connected and suitably oriented.*

Remarks. Of course, the above conclusion means that $\mathcal{X}$ is a Riemann surface where each h_ℓ extends holomorphically.

Proof. If $\mathcal{X}$ exists with the required properties, $\delta = h(\gamma)$ is in the sense of current the boundary of $h(\mathcal{X}) \backslash \delta$ which is a complex curve of finite volume of $\mathbb{C}^n \backslash \delta$. Cauchy theorem implies then that the moments condition (6.13) is verified. If this condition is satisfied, [18] produces a holomorphic 1-chain Y such that $dY = [\delta]$. Since γ is connected, we know that $Y = [\mathcal{Y}]$ for a suitable orientation of γ. A normalization of $\overline{\mathcal{Y}}$ constructed as in the proof of th. 3a gives a suitable $\mathcal{X}$. ∎

7. Main hypothesis, proof of prop. 0

Under the hypothesis of proposition 0, the set of values for which f_1 has several preimages is finite and each of these values has a finite number of preimages. For our aim, it is thus sufficient to modify $f_2 = (Lu_2)/(Lu_0)$ in a finite number of specified points. This can be derived from general theory but we give here an explicit construction more in the spirit of numerical application. For the sake of the simplicity of its writing, it is given when exactly two points p_1 and p_2 of γ have the same value by f.

Thus, $(Nu_2(p_1), Nu_2(p_2))$ is a solution for

$$S(u_2): b_2X - b_1Y = +\Theta_a \quad \& \quad a_2X - a_1Y = -\Theta_b u_2$$

where $b_\alpha = Nu_0(p_\alpha)$, $a_\alpha = Tu_0(p_\alpha)$ $(\alpha = 1, 2)$ and $\Theta_c u_2 = c_2 Tu_2(p_1) - c_1 Tu_2(p_2)$. Since $N : C^2(\gamma) \to C^0(\gamma)$ is continuous and $C^\omega(\gamma)$ dense in $C^2(\gamma)$, it is sufficient for our goal to construct $v_2 \in C^2(\gamma)$ arbitrarily close to u_2 in $C^2(\gamma)$ with same tangential derivatives at p_1 and p_2 but for which $\Phi = v_2 - u_2$, $(N\Phi(p_1), N\Phi(p_2))$ does not solve $S(0)$.

Fix δ in $]0,1[$ and choose for $\alpha \in \{1,2\}$ an odd function $\xi_\alpha \in C_0^2([-\delta,\delta], \mathbb{R})$ such that $h_\alpha = \xi'_\alpha(0) > 0$; set $m_\alpha = \max|\xi''_\alpha|$. Let $\varphi : \gamma \to \mathbb{R}$ be the function equal to $\xi_1 \circ \zeta_1$ on U_1, to $\xi_2 \circ \zeta_2$ on U_2 and zero otherwise ; φ has a primitive Φ which vanishes outside $\zeta_1^{-1}([-\delta, +\delta]) \cup \zeta_2^{-1}([-\delta, +\delta])$.

Let g be a Green function for $\mathcal{X}$ (g is a smooth symmetric function on $\mathcal{X}^2$ outside its diagonal Δ, for each $z \in \mathcal{X}$, $g(z,.)$ is harmonic in $\mathcal{X} \backslash \{z\}$, has singularity $\frac{1}{2\pi} \ln \operatorname{dist}(z,.)$ and is zero on $\gamma = b\mathcal{X}$). If $p_0 \in \overline{X}$, we let ω_{p_0} be the (well defined) global 1-form $\omega_{p_0} = d(g_{p_0} + ih_{p_0})$ where h_{p_0} denote a local harmonic conjugate of $g_{p_0} = g(., p_0)$. Then ω is smooth outside the diagonal Δ of $\overline{X} \times \overline{X}$, has a singularity or order 1 on Δ and the harmonic extension $\widetilde{u}$ of $u \in C^0(\gamma)$ to X is given by

$$\widetilde{u}(p_0) = \int_\gamma u(p)\,\omega_{p_0}(p)\,. \tag{7.1}$$

Let (U_1, ζ_1) and (U_2, ζ_2) be disjoint $\overline{X}$-chart centered respectively in p_1 and p_2 such that for open set V_α of $\mathbb{C}$, $\zeta_\alpha(U_\alpha) = \{\mathrm{Im} \geqslant 0\} \cap V_\alpha \overset{déf}{=} U_\alpha$ and $\zeta_\alpha(\gamma \cap U_\alpha) = \{\mathrm{Im} = 0\} \cap V_\alpha =]-1, 1[$. If $p_0 \in U_\alpha$ and $z_0 = \zeta(p_0)$, it follows from the properties of the Green function that in $U_\alpha \backslash \{z_0\}$

$$(7.2) \qquad \left(g_{p_0} \circ \zeta_\alpha^{-1}\right)(z, z_0) = \ln\left|\frac{z - \overline{z_0}}{z - z_0}\right| + R_\alpha(z, z_0)$$

where R_α is harmonic in each variables. When U_α is a half-disc, a case to which it is always possible to go back, (7.2) implies

$$(7.3) \qquad (\zeta_\alpha)_* \omega_{p_0} = \left[\frac{1}{z - \overline{z_0}} - \frac{1}{z - z_0} + \eta_\alpha(z, z_0)\right] dz$$

where η_α is harmonic in each variable. Shrinking Γ if necessary, we assume that Γ is a union of smooth level sets of ρ so the outward normal unit vector ν_{p_0} to $\{\rho < \rho(p_0)\}$ is well defined for every p_0 in Γ. Since ζ_1 is holomorphic, (7.1) and (7.2) imply that for $p_0 \in U_1$ and $z_0 = \zeta_1(p_0)$,

$$(7.4) \quad \begin{aligned} \frac{\partial \widetilde{\Phi}}{\partial \nu_{p_0}}(p_0) \quad = \quad & \int_{-\delta}^{+\delta} \left[\frac{\Xi_1(x)}{(x - x_0 + iy_0)^2} + \frac{\Xi_1(x)}{(x - x_0 - iy_0)^2}\right] dx + \\ & + i\int_{-\delta}^{+\delta} \Xi_1(x) \frac{\partial \eta_1}{\partial y_0}(x, p_0) + i\int_{-\delta}^{+\delta} \Xi_2 \frac{\partial}{\partial y_0} (\zeta_2)_* \omega_{p_0} \end{aligned}$$

where $\Xi_\alpha = \Phi \circ \zeta_\alpha$, $\alpha = 1, 2$. After an integration by parts in the first integral of (7.4) and having taken limit when $U_1 \ni p_0 \to p_1$, we get

(7.5)

$$N\Phi(p_1) = PV\int_{-\delta}^{+\delta} \frac{\xi_1(x)}{x} dx + \int_{-\delta}^{+\delta} \Xi_1(x) \frac{\partial \eta_1}{\partial y_0}(x, p_1) + i\int_{-\delta}^{+\delta} \Xi_2 \frac{\partial}{\partial y_0} (\zeta_2)_* \omega_{p_0}$$

where PV stand for principal value. The inequality $\left|\xi_1(x) - x\xi_1'(0)\right| \leqslant m_1 x^2$ implies

$$PV\int_{-\delta}^{+\delta} \frac{\xi_1(x)}{x} dx = 2\delta h_1 + \delta^2 m_1 O(1)$$

with $|O(1)| \leqslant 1$. The same inequality also implies that $|\Xi(x)| \leqslant \delta^2 h_1 + m_1 \delta^3$. Hence, the module of the second integral of (7.5) is less than $\delta^2 h_1 + m_1 \delta^3$ up to a multiplicative constant depending only of $(\gamma, \mathcal{U}_1, \mathcal{U}_2)$. The third integral of (7.5) is less than $\delta^2 h_2 + m_2 \delta^3$ up to the same kind of multiplicative constant. Set $m_\alpha \delta \leqslant h_2$ $(\alpha = 1, 2)$ as an additional condition. Then with similar calculus but for p_2, we get finally

$$\left\{\begin{array}{l} b_2 N\Phi(p_1) - b_1 N\Phi(p_2) = 2\delta\left[b_2 h_1 - b_1 h_2\right] + (h_1 + h_2)\delta^2 O(1) \\ a_2 N\Phi(p_1) - a_1 N\Phi(p_2) = 2\delta\left[a_2 h_1 - a_1 h_2\right] + (h_1 + h_2)\delta^2 O(1) \end{array}\right. .$$

where $|O(1)|$ are less than constants depending only of $(\gamma, \mathcal{U}_1, \mathcal{U}_2)$. Since $(a_1, b_1) \neq (0, 0)$, it is thus possible to choose (h_1, h_2, δ) arbitrarily small so that at least one of the left members of the above equalities is non zero. $\square$

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