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# CROSSING AND ALIGNMENTS OF PERMUTATIONS

SYLVIE CORTEEL

ABSTRACT. We derive the continued fraction form of the generating function of some new  $q$ -analogs of the Eulerian numbers  $\hat{E}_{k,n}(q)$  introduced by Lauren Williams building on work of Alexander Postnikov. They are related to the number of alignments and weak exceedances of permutations. We show how these numbers are related to crossing and generalized patterns of permutations. We generalize to the case of decorated permutations. Finally we show how these numbers appear naturally in the stationary distribution of the ASEP model.

## 1. INTRODUCTION

The purpose of this paper is to derive the continued fraction form of generating function of some new  $q$ -analogs of Eulerian numbers  $\hat{E}_{k,n}(q)$  introduced by Lauren Williams [16] building on work of Alexander Postnikov [13] and to link them to generalized patterns. These numbers are related to the number of alignments and weak exceedances of permutations and come from the enumeration of totally positive Grassmann cells. We refer to [13] and [16] for details. We start by some definitions.

Let  $\sigma = (\sigma(1), \dots, \sigma(n))$  be a permutation of  $[n]$ . Then let

- $A_+(i) = \{j \mid j < i \leq \sigma(i) < \sigma(j)\}$ ;  $A_+(\sigma) = \sum_{i=1}^n |A_+(i)|$ .
- $A_-(i) = \{j \mid j > i > \sigma(i) > \sigma(j)\}$ ;  $A_-(\sigma) = \sum_{i=1}^n |A_-(i)|$ .
- $A_{+,-}(i) = \{j \mid j \leq \sigma(j) < \sigma(i) < i\} \cup \{j \mid \sigma(i) < i < j \leq \sigma(j)\}$ ;  
 $A_{+,-}(\sigma) = \sum_{i=1}^n |A_{+,-}(i)|$ .

For example let  $\sigma = (4, 7, 3, 6, 2, 1, 5)$  then  $A_+(\sigma) = 3$ ,  $A_-(\sigma) = 1$ , and  $A_{+,-}(\sigma) = 2$ .

We can also define these parameters using the permutation diagram. If  $\sigma$  is a permutation of  $[n]$ . We draw a line and put the numbers from 1 to  $n$  and we draw an edge from  $i$  to  $\sigma(i)$  above the line if  $i \leq \sigma(i)$  and under the line otherwise. Then  $A_+(\sigma)$  is the number of pairs of nested edges above (resp. under) the line and  $A_{+,-}(\sigma)$  is the number of pairs of edges such that one is above the line and the other under and such that their support do not intersect.



FIGURE 1. The permutation diagram of  $\sigma = (4, 7, 3, 6, 2, 1, 5)$

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The permutation diagram of  $\sigma = (4, 7, 3, 6, 2, 1, 5)$  is on Figure 1. The pairs of edges contributing to  $A_+(\sigma)$  are  $\{(1, 4), (3, 3)\}$ ,  $\{(2, 7), (3, 3)\}$  and  $\{(2, 7), (4, 6)\}$ . The pair of edges contributing to  $A_-(\sigma)$  is  $\{(5, 2), (6, 1)\}$ . The pairs of edges contributing to  $A_{+,-}(\sigma)$  are  $\{(7, 5), (1, 4)\}$  and  $\{(7, 5), (3, 3)\}$ .

**Definition 1.** *The number of alignements of a permutation  $\sigma$  is equal to*

$$A_+(\sigma) + A_-(\sigma) + A_{+,-}(\sigma).$$

This definition looks a bit different but is equivalent to the definition in [16]. The number of weak exceedances of a permutation  $\sigma$  is the cardinal of the set  $\{j \mid \sigma(j) \geq j\}$ . In [16] the following result was proved :

**Proposition 1.** *The number of permutations of  $[n]$  with  $k$  weak exceedances and  $\ell$  alignments is the coefficient of  $[q^{(k-1)(n-k)-\ell}]$  in*

$$\hat{E}_{k,n}(q) = q^{k-k^2} \sum_{i=0}^{k-1} (-1)^i [k-i]_q^n q^{k(i-1)} \left( \binom{n}{i} q^{k-i} + \binom{n}{i-1} \right).$$

where  $[k]_q = 1 + q + \dots + q^{n-1}$ .

These numbers have the property that if  $q = 1$  they are the eulerian numbers, if  $q = 0$  they are the Narayana numbers and if  $q = -1$  they are the binomial coefficients. See [16] for details.

Lauren Williams proved in [16] that  $E(q, x, y) = \sum_{n,k} q^{n-k} \hat{E}_{k,n}(q) y^k x^n$  is equal to

$$\sum_{i=0}^{\infty} \frac{y^i (q^{2i+1} - y)}{q^{i^2+i+1} (q^i - q^{i+1} [i]_q x + [i]_q xy)}.$$

Here we exhibit the generating function  $\hat{E}(q, x, y) = \sum_{n,k} \hat{E}_{k,n}(q) y^k x^n$  in a continued fraction form.

**Theorem 1.**

$$\hat{E}(q, x, y) = \frac{1}{1 - b_0 x - \frac{\lambda_1 x^2}{1 - b_1 x - \frac{\lambda_2 x^2}{1 - b_2 x - \frac{\lambda_3 x^2}{\dots}}}}$$

with  $b_n = y[n+1]_q + [n]_q$  and  $\lambda_n = y[n]_q^2$  and  $[n]_q = 1 + q + \dots + q^{n-1}$ .

To prove this theorem we introduce the notion of crossings. Let

- $C_+(i) = \{j \mid j < i \leq \sigma(j) < \sigma(i)\}$ ;  $C_+(\sigma) = \sum_{i=1}^n |C_+(i)|$ .
- $C_-(i) = \{j \mid j > i > \sigma(j) > \sigma(i)\}$ ;  $C_-(\sigma) = \sum_{i=1}^n |C_-(i)|$ .

We define the number of crossings of a permutation  $\sigma$  to be  $C_-(\sigma) + C_+(\sigma)$ .

We can again define those parameters using the permutation diagram.  $C_+(\sigma)$  is the number of pairs of edges above the line that intersect and  $C_-(\sigma)$  is the number of pairs of edges under the line that cross.

For  $\sigma = (4, 7, 3, 6, 2, 1, 5)$  we have  $C_+(\sigma) = 2$  and  $C_-(\sigma) = 1$ . The pairs of edges contributing to  $C_+(\sigma)$  are  $\{(1, 4), (4, 6)\}$  and  $\{(1, 4), (2, 7)\}$ . The pair of edges contributing to  $C_-(\sigma)$  is  $\{(7, 5), (6, 1)\}$ .

The proof of the theorem is direct with the following two propositions.

**Proposition 2.** *The coefficient  $[x^n y^k q^\ell]$  in  $\hat{E}(q, x, y)$  is the number of permutations of  $[n]$  with  $k$  weak excedances with  $\ell$  crossings.*

**Proposition 3.** *For any permutation with  $k$  weak excedances the number of crossings plus the number of alignments is  $(k - 1)(n - k)$ .*

We prove these two propositions in Section 2. The proof of the first one uses a bijection of Foata and Zeilberger between permutations and weighted bicolored Motzkin paths [9]. The proof of the second one uses basic properties of permutations. Then in Section 3 we use a bijection due to Françon and Viennot [10] and the bijection of Foata and Zeilberger and use results from [4, 5] to link with generalized patterns. One result is :

**Proposition 4.** *There is a one-to-one correspondance between permutations with  $k$  weak excedances and  $(n - k)(k - 1) - \ell$  alignments and permutations with  $k - 1$  descents and  $\ell$  occurrences of the generalised pattern  $13 - 2$ .*

The link between alignments and generalized patterns was conjectured by Steingrímsson and Williams [15]. In Section 4 we generalize to the case of decorated permutations. In Section 5 we show how these numbers appear naturally in the stationary distribution of the ASEP model.

## 2. PROOF OF PROPOSITION 2 AND 3

**2.1. Proof of Proposition 2.** We use a bijection of Foata and Zeilberger [9] between permutations and weighted bicolored Motzkin paths. We could also use the bijection of Biane [1]. We refer to [5] for a compact definition of these bijections.

A bicolored Motzkin path of length  $n$  is a sequence  $c = (c_1, \dots, c_n)$  such that  $c_i \in \{N, S, E, \bar{E}\}$  for  $1 \leq i \leq n$  such that if  $h_i = \{j < i \mid c_j = N\} - \{j < i \mid c_j = S\}$  then  $h_i \geq 0$  for  $1 \leq i \leq n$  and  $h_{n+1} = 0$ .

To any permutation  $\sigma$  we associate a pair  $(c, w)$  made of a bicolored Motzkin path  $c = (c_1, c_2, \dots, c_n)$  and a weight  $w = (w_1, \dots, w_n)$ . The path is created using the following rules :

- $c_i = N$  if  $i < \sigma(i)$  and  $i < \sigma^{-1}(i)$
- $c_i = E$  if  $i < \sigma(i)$  and  $i > \sigma^{-1}(i)$
- $c_i = \bar{E}$  if  $i > \sigma(i)$  and  $i < \sigma^{-1}(i)$
- $c_i = S$  if  $i > \sigma(i)$  and  $i > \sigma^{-1}(i)$

The weight is created using the following rules :

- $w_i = yp^{|A_+(i)|} q^{|C_+(i)|}$  if  $c_i = N, E$ .
- $w_i = p^{|A_-(i)|} q^{|C_-(i)|}$  if  $c_i = S, \bar{E}$ .

For example  $\sigma = (4, 1, 5, 6, 2, 3)$  gives the path  $(N, \bar{E}, N, E, S, S)$  and the weight  $(y, 1, yp, yp, p, 1)$ .

We now need

**Lemma 2.** *If  $i \leq \sigma(i)$  then*

$$|C_+(i)| = h_i - |A_+(i)|;$$

*and if  $i > \sigma(i)$  then*

$$|C_-(i)| = h_i - 1 - |A_-(i)|.$$

**Proof.** It is easy to prove by induction that

$$h_i = |\{j < i \mid \sigma(j) \geq i\}| = |\{j \geq i \mid \sigma(j) < i\}|.$$

Using the definitions given in Section 1, if  $i \leq \sigma(i)$  then

$$A_+(i) \cup C_+(i) = \{j < i \mid \sigma(j) \geq i\}$$

and if  $i > \sigma(i)$  then

$$A_-(i) \cup C_-(i) = \{j > i \mid \sigma(j) < i\} = \{j \geq i \mid \sigma(j) < i\} \setminus \{i\}.$$

This implies the result.  $\square$

Let  $\mathcal{P}_n$  be the set of pairs  $(c, w)$  obtained from permutations of  $[n]$ . Using the machinery developed in [8, 14], we get directly that if :

$$\hat{E}(q, p, x, y) = \sum_n x^n \sum_{(c, w) \in \mathcal{P}_n} \prod_{i=1}^n w_i.$$

then

$$(1) \quad \hat{E}(q, p, x, y) = \frac{1}{1 - b_0x - \frac{\lambda_1 x^2}{1 - b_1x - \frac{\lambda_2 x^2}{1 - b_2x - \frac{\lambda_3 x^2}{\dots}}}}$$

with  $b_n = y[n+1]_{p,q} + [n]_{p,q}$  and  $\lambda_n = y[n]_{p,q}^2$ .

This gives a generalization of Proposition 2. We define the number of nestings of a permutation  $\sigma$  to be  $A_+(\sigma) + A_-(\sigma)$ .

**Proposition 5.** *The coefficient of  $[x^n y^k q^\ell p^m]$  in  $\hat{E}(q, p, x, y)$  is the number of permutations  $\sigma$  of  $[n]$  with  $k$  weak exceedances,  $\ell$  crossings and  $m$  nestings.*

Note that this bijection implies that

**Proposition 6.** *The number of permutations with  $k$  weak exceedances and  $\ell$  crossings and  $m$  nestings is equal to the number of permutations with  $k$  weak exceedances and  $\ell$  nestings and  $m$  crossings.*

Similar results are known for set partitions and matchings [11, 3].

**2.2. Proof of Proposition 3.** We suppose that  $\sigma$  is a permutation of  $[n]$  with  $k$  weak exceedances. For any  $i$  with  $1 \leq i < n$ , we first define :

- $B_+(i) = \{j \mid j < i \leq \sigma(j)\}$
- $B_-(i) = \{j \mid \sigma(j) < i \leq j\}$

Note that  $h_i = |B_+(i)| = |B_-(i)|$  and that for  $i > \sigma(i)$ ,  $A_-(i) \cup C_-(i) \cup \{i\} = B_-(i)$ . Therefore

$$(2) \quad A_-(\sigma) + C_-(\sigma) = \sum_{i > \sigma(i)} (|B_-(i)| - 1) = k - n + \sum_{i > \sigma(i)} |B_+(i)|$$

For  $i \leq \sigma(i)$ , let

$$E_+(i) = \{j \mid i \in C_+(j)\} = \{j \mid i < j \leq \sigma(i) < \sigma(j)\}.$$

It is easy to see that :

$$E_+(i) \cup A_+(i) = B_+(\sigma(i)).$$

Therefore

$$\begin{aligned}
A_+(\sigma) + C_+(\sigma) &= \sum_{i \leq \sigma(i)} |C_+(i)| + |A_+(i)| \\
&= \sum_{i \leq \sigma(i)} |E_+(i)| + |A_+(i)| \\
&= \sum_{i \leq \sigma(i)} |B_+(\sigma(i))| \\
&= \sum_{i > \sigma(i)} |D_+(i)|,
\end{aligned}$$

where

$$D_+(i) = \{j \mid j \leq \sigma(j) \text{ and } i \in B_+(\sigma(j))\}.$$

It is easy to see that for  $i > \sigma(i)$ ,  $D_+(i) = \{j \mid j \leq \sigma(j) \text{ and } \sigma(i) < \sigma(j) < i\}$  and therefore that  $B_+(i) \cup D_+(i) \cup A_{+,-}(i) = \{j \mid j \geq \sigma(j)\}$ . As they are pairwise disjoint then

$$|B_+(i)| + |D_+(i)| + |A_{+,-}(i)| = k.$$

Combining equations (2) and (3) we get that  $A_+(\sigma) + C_+(\sigma) + A_-(\sigma) + C_-(\sigma) + A_{+,-}(\sigma) = k - n + \sum_{i > \sigma(i)} |D_+(i)| + |B_+(i)| + |A_{+,-}(i)|$ . This concludes the proof of Proposition 3.  $\square$

### 3. LINK WITH GENERALIZED PATTERNS

Continued fractions like the one presented in equation (1) were studied combinatorially in [4, 5]. A descent in a permutation is an index  $i$  such that  $\sigma(i) > \sigma(i+1)$ . An ascent in a permutation is an index  $i$  such that  $\sigma(i) < \sigma(i+1)$ . The pattern 31-2 (resp. 2-31, 13-2) occurs in  $\sigma$  if there exist  $i < j$  such that  $\sigma(i) > \sigma(j) > \sigma(i+1)$  (resp.  $\sigma(j+1) < \sigma(i) < \sigma(j)$ ,  $\sigma(i+1) > \sigma(j) > \sigma(i)$ ).

Theorem 10 in [5] associated with Theorem 22 in [4] tells us that :

**Proposition 7.** *The coefficient of  $[t^n x^k q^\ell p^m]$  in  $\hat{E}(t, x, q, p)$  is the number of permutations  $\sigma$  of  $[n]$  with  $n - k$  descents,  $\ell$  occurrences of the patterns 31 - 2 and  $m$  occurrences of the pattern 2 - 31.*

This can also be proved bijectively thanks to a bijection of Françon and Viennot [10]. We present now that bijection. See also [5].

Given a permutation  $\sigma = (\sigma(1), \dots, \sigma(n))$ , we set  $\sigma(0) = 0$  and  $\sigma(n+1) = n+1$ . Let  $\sigma(j) = i$ . Then  $i$  is

- a valley if  $\sigma(j-1) > \sigma(j) < \sigma(j+1)$
- a double ascent if  $\sigma(j-1) < \sigma(j) < \sigma(j+1)$
- a double descent if  $\sigma(j-1) > \sigma(j) > \sigma(j+1)$
- a peak if  $\sigma(j-1) < \sigma(j) > \sigma(j+1)$
- the beginning (resp. end) of a descent if  $\sigma(j) > \sigma(j+1)$  (resp.  $\sigma(j-1) > \sigma(j)$ )
- the beginning (resp. end) of an ascent if  $\sigma(j) < \sigma(j+1)$  (resp.  $\sigma(j-1) < \sigma(j)$ ).

If  $\sigma(j) = i$  then we define  $31 - 2(i)$  (resp.  $2 - 31(i)$ ) to be the number of indices  $k < j$  (resp.  $k > j$ ) such that  $\sigma(k - 1) > \sigma(j) > \sigma(k)$ .

To any permutation, we associate a pair  $(c, w)$  made of a bicolored Motzkin path  $c = (c_1, c_2, \dots, c_n)$  and a weight  $w = (w_1, \dots, w_n)$ . The path is created using the following rules :

- $c_i = N$  if  $i$  is a valley
- $c_i = E$  if  $i$  is a double ascent
- $c_i = \bar{E}$  if  $i$  is a double descent
- $c_i = S$  if  $i$  is a peak

The weight is created using the following rules :

- $w_i = yp^{31-2(i)}q^{2-31(i)}$  if  $c_i = N, E$ .
- $w_i = p^{31-2(i)}q^{2-31(i)}$  if  $c_i = S, \bar{E}$ .

For example  $\sigma = (6, 2, 1, 5, 3, 4)$  gives the path  $(N, \bar{E}, N, E, S, S)$  and the weight  $(y, 1, yp, yp, y, 1)$ .

Now we prove the following Lemma

**Lemma 3.** *For any  $i$*

$$31 - 2(i) + 2 - 31(i) = \begin{cases} h_i & \text{if } i \text{ is the beginning of an ascent} \\ h_i - 1 & \text{if } i \text{ is the beginning of a descent} \end{cases}$$

**Proof.** We prove this lemma by induction. If  $i$  is equal to 1 then  $i$  is the beginning of an ascent and  $31 - 2(1) + 2 - 31(1) = 0 = h_1$ . If  $i > 1$  then  $31 - 2(i) + 2 - 31(i) = 31 - 2(1) + 2 - 31(i - 1) + v$  where  $v$  is zero, one or minus one. It is easy to see that  $v$  is one if  $i - 1$  is the end of a descent and  $i$  is the beginning of an ascent,  $v$  is minus one if  $i - 1$  is the end of an ascent and  $i$  is the beginning of a descent and 0 otherwise. That gives exactly the lemma.  $\square$

Let  $\mathcal{P}_n$  be the set of pairs  $(c, w)$  obtained from permutations of  $[n]$ . Using the machinery developed in [8, 14], we get directly that  $\sum_n x^n \sum_{(c,w) \in \mathcal{P}_n} \prod_{i=1}^n w_i$  is  $\hat{E}(q, p, x, y)$  defined in equation (1).

Combining the bijection of Françon and Viennot and the inverse of bijection of Foata and Zeilberger [9] we get that

**Proposition 8.** *The number of permutations of  $[n]$  with  $k$  descents,  $\ell$  occurrences of the patterns  $31 - 2$  and  $m$  occurrences of the pattern  $2 - 31$  is equal to number of permutations  $\sigma$  of  $[n]$  with  $n - k$  weak exceedances and  $\ell$  crossings and  $m$  nestings.*

We also propose the direct mapping. Starting from a permutation  $\sigma$  with  $k$  descents,  $\ell$  occurrences of the patterns  $31 - 2$  and  $m$  occurrences of the pattern  $2 - 31$ , we form a permutation  $\tau$  with  $k$  descents,  $\ell$  occurrences of the patterns  $31 - 2$  and  $m$  occurrences of the pattern  $2 - 31$ .

We first form two two-rowed arrays  $f$  and  $g$ . The first line of  $f$  contains all the entries of  $\sigma$  that are the beginning of a descent. They are sorted in increasing order. The second line of  $f$  contains all the entries of  $\sigma$  that are the end of a descent. They are sorted such that if  $i$  is in that row then  $2 - 31(i)$  in  $\sigma$  is the number of entries of the row to the right of  $i$  smaller than  $i$ . The second line of  $f$  contains all the

entries of  $\sigma$  that are the end of a descent sorted in increasing order. The first line of  $g$  contains all the entries of  $\sigma$  that are the beginning of an ascent and that are sorted in increasing order. The second line of  $g$  contains all the entries of  $\sigma$  that are not the end of a descent. They are sorted such that if  $i$  is in that row then  $2 - 31(i)$  in  $\sigma$  is the number of entries of the row to the left of  $i$  greater than  $i$ . We create the permutation  $\tau$  which is the union of  $f$  and  $g$ .

For example if  $\sigma = (5, 1, 7, 4, 3, 6, 8, 2)$  then the  $2 - 31$  sequence is

$$(2 - 31(1), \dots, 2 - 31(8)) = (0, 0, 1, 1, 2, 1, 1, 0).$$

Then

$$f = \begin{pmatrix} 4, 5, 7, 8 \\ 1, 3, 4, 2 \end{pmatrix}$$

and

$$g = \begin{pmatrix} 1, 2, 3, 6 \\ 8, 6, 5, 7 \end{pmatrix}$$

$$\text{Then } \tau = \begin{pmatrix} 1, 2, 3, 4, 5, 6, 7, 8 \\ 8, 6, 5, 1, 3, 7, 4, 2 \end{pmatrix} = (8, 6, 5, 1, 3, 7, 4, 2).$$

We conclude this Section by proving Proposition 4. Given a permutation  $\sigma = (\sigma(1), \dots, \sigma(n))$ . Let  $\pi = (\sigma(n), \dots, \sigma(1))$ . If  $\sigma$  has  $k - 1$  ascents (or  $n - k$  descents) and  $\ell$  occurrences of the pattern  $31 - 2$ , then  $\pi$  has  $k - 1$  descents and  $\ell$  occurrences of the pattern  $2 - 13$ . With the previous proposition this gives Proposition 4.

#### 4. GENERALIZATION FOR DECORATED PERMUTATIONS

We can also derive the generating function of

$$A_{k,n}(q) = \sum_{i=0}^{k-1} \binom{n}{i} E_{k,n-i}.$$

These were introduced in [16].

The following corollary is an easy consequence of Theorem 1. Let  $A(q, x, y) = \sum_{n,k} A_{k,n}(q) x^n y^k$ .

**Corollary 4.**

$$A(q, x, y) = \frac{1}{1 - b_0 x - \frac{\lambda_1 x^2}{1 - b_1 x - \frac{\lambda_2 x^2}{1 - b_2 x - \frac{\lambda_3 x^2}{\dots}}}}$$

with  $b_n = (1 + y)[n + 1]_q$  and  $\lambda_n = yq[n]_q^2$ .

Lauren Williams [15] proved that

$$A(q, x, y) = \frac{-y}{1 - q} + \sum_{i \geq 1} \frac{y^i (q^{2i+1} - y)}{q^{i^2+i+1} (q^i - q^i [i + 1]_q x + [i]_q xy)}$$

It would be interesting to have a direct proof of the identity of the formal series form and the continued fraction form of these generating functions [15].



We can also interpret these results combinatorially. In [16] the coefficient of  $[q^{(n-k)k-\ell}]$  in  $A_{k,n}(q)$  is interpreted in terms of decorated permutations with  $k$  weak excedances and  $\ell$  alignments. Let us define these notions. Decorated permutations are permutations where the fixed points are bicolored [13]. We color these fixed points by colors  $\{+, -\}$ . We say that  $i \leq_+ \sigma(i)$  if  $i < \sigma(i)$  or  $i = \sigma(i)$  and colored with color  $+$ . We say that  $i \geq_- \sigma(i)$  if  $i > \sigma(i)$  or  $i = \sigma(i)$  and colored with color  $-$ .

For a decorated permutation  $\sigma$  and  $i$ , let

- $A_+(i) = \{j \mid j < i \leq_+ \sigma(i) < \sigma(j)\}$
- $A_-(i) = \{j \mid j > i \geq_- \sigma(i) > \sigma(j)\}$
- $A_{+,-}(i) = \{j \mid i \geq_- \sigma(i) > \sigma(j) \geq j\} \cup \{j \mid \sigma(j) \geq j > i \geq_- \sigma(i)\}$
- $C_+(i) = \{j \mid i < j \leq \sigma(i) < \sigma(j)\}$
- $C_-(i) = \{j \mid j > i > \sigma(j) > \sigma(i)\}$

As for permutations we define  $A_{\pm}(\sigma) = \sum_i A_{\pm}(i)$ .

With these notions, we can again define the number of alignments (resp. nestings, crossings) of a decorated permutation  $\sigma$  as  $A_+(\sigma) + A_-(\sigma) + A_{+,-}(\sigma)$  (resp.  $A_+(\sigma) + A_-(\sigma)$ ,  $C_+(\sigma) + C_-(\sigma)$ ). The number of weak excedances (resp. descendants) of a decorated permutation is the cardinality of the set  $\{i \mid i \geq_+ \sigma(i)\}$  (resp.  $\{i \mid i < \sigma(i)\}$ ).

Let

$$A(q, p, x, y) = \frac{1}{1 - b_0x - \frac{\lambda_1 x^2}{1 - b_1x - \frac{\lambda_2 x^2}{1 - b_2x - \frac{\lambda_3 x^2}{\dots}}}}$$

with  $b_n = (1 + y)[n + 1]_{p,q}$  and  $\lambda_n = yq[n]_{p,q}^2$ .

A direct generalization of the bijection à la Foata-Zeilberger on decorated permutations gives :

**Proposition 9.** *The coefficient of  $[x^n y^k q^\ell p^m]$  in  $A(q, p, x, y)$  is the number of decorated permutations  $\sigma$  of  $[n]$  with  $k$  weak excedances,  $\ell$  is the sum of the crossings and the descendants and  $m$  nestings.*

We can also make a direct link with the alignments :

**Proposition 10.** *For any decorated permutation  $\sigma$  with  $k$  weak excedances*

$$A_+(\sigma) + A_-(\sigma) + C_+(\sigma) + C_-(\sigma) + A_{+,-}(\sigma) + |\{j \mid j > \sigma(j)\}| = (n - k)k.$$

**Proof.** The proof is omitted as it follows exactly the same steps as the proof of Proposition 3.  $\square$

We leave as an open question to derive the bijection à la Françon-Viennot on decorated permutations to interpret the  $A_{k,n}(q)$  in terms of descents and generalized patterns.

## 5. LINK WITH THE ASEP MODEL

The ASEP model [12, 6] consists of black particles entering a row of  $n$  cells, each of which is occupied by a black particle or vacant. A particle may enter the system from the left hand side, hop to the right or to the left and leave the system from the

right hand side, with the constraint that a cell contains at most one particle. We will say that the empty cells are filled with white particles  $\circ$ . A basic configuration is a row of  $n$  cells, each containing either a black  $\bullet$  or a white  $\circ$  particle. Let  $\mathcal{B}_n$  be the set of basic configurations of  $n$  particles. We write these configurations as though they are words of length  $n$  in the language  $\{\circ, \bullet\}^*$ .

The ASEP defines a Markov chain  $P$  defined on  $\mathcal{B}_n$  with the transition probabilities  $\alpha$ ,  $\beta$ , and  $q$ . The probability  $P_{X,Y}$ , of finding the system in state  $Y$  at time  $t+1$  given that the system is in state  $X$  at time  $t$  is defined by:

- If  $X = A \bullet \circ B$  and  $Y = A \circ \bullet B$  then

$$(3a) \quad P_{X,Y} = 1/(n+1); \quad P_{Y,X} = q/(n+1)$$

- If  $X = \circ B$  and  $Y = \bullet B$  then

$$(3b) \quad P_{X,Y} = \alpha/(n+1).$$

- If  $X = B \bullet$  and  $Y = B \circ$  then

$$(3c) \quad P_{X,Y} = \beta/(n+1).$$

- Otherwise  $P_{X,Y} = 0$  for  $Y \neq X$  and  $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$ .

See an example for  $n = 2$  in Figure 2.

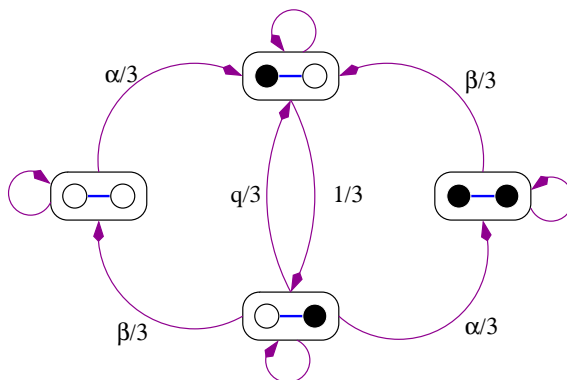


FIGURE 2. The chain  $P$  for  $n = 2$ .

This Markov chain has a unique stationary distribution [6]. The case  $q = 0$  was studied combinatorially by Enrica Duchi and Gilles Schaeffer [7]. In [2] another combinatorial approach was taken to treat the general case.

**Definition 2.** [2] Let  $\mathcal{P}(n)$  be the set of bicolored Motzkin paths of length  $n$ . The weight of the path in  $\mathcal{P}(0)$  is 1. The weight of any path  $p$  denoted by  $w(p)$  is the product of the weights of its steps. The weight of a step,  $p_i$ , starting at height  $h$  is given by:

$$\begin{aligned} \text{if } p_i = N & \quad \text{then} \quad w(p_i) = [h+1]_q \\ \text{if } p_i = \bar{E} & \quad \text{then} \quad w(p_i) = [h]_q + q^h/\alpha \\ \text{if } p_i = E & \quad \text{then} \quad w(p_i) = [h]_q + q^h/\beta \\ \text{if } p_i = S & \quad \text{then} \quad w(p_i) = [h]_q + q^h/(\alpha\beta) - q^{h-1}(1/\alpha - 1)(1/\beta - 1). \end{aligned}$$

Given a path  $p$ ,  $\theta(p)$  is the basic configuration such that each  $\bar{E}$  and  $S$  step is changed to  $\circ$  and each  $E$  and  $N$  step is changed to  $\bullet$ . Let

$$(5) \quad W(X) = \sum_{p \in \theta^{-1}(X)} w(p)$$

and

$$(6) \quad Z_n = \sum_{X \in B_n} W(X)$$

**Theorem 5.** [2] *At the steady state, the probability that the chain is in the basic configuration  $X$  is*

$$\frac{W(X)}{Z_n}$$

We can use these results and observations from the previous sections to get :

**Theorem 6.** *If  $\alpha = \beta = 1$ , at the steady state, the probability that the chain is in a basic configuration with  $k$  particles is*

$$\frac{\hat{E}_{k+1, n+1}(q)}{Z_n}.$$

Before proving that theorem, we need a Lemma

**Lemma 7.** *There is a weight preserving bijection between  $\mathcal{P}(n, k)$  the set of bicolored Motzkin paths of length  $n$  where the weight of any step starting at height  $h$  is  $[h+1]_q$  and where  $k$  is the number of steps  $N$  plus the number of steps  $E$  and  $\mathcal{P}'(n+1, k+1)$  the set of Motzkin paths of length  $n+1$  where the weight of any step starting at height  $h$  is  $[h+1]_q$  if the step is  $N$  or  $E$  and  $[h]_q$  otherwise and where  $k+1$  is the number of steps  $N$  plus the number of steps  $E$ .*

**Proof.** This is a general version of the classical bijection between bicolored Motzkin path of length  $n$  and bicolored Motzkin path of length  $n+1$  where one of the horizontal steps can not appear at height 0. The proof consists of transforming each step  $N$  (resp.  $E$ ,  $\bar{E}$ ,  $S$ ) weighted by  $w$  by two steps  $(N, N)$  (resp.  $(N, S)$ ,  $(S, N)$ ,  $(S, S)$ ) each of them weighted by  $w/2$ . Then we add a  $N$  step at the beginning and add a step  $S$  at the end both with weight  $1/2$  and apply the reverse map.  $\square$

**Proof of the Theorem.** We want to compute for  $\alpha = \beta = 1$

$$W(k, n) = \sum_{\substack{X \in B_n \\ X \text{ has } k \text{ particles}}} W(X) = \sum_{p \in \mathcal{P}(n, k)} w(p).$$

Now we use the previous lemma and get

$$W(k, n) = \sum_{p \in \mathcal{P}'(n+1, k+1)} w(p).$$

Using [8, 14] and the definition of the weight of the steps of  $\mathcal{P}'(n+1, k+1)$ , we conclude that :

$$1 + \sum_{n \geq 0} x^{n+1} \sum_{k=0}^n y^{k+1} W(k, n) = \hat{E}(q, x, y)$$

and therefore that  $W(k, n) = \hat{E}_{k+1, n+1}(q)$ .  $\square$

## 6. CONCLUSION

Several open problems naturally arise :

- Can we define generalized patterns for decorated permutations?
- Can we generalize these  $q$ -Eulerian numbers to understand the ASEP when  $\alpha \neq 1$  or  $\beta \neq 1$ ?
- Can we extend the definition of  $k$ -crossing and  $k$ -nesting that were defined for matchings and set partitions [3] ?

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