



HAL
open science

k-core decomposition: a tool for the visualization of large scale networks

José Ignacio Alvarez-Hamelin, Luca Dall'Asta, Alain Barrat, Alessandro Vespignani

► **To cite this version:**

José Ignacio Alvarez-Hamelin, Luca Dall'Asta, Alain Barrat, Alessandro Vespignani. k-core decomposition: a tool for the visualization of large scale networks. 2005. hal-00004807v1

HAL Id: hal-00004807

<https://hal.science/hal-00004807v1>

Preprint submitted on 27 Apr 2005 (v1), last revised 3 Oct 2005 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

k -core decomposition: a tool for the visualization of large scale networks

Ignacio Alvarez-Hamelin^{1*}, Luca Dall'Asta^{1†}, Alain Barrat^{1‡}, and Alessandro Vespignani^{2§}

¹ Laboratoire de Physique Théorique,
(UMR du CNRS 8627)
Bâtiment 210, Université de Paris-Sud,
91405 ORSAY Cedex, France

² School of Informatics and
Department of Physics,
Indiana University,
Bloomington, IN 47408, USA

Abstract

We use the k -core decomposition to visualize large scale complex networks. This decomposition, based on a recursive pruning of the least connected vertices, allows to disentangle the hierarchical structure of networks by progressively focusing on their central cores. By using this strategy we develop a general visualization algorithm that can be used to compare the structure of various networks and highlight their hierarchical structure. The low computational complexity of the algorithm $\mathcal{O}(n)$, where n is the size of the network, makes it suitable for the visualization of very large networks. We apply the proposed visualization tool to several real and synthetic graphs, showing its utility in finding specific structural fingerprints of computer generated and real world networks.

Keywords: visualization, k-cores, complex networks

1 Introduction

In recent times, the possibility of accessing, handling and mining large-scale networks datasets has revamped the interest in their investigation and theoretical characterization along with the definition of new modeling frameworks. In particular, mapping projects of the World Wide Web (WWW) and the physical Internet offered the first chance to study topology and traffic of large-scale networks. Gradually other studies followed describing population networks of practical interest in social science, critical infrastructures and epidemiology [1, 4, 14, 25]. The study of large scale networks, however, face us with an array of new challenges. The definition of centrality, hierarchies and structural organizations are hindered by the large size of these networks and the complex interplay of connectivity patterns, traffic flows and geographical, social and economical attributes characterizing their basic elements. In this context, a large research effort is devoted to provide effective visualization and analysis tools able to cope with graphs whose size may easily reach millions of vertices.

*Also with Facultad de Ingeniería, Universidad de Buenos Aires, Paseo Colón 850, C 1063 ACV Buenos Aires, Argentina. E-mail: Ignacio.Alvarez-Hamelin@th.u-psud.fr

†E-mail: Luca.Dallasta@th.u-psud.fr

‡E-mail: Alain.Barrat@th.u-psud.fr

§Also with Laboratoire de Physique Théorique, Bâtiment 210, Université de Paris-Sud, 91405 ORSAY Cedex France. E-mail: alexv@indiana.edu

In this paper, we propose a visualization algorithm based on the k -core decomposition able to uncover in a two-dimensional layout several topological and hierarchical properties of large scale networks. The k -core decomposition [9] consists in identifying particular subsets of the graph, called k -cores, each one obtained by recursively removing all the vertices of degree smaller than k , until the degree of all remaining vertices is larger than or equal to k . Larger values of coreness clearly correspond to vertices with larger degree and more central position in the network's structure.

When applied to the graphical analysis of real and computer-generated networks, this visualization tool allows the identification of networks' fingerprints, according to properties such as hierarchical arrangement, degree correlations and centrality, etc. The distinction between networks with seemingly similar properties is achieved by inspecting the different layout generated by the visualization algorithm. In addition, the running time of the algorithm grows only linearly with the size of the network, granting the scalability needed for the visualization of very large scale networks. The proposed visualization algorithm appears therefore as a convenient method for the general analysis of large scale complex networks and the study of their architecture. The presented visualization algorithm is publicly available [20].

The paper is organized as follows: after a brief survey on k -core studies (section 2), the basic definitions are introduced in section 3; the graphical algorithms are exposed in section 4, while section 5 is devoted to the application of the algorithm to the visualization of various real and computer-generated networks.

2 Related work

The notion of k -cores has been recently used in biologically related contexts, where it was applied to the analysis of protein interaction networks [5] or in the prediction of protein functions [2, 26]. An interesting application in the area of networking has been provided by Gaertler *et al.* [17], where the k -core decomposition is used for filtering out peripheral Autonomous Systems (ASes) in the case of Internet maps.

While a large number of algorithms aimed at the visualization of large scale networks have been developed (e.g., see [21]), only a few consider explicitly the k -core decomposition. Vladimir Batagelj *et al.* [8] studied the k -cores decomposition applied to visualization problems, introducing some graphical tools to analyse the cores, mainly based on the visualization of the adjacency matrix of certain k -cores. To the best of our knowledge, the algorithm presented by Baur *et al.* in the paper "Drawing the AS Graph in 2.5 Dimensions" [10], is the only one completely based on a k -core analysis and directly targeted at the study of large information networks. This algorithm uses a spectral layout to place vertices having the largest coreness. A combination

of barycentric and iteratively directed-forces allows to place the vertices of each coreness set, in decreasing order. Finally, the network is drawn in three dimensions, using the z axis to place each coreness set in a distinct horizontal layer. It is important to stress that the spectral layout is not able to distinguish two or more disconnected components. The algorithm by Baur *et al.* is also tuned for representing AS graphs and its total complexity depends on the size of the highest k -core (see [11] for more details on spectral layout), making the computation time of this proposal largely variable. In this respect, the algorithm presented here is considerably different in that it can represent networks in which k -cores are composed by several connected components. Finally, the algorithm parameters can be universally defined (see section 5), yielding a fast and general tool for analyzing all types of networks.

3 k -core decomposition: main definitions

Let us consider a graph $G = (V, E)$ of $|V| = n$ vertices and $|E| = e$ edges; a k -core is defined as follows [9]:

Definition 1 A subgraph $H = (C, E|C)$ induced by the set $C \subseteq V$ is a k -core or a core of order k iff $\forall v \in C$: $\text{degree}_H(v) \geq k$, and H is the maximum subgraph with this property.

A k -core of G can therefore be obtained by recursively removing all the vertices of degree less than k , until all vertices in the remaining graph have at least degree k .

Furthermore, we will use the following definitions:

Definition 2 A vertex i has coreness c if it belongs to the c -core but not to $(c + 1)$ -core. We denote by c_i the coreness of vertex i .

Definition 3 A coreness set C_c is composed by all the vertices whose coreness is c . The maximum value c such that C_c is not empty is denoted c_{\max} . The k -core is thus the union of all coreness sets C_c with $c \geq k$.

Definition 4 Each connected set of vertices having the same coreness c is a cluster Q^c .

Each coreness set C_c is thus composed by clusters Q_m^c , such that $C_c = \cup_{1 \leq m \leq q_{\max}^c} Q_m^c$, where q_{\max}^c is the number of clusters in C_c .

Two steps are necessary to perform the k -core decomposition of a graph. First a list of the vertices with their respective neighbors is prepared. The recursive pruning algorithm is applied. Building the list of n vertices with their degree takes a time $\mathcal{O}(n)$. Starting from the lowest degree value k_{\min} , all the vertices of degree equal to k_{\min} are then recursively cut out. Pruning a neighbor j of a vertex i of degree $k_{\min} + 1$ means that the degree of i is decreased to k_{\min} , so that i is subsequently pruned as well. This is what is meant by the expression “recursively cutting out”. The first coreness set (of coreness k_{\min}) contains all vertices removed during this process. When the remaining graph does not contain any vertex of degree k_{\min} , the algorithm repeats the procedure by recursively removing vertices of degree $k_{\min} + 1$, thus constructing the coreness set of coreness $k_{\min} + 1$. This process is repeated until no vertices are left, thus obtaining the coreness sets c_{\max} . The construction of the coreness sets takes a time $\mathcal{O}(e)$ (where e is the number of edges), because removing a vertex implies cutting the edges between this vertex and its neighbors. The building

Algorithm 1

```

1  input: vectors of coreness  $\mathcal{C}$  and cluster  $\mathcal{Q}$ , and  $\mathcal{T}$ , indexed by
   vertex  $i$ 
2  for each vertex  $i$  do
3    if  $c_i == c_{\max}$  then
4      set  $\rho_i$  and  $\alpha_i$  according to a uniform distribution in the
       disk of radius  $u$  ( $u$  is the core representation unit size)
5    else
6      set  $\rho_i$  and  $\alpha_i$  according to Eqs. 1 and 2
7  return  $\rho$  and  $\alpha$  vectors

```

Table 1: Algorithm for representing networks using k -cores decomposition

of all coreness sets thus implies that all edges are removed one after the other in the process. In summary, the total time to perform the decomposition is $\mathcal{O}(n + e)$. In order to build the clusters, each vertex should verify the coreness of its neighbors, which takes $2 \cdot e$ steps in the worse case. Finally, the total time complexity is $\mathcal{O}(n + e)$ for a general graph.

4 Graphical representation

The visualization algorithm we propose places vertices in 2 dimensions, the position of each vertex depending on its coreness and on the coreness of its neighbors. A color code allows for the identification of core numbers, while the vertex’s original degree is immediately provided by its size that depends logarithmically on the degree. For the sake of clarity, our algorithm represent a small percentage of the edges, chosen uniformly at random. As mentioned, a central role in our visualization method is played by multi-components representation of k -cores. In the most general situation, indeed, the recursive removal of vertices having degree less than a given k can break the original network into various connected components, each of which might even be once again broken by the subsequent decomposition. Our method takes into account this possibility, but for the sake of simplicity, we will first present the algorithm in the simplified case (Table 1), in which none of the k -cores is fragmented. Then, this algorithm will be used as a subroutine for treating the general case (Table 2).

4.1 Drawing algorithm for k -cores with single connected component

The network under study is represented by a graph $G = \{V, E\}$, where V is the set of vertices and E is the set of links.

k -core decomposition. The coreness of each vertex is computed (according to the procedure described in section 3) and stored in a vector \mathcal{C} , along with the coreness sets C_c and the maximum coreness value c_{\max} . Each coreness set is then decomposed into clusters Q_m^c of connected vertices, and each vertex i is labeled by its coreness c_i and by a number q_i representing the cluster it belongs to.

The two dimensional graphical layout. The visualization is obtained assigning to each vertex i a couple of radial coordinates (ρ_i, α_i) : the radius ρ_i is a function of the coreness of the vertex i and of its neighbors; the angle α_i depends on the cluster number q_i . In this way, coreness sets are displayed as layers with the form of circular shells, the innermost one corresponding to the set of vertices with

highest coreness. A vertex i belongs to the $c_{\max} - c_i$ layer from the center.

More precisely, ρ_i is computed according to the following formula:

$$\rho_i = (1 - \epsilon)(c_{\max} - c_i) + \frac{\epsilon}{|V_{c_j \geq c_i}(i)|} \sum_{j \in V_{c_j \geq c_i}(i)} (c_{\max} - c_j), \quad (1)$$

$V_{c_j \geq c_i}(i)$ is the set of neighbors of i having coreness c_j larger or equal to c_i . The parameter ϵ controls the possibility of rings overlapping, and is one of the only three external parameters required to tune image's rendering.

Inside a given coreness set, the angle α_i of a vertex i is computed as follow:

$$\alpha_i = 2\pi \sum_{1 \leq m < q_i} \frac{|Q_m|}{|C_{c_i}|} + \mathbf{N} \left(\frac{|Q_{q_i}|}{2|C_{c_i}|}, \pi \cdot \frac{|Q_{q_i}|}{|C_{c_i}|} \right), \quad (2)$$

where Q_{q_i} and C_{c_i} are respectively the cluster q_i and c_i -coreness set to which vertex i belongs, \mathbf{N} is a normal distribution of mean $\frac{|Q_{q_i}|}{2|C_{c_i}|}$ and width $2\pi \cdot \frac{|Q_{q_i}|}{|C_{c_i}|}$. Since we are interested in distinguishing different clusters in the same coreness set, the first term on the right side of Eq. 2, referring to clusters with $m < q_i$, allows to allocate a correct partition of the angular sector to each cluster. The second term on the right side of Eq. 2, on the other hand, specifies a random position for the vertex i in the sector assigned to the cluster Q_{q_i} .

Colors and size of vertices. Colors are assigned according to the coreness: vertices with coreness 1 are violet, and the maximum coreness vertices are red, following the rainbow color scale. Finally, the diameter of each vertex corresponds to the logarithm of its degree, giving a further information on vertex's properties. Note that the vertices with largest coreness are placed uniformly in a disk of radius u , which is the unit length (u equals 1 for this reduced algorithm).

The complete algorithm is presented in Table 1. In particular, vector \mathcal{Q} collects the cluster numbers $\{q_i\}$ of all vertices, and table \mathcal{T} contains the following pair of elements, indexed by the coreness c and cluster label q

$$\mathcal{T}(c, q) = \left(\sum_{1 \leq m < q} \frac{|Q_m|}{|C_c|}, \frac{|Q_q|}{|C_c|} \right). \quad (3)$$

These input quantities, used in Eq. 2, can be computed during the k -core decomposition, when the cluster labels are assigned.

4.2 Extended algorithm using k -cores components

The algorithm presented in the previous section can be used as the basic routine to define an extended algorithm aimed at the visualization of networks whose any given coreness set is fragmented; i.e. made by more than one connected component. This issue is solved by assigning to each connected component of a k -core a center and a size, which depends on the relative sizes of the various components. Larger components are put closer to the global center of the representation (which has Cartesian coordinates $(0, 0)$), and have larger sizes.

The algorithm begins with the center at the origin $(0, 0)$. Whenever a connected component of a k -core, whose center p had coordinates (X_p, Y_p) , is broken into several components by removing all vertices of degree k , i.e. by applying the next

decomposition step, a new center is computed for each new component. The center of the component h has coordinates (X_h, Y_h) , defined by

$$X_h = X_p + \delta(c_{\max} - c_h) \cdot u_p \cdot \varrho_h \cdot \cos(\phi_h) \quad (4)$$

$$Y_h = Y_p + \delta(c_{\max} - c_h) \cdot u_p \cdot \varrho_h \cdot \sin(\phi_h), \quad (5)$$

where δ scales the distance between components, c_{\max} is the maximum coreness and c_h is the core number of component h (the components are numbered by $h = 1, \dots, h_{\max}$ in an arbitrary order), u_p is the unit length of its parent component, ϱ_h and ϕ_h are the radial and angular coordinates of the new center with respect to the parent center (X_p, Y_p) . We define ϱ_h and ϕ_h as follows:

$$\varrho_h = 1 - \frac{|S_h|}{\sum_{1 \leq j \leq h_{\max}} |S_j|} \quad (6)$$

$$\phi_h = \phi_{ini} + \frac{2\pi}{\sum_{1 \leq j \leq h_{\max}} |S_j|} \sum_{1 \leq j \leq h} |S_j|, \quad (7)$$

where S_h is the set of vertices in the component h , $\sum_j |S_j|$ is the sum of the sizes of all components having the same parent component. In this way, larger components will be closer to the original parent component's center p .

The angle ϕ_h has two contributions. The initial angle ϕ_{ini} is chosen uniformly at random¹, while the angle sector is the sum of component angles whose number is less than or equal to the actual component number h .

Finally, the unit length u_h of a component h is computed as

$$u_h = \frac{|S_h|}{\sum_{1 \leq j \leq h_{\max}} |S_j|} \cdot u_p, \quad (8)$$

where u_p is the unit length of its parent component. Larger unit length and size are therefore attributed to larger components.

For each vertex i , radial and angular coordinates are computed by equations 1 and 2 as in the previous algorithm. These coordinates are then considered as relative to the center (X_h, Y_h) of the component to which i belongs. The position of i is thus given by

$$x_i = X_h + \gamma \cdot u_h \cdot \rho_i \cdot \cos(\alpha_i) \quad (9)$$

$$y_i = Y_h + \gamma \cdot u_h \cdot \rho_i \cdot \sin(\alpha_i) \quad (10)$$

where γ is a parameter controlling component's diameter.

The global algorithm is formally presented in Table 2. The main loop is composed by the following functions. First, the function $\{(end, \mathcal{C}) \leftarrow \text{make_core } k\}$ recursively removes all vertices of degree $k - 1$, obtaining the k -core, and stores into \mathcal{C} the coreness $k - 1$ of the removed vertices. The boolean variable end is set to *true* if the k -core is empty, otherwise it is set to *false*. The function $\{(\mathcal{Q}, \mathcal{T}) \leftarrow \text{compute_clusters } k - 1\}$ operates the decomposition of the $(k - 1)$ -coreness set into clusters, storing for each vertex the cluster label into the vector \mathcal{Q} , and filling table \mathcal{T} (see Eq. 3). The possible decomposition of the k -core into connected components is determined by function $\{\mathcal{S} \leftarrow \text{compute_components } k\}$, that also collects into a vector \mathcal{S} the number of vertices contained in each component. At the following step, functions $\{(X, Y) \leftarrow \text{compute_origin_coordinates_cmp } k\}$ and $\{U \leftarrow \text{compute_unit_size_cmp } k\}$ get, respectively, the center and size of each component of the k -core, gathering them

¹Note that if the ϕ_{ini} is fixed, all the centers of the various components are aligned in the final representation.

Algorithm 2

```

1   $k := 1$  and  $end := false$ 
2  while not  $end$  do
3     $(end, C) \leftarrow make\_core\ k$ 
4    if  $k > 1$   $(Q, T) \leftarrow compute\_clusters\ k - 1$ 
5     $S \leftarrow compute\_components\ k$ 
6     $(X, Y) \leftarrow compute\_origin\_coordinates\_cmp\ k$  (Eqs. from 4 to 7)

7     $U \leftarrow compute\_unit\_size\_cmp\ k$  (Eq. 8)
8     $k := k + 1$ 
9     $(\rho, \alpha) \leftarrow \mathbf{Algorithm\ 1}$  with  $C, Q, T$  and  $U$ 
10    $(\mathcal{X}, \mathcal{Y}) \leftarrow compute\_final\_coordinates\ \rho\ \alpha\ U\ X\ Y$  (Eqs. 9 and 10)

```

Table 2: Extended algorithm for the representation of networks using k -cores decomposition

Name	# vertices	$\langle k \rangle$	% dsp. links
ER	10^3	10	65
BA	10^4	4	1.3
RBA	10^3	10.8	65
FKP _{ext}	10^3	4	10

Table 3: Properties of generated networks; the percentage of displayed links is large because most edges are between vertices belonging to the highest k -core and do not perturb the layout.

in vectors X, Y and U . Finally, the coordinates of each vertex are computed and stored in the vectors \mathcal{X} and \mathcal{Y} .

5 Results from computer-generated and real networks

The algorithm presented in the previous section allows to visualize the k -core hierarchy of a very large variety of networks. The three parameters ϵ, γ and δ may be tuned to provide an optimal visualization: ϵ controls the possibility for various layers to overlap, δ determines the distance between various connected components, and γ gives the component diameter. In order to show how information on networks' architecture can be gathered with our graphical representation, we present the results of the visualization of a set of large graphs, corresponding to various computer-generated and real networks,

For the sake of clarity, we draw only a randomly chosen fraction of the huge total number of edges, while vertices are all shown and colored according to their coreness. Furthermore, edges are colored in two halves, each half according to the color of the corresponding extremity. Finally, the external parameters of the algorithm are set to the values $\epsilon = 0.18$, $\delta = 1.3$ and $\gamma = 1.5$, that give a very clear and readable layout.

In the following we will outline the various properties of the networks analyzed and we will discuss how the visualization layout allows the discrimination of signatures and fingerprints associated to the different topological and hierarchical properties of the networks.

5.1 Computer-generated networks

The characteristics of the computer-generated networks are listed in table 3. We display the number of vertices, their average degree $\langle k \rangle$ and the percentage of represented links.

The Erdős-Rényi (ER) model [15], with poissonian degree distribution, and very small density of triangles, is a typical

Name	# vertices	$\langle k \rangle$	% dsp. links	date
AS	11174	4.19	1	4 - 2001
AS+	11461	5.71	1	5 - 2001
IR	228263	2.15	0.1	2000
IR_CAIDA	192244	6.62	0.05	5 - 2003
WWW	10^6	3.56	0.01	2002
Cond-Mat	15616	5.51	0.5	1995 - 1998
WWA	3880	9.7	1	2002

Table 4: Properties of real networks

example of graphs with a characteristic value for the degree (the average value $\langle k \rangle$). Since an ER graph can consist of more than one connected component, we consider only the largest of these components.

Since many real-world networks have been shown to display a very heterogeneous topology as measured by broad degree distributions, many models and mechanisms have been proposed to construct heterogeneous networks. The most famous is the Barabási-Albert (BA) model [6], which considers growing networks according to the preferential attachment mechanism: each new vertex is connected to m already existing vertices chosen with a probability proportional to their starting degree. The Heuristically Optimized Trade-off model of Fabrikant, Koutsoupias and Papadimitriou [16] (FKP model), is another well-known model for heterogeneous networks; here, we consider an extension of this model proposed by Alvarez-Hamelin and Schabanel [3]. This model of growing network embedded in a two-dimensional space is essentially based on a competition criterium: a new vertex n is connected to already existing vertices in a way to optimize a combination of its distance on the graph to a predefined "center" and the Euclidean length of the connections. Both models produce graphs with power-law degree distributions, thus characterized by a very large variety of degree values.

5.2 Real world networks

The principal characteristics of the real networks are given in table 4.

Internet's maps: In real mapping projects, information about Internet's topology are collected studying the network at two different levels of granularity: the autonomous systems (AS) level and the router (IR) level. The autonomous system level of the Internet is represented by collected routes of the *Oregon route-views* project [24]; its extended version AS+, enriched with a large quantity of links, and obtained using informations from peering relationships of the autonomous systems, is presented in Chen *et al.* [12]. Both graphs date back to May 26, 2001.

For the Internet router level we also consider two different graphs: the one assembled by the explorations of Govindan and Tangmunarunkit [18] in 2000, here called IR graph, and the IR_CAIDA graph collected by the CAIDA project [13] between April 21st and May 8th, 2003. The differences are due to the diversity of exploration methods: the IR_CAIDA map was obtained by sending `traceroute` probes from many sources (i.e. 23 variously placed) to many destinations, while the IR map was obtained from single-source probing, but using other heuristics in order to obtain also lateral connections.

World Wide Web (WWW): Because of the huge size of the World Wide Web, of its dynamical nature and of the different methodologies of explorations, no complete and

unique map of WWW can be obtained. We present here a portion of the .fr domain, composed by 1 million of web-pages (vertices) beginning by the string "www". We limit the application of the algorithm to 1 million pages, because its actual implementation depends on the RAM memory size, which was 1GB for the presented figures. Since the actual definition of k -cores is made for undirect graphs, we consider the WWW as undirected, aiming to compare this network with others undirected networks. We are also working on an extension of k -core decomposition for the directed case.

Scientific Collaboration Network (Cond-Mat): this network considers scientists who have authored manuscripts submitted to the e-print archive relative to condensed matter physics between 1995 and 1998 (<http://xxx.lanl.gov/archive/cond-mat>). Scientists are identified with vertices and an edge exists between two scientists if they have co-authored at least one paper. The topological properties of this network and other similar networks of scientific collaborations have been studied in Ref. [22, 23].

World-Wide Airport Network (WWA): We analyze the International Air Transportation Association (IATA)² database containing the world list of airports pairs connected by direct flights for the year 2002. The resulting network counts $n = 3880$ vertices (airports) and $e = 18810$ edges (direct flight connection between two airports). Graph's topology has been recently studied [19, 7], showing scale-free properties on a definite range of degree values.

5.3 Visualization and analysis of computer-generated graphs

An instance of the visualization of Erdős Rényi random graphs is provided by Figure 1: the minimum and maximum coreness are clearly related to the average degree $\langle k \rangle = 10$. The large central mass is the result of the very homogeneous topology; the vertex degree have only small fluctuations, thus most vertices belong to the highest k -core.

The Barabási-Albert network with fixed $m = 2$, reported in Fig. 2, produces another peculiar representation. Indeed, although this graph displays a very heterogeneous vertex degree distribution, its k -core decomposition is trivial; only few layers at very small coreness are visible. The construction mechanism provides a simple explanation. Each new vertex enters the system with degree m , but at the following time steps new vertices may connect to it, increasing its degree. Inverting the procedure, we obtain exactly the k -core decomposition. The minimum degree is m , therefore all coreness sets C_c with $c < m$ are empty. Recursively pruning all vertices of degree m , one first removes the last vertex, then the one added at the preceding step, whose degree is now reduced to the initial value m , and so on, up to the initial vertices which may have larger degree. Hence, all vertices except the initial ones belong to the coreness set of coreness m . This somewhat pathological property holds for all growing networks with fixed attaching rate for new vertices, but is easily removed by randomly varying m during the growth: this yields a "Random BA" (RBA) network. Figure 3 shows a graph generated varying m randomly between 1 and 10 (at each time step). The layout's structure remains however quite simple, only few coreness sets being populated, and most of the vertices have the largest coreness value. Moreover, each layer is almost ring-like, rather than a real circular shell of some width, suggesting that the vertices of a given

coreness set have a similar number of neighbors in the sets of larger coreness.

The Extended FKP model is an attempt to model Internet, therefore it is remarkable that, at a first glance, the result (Fig. 4) is rather similar to that obtained for the AS graphs (see next subsection). At a deeper insight, one observes that some vertices in medium k -cores have a large degree, while in the ASes large degree vertices are in the innermost layers. In spite of this, the link structure seems to indicate a standard hierarchy, because the connectivity between different layers plays an important role. It is worth remarking that this network has only 1000 vertices because of the strong limitations due to the extremely long generation time, and it would be interesting to check the present finding for larger samples.

5.4 Visualization and analysis of real networks

Figures 5 and 6 display the representation of the autonomous system graphs AS and AS+. All coreness layers are populated, and, in a given coreness set, the vertices are distributed on a relatively large range of the radial coordinate, which means that their neighborhoods are variously composed. It is worth noting that the coreness and the degree are very correlated, with a clear hierarchical structure. Links go principally from one coreness set to another, although there are of course also intra-layer links. The hierarchical structure exhibited by our analysis of the autonomous system level is a striking property; for instance, one might exploit it for showing that in the Internet high-degree vertices are naturally (as an implicit result of the self-organizing growing) placed in the innermost structure.

At high resolution, i.e. at the IR level, Internet's properties are less structured, which is reflected in Figures 7 and 8, in which a completely different scenario emerges: external layers, of lowest coreness, contain vertices with large degree. There are 20 vertices for the IR case, and 5 vertices for the IR_CAIDA graph, which have a degree larger than 100 and coreness less than 6. In contrast, the AS and AS+ have only two vertices with degree larger than 100, and coreness 5 and 13 for AS, and 10 for AS+. The correlation between coreness and degree is thus clearly of a very different nature in the maps of Internet obtained at different granularities i.e. routers or autonomous systems.

The lowest coreness sets, containing vertices that are very external, are displayed as quite broad shells, meaning that the corresponding vertices have neighbors with coreness covering a large range of values. The larger coreness sets, in particular for the IR_CAIDA network, are thin rings, as for the RBA, which means that the neighbors of the vertices in a given layer have similar coreness.

It is worth remarking how the present visualization allows the distinction of networks which appears very similar on the basis of the sole statistical properties. Indeed, we can notice that the IR map is quite different from the IR_CAIDA map. This difference likely finds its origin in the different exploration methods used to gather the two data sets. The IR map has been obtained from one source monitor using, using *source routing* to detect lateral connectivity. The IR_CAIDA map, instead, is the merger of data gathered by 23 different probing monitors. On one hand, it is likely that the high cores of the IR network are composed by routers with *source routing* activated (approximately 8% of the total routers, see Ref. [18]). These routers sampled destinations unevenly resulting in a less regular layout. On the other hand, the role of sources in the IR_CAIDA map is

²<http://www.iata.org>.

symmetric, that is all the monitor sent traceroute probes to all destinations, resulting in a very regular layout.

The appearance of the WWA (Fig. 9) is highly hierarchical: large degree vertices are very entangled in the core of the network. Some similarities with the AS graphs emerge, but the major difference resides in the large number of connections between vertices with similar coreness.

Figures 10 and 11 represent the graphs of the Cond-Mat archive and the World-Wide-Web respectively. Their particularity resides in the existence of various fragmented components. Let us consider the Cond-Mat graph: we can distinguish three different fragments for orange and red vertices. One of these fragments is very small (it is far away from the center because of its size) and it has few green vertices attached. The largest of these fragments on the other hand does not contain vertices with the largest coreness (red vertices): this particularity is detected by the fact that its vertices are orange and that the center of the corresponding ring is empty. The last one is of medium size and contains vertices with the highest coreness. It is also interesting that even for small coreness, there are various small components but only one giant component. This is shown by some vertices in blue inside the violet layer. Links arrangement as well as vertices degree suggest a standard hierarchy, where the predominant connections are between different layers. This means that larger coreness layers have vertices with an elevated degree. On the other hand, the WWW graph is an interesting example of how also two components of approximately the same size (here at the coreness 12 level) can be well represented using the algorithm.

In summary, heterogeneous real networks present different kinds of k -core hierarchies, with one or more connected components, for which the proposed visualization tool allows an easy access and characterization. A detailed discussion of each network and the relative fingerprints are beyond the scope of this paper and are left to the specific investigation of the corresponding network.

6 Conclusions

In this paper, we have proposed a general visualization tool for large scale graphs. Exploiting k -core decomposition, and the natural hierarchical structures emerging from it, our algorithm allows a layout, that possesses both the simplicity of a 2D representation with a considerable amount of information encoded. One can easily read basic features of the graph (degree, hierarchical structure, position of the highest degree vertices, etc.) as well as more complicated properties, e.g. the relation between a vertex and the hierarchical position of its neighbors. Our results show the possibility of gaining clear insights on the architecture of many real world and computer-generated networks by a visualization based on the rationalization of the corresponding graph. In conclusion, the present visualization strategy is a useful tool for a distinction between networks with different topological properties and structural arrangement, but it may be also used for determining if a certain model is in good agreement with real data, providing a further interesting tool for models validation. Finally, we also provide a publicly available tool for visualizing networks [20].

Acknowledgments: We gratefully acknowledge Mark Newman of University of Michigan, for providing the **Cond_Mat** network. We also gratefully acknowledge Fabien Mathieu of LIRMM at Montpellier, France, for providing the .fr portion of the WWW graph. Finally, we acknowledge to IATA for the airports network (**WWA**). This work

has been partially funded by the European Commission - Fet Open project COSIN IST-2001-33555 and contract 001907 (DELIS).

References

- [1] R. Albert and A.-L. Barabási. Statistical mechanics of complex networks. *Rev. Mod. Phys.*, 74:47–, 2000.
- [2] Md. Altaf-Ul-Amin, K. Nishikata, T. Koma, T. Miyasato, Y. Shinbo, Md. Arifuzzaman, C. Wada, M. Maeda, T. Oshima, H. Mori, and S. Kanaya. Prediction of Protein Functions Based on K-Cores of Protein-Protein Interaction Networks and Amino Acid Sequences. *Genome Informatics*, 14:498–499, 2003.
- [3] J. I. Alvarez-Hamelin and N. Schabanel. An Internet Graph Model Based on Trade-Off Optimization. *Eur. Phys. J. B, special issue on “Applications of networks”*, 38(2):231–237, march II 2004.
- [4] L. A. N. Amaral, A. Scala, M. Barthélemy, and H. E. Stanley. Classes of small world networks. *Proc. Natl. Acad. Sci. (USA)*, 97:11149–11152, 2000.
- [5] G. D. Bader and C. W. V. Hogue. An automated method for finding molecular complexes in large protein interaction networks. *BMC Bioinformatics*, 4(2), 2003.
- [6] A.-L. Barabási and R. Albert. Emergence of scaling in random networks. *Science*, 286:509–512, 1999.
- [7] A. Barrat, M. Barthélemy, R. Pastor-Satorras, and A. Vespignani. The architecture of complex weighted networks. *Proc. Natl. Acad. Sci. (USA)*, 101:3747, 2004.
- [8] V. Batagelj, A. Mrvar, and M. Zaversnik. Partitioning Approach to Visualization of Large Networks. In *Graph Drawing '99, Castle Stirin, Czech Republic, LNCS 1731*, pages 90–98, 1999.
- [9] V. Batagelj and M. Zaversnik. Generalized Cores. *CoRR*, cs.DS/0202039, 2002.
- [10] M. Baur, U. Brandes, M. Gaertler, and D. Wagner. Drawing the AS Graph in 2.5 Dimensions. In *12th International Symposium on Graph Drawing, Springer-Verlag editor*, pages 43–48, 2004.
- [11] U. Brandes and S. Cornelsen. Visual Ranking of Link Structures. *Journal of Graph Algorithms and Applications*, 7(2):181–201, 2003.
- [12] Q. Chen, H. Chang, R. Govindan, S. Jamin, S. Shenker, and W. Willinger. The Origin of Power Laws in Internet Topologies Revisited. In *IEEE Infocom 2002*, 2002.
- [13] Router-Level Topology Measurements” ”Cooperative Association for Internet Data Analysis. http://www.caida.org/tools/measurement/skitter/router_topology/.
- [14] S. N. Dorogovtsev and J. F. F. Mendes. *Evolution of networks: From biological nets to the Internet and WWW*. Oxford University Press, 2003.
- [15] P. Erdős and A. Rényi. On random graphs I. *Publ. Math. (Debrecen)*, 6:290–297, 1959.
- [16] A. Fabrikant, E. Koutsoupias, and C. H. Papadimitriou. Heuristically Optimized Trade-Offs: A New Paradigm for Power Laws in the Internet. *LNCS*, 2380:110–, Jun 2002.
- [17] M. Gaertler and M. Patrignani. Dynamic Analysis of the Autonomous System Graph. In *IPS 2004, International Workshop on Inter-domain Performance and Simulation, Budapest, Hungary*, pages 13–24, 2004.
- [18] R. Govindan and H. Tangmunarunkit. Heuristics for Internet Map Discovery. In *IEEE INFOCOM 2000*, pages 1371–1380, Tel Aviv, Israel, March 2000. IEEE.
- [19] R. Guimera and L. A. N. Amaral. Modeling the world-wide airport network. *Eur. Phys. J. B*, 38:381–385, 2004.
- [20] LArge NETwork VISualization tool. <http://xavier.informatics.indiana.edu/lanet-vi/>.

- [21] COSIN: networks visualization. <http://i11www.ilkd.uni-karlsruhe.de/cosin/tools/index.php>.
- [22] M. E. J. Newman. Scientific collaboration networks. I. Network construction and fundamental results. *Phys. Rev. E*, 64:016131, 2001.
- [23] M. E. J. Newman. Scientific collaboration networks. II. Shortest paths, weighted networks, and centrality. *Phys. Rev. E*, 64:016132, 2001.
- [24] University of Oregon Route Views Project. <http://www.routeviews.org/>.
- [25] R. Pastor-Satorras and A. Vespignani. *Evolution and structure of the Internet: A statistical physics approach*. Cambridge University Press, 2004.
- [26] S. Wuchty and E. Almaas. Peeling the Yeast protein network. *Proteomics*. 2005 Feb;5(2):444-9., 5(2):444-449, 2005.

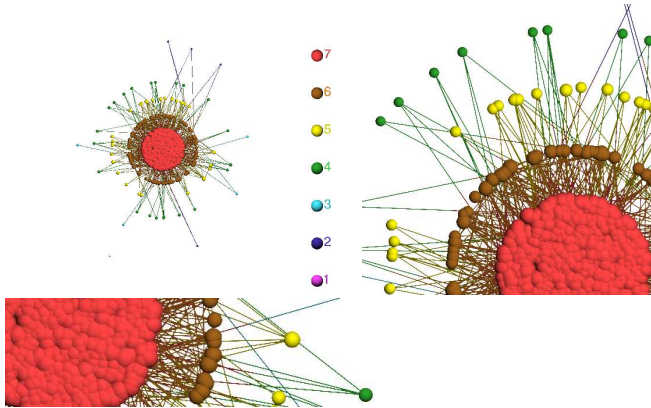


Figure 1: Graphical representation of an ER graph with $\langle k \rangle = 10$ and 1000 vertices

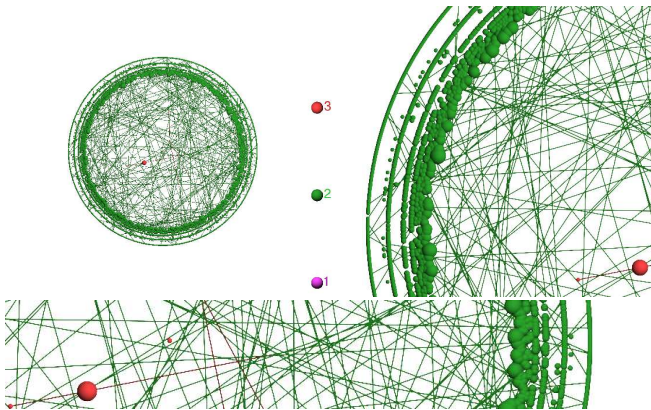


Figure 2: Graphical representation of a BA network with $\langle k \rangle = 4$ and 10000 vertices

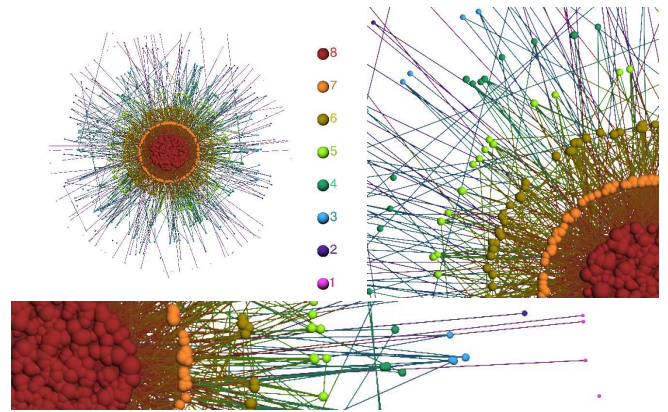


Figure 3: Graphical representation of a BA network with a random $m \in [1 : 10]$, and $n=1000$

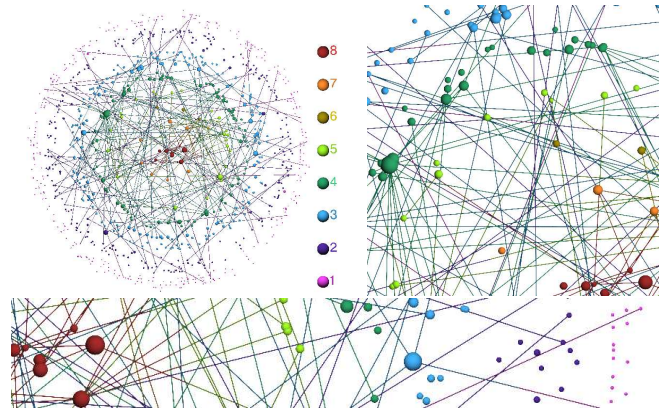


Figure 4: Graphical representation of the extended Heuristically Optimized Trade-off (FKP model) ($n = 1000$)

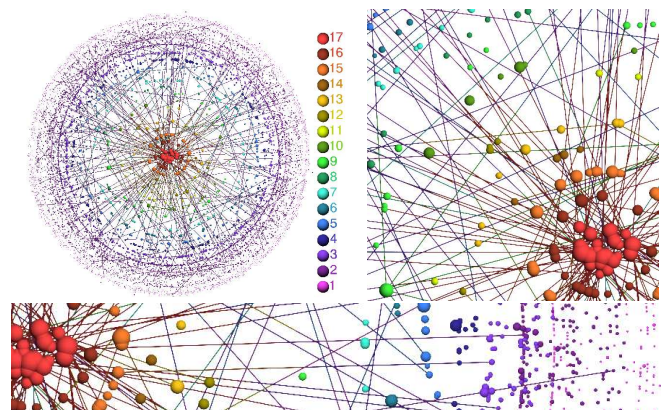


Figure 5: Graphical representation of the AS graph

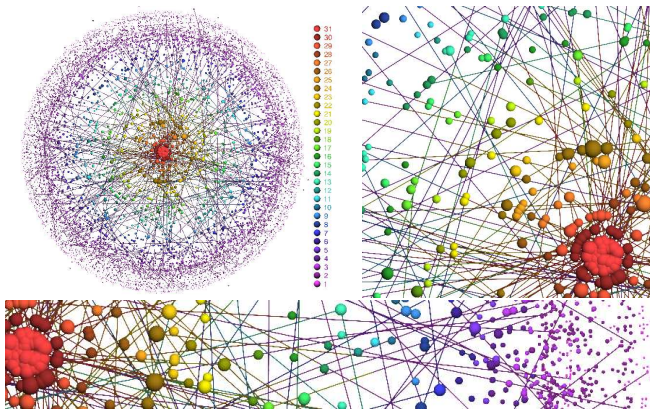


Figure 6: Graphical representation of the AS+ graph

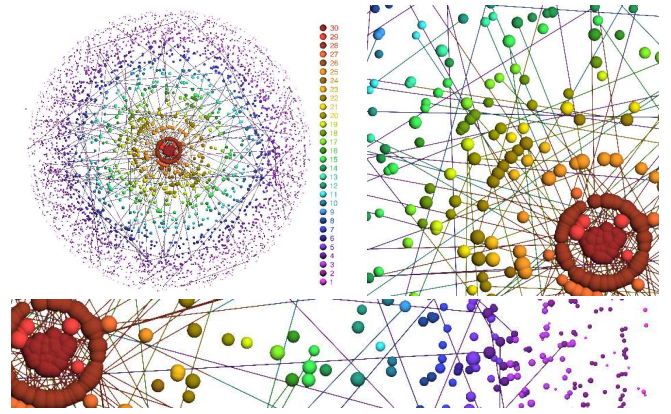


Figure 9: Graphical representation of the Airports network

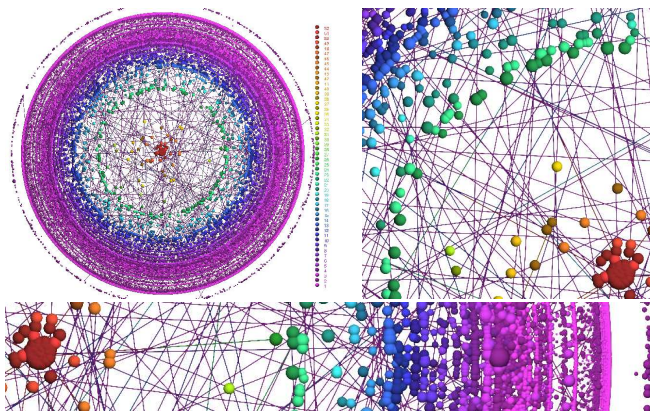


Figure 7: Graphical representation of the IR graph

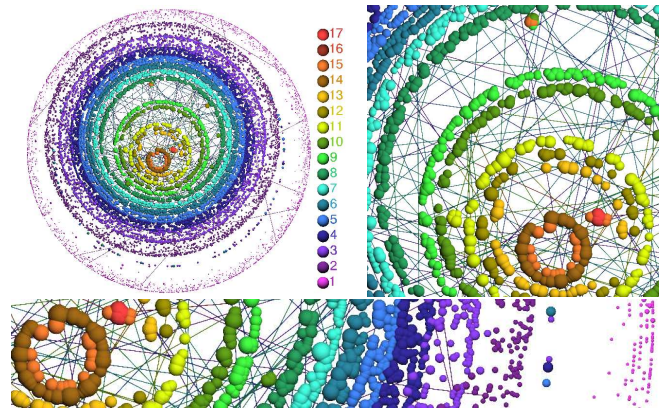


Figure 10: Graphical representation of the scientific collaboration network (Cond-Mat)

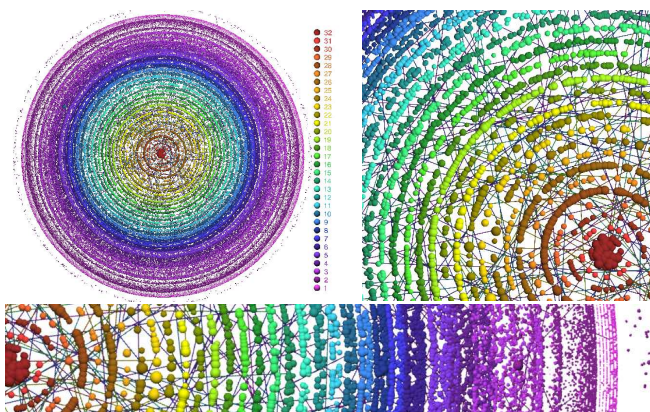


Figure 8: Graphical representation of the CAIDA IR graph

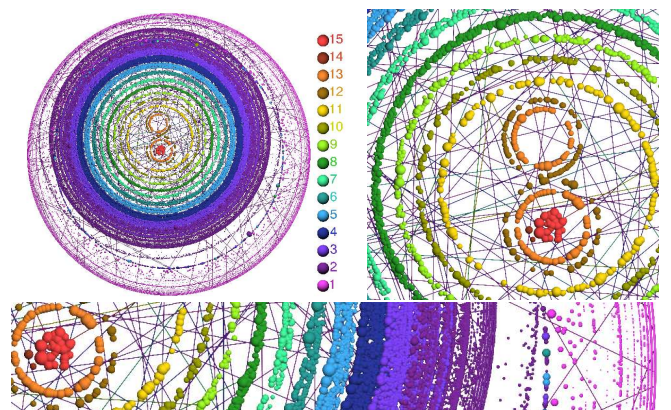


Figure 11: Graphical representation of a fraction of the .fr domain of the WWW