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A NEW PROOF OF JAMES' SUP THEOREM

MARIANNE MORILLON

Abstract. We provide a new proof of James' sup theorem for (non necessarily separable) Banach spaces. One of the ingredients is the following generalization of a theorem of Hagler and Johnson ([5]) : *"If a normed space E does not contain any asymptotically isometric copy of $\ell^1(\mathbb{N})$, then every bounded sequence of E' has a normalized block sequence pointwise converging to 0".*

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1. Preliminaries

1.1. **Introduction.** Given a normed space E, we denote by Γ_E its closed unit ball, by E' the continuous dual of E (endowed with its dual norm), and by E'' the second continuous dual of E. Say that the normed space E is *onto-reflexive* if the canonical mapping $j_E : E \to E''$ is onto. Say that E is *J-reflexive* (James-reflexive) if Γ_E does not contain any sequence $(a_n)_{n\in\mathbb{N}}$ satisfying

$$\inf_{n\in\mathbb{N}} distance(span\{a_i : \ i < n\}, conv\{a_i : i \geq n\}) > 0$$

Say that E is *sup-reflexive* if every $f \in E'$ attains its upper bound on Γ_E. It is known that, with the Axiom of Choice, for a given Banach space the following notions are equivalent : onto-reflexivity, weak compactness of the closed unit ball, J-reflexivity, sup-reflexivity, Eberlein-Smulyan property ("Every bounded sequence has a weakly convergent subsequence") ... Both implications "J-reflexivity $\Rightarrow$ onto-reflexivity" and "sup-reflexivity $\Rightarrow$ J-reflexivity" are due to James ; the first implication has a recent short proof (see [14]), but classical proofs of the "reflexivity" of sup-reflexive Banach spaces (James' sup theorem) are rather intricate (see [9], [10], [16]). There exist simpler proofs of this theorem *under some restrictions* : for example, there is a short proof, relying on Simons' inequality, that *separable* sup-reflexive Banach spaces are onto-reflexive (see [4]). In this paper, we provide a new proof of James' sup theorem.

1.2. **Presentation of the results.** We work in Zermelo-Fraenkel set-theory without choice **ZF**, and expressly mention the two (weak) forms of the Axiom of Choice (**AC**) that are used in our proofs, namely : the axiom of *Hahn-Banach* (**HB**) and the axiom of *Dependent Choices* (**DC**) (see Section 5).

Given a real vector space E and a sequence $(x_n)_{n\in\mathbb{N}}$ of E, a sequence $(b_n)_{n\in\mathbb{N}}$ of E is a *block sequence* of $(x_n)_n$ if there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of pairwise disjoint finite subsets of $\mathbb{N}$ and a sequence $(\lambda_i)_{i\in\mathbb{N}}$ of real numbers such that for every $n \in \mathbb{N}$, $b_n = \sum_{i\in F_n} \lambda_i x_i$; if for each $n \in \mathbb{N}$, $\sum_{i\in F_n} |\lambda_i| = 1$, the block sequence $(b_n)_{n\in\mathbb{N}}$ is said to be *normalized* ; if, in addition, for every $i \in \cup_{n\in\mathbb{N}} F_n$, $\lambda_i \geq 0$, say that the block sequence is *convex* : thus, every infinite subsequence is a convex block sequence. The sequence $(F_n)_n$ is called a sequence of *supports* of the block sequence $(b_n)_n$. Say that a topological space is *sequentially compact* if every sequence of this space has an infinite subsequence which converges. Say that a subset C of a topological vector space E is *block compact* if every sequence of C has a normalized block sequence $(b_n)_{n\in\mathbb{N}}$ which converges in E (hence the normalized block sequence $(\frac{b_{2n}-b_{2n+1}}{2})_{n\in\mathbb{N}}$ converges to 0). Say that C is *convex block compact* if every sequence of C has a convex block sequence which converges in E.

Using Simons' inequality, we begin by proving in **ZF+HB** that *every sup-reflexive normed space with a *weak convex block compact dual ball is*

J-reflexive (Corollary 2-1 of Section 2.1) ; in particular, this yields a new proof (indeed in **ZF**) of the J-reflexivity of sup-reflexive separable normed spaces (Corollary 2-2). Then, in Section 2.2, using Simons' inequality and Rosenthal's ℓ^1-theorem, we prove the following "generalization" :

Theorem 1 (**HB**). *Given a sup-reflexive normed space E, if $\Gamma_{E'}$ is *weak block compact, then E is J-reflexive.*

We then prove (Section 3) :

Theorem 2 (**DC**). *If a normed space E does not contain any asymptotically isometric copy of $\ell^1(\mathbb{N})$, then, $\Gamma_{E'}$ is *weak block compact.*

Here, say that E contains an *asymptotically isometric copy of* $\ell^1(\mathbb{N})$ if there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of Γ_E and some sequence $(\delta_n)_{n\in\mathbb{N}}$ of $]0,1[$ converging to 1 satisfying the following inequality for every finite sequence $(\lambda_i)_{0\leq i\leq n}$ of $\mathbb{R}$:

$$\sum_{0\leq i\leq n} \delta_i|\lambda_i| \leq \left\| \sum_{0\leq i\leq n} \lambda_i a_i \right\|$$

Theorem 2 generalizes a result due to Hagler and Johnson ([5], 1977), where the normed space contains no *isomorphic* copy of $\ell^1(\mathbb{N})$.

We finally prove (Section 4) :

Theorem 3 (**HB**). *A sup-reflexive normed space does not contain any asymptotically isometric copy of $\ell^1(\mathbb{N})$.*

This result is a straightforward generalization of a short theorem due to James (see [8, Theorem 2 p. 209]).

Thus we get the following new proof of James' sup theorem :

Corollary 1 (**HB+DC**). *Every sup-reflexive normed space is J-reflexive.*

Proof. Let E be a sup-reflexive normed space. According to Theorem 3, the space E does not contain any asymptotically isometric copy of $\ell^1(\mathbb{N})$; so, with Theorem 2, $\Gamma_{E'}$ is *weak block compact ; whence, by Theorem 1, E is J-reflexive. □

1.3. **Questions.** Given a normed space E, obviously,

$\Gamma_{E'}$ *weak sequentially compact $\Rightarrow$

$\Gamma_{E'}$ *weak convex block compact $\Rightarrow \Gamma_{E'}$ *weak block compact

The first implication is not reversible in **ZFC** since there exists a Banach space E such that $\Gamma_{E'}$ endowed with the *weak topology is convex block compact and not sequentially compact (see [6]) : notice that the construction of the space built there depends on a well-order on $\mathbb{R}$.

Question 1. Does there exist in **ZF** a Banach space E such that $\Gamma_{E'}$ is *weak convex block compact and not *weak sequentially compact ?

Question 2. Does there exist (in **ZF** or **ZFC**) a Banach space E such that $\Gamma_{E'}$ is *weak block compact and not *weak convex block compact ?

Question 3. According to a theorem of Valdivia (see [18]) for locally convex spaces, who refers to Bourgain and Diestel (see [2]) for Banach spaces, themselves referring to Bourgain-Fremlin-Talagrand (see [3]), **DC** implies that *"the dual ball of a normed space not containing any isomorphic copy of $\ell^1(\mathbb{N})$ is *weak convex block compact"*. Does this result persist (in **ZF** or **ZFC**) for normed spaces which do not contain asymptotically isometric copies of $\ell^1(\mathbb{N})$?

2. Spaces with a *weak block compact dual ball

2.1. Spaces with a *weak convex block compact dual ball.

Theorem (**Simons' inequality** [17]). *Let S be a set and $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence of $\ell^\infty(S)$. Denote by Λ the set of sequences $(\lambda_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ satisfying $\sum_{n\in\mathbb{N}} \lambda_n = 1$. Assume that for every $(\lambda_n)_{n\in\mathbb{N}} \in \Lambda$, the infinite convex combination $\sum_{n\in\mathbb{N}} \lambda_n x_n$ attains its upper bound on S. Then,*

$$\inf\{\sup_S \sum_{n\in\mathbb{N}} \lambda_n x_n \ : \ (\lambda_n)_{n\in\mathbb{N}} \in \Lambda\} \leq \sup_S \limsup_{n\in\mathbb{N}} x_n$$

Notice that the proof given in [17] (see also [13]) is choiceless : use convex combinations with finite supports and rational coefficients.

Given a normed space E, and some real number $\vartheta > 0$, say that a sequence $(a_n)_{n\in\mathbb{N}}$ of E is a *ϑ-sequence* if $\inf_{n\in\mathbb{N}} d(span\{a_i : \ i < n\}, conv\{a_i : i \geq n\}) \geq \vartheta$. Given a ϑ-sequence $(a_n)_{n\in\mathbb{N}}$ of Γ_E, and denoting by V the vector space $span(\{a_n : n \in \mathbb{N}\})$, there is a sequence $(f_n)_{n\in\mathbb{N}}$ of $\Gamma_{V'}$ satisfying $f_n(a_i) = 0$ if $i < n$ and $f_n(a_i) \geq \vartheta$ if $n \leq i$: in this case, say that $(a_n, f_n)_{n\in\mathbb{N}}$ is a *ϑ-triangular* sequence of E. Using **HB** (or rather its "multiple form", see Section 5), there exists a sequence $(\tilde{f}_n)_{n\in\mathbb{N}}$ of $\Gamma_{E'}$ such that each $\tilde{f}_n$ extends f_n : in this case, say that $(a_n, f_n)_{n\in\mathbb{N}}$ is an *extended* triangular sequence of E. Thus, a normed space E is J-reflexive if and only if it has no ϑ-triangular sequence for any $\vartheta > 0$.

Lemma 1. *Given a sup-reflexive space E, some $\vartheta > 0$, and some extended ϑ-triangular sequence $(a_n, f_n)_{n\in\mathbb{N}}$ of E, no convex block sequence of $(f_n)_{n\in\mathbb{N}}$ pointwise converges.*

Proof. Seeking a contradiction, assume that some convex block sequence $(b_n)_{n\in\mathbb{N}}$ pointwise converges to some f. Without loss of generality, we may assume that the sequence of supports $(F_n)_{n\in\mathbb{N}}$ satisfies $F_0 < F_1 < \cdots < F_n < \ldots$. Observe that for every $n \in \mathbb{N}$, $f(a_n) = 0$. Then denoting by h_n the mapping $\frac{b_n - f}{2}$, and by d_n the last element of F_n, the sequence $(a_{d_n}, h_n)_{n\in\mathbb{N}}$ is $\frac{\vartheta}{2}$-triangular. Using Simons' inequality and the assumption of sup-reflexivity of E, there exists some finite convex combination $g :=$

$\sum_{n\in F}\lambda_n h_n$ of $(h_n)_{n\in\mathbb{N}}$ such that $\sup_{\Gamma_E} g \leq \frac{\vartheta}{4}$; but for any integer $N > \max F$, $g(a_N) = \sum_{i\in F}\lambda_i h_i(a_N) \geq \frac{\vartheta}{2}$: contradiction ! □

Corollary 2. *Let E be a sup-reflexive normed space.*

(1) *If $\Gamma_{E'}$ is *weak convex block compact, then E has no extended triangular sequence, in which case, with* **HB**, *E is J-reflexive.*
(2) *If E is separable, then E is J-reflexive.*

Proof. Point 1. reformulates Lemma 1. For Point 2 : if E is separable, its dual ball, which is homeomorphic with a closed subset of $[-1,1]^{\mathbb{N}}$, is *weak sequentially compact ; so with Point 1, E has no extended triangular sequence. Besides the "multiple" version of **HB** is provable in **ZF** for separable normed spaces, hence E has no triangular sequences either, whence it is J-reflexive. □

Remark 1. Using Valdivia's theorem (see Question 3), it follows from **DC** that *Sup-reflexive spaces not containing isomorphic copies of $\ell^1(\mathbb{N})$ do not have extended triangular sequences.*

2.2. **Spaces with *weak block compact dual ball.** Say that a bounded sequence $(f_n)_{n\in\mathbb{N}}$ of a normed space E is *equivalent to the canonical basis of $\ell^1(\mathbb{N})$* if there exists some real number $M > 0$ satisfying $M\sum_{n\in\mathbb{N}}|\lambda_n| \leq \left\|\sum_{n\in\mathbb{N}}\lambda_n f_n\right\|$ for every $(\lambda_n)_n \in \ell^1(\mathbb{N})$: if in addition, $\forall n\in\mathbb{N}\ \|f_n\| \leq 1$, say that $(f_n)_{n\in\mathbb{N}}$ is *M-equivalent* to the canonical basis of $\ell^1(\mathbb{N})$.

Theorem (Rosenthal's ℓ^1-theorem). *Given a set X and a bounded sequence $(f_n)_{n\in\mathbb{N}}$ of $\ell^\infty(X)$, there exists a subsequence of $(f_n)_{n\in\mathbb{N}}$ which pointwise converges, or there exists a subsequence which is equivalent to the canonical basis of $\ell^1(\mathbb{N})$.*

Rosenthal's Theorem is a choiceless consequence (see Kechris, [11, p.135-136]) of the following choiceless result (see for example Avigad, [1], 1996) :

Theorem (Cohen, Ehrenfeucht, Galvin (1967)). *Every open subset of $[\mathbb{N}]^{\mathbb{N}}$ (the set of infinite subsets of $\mathbb{N}$ endowed with the product topology) is Ramsey.*

Proof of Theorem 1. Seeking a contradiction, assume that some sup-reflexive normed space E is not J-reflexive, though $\Gamma_{E'}$ is *weak block compact. Non J-reflexivity of E yields some ϑ-triangular sequence $(a_n, f_n)_{n\in\mathbb{N}}$ with $\vartheta > 0$. Using **HB**, extend each f_n to some $\tilde{f}_n \in \Gamma_{E'}$. Then, with Lemma 1, no infinite subsequence of $(\tilde{f}_n)_{n\in\mathbb{N}}$ pointwise converges, so, using Rosenthal's ℓ^1-theorem, there exists some infinite subsequence $(\tilde{f}_n)_{n\in A}$ and some $M > 0$ such that the bounded sequence $(\tilde{f}_n)_{n\in A}$ is M-equivalent to the canonical basis of $\ell^1(\mathbb{N})$. Now, by *weak block compactness of $\Gamma_{E'}$, $(\tilde{f}_n)_{n\in A}$ has a normalized block sequence $(b_n)_{n\in\mathbb{N}}$ *weakly converging to 0. Using Simons'

inequality, there exists some finite convex combination $g := \sum_{i\in F} \lambda_i b_i$ of $(b_n)_{n\in\mathbb{N}}$ such that $\|g\| = \sup_{\Gamma_E} g \leq \frac{M}{2}$; but, since the block sequence $(b_n)_{n\in\mathbb{N}}$ is normalized, it is also M-equivalent to the canonical basis of $\ell^1(\mathbb{N})$, hence $\|g\| \geq M \sum_{i\in F} |\lambda_i| = M$: the contradiction ! $\square$

3. Extension of a theorem by Hagler and Johnson

Notation 1 ([5])**.** If $(b_n)_{n\in\mathbb{N}}$ is a normalized block sequence of a sequence $(x_n)_{n\in\mathbb{N}}$ of a real vector space, we write $(b_n)_n \prec (x_n)_n$.
Given a set X, for every bounded sequence $(f_n)_{n\in\mathbb{N}}$ of $\ell^\infty(X)$, and every subset K of X, let

$$\delta_K(f_n)_n := \sup_K \limsup f_n$$

$$\varepsilon_K(f_n)_n := \inf\{\delta_K(h_n)_n : \ (h_n)_n \prec (f_n)_n\}$$

Remark 2. If for every $n \in \mathbb{N}$, $f_n[K] = f_n[-K]$, then $\delta_K(f_n)_n = 0$ if and only if $(f_n)_n$ pointwise converges to 0 on K. Observe that if for some $n_0 \in \mathbb{N}$, $(h_n)_{n\geq n_0}$ is a normalized block sequence of $(f_n)_n$, then $\delta_K(h_n) \leq \delta_K(f_n)$ and $\varepsilon_K(f_n)_n \leq \varepsilon_K(h_n)$.

Given a metric space (X, d), for every $x \in X$ and every real number $r > 0$ we denote by $B(x, r)$ the open ball $\{y \in X : \ d(x, y) < r\}$.

Lemma 2 (**quantifier permuting**)**.** *Let (K, d) be a precompact metric space, $\lambda \in \mathbb{R}_+^*$, and $(f_n)_{n\in\mathbb{N}}$ be a sequence of λ-Lipschitz real mappings on K. If $\delta_K(f_n)_n \leq 1$ then, for every $\varepsilon \in \mathbb{R}_+^*$, there exists $N \in \mathbb{N}$ satisfying $\forall n \geq N$ $sup_K f_n \leq 1 + \varepsilon$.*

Proof. Let $\eta \in]0, \varepsilon[$. Given some $x \in K$, there exists some finite subset F_x of $\mathbb{N}$ satisfying $\forall n \in \mathbb{N}\backslash F_x$ $f_n(x) < 1 + \eta$; thus, denoting by ρ the positive number $\frac{\varepsilon-\eta}{\lambda}$, for every $n \in \mathbb{N}\backslash F_x$, for every $y \in B(x, \rho)$, $f_n(y) < 1 + \varepsilon$. Now the precompact set K is contained in a finite union of the form $\bigcup_{1\leq i\leq N} B(x_k, \rho)$. Let F be the finite set $\bigcup_{1\leq k\leq n} F_{x_k}$. Then, for every $y \in K$, given $k \in \{1..n\}$ such that $d(x_k, y) < \rho$, for every $i \in \mathbb{N}\backslash F$, $f_i(y) < f_i(x_k) + \lambda\rho \leq 1 + \eta + (\varepsilon - \eta) = 1 + \varepsilon$. $\square$

Lemma 3 ([5, proof of Theorem 1])**.** *Given a set X and a bounded sequence $(f_n)_{n\in\mathbb{N}}$ of $\ell^\infty(X)$, there exists a normalized block sequence $(b_n)_{n\in\mathbb{N}}$ of $(f_n)_n$ such that $\varepsilon_X(b_n)_n = \delta_X(b_n)_n$.*

Proof. Diagonalization. Choose some normalized block sequence $(h_n^0)_{n\in\mathbb{N}}$ of $(f_n)_n$ such that $\delta_X(h_n^0)_n \leq \varepsilon_X(f_n)_n + \frac{1}{2^0}$, and then, for every $i \in \mathbb{N}$, inductively choose some normalized block sequence $(h_n^{i+1})_{n\in\mathbb{N}}$ of $(h_n^i)_n$ such that $\delta_X(h_n^{i+1})_n \leq \varepsilon_X(h_n^i)_n + \frac{1}{2^{i+1}}$. For every $n \in \mathbb{N}$, let $b_n := h_n^n$: then $(b_n)_n \prec (f_n)_n$; moreover, given a normalized block sequence $(k_n)_{n\in\mathbb{N}}$ of

$(b_n)_n$, for every $i \in \mathbb{N}$, $\delta_X(b_n)_n \leq \delta_X(h_n^{i+1})_n \leq \varepsilon_X(h_n^i)_n + \frac{1}{2^{i+1}} \leq \delta_X(k_n)_n + \frac{1}{2^{i+1}}$ whence $\delta_X(b_n)_n \leq \varepsilon_X(b_n)_n$. $\square$

Lemma 3 is valid in **ZF** : no choice is needed here since block can be built with rational coefficients.

Notation 2. For every integer $n \in \mathbb{N}$, we denote by $\mathcal{S}_n$ the set $\{0,1\}^n$ of finite sequences of $\{0,1\}$ with length n ; let $\mathcal{S}$ be the set of all finite sequences of $\{0,1\}$. Given an infinite subset A of $\mathbb{N}$, we denote by $i \mapsto i_A$ the increasing mapping from $\mathbb{N}$ onto A.

Say that a family $(A_\sigma)_{\sigma\in\mathcal{S}}$ of infinite subsets of $\mathbb{N}$ is a *tree* (of subsets of $\mathbb{N}$) if for every $\sigma \in \mathcal{S}$, $A_{\sigma\frown 0}$ and $A_{\sigma\frown 1}$ are disjoint subsets of A_σ.

Proof of Theorem 2. We essentially follow the proof of Hagler and Johnson, extending it with the help of Lemma 2. Assuming that E is a normed space, and that $(g_n)_{n\in\mathbb{N}}$ is a bounded sequence of E' without any normalized block sequence pointwise converging to 0, we are to show that E contains an *asymptotically isometric* copy of $\ell^1(\mathbb{N})$. Using Lemma 3, the sequence $(g_n)_{n\in\mathbb{N}}$ has a normalized block sequence $(f_n)_{n\in\mathbb{N}}$ satisfying $\varepsilon_{\Gamma_E}(f_n)_n = \delta_{\Gamma_E}(f_n)_n > 0$. Dividing each f_n by $\varepsilon_{\Gamma_E}(f_n)_n$, we may assume that $\varepsilon_{\Gamma_E}(f_n)_n = \delta_{\Gamma_E}(f_n)_n = 1$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of $]0,1/3[$ decreasing to 0 ; for every $n \in \mathbb{N}$, let $\varepsilon_n := \frac{u_n}{2^n}$ and let $\delta_n := 1 - \varepsilon_n$; thus $(\varepsilon_n)_{n\in\mathbb{N}}$ also decreases to 0. Using **DC**, we will build a tree $(A_\sigma)_{\sigma\in\mathcal{S}}$ rooted at $A_\varnothing := \mathbb{N}$, and a sequence $(\omega_n)_{n\geq 1}$ of Γ_E satisfying for every $n \geq 1$, $\sigma = (\alpha_1, \ldots, \alpha_n) \in \mathcal{S}_n$, and $i \in A_\sigma$,

$$\|f_i\restriction span\{\omega_1,\ldots,\omega_n\}\| \leq 1+\varepsilon_n \text{ and } \begin{cases} f_i(\omega_n) \geq 1-3u_n & \text{if } \alpha_n = 1 \\ f_i(\omega_n) \leq -1+3u_n & \text{if } \alpha_n = 0 \end{cases}$$

Then, with

$$P_n := \{f \in E' : f(\omega_n) \geq 1-3u_n \text{ and } \|f\restriction span\{\omega_1,\ldots,\omega_n\}\| \leq 1+\varepsilon_n\}$$
$$Q_n := \{f \in E' : f(\omega_n) \leq -1+3u_n \text{ and } \|f\restriction span\{\omega_1,\ldots,\omega_n\}\| \leq 1+\varepsilon_n\}$$

it will follow that $(P_n, Q_n)_{n\geq 1}$ is *independent* (for every disjoint finite subsets F, G of $\mathbb{N}\backslash\{0\}$, $\bigcap_{n\in F} P_n \cap \bigcap_{n\in G} Q_n$ is non-empty), whence the sequence $(\omega_n)_{n\geq 1}$ of Γ_E is asymptotically isometric to the canonical basis of $\ell^1(\mathbb{N})$: indeed, given real numbers $\lambda_1, \ldots, \lambda_n$, letting $f \in \bigcap_{\{i:\lambda_i>0\}} P_i \cap \bigcap_{\{i:\lambda_i<0\}} Q_i$,

$$\begin{aligned}\|f\restriction span\{\omega_1,\ldots,\omega_n\}\| \left\| \sum_{1\leq i\leq n} \lambda_i\omega_i \right\| &\geq f\big(\sum_{1\leq i\leq n} \lambda_i\omega_i\big) \\ &\geq \big(\sum_{\{i:\lambda_i>0\}} \lambda_i(1-3u_i) + \sum_{\{i:\lambda_i<0\}} \lambda_i(-1+3u_i)\big) \geq \sum_{1\leq i\leq n} |\lambda_i|(1-3u_i)\end{aligned}$$

whence $\left\|\sum_{1\leq i\leq n}\lambda_i\omega_i\right\| \geq \frac{1}{1+\varepsilon_n}\sum_{1\leq i\leq n}|\lambda_i|(1-3u_i) \geq \sum_{1\leq i\leq n}|\lambda_i|\frac{1-3u_i}{1+\varepsilon_i}$, with the sequence $(\frac{1-3u_i}{1+\varepsilon_i})_{i\in\mathbb{N}}$ of $]0,1[$ converging to 1.

Building ω_{n+1} and $(A_\sigma)_{\sigma\in\mathcal{S}_{n+1}}$ from $(A_\sigma)_{\sigma\in\mathcal{S}_n}$ and $(\omega_i)_{1\leq i\leq n}$. Given, for every $\sigma\in\mathcal{S}_n$, two infinite disjoint subsets L_σ and R_σ of A_σ, consider the normalized block sequence $(h_i^n)_{i\in\mathbb{N}}$ with $h_i^n := \frac{1}{2^n}\sum_{\sigma\in\mathcal{S}_n}\frac{f_{i_{R_\sigma}}-f_{i_{L_\sigma}}}{2}$. Since $\delta_{\Gamma_E}(h_i^n)_i \geq 1$, there is some $\omega_{n+1}\in\Gamma_E$ satisfying $\limsup_i h_i^n(\omega_{n+1}) > \delta_{n+1}$ and in particular, the set $J := \{i\in\mathbb{N} : h_i^n(\omega_{n+1}) > \delta_{n+1}\}$ is infinite. Since the closed unit ball K of the finite dimensional space $span\{\omega_1,\ldots,\omega_{n+1}\}$ is compact and $\delta_K(f_i)_i \leq \delta_{\Gamma_E}(f_i)_i \leq 1$, Lemma 2 implies the existence of some $N_0\in\mathbb{N}$ satisfying $\forall i\geq N_0\ \|f_i\restriction span\{\omega_1,\ldots,\omega_{n+1}\}\| \leq 1+\varepsilon_{n+1}$. Let $J' := \{i\in J : i\geq N_0\}$. Now, given any $\sigma\in\mathcal{S}_n$, for every $i\in J'$, notice that, below, $i_{R_\tau}, i_{L_\tau} \geq i \geq N_0$:

$$\begin{aligned}\frac{f_{i_{R_\sigma}}(\omega_{n+1})-f_{i_{L_\sigma}}(\omega_{n+1})}{2} &= 2^n h_i^n(\omega_{n+1}) - \sum_{\tau\in\mathcal{S}_n,\tau\neq\sigma}\frac{f_{i_{R_\tau}}(\omega_{n+1})-f_{i_{L_\tau}}(\omega_{n+1})}{2}\\ &\geq 2^n\delta_{n+1} - (2^n-1)(1+\varepsilon_{n+1})\\ &= 2^n(1-\varepsilon_{n+1}) - (2^n-1)(1+\varepsilon_{n+1})\\ &= 1-(2^{n+1}-1)\varepsilon_{n+1} \geq 1-2^{n+1}\varepsilon_{n+1} = 1-u_{n+1}\end{aligned}$$

whence

$$\begin{aligned}f_{i_{R_\sigma}}(\omega_{n+1}) &\geq 2(1-u_{n+1}) + f_{i_{L_\sigma}}(\omega_{n+1})\\ &\geq 2(1-u_{n+1}) - (1+\varepsilon_{n+1})\\ &= 1-u_{n+1}(2+\frac{1}{2^{n+1}}) \geq 1-3u_{n+1}\end{aligned}$$

likewise, $f_{i_{L_\sigma}}(\omega_{n+1}) \leq -1+3u_{n+1}$; then, let $A_{\sigma\frown 0} := \{i_{L_\sigma} : i\in J'\}$, and $A_{\sigma\frown 1} := \{i_{R_\sigma} : i\in J'\}$. □

4. No asymptotically isometric copy of $\ell^1(\mathbb{N})$ in sup-reflexive spaces

Notice that any asymptotically isometric copy $(a_n)_{n\in\mathbb{N}}$ of $\ell^1(\mathbb{N})$ in a normed space E is linearly independent (indeed, it is a Schauder sequence of E).

Proof of Theorem 3. Assume the existence of some sequence $(a_n)_{n\in\mathbb{N}}$ of Γ_E, asymptotically isometric with the canonical basis of $\ell^1(\mathbb{N})$, witnessed by a sequence of coefficients $(\delta_i)_{i\in\mathbb{N}}$ of $]0,1]$ converging to 1. Let $V := span\{a_n : n\in\mathbb{N}\}$. For every $n\in\mathbb{N}$, consider the linear mapping $g_n : V\to\mathbb{R}$ such that $g_n(a_i) = -\delta_i$ if $i<n$ and $g_n(a_i) = \delta_i$ if $n\leq i$; then, for every $(\lambda_i)_{i\in\mathbb{N}}\in\mathbb{R}^{(\mathbb{N})}$ ($\lambda_i = 0$ for all but finitely many i's), $|g_n(\sum_{j=0}^\infty\lambda_j a_j)| = |\sum_{j<n}-\lambda_j\delta_j + \sum_{j\geq n}\lambda_j\delta_j| \leq \sum_{j=0}^\infty|\lambda_j|\delta_j \leq \left\|\sum_{j=0}^\infty\lambda_j a_j\right\|$ whence g_n is continuous and $\|g_n\|\leq 1$; besides for every integer $i\geq n$, $g_n(a_i) = \delta_i$, whence

$\lim_{i\to+\infty} g_n(a_i) = 1$; so $\|g_n\| = 1$. Using **HB**, for each $n \in \mathbb{N}$, extend g_n to some $\tilde{g}_n \in S_{E'}$. Let W be the vector subspace of elements $x \in E$ such that $\big(\tilde{g}_n(x)\big)_{n\in\mathbb{N}}$ converges. Then the linear mapping $g := \lim_n \tilde{g}_n$ is continuous with norm ≤ 1 on W ; extend it to some element $\tilde{g} \in \Gamma_{E'}$. Now consider some sequence $(\alpha_i)_{i\in\mathbb{N}}$ of $]0,1[$ such that $\sum_{i\in\mathbb{N}} \alpha_i = 1$ and let $h := \sum_{k\in\mathbb{N}} \alpha_k \tilde{g}_k - \tilde{g}$. Clearly, $\|h\| \leq 2$. Besides, for every $n \in \mathbb{N}$, $h(a_n) = \sum_{k\leq n} \alpha_k \delta_n - \sum_{k>n} \alpha_k \delta_n + \delta_n = 2\delta_n \sum_{k\leq n} \alpha_k$, thus $\lim_n h(a_n) = 2$. So $\|h\| = 2$. By sup-reflexivity of E, let $u \in \Gamma_E$ such that $h(u) = 2$. Observe that $\tilde{g}(u) = -1$, and for every $k \in \mathbb{N}$, $\tilde{g}_k(u) = 1$ (notice that for each k, $\alpha_k \neq 0$) ; now $u \in W$ and $g(u) = \lim_k \tilde{g}_k(u) = 1$, contradicting $\tilde{g}(u) = -1$! □

5. Comments in set-theory without choice **ZF**

Recall the Axiom of Choice :

(**AC**). *Given a family $(A_i)_{i\in I}$ of non-empty sets, there exists a mapping $f : I \to \cup_{i\in I} A_i$ satisfying $f(i) \in A_i$ for every $i \in I$.*

and the two following weak forms of the Axiom of Choice :

(**DC**). **(Dependent Choices)** *Given a non-empty set E and a binary relation $R \subseteq E \times E$ satisfying $\forall x \in E\ \exists y \in E\ xRy$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of E satisfying $\forall n \in \mathbb{N} x_n R x_{n+1}$.*

(**HB**). **(Hahn-Banach)** *Given a real vector space E, a sublinear mapping $p : E \to \mathbb{R}$, a vector subspace F of E and a linear mapping $f : F \to \mathbb{R}$ satisfying $f \leq p\restriction F$, there exists a linear mapping $\tilde{f} : E \to \mathbb{R}$ extending f and satisfying $\tilde{f} \leq p$.*

It is known (see [7]) that **ZF+DC** does not prove **HB**, that **ZF+HB** does not prove **DC**, and that **ZF+HB+DC** does not prove **AC**. The axiom **HB** is known to be equivalent to its *multiple form* (see [7]) : *Given a family $(E_i)_{i\in I}$ of real vector spaces, a family $(p_i)_{i\in I}$ of sublinear mappings $p_i : E_i \to \mathbb{R}$, a family $(F_i)_{i\in I}$ of vector subspaces $F_i \subseteq E_i$ and a family $(f_i)_{i\in I}$ of linear mappings $f_i : F_i \to \mathbb{R}$ satisfying $f_i \leq p_i\restriction F_i$ for every $i \in I$, there exists a family $(\tilde{f}_i)_{i\in I}$ of linear mappings $\tilde{f}_i : E_i \to \mathbb{R}$ extending f_i and satisfying $\tilde{f}_i \leq p_i$ for every $i \in I$.*

Question 4. All notions of reflexivity we reviewed till now (J-reflexivity, onto-reflexivity, convex-reflexivity, compact-reflexivity, Smulian-reflexivity, sup-reflexivity, ...-see [12]-) are equivalent in **ZF+DC+HB**. Is there some "classical" notion of reflexivity which is not equivalent to "sup-reflexivity" in **ZF+DC+HB** ?

Question 5. In **ZF** (see Corollary 2 of Section 2.1), separable sup-reflexive Banach spaces are J-reflexive. More generally, all notions of reflexivity for *separable* Banach spaces we reviewed till now are equivalent in **ZF** either to

onto-reflexivity or to J-reflexivity, the former being weaker than the latter, and these two notions being not equivalent since there exists a model of **ZF**+**DC** (see [15]) where the continuous dual of $\ell^\infty(\mathbb{N})$ is $\ell^1(\mathbb{N})$: in such a model, the (non J-reflexive, separable) Banach space $\ell^1(\mathbb{N})$ is onto-reflexive. Is there some "classical" notion of reflexivity which, for separable Banach spaces, is equivalent in **ZF**, neither to J-reflexivity nor to onto-reflexivity ?

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