The action of the Frobenius map on rank 2 vector bundles over a supersingular genus 2 curve in characteristic 2.

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1 Introduction

Let X be a smooth proper genus g curve over a field k of characteristic $p > 0$. Denote by $\mathrm{M}_X(r)$ the moduli space of semi-stable vector bundles of rank r and trivial determinant. If X_1 is the curve $X \times_{k,\sigma} k$, where $\sigma : k \to k$ is the Frobenius of k, the k-linear relative Frobenius $F : X \to X_1$ induces by pull-back a (rational) map $V : M_{X_1}(r) \dashrightarrow M_X(r)$. We are interested in determining e.g. surjectivity and degree of V, density of Frobenius-stable bundles, loci of Frobenius-destabilized bundles. For general (g, r, p), not much seems to be known (see the introductions of [LP1] and [LP2] for an overview of this subject).

When $g = 2$, $r = 2$, the classical isomorphism $D : M_X \to |2\Theta| \cong \mathbb{P}^3$ (see [NR] for the complex case) remains valid for an algebraically closed field of positive characteristic (see [LP1], section 5 for a sketch of proof) and we have the commutative diagram

$$\begin{array}{ccc} \mathrm{M}_{X_1}(2) & \xrightarrow{V} & \mathrm{M}_X(2) \\ \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} \\ |2\Theta_1| & \xrightarrow{\widetilde{V}} & |2\Theta| \end{array}$$

Furthermore, the semi-stable boundary of the moduli space $M_X(2)$ identifies via D with the Kummer quartic surface Kum_X, which is canonically contained in the linear system $|2\Theta|$, and $\widetilde{V}$ maps Kum_{X_1} to Kum_X. In [LP2], it is shown that $\widetilde{V}$ is given by polynomials of degree p and always has base-points.

When $p = 2$ and X is an ordinary curve, [LP1] determined the quadric equations of $\widetilde{V}$ in terms of the generalized theta constants of the curve X and thus could answer the above mentioned questions. In [LP2], they could give the equations of $\widetilde{V}$ in case of a non-ordinary curve X with Hasse-Witt invariant equal to 1 by specializing a family $\mathcal{X}$ of genus 2 curve parameterized by a discrete valuation ring with ordinary generic fiber and special fiber isomorphic to X. In particular, they determined the coefficients of the quadrics of $\widetilde{\mathcal{V}}_\eta$, which coincide with the Kummer quartic surface coefficients, in terms of the coefficients of an affine equation for the birational model of the ordinary curve $\mathcal{X}_\eta$.

In this paper, we complete the study of the $(g, r, p) = (2, 2, 2)$ case by giving the equations of $\widetilde{V}$ in case of a supersingular curve X (Theorem 4.2). We adapt the strategy of [LP2]. Namely, we consider a family $\mathcal{X}$ of genus 2 curve of genus 2 curve parameterized by a discrete valuation ring R with ordinary generic fiber and special fiber isomorphic to X. In order to find an R-base of the free

R-module $\mathcal{W} := H^0(\mathcal{J}\mathcal{X},\, 2\Theta)$ and to express the canonical theta functions of order 2 in that base, we pull back 2Θ on the product $\mathcal{X} \times \mathcal{X}$ via the Abel-Jacobi map $\mathcal{X} \times \mathcal{X} \to \mathcal{J}\mathcal{X}$ and make use of an explicit theorem of the square for hyperelliptic jacobians (see [AG]). As for the two other cases, we can easily deduce a full description of the Verschiebung $V : \mathrm{M}_{X_1}(2) \to \mathrm{M}_X(2)$ (Proposition 4.3).

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2 Deformation of genus 2 curves

2.1 Specializing an ordinary curve

Let k be a characteristic 2 algebraically closed field, R the discrete valuation ring $k[[s]]$ and K its fraction field. We consider proper and smooth curve X, of genus 2, supersingular. After [L], we know that there exists a unique $\mu \in k$ such that X is birationnally equivalent to the plane curve given by the equation

$$y^2 + y \quad = \quad x^5 + \mu^2 x^3 \tag{2.1}$$

The projection $(x,\, y) \mapsto x$ is the restriction of the ramified double cover $\pi : X \to \mathbb{P}^1_k = |K_X|$ (where K_X is the canonical divisor of X) with a single Weierstrass point, namely ∞.

Let's choose an element ω of $R - \{0,\, 1\}$ and denote by $\mathcal{X}$ the R-scheme defined by the two affine charts

$$y^2 + (s^2x + 1)(s^2\omega^2 x + 1)y = x^5 + \mu^2 x^3 \tag{2.2}$$

and

$$\widetilde{y}^2 + \widetilde{x}(s^2 + \widetilde{x})(s^2\omega^2 + \widetilde{x})\widetilde{y} = \widetilde{x} + \mu^2\widetilde{x}^3$$

glued by the isomorphism given by $x \mapsto \widetilde{x}^{-1}$ and $y \mapsto \widetilde{y}(\widetilde{x}^{-3})$. $\mathcal{X}$ is a proper and smooth R-scheme and again, the projection $(x,\, y) \mapsto x$ is the restriction of an R-morphism $\mathcal{X} \to \mathbb{P}^1_R$, still denoted by π. It's convenient to introduce the notations

$$\begin{cases} q(x) = (s^2x + 1)(s^2\omega^2 x + 1) = s^4\omega^2 x^2 + s^2(1 + \omega^2)x + 1 \\ p(x) = x^5 + \mu^2 x^3 \end{cases}$$

The special fiber $\mathcal{X}_0$ is isomorphic to X, the generic fiber $\mathcal{X}_\eta$ is a proper and smooth ordinary curve of genus 2 over K, and $\pi_\eta : \mathcal{X}_\eta \to \mathbb{P}^1_K$ is a double ramified cover with Weierstrass points 0_η (with coordinate $1/s^2$), 1_η (with coordinate $1/s^2\omega^2$) and ∞. Thus, we have $\mathcal{J}\mathcal{X}[2]_\eta \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mu_2^2/K$ and $\mathcal{J}\mathcal{X}[2]_0$ is a self dual local-local group scheme over k of dimension 2 and height 4.

One can define (over R) the Abel-Jacobi map

$$\begin{array}{rccc} AJ : & \mathcal{X} \times \mathcal{X} & \to & \mathcal{J}\mathcal{X} \\ & (x,\, y) & \mapsto & \mathcal{O}(x + y) \otimes K_{\mathcal{X}}^{-1} \end{array}$$

as the compositum of the quotient map $\mathcal{X}\times\mathcal{X}\to \mathrm{Sym}^2\mathcal{X}=\mathrm{Div}^2(\mathcal{X})$ under the natural action of $\mathfrak{S}_2$ by the natural map $\mathrm{Div}^2\mathcal{X}\to\mathcal{J}\mathcal{X}$. Using Riemann-Roch, it is surjective, separable of degree 2. Denote by

$$[0]_\eta = AJ(1_\eta+\infty);\ [1]_\eta = AJ(0_\eta+\infty);\ [\infty]_\eta = AJ(0_\eta+1_\eta)$$

the three non-zero elements of $\mathcal{J}\mathcal{X}[2]^{et}_\eta$ and notice that they specialize to 0.

2.2 Standard birational model

There exists a unique triplet $(a,\,b,\,c)$ of elements in the algebraic closure $\bar{K}$ of K such that $\mathcal{X}_\eta$ is birationally equivalent to the plane curve given by the equation

$$w^2+z(z+1)w = z(z+1)(az^3+(a+b)z^2+cz+c) \tag{2.3}$$

and such that, with these coordinates, $0_\eta=0$, $1_\eta=1$ and $\infty=\infty$. These latter condition amounts to an ordering of the Weierstrass points of $\mathcal{X}_\eta$, and this is equivalent to an isomorphism ${\mathbb{F}_2}^2\xrightarrow{\sim}\mathcal{J}\mathcal{X}[2]^{et}_\eta$, ie a level-2-structure on X_η (see [DO] for the characteristic 0 case).

One can determine explicitly this coordinate change. As x and z are rational coordinates on $\mathbb{P}^1_K$, the function $z=z(x)$ has to induce a K-isomorphism of $\mathbb{P}^1_K$, which fixes ∞, sends $1/s^2$ on 0 and $1/s^2\omega^2$ on 1. Thus, we have

$$z(x)=\frac{\omega^2}{1+\omega^2}(s^2x+1) \quad\Leftrightarrow\quad x(z)=\frac{1+\omega^2}{s^2\omega^2}z+\frac{1}{s^2} \tag{2.4}$$

In particular, one has $q(x)=\dfrac{(1+\omega^2)^2}{\omega^2}z(z+1)$ and replacing in (2.2), one sees that the function $y(z,\,w)$ has to be of the form

$$y(z,\,w)=\frac{(1+\omega^2)^2}{\omega^2}[w+(\alpha z^2+\beta z+\gamma)] \tag{2.5}$$

with α, β and γ in $\bar{K}$. Inversely, we have

$$w(x,\,y)=\frac{\omega^2}{(1+\omega^2)^2}\left[y+\alpha\,q(x)+(1+\omega^2)\left[(\alpha+\beta)s^2x+\left((\alpha+\beta+\gamma)+\frac{\gamma}{\omega^2}\right)\right]\right] \tag{2.6}$$

2.1. *Remark.* At the Weierstrass point 0_η, (2.2) gives that $y(0_\eta)^2=p\left(\dfrac{1}{s^2}\right)$ and (2.3) gives that $w(0_\eta)=0$. Therefore, one has, using (2.5), that

$$\gamma^2=\frac{\omega^4}{(1+\omega^2)^4}p\left(\frac{1}{s^2}\right) \;=\; \frac{\omega^4}{s^{10}}\frac{1+\mu^2s^4}{(1+\omega^2)^4} \tag{2.7}$$

Similarly with the Weierstrass point 1_η, one gets that

$$(\alpha+\beta+\gamma)^2=\frac{\omega^4}{(1+\omega^2)^4}p\left(\frac{1}{s^2\omega^2}\right) \;=\; \frac{1}{s^{10}\omega^6}\frac{1+\mu^2s^4\omega^4}{(1+\omega^2)^4}$$

For further use, we give

$$(\alpha+\beta)^2 \;=\; \frac{1}{s^{10}\omega^6}\left[\frac{1+\omega^{10}}{(1+\omega^2)^4}+\mu^2s^4\omega^4\frac{1+\omega^6}{(1+\omega^2)^4}\right] \tag{2.8}$$

$$(\alpha+\beta+\gamma)^2+\frac{\gamma^2}{\omega^4} \;=\; \frac{1}{s^{10}\omega^6}\left[\frac{1+\omega^6}{(1+\omega^2)^4}+\mu^2s^4\omega^4\frac{1}{(1+\omega^2)^3}\right] \tag{2.9}$$

$$(\alpha+\beta)^2+(1+\omega^2)^2\left((\alpha+\beta+\gamma)^2+\frac{\gamma^2}{\omega^4}\right) \;=\; \frac{1}{s^{10}\omega^2}\frac{1+\mu^2s^4\omega^2}{(1+\omega^2)^3} \tag{2.10}$$

Replacing x and y by their expressions (2.4) and (2.5), we deduce from (2.2) the equation

$$w^2 + z(z+1)w) = \frac{\omega^4}{(1+\omega^2)^4}\left[[(\alpha+\alpha^2)z^4 + (\alpha+\beta)z^3 + (\beta^2+\beta+\gamma)z^2 + \gamma z + \gamma^2] + p(x(z))\right]$$

Identifying the right-hand term of this equality and the right-hand term of (2.3) (both are degree 5 polynomials), we get 6 equalities, the first five of which are

$$\begin{cases} a = \dfrac{1+\omega^2}{s^{10}\omega^6} \\ b = \alpha^2 + \alpha + \dfrac{1}{s^{10}\omega^4} \\ a+b+c = \alpha+\beta + \dfrac{\mu^2}{s^6\omega^2(1+\omega^2)} \\ 0 = \beta^2+\beta+\gamma + \dfrac{\mu^2}{s^6(1+\omega^2)^2} \\ c = \gamma + \dfrac{\omega^2(1+\mu^2 s^4)}{s^{10}(1+\omega^2)^3} \end{cases} \tag{2.11}$$

Equation (2.7) gives γ and then, the last equality above gives

$$c = \frac{1}{s^{10}}\frac{\omega^2}{(1+\omega)^6}(1+\nu_0) \text{ with } \nu_0 = \mu^2 s^4 + s^5(1+\mu s^2)(1+\omega^2)). \tag{2.12}$$

Using the expressions of a and c, as well as the expression (2.9) of $\alpha+\beta$, the third equality gives

$$b = \frac{1}{s^{10}}\frac{1}{\omega^6(1+\omega)^6}(1+\nu_1) \text{ with } \nu_1 = \mu^2 s^4\omega^4 + s^5\omega^3(1+\omega^2)(1+\mu s^2\omega^2)). \tag{2.13}$$

2.2. *Remark.* Because of the choice of the Weierstrass points $1/s^2$ and $1/s^2\omega^2$ (which belong to $K^2 \subset K$), the scalars a, b and c lie in fact in K (but not in K^2). The same phenomenon occurs with the parameter μ (hence the choice of μ^2 in (2.1)).

2.3. *Remark.* On the other hand, α and β lie in the finite extension of $K \subset K^{1/2}$. Furthermore, α and β are not uniquely defined but their sum $\alpha+\beta$ is (and belongs to K). As it can be seen in the expression (2.6) of $w(x,y)$, this indeterminacy corresponds to the hyperelliptic involution of the curve $\mathcal{X}_\eta$ (which is given by the transformation $(x,\, y) \mapsto (x,\, y+q(x))$).

3 The space $H^0(\mathcal{J}\mathcal{X},\, \mathcal{O}(2\Theta))$ for a family of genus 2 curve $\mathcal{X}/R$

Let L be a field and C a smooth proper curve of genus 2 over L. If JC is the associated jacobian variety, the Abel-Jacobi map $AJ : C \times C \to JC$ is a surjective, separable of degree 2, morphism of L-varieties. The canonical divisor K_C gives a 2 sheets ramified cover $\pi : C \to \mathbb{P}^1_L$, and denote by $\bar{p}$ the involutive conjugate of any point p in C.

It's classical that the natural map $\mathrm{Sym}^2 C = \mathrm{Div}^2(C) \to JC$ identifies with the blowing-up $\widetilde{JC} \to JC$ of JC at the origin : $\mathrm{Div}^2(C) \to JC$ is a birational morphism of (nonsingular, projective)

surfaces and, using Riemann-Roch theorem for C, we see that $\bar{\Delta}$ is the only irreducible curve in $\mathrm{Div}^2(C)$ contracted to a point in JC, namely the origin. Using [H] (Chp. V, Cor. 5.4.), we find that $\mathrm{Div}^2(C) \to JC$ is a monoidal transformation, which is necessarily the monoidal transformation $\widetilde{JC} \to JC$ with center the origin since $\mathrm{Div}^2(C) \to JC$ factors through it (*ibid.* Prop. 5.3.).

Suppose there exists a L-rational ramification point ∞. Then, one can embed (non canonically) the curve C in JC by associating to each point $p \in C$ the degree 0 line bundle $\mathcal{O}(p - \infty)$. Denote by Θ the corresponding divisor on JC. Its support is the set $\{j \in JC / H^0(j \otimes \mathcal{O}_C(\infty)) \neq 0\}$ and it is well known that $\mathcal{O}(\Theta)$ is a principal polarization on the jacobian JC. Consequently, we have $\dim H^0(JC,\, \mathcal{O}(2\Theta)) = 4$.

When considering the family of genus 2 curve $\mathcal{X} \to \mathrm{Spec}\, R$ constructed in the previous section, we can extend the construction and therefore, $\mathcal{W} := H^0(\mathcal{J}\mathcal{X},\, \mathcal{O}(2\Theta))$ is a rank 4 free R-module.

3.1 Canonical theta functions

In the case of an ordinary genus 2 curve over an algebraically closed field of characteristic 2, π has three ramification points. Up to an automorphism of $\mathbb{P}^1_L$, one can assume that they are $0, 1$ and ∞, and that C is birational to a plane curve of the standard form (3), namely

$$w^2 + z(z+1)w = z(z+1)(az^3 + (a+b)z^2 + cz + c)$$

with a, b, c in L. Thus, the three non-zero 2-torsion points of $JC[2]$ are

$$[0] = AJ(1+\infty);\ [1] = AJ(0+\infty);\ [\infty] = AJ(0+1)$$

Furthermore, JC is given a canonical polarization Θ_B defined (see [R]) by means of the canonical theta-characteristic $B \cong \mathcal{O}_C(0+1+\infty) \otimes K_C^{-1}$.

The 4-dimensional L-vector space $W := H^0(JC,\, \mathcal{O}(2\Theta_B))$ is the unique irreducible representation of weight 1 of the Heisenberg group $G(\mathcal{O}(2\Theta_B))$, obtained as a central extension

$$0 \to \mathbb{G}_m \to G(\mathcal{O}(2\Theta_B)) \to JC[2] \to 0$$

(cf [Mu2] for the general theory of Heisenberg groups and [Sek] for the characteristic 2 case).

Taking a non-zero section θ of $H^0(JC,\, \mathcal{O}(\Theta_B))$ and setting

$$X_B = \theta^2;\ X_0 = [0]^* X_B;\ X_1 = [1]^* X_B;\ X_\infty = [\infty]^* X_B$$

one obtains a base (unique up to scalar if we ask for these conditions) of W (cf [LP1], section 2). As $\mathrm{Supp}(X_B) = \{j \in JC / H^0(j \otimes B) \neq 0\}$, we have $\mathrm{Supp}(X_\infty) = \{j \in JC / H^0(j \otimes [\infty] \otimes B) \neq 0\}$. But $[\infty] \otimes B = \mathcal{O}_C(0+1) \otimes \mathcal{O}_C(0+1+\infty) \otimes K_C^{-2}$ and using the fact that 0 and 1 are Weierstrass points of C, one finds $[\infty] \otimes B = \mathcal{O}_C(\infty)$. Thus, $\mathrm{Supp}(X_\infty) = \mathrm{Supp}(\Theta)$ and finally $2\Theta = [\infty]^*(2\Theta_B)$.

Following [LP2] (lemma 3.3), one can express, using the Abel-Jacobi map and an explicit theorem of the square ([AG]), the rational functions $\dfrac{X_B}{X_\infty}$, $\dfrac{X_0}{X_\infty}$, $\dfrac{X_1}{X_\infty} \in \bar{k}(JC) \subseteq \bar{k}(C \times C)$, . We have the following equalities in $\bar{k}(C \times C)$:

$$F_B := AJ^*\left(\frac{X_B}{X_\infty}\right) = \alpha_B \frac{(W_1 + W_2)^2}{P(z_1,\, z_2)} \text{ where } \begin{cases} W_i = \dfrac{w_i}{z_i(z_i+1)} \\ P(z_1,\, z_2) = \dfrac{(z_1+z_2)^2}{z_1 z_2 (z_1+1)(z_2+1)} \end{cases}$$

$$F_0 := AJ^*\left(\frac{X_0}{X_\infty}\right) = \alpha_0 z_1 z_2, \qquad F_1 := AJ^*\left(\frac{X_1}{X_\infty}\right) = \alpha_1 (z_1+1)(z_2+1)$$

with non-zero scalars α_B, α_0 and α_1 explicitly determined in terms of the scalars a, b, c appearing in the equation of the standard birational model of C, namely

$$\alpha_B = \frac{1}{\sqrt{bc}}; \qquad \alpha_0 = \sqrt{\frac{a}{c}}; \qquad \alpha_1 = \sqrt{\frac{a}{b}}$$

3.2 Case of the family $\mathcal{X} \to \mathbf{Spec}\, R$

The latter results only apply over the generic fiber $\mathcal{X}_\eta$ but we can use the coordinates change formulae (2.4) and (2.6) to express the rational functions F_B, F_0 and F_1 in terms of the R-coordinates $(x_i,\, y_i)$ instead of the K-coordinates $(z_i,\, w_i)$. First, using the expressions (2.11), (2.12) and (2.13) of coefficients a, b and c, and putting τ_0 and τ_1 for $\sqrt{\frac{1}{1+\nu_0}}$ and $\sqrt{\frac{1}{1+\nu_1}}$ respectively (which belong to $R[\sqrt{s}]$), we have

$$\sqrt{\frac{a}{c}} = \frac{1+\omega^4}{\omega^4}\tau_0; \qquad \sqrt{\frac{a}{b}} = (1+\omega^4)\tau_1; \qquad \sqrt{\frac{1}{bc}} = s^{10}\omega^2(1+\omega^2)\tau_0\tau_1 \tag{3.1}$$

Then, straightforward computations give

$$F_0 = \tau_0(1 + s^2(x_1+x_2) + s^4 x_1 x_2); \qquad F_1 = \tau_1(1 + s^2\omega^2(x_1+x_2) + s^4\omega^4 x_1 x_2) \tag{3.2}$$

and

$$\begin{aligned} F_B \;=\; & \tau_0\tau_1 \left[s^6\omega^2(1+\omega^2)\frac{1}{q(x_1)q(x_2)} \left(\frac{q(x_2)y_1 + q(x_1)y_2}{x_1+x_2}\right)^2 \right. \\ & + s^{18}\omega^6(1+\omega^2)^3(\alpha+\beta)^2\frac{(x_1x_2)^2}{q(x_1)q(x_2)} \\ & + \left(s^{14}\omega^6(1+\omega^2)^3\left((\alpha+\beta+\gamma)^2 + \frac{\gamma^2}{\omega^4}\right)\right)\frac{(x_1+x_2)^2}{q(x_1)q(x_2)} \\ & \left. + s^{10}\omega^2(1+\omega^2)^3\left((\alpha+\beta)^2 + (1+\omega^2)^2\left((\alpha+\beta+\gamma)^2+\frac{\gamma^2}{\omega^4}\right)\right)\frac{1}{q(x_1)q(x_2)}\right] \end{aligned}$$

We compute the coefficients of the last three terms, using equalities (2.8), (2.9) and (2.10) and finally obtain

$$\begin{aligned} F_B \;=\; & \tau_0\tau_1 \left[s^6\omega^2(1+\omega^2)\frac{1}{q(x_1)q(x_2)} \left(\frac{q(x_2)y_1 + q(x_1)y_2}{x_1+x_2}\right)^2 \right. \\ & + s^8\left(\frac{1+\omega^{10}}{1+\omega^2} + \mu^2 s^4\omega^4\frac{1+\omega^6}{1+\omega^2}\right)\frac{(x_1x_2)^2}{q(x_1)q(x_2)} \\ & \left. + s^4\left(\frac{1+\omega^6}{1+\omega^2} + \mu^2 s^4\omega^4\right)\frac{(x_1+x_2)^2}{q(x_1)q(x_2)} + (1+\mu^2 s^4\omega^2)\frac{1}{q(x_1)q(x_2)}\right] \end{aligned} \tag{3.3}$$

3.1. *Remark.* Notice that, as the product $q(x_1)q(x_2)$ specializes to 1 over the special fiber, the three rational functions $\frac{X_0}{X_\infty}$, $\frac{X_1}{X_\infty}$ and $\frac{X_B}{X_\infty}$ specialize to 1 over the special fiber. This parallelizes the specialization of the 2-torsion points $[0]_\eta$, $[1]_\eta$ and $[\infty]_\eta$ to 0.

3.3 Finding an R-base

Let's consider again a genus 2 curve C over an arbitrary field L. Denote by $\imath$ the hyperelliptic involution that permute the two sheets of the ramified double cover $\pi : C \to \mathbb{P}^1_L = |K_C|$ given by

the canonical map and assume that π is given a L-rational ramification point ∞. In the sequel, we will use the Abel-Jacobi map $AJ : C \times C \to JC$ and the divisor Θ on JC defined as the image of C in JC via the embedding $p \mapsto \mathcal{O}_C(p - \infty)$.

Lemma 3.2 *The pull-back $AJ^*(\Theta)$ is the divisor $(\{\infty\} \times C + C \times \{\infty\}) + \bar{\Delta}$ on $C \times C$, where $\bar{\Delta} = (Id \times \imath)^*\Delta$ and Δ is the diagonal in the product $C \times C$.*

Proof : Let M (resp. N) be the prime divisor of $\mathrm{Sym}^2 C$ with support the set $\{(p+\infty),\, p \in C\}$ (resp. the set $\{(p + \bar{p}),\, p \in C\}$). From [AG], we know that $b^*(\Theta) = M + nN$, where n is a non negative integer, and that $\sigma^*(M) = C \times \{\infty\} + \{\infty\} \times C$. Furthermore, as $\sigma^{-1}(N) = \bar{\Delta}$, $\sigma^*(N) = k\bar{\Delta}$, where k is an non negative integer satisfying the equation $\deg(\sigma)(N)^2 = k^2(\bar{\Delta})^2$. On one hand, since N is the exceptional curve of the blowing-up $b : \mathrm{Sym}^2 C \to JC$, we have $(N)^2 = -1$. On the other hand, we have $(\bar{\Delta})^2 = \deg_{\bar{\Delta}}(\mathcal{O}(\bar{\Delta}) \otimes \mathcal{O}_{\bar{\Delta}})$. But $\mathcal{O}(\bar{\Delta}) \otimes \mathcal{O}_{\bar{\Delta}} = \mathcal{N}_{\bar{\Delta}/C\times C}$ and taking determinants in the short exact sequence of $\mathcal{O}_{\bar{\Delta}}$-coherent sheaves

$$0 \to \mathcal{N}^{-1}_{\bar{\Delta}/C\times C} \to \Omega^1_{C\times C} \otimes \mathcal{O}_{\bar{\Delta}} \to \omega_{\bar{\Delta}} \to 0$$

we obtain $\omega_C \otimes \imath^*\omega_C \cong \omega_C^2 \cong \mathcal{N}^{-1}_{\bar{\Delta}/C\times C} \otimes \omega_{\bar{\Delta}}$ hence

$$\omega^{-1}_{\bar{\Delta}} \cong \mathcal{N}_{\bar{\Delta}/C\times C} \tag{3.4}$$

Therefore, $(\bar{\Delta})^2 = 2 - 2g_C = -2$, hence $k = 1$ and $AJ^*(\Theta) = C \times \{\infty\} + \{\infty\} \times C + n\bar{\Delta}$. One can compute self-intersection again to determine n. We write

$$\begin{aligned} \deg(AJ)(\Theta)^2 = (C \times \{\infty\})^2 \quad &+ \quad (\{\infty\} \times C)^2 + n^2(\bar{\Delta})^2 \\ &+ \quad 2[(C \times \{\infty\}).(\{\infty\} \times C) + n((C \times \{\infty\}).\bar{\Delta} + (\{\infty\} \times C).\bar{\Delta})] \end{aligned}$$

It's clear that $(C \times \{\infty\})^2$ (resp. $(\{\infty\} \times C)^2$) equals to zero for $C \times \{\infty\}$ (resp. $\{\infty\} \times C$) being algebraically equivalent to a divisor that doesn't meet $C \times \{\infty\}$ (resp. $\{\infty\} \times C$). It's clear as well that the intersection products $(C \times \{\infty\}).(\{\infty\} \times C)$, $(C \times \{\infty\}).\bar{\Delta}$ et $(\{\infty\} \times C).\bar{\Delta}$ equal to 1 since the two divisors in each pairs obviously meet transversally in a single point, namely (∞, ∞). Furthermore, using Riemann-Roch theorem for an abelian variety of dimension 2 and a principal (hence ample) divisor ([Mu1]), we have $(\Theta)^2 = 2$. Replacing, we obtain $n = 1$ $\square$

Define by p_i the canonical projection $C \times C \to C$ on the i-th factor ($i = 1, 2$). Because the canonical divisor of C is 2∞ and because $\Omega^1_{C\times C} \cong p_1^*(\omega_C) \oplus p_2^*(\omega_C)$, the canonical divisor $K_{C\times C}$ equals $2(C \times \{\infty\}) + 2(\{\infty\} \times C)$. Therefore, the previous lemma allows us to see the L-vector space $H^0(JC, \mathcal{O}(2\Theta))$ as the linear subspace of $H^0(C \times C, \mathcal{O}(K_{C\times C} + 2\bar{\Delta}))$ consisting in the set of symmetric sections (under the action of $\mathfrak{S}_2$) that take constant value along $\bar{\Delta}$.

Lemma 3.3 *The natural inclusion $H^0(C \times C, K_{C\times C}) \hookrightarrow H^0(C \times C, \mathcal{O}(K_{C\times C} + 2\bar{\Delta}))$ induces three linearly independent sections 1, $x_1 + x_2$ and x_1x_2 of $H^0(JC, \mathcal{O}(2\Theta))$.*

Proof : Consider the short exact sequence of $\mathcal{O}_{C\times C}$-coherent sheaves

$$0 \to \mathcal{O}(-\bar{\Delta}) \to \mathcal{O}_{C\times C} \to \mathcal{O}_{\bar{\Delta}} \to 0 \tag{3.5}$$

Tensoring with $\mathcal{O}(K_{C\times C}+2\bar{\Delta})$ and taking cohomology groups, one gets the exact sequence of L-vector spaces

$$0\to H^0(\mathcal{O}(K_{C\times C}+\bar{\Delta}))\to H^0(\mathcal{O}(K_{C\times C}+2\bar{\Delta}))\to H^0(\mathcal{O}(K_{C\times C}+2\bar{\Delta})\otimes\mathcal{O}_{\bar{\Delta}})$$

But $\mathcal{O}(K_{C\times C}+2\bar{\Delta})\otimes\mathcal{O}_{\bar{\Delta}}\cong\big(\mathcal{O}(K_{C\times C})\otimes\mathcal{O}(\bar{\Delta})\otimes\mathcal{O}_{\bar{\Delta}}\big)\otimes_{\mathcal{O}_{\bar{\Delta}}}\big(\mathcal{O}(\bar{\Delta})\otimes\mathcal{O}_{\bar{\Delta}}\big)$. The first term of this tensorial product is isomorphic to $\omega_{\bar{\Delta}}$ and the second to $\mathcal{N}_{\bar{\Delta}/C\times C}$. Thus, using the isomorphism (3.4), we obtain the structural sheaf $\mathcal{O}_{\bar{\Delta}}$ and we have an exact sequence

$$0\to H^0(\mathcal{O}(K_{C\times C}+\bar{\Delta}))\to H^0(\mathcal{O}(K_{C\times C}+2\bar{\Delta}))\to L$$

Tensoring (3.5) with $\mathcal{O}(K_{C\times C}+\bar{\Delta})$ and taking cohomology groups, one gets the exact sequence of L-vector spaces

$$0\to H^0(\mathcal{O}(K_{C\times C}))\to H^0(\mathcal{O}(K_{C\times C}+\bar{\Delta}))\to H^0(\mathcal{O}(K_{C\times C}+\bar{\Delta})\otimes\mathcal{O}_{\bar{\Delta}})\xrightarrow{\delta}H^1(\mathcal{O}(K_{C\times C}))$$

with $\mathcal{O}(K_{C\times C}+\bar{\Delta})\otimes\mathcal{O}_{\bar{\Delta}}\cong\omega_{\bar{\Delta}}$. By Serre duality, the dual exact sequence corresponds to the long exact sequence of cohomology groups associated to the short exact sequence (3.5). Therefore, using Serre duality again and Kunnëth isomorphism, one has the following commutative diagram of L-vector space morphisms

$$\begin{array}{ccc}
H^1(\mathcal{O}(K_{C\times C}))^\vee & \xrightarrow{\delta^\vee} & H^0(\omega_{\bar{\Delta}})^\vee \\
\uparrow\wr & & \uparrow\wr \\
H^1(\mathcal{O}_{C\times C}) & \longrightarrow & H^1(\mathcal{O}_{\bar{\Delta}}) \\
\uparrow\wr & & \uparrow\wr \\
H^1(\mathcal{O}_C)\oplus H^1(\mathcal{O}_C) & \xrightarrow{Id+H^1(\imath)} & H^1(\mathcal{O}_C)
\end{array}$$

As the bottom horizontal arrow is surjective, $\delta^\vee$ is surjective and δ is injective. Thus, the morphism $H^0(\mathcal{O}(K_{C\times C}))\to H^0(\mathcal{O}(K_{C\times C}+\bar{\Delta}))$ is an isomorphism and using the Kunnëth isomorphism again, one gets the following exact sequence of L-vector spaces

$$0\to H^0(\omega_C)\otimes H^0(\omega_C)\to H^0(\mathcal{O}(K_{C\times C}+2\bar{\Delta}))\to L$$

Denote by x the rational coordinate function of $\mathbb{P}^1_L$ with pole at ∞. Considering $H^0(\omega_C)$ as a sub-L-space of the function field $\bar{k}(C)$ by means of the differential dx, it has a base $\{1,\,x\}$. Thus, considering $H^0(\omega_C)\otimes H^0(\omega_C)$ as a sub-L-space of the function field $\bar{k}(C\times C)$, we find that $\big(H^0(\omega_C)\otimes H^0(\omega_C)\big)^{\mathfrak{S}_2}$ has a base $\{1,\,x_1+x_2,\,x_1x_2\}$. Note that the three corresponding sections of $H^0(\mathcal{O}(K_{C\times C}+2\bar{\Delta}))^{\mathfrak{S}_2}$ are in the kernel of the evaluation $H^0(\mathcal{O}(K_{C\times C}+2\bar{\Delta}))\to L\cong H^0(\mathcal{O}_{\bar{\Delta}})$ so they have constant value (equal to zero) along $\bar{\Delta}$, hence define sections of $H^0(JC,\,\mathcal{O}(2\Theta))$ □

3.4. *Remark.* Consider the exact sequence (3.5), tensored by $\mathcal{O}(K_{C\times C}+2\bar{\Delta})$ and localized at the generic point $\bar{\Delta}$ of the irreducible subscheme $\bar{\Delta}$ of $C\times C$. If A denotes the discrete valuation ring $\mathcal{O}_{C\times C,\,\bar{\Delta}}$, if t is a element of A that generates the maximal ideal, we find the exact sequence

$$0\to t^{-1}.A\to t^{-2}.A\to\bar{k}(\bar{\Delta})\to 0$$

Now, given $g\in H^0(\mathcal{O}(K_{C\times C}+2\bar{\Delta}))$, we can look at it as an element of $t^{-2}.A$ and write it as $a_{-2}t^{-2}+a_{-1}t^{-1}+\sum_{n\geq 0}a_nt^n$, where the a_i are *a priori* elements of the residue field $\bar{k}(\bar{\Delta})$. Its

class in $\bar{k}(\bar{\Delta})$ is a_{-2}, which lies in fact in L since the following diagram (where the vertical arrows are localization) commutes

$$\begin{array}{ccc} H^0(\mathcal{O}(K_{C\times C}+2\bar{\Delta})) & \longrightarrow & L \\ \downarrow & & \downarrow \\ t^{-2}.A & \longrightarrow & \bar{k}(\bar{\Delta}) \end{array}$$

Therefore, the morphism $H^0(\mathcal{O}(K_{C\times C}+2\bar{\Delta}))\to L$ amounts, in some sense, to the computation of a residue.

3.5. *Remark.* As $H^0(JC,\,\mathcal{O}(2\Theta))$ is of dimension 4, the morphism $H^0(\mathcal{O}(K_{C\times C}+2\bar{\Delta}))\to L$ cannot be zero and there must exist a symmetric rational function f on $C\times C$ such that

(1) $\mathrm{div}(f)+K_{C\times C}+2\bar{\Delta}$ is effective ;

(2) f has a pole of order 2 along $\bar{\Delta}$.

Thus, we'll have a base $\{1,\,x_1+x_2,\,x_1x_2,\,f\}$ of the vector space $H^0(JC,\,\mathcal{O}(2\Theta))$.

3.6. *Remark.* These two lemmas extend to the relative case of the family of genus 2 curves $\mathcal{X}\to\mathrm{Spec}\,R$, and we find three linearly independent sections 1, x_1+x_2 and x_1x_2 of $\mathcal{W}$.

We now restrict to the case of the family $\mathcal{X}\to\mathrm{Spec}\,R$. Note that F_B certainly satisfies condition (1) in the remark above : Over the generic point, it is the pull-back by AJ of the rational function $\dfrac{X_B}{X_\infty}$, where X_B and X_∞ are two sections of the line bundle $\mathcal{O}(2\Theta)$, which corresponds, by lemma (3.2), to the divisor $K_{\mathcal{X}_\eta\times\mathcal{X}_\eta}+2\bar{\Delta}$. Furthermore, the origin of $\mathcal{J}\mathcal{X}_\eta$ does not belong to $\mathrm{Supp}(\Theta_B)$: if it did, we would have $H^0(\mathcal{X}_\eta,B)\geq 1$ and there would exist a point p in $\mathcal{X}_\eta$ such that the divisors $0_\eta+1_\eta$ and $p+\infty$ were linearly equivalent, which is impossible since the three Weierstrass points of $\mathcal{X}_\eta$ are pairwise different. One the other hand, $H^0(\mathcal{X}_\eta,\mathcal{O}(\infty))\geq 1$ so $\dfrac{X_B}{X_\infty}$ has a pole at the origin, and by pull-back, F_B satisfies condition (2) over the generic fiber.

Of course, using lemma (3.3) and the remark (3.5), any linear combination

$$\mu\left(\frac{1}{\tau_0\tau_1}F_B+(\lambda+\lambda_\Sigma(x_1+x_2)+\lambda_\Pi(x_1x_2))\right)$$

with μ, λ, λ_Σ and λ_Π in K such that it belongs to the ring

$$A=R[x_1,\,x_2,\,y_1,\,y_2,\,(x_1+x_2)^{-1},\,(q(x_1)q(x_2))^{-1}]$$

will fulfill condition (1) over both generic and special fibers and condition (2) over generic fiber.

Using (3.3), recall that

$$\begin{aligned}\frac{1}{\tau_0\tau_1}F_B \;=\;& s^6\omega^2(1+\omega^2)\frac{1}{q(x_1)q(x_2)}\left(\frac{q(x_2)y_1+q(x_1)y_2}{x_1+x_2}\right)^2\\ &+s^8\left(\frac{1+\omega^{10}}{1+\omega^2}+\mu^2s^4\omega^4\frac{1+\omega^6}{1+\omega^2}\right)\frac{(x_1x_2)^2}{q(x_1)q(x_2)}\\ &+s^4\left(\frac{1+\omega^6}{1+\omega^2}+\mu^2s^4\omega^4\right)\frac{(x_1+x_2)^2}{q(x_1)q(x_2)}+(1+\mu^2s^4\omega^2)\frac{1}{q(x_1)q(x_2)}\end{aligned}$$

As $\dfrac{1}{q(x_1)q(x_2)}\left(\dfrac{q(x_2)y_1+q(x_1)y_2}{x_1+x_2}\right)^2$ obviously has a pole of order 2 along $\bar{\Delta}$ over both generic and special fibers, we look for λ, λ_Σ and λ_Π in K such that we can take $\mu = \dfrac{1}{s^6\omega^2(1+\omega^2)}$. Looking at the class of $\dfrac{1}{\tau_0\tau_1}F_B$ in A/s^6, we find

$$\frac{1}{q(x_1)q(x_2)}\left[s^4\frac{1+\omega^6}{1+\omega^2}(x_1+x_2)^2+(1+\mu^2s^4\omega^2)\right]$$

On the other hand, $(\lambda+\lambda_\Sigma(x_1+x_2)+\lambda_\Pi(x_1x_2))q(x_1)q(x_2)$ equals

$$\lambda+(\lambda_\Sigma+\lambda s^2(1+\omega^2))(x_1+x_2)+(\lambda_\Pi+\lambda s^4(1+\omega^4))(x_1x_2)+(\lambda s^4\omega^2+\lambda_\Sigma s^2(1+\omega^2))(x_1+x_2)^2$$

in A/s^6. Thus,we see that

$$\frac{1}{\tau_0\tau_1}F_B+(1+\mu^2s^4\omega^2)+s^2(1+\omega^2)(x_1+x_2)+s^4(1+\omega^4)(x_1x_2)$$

belongs to s^6A. A few more calculations show that it is in fact in $s^6\omega^2(1+\omega^2)A$ and we set

$$f=\frac{1}{s^6\omega^2(1+\omega^2)}\left[\frac{1}{\tau_0\tau_1}F_B+(1+\mu^2s^4\omega^2)+s^2(1+\omega^2)(x_1+x_2)+s^4(1+\omega^4)(x_1x_2)\right]$$

Therefore,

$$F_B=\tau_0\tau_1((1+\mu^2s^4\omega^2)+s^2(1+\omega^2)(x_1+x_2)+s^4(1+\omega^4)(x_1x_2)+s^6\omega^2(1+\omega^2)f) \qquad (3.6)$$

Proposition 3.7 *The projective space $\mathbb{P}\mathcal{W}\cong\mathbb{P}^3_R$ has homogeneous coordinates $\{z_1, z_2, z_3, z_\infty\}$ in terms of which, over the generic point $\operatorname{Spec}K\to\operatorname{Spec}R$, the canonical theta coordinates $\{x_B, x_0, x_1, x_\infty\}$ are given, up to scalar, by the formulae*

$$\begin{cases} x_B=z_1 \\ x_0=\tau_1(z_1+s^2\omega^2z_2+s^4\omega^4z_3) \\ x_1=\tau_0(z_1+s^2z_2+s^4z_3) \\ x_\infty=\tau_0\tau_1((1+\mu^2s^4\omega^2)z_1+s^2(1+\omega^2)z_2+s^4(1+\omega^4)z_3+s^6\omega^2(1+\omega^2)z_\infty) \end{cases}$$

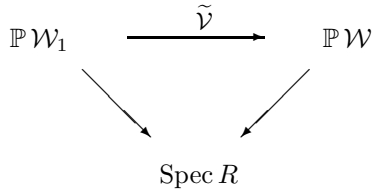
Proof : The previous calculations show that one can form an R-base $\{Z_\bullet\}$ of $\mathcal{W}$ by setting

$$\begin{array}{ll} Z_\infty=X_\infty & Z_1=f\,X_\infty \\ Z_2=x_1x_2X_\infty & Z_3=(x_1+x_2)X_\infty \end{array}$$

The expressions of the $X_\bullet$ in terms of the $Z_\bullet$ are then given by the equations (3.2) and (3.6). By duality, we easily deduce the corresponding base change for coordinates □

4 Equations of $\widetilde{V}$ for a supersingular genus 2 curve in characteristic 2

In [LP1] (section 5), it is shown that the morphism $D: M_X\to\mathbb{P}^3=|2\Theta|$ is, as in the complex case (see [NR]), an isomorphism when X is an ordinary genus 2 curve over a characteristic 2 algebraically closed field. As asserted in [LP2], this identification extends to the relative case $\mathcal{X}\to\operatorname{Spec}R$, so that Frobenius morphism $\mathcal{X}\to\mathcal{X}_1$ induces, by pull-back, a rational map



where $\mathcal{W}_1$ stands for the 2-twist $H^0(\mathcal{JX}_1, \mathcal{O}(2\Theta_1))$ of $\mathcal{W}$.

Over the generic point η, [LP1] (Proposition 3.1) gives the form of the rational map

$$\begin{array}{rcl} \widetilde{\mathcal{V}}_\eta : \mathbb{P}\,(\mathcal{W}_1)_\eta & \to & \mathbb{P}\,(\mathcal{W})_\eta \\ x = (x_\bullet) & \mapsto & (\lambda_B P_B(x) :\, \lambda_0 P_0(x) :\, \lambda_1 P_1(x) :\, \lambda_\infty P_\infty(x)) \end{array}$$

where the $x_\bullet$ are the theta coordinates of the spaces $(\mathcal{W}_1)_\eta$ and $(\mathcal{W})_\eta$ (which correspond via the K-semi-linear isomorphism $i^* : (\mathcal{W})_\eta \to (\mathcal{W}_1)_\eta$), where the $P_\bullet$ are the quadrics

$$\begin{array}{ll} P_B(x) = x_B^2 + x_0^2 + x_1^2 + x_\infty^2; & P_0(x) = x_B x_0 + x_1 x_\infty; \\ P_1(x) = x_B x_1 + x_0 x_\infty; & P_\infty(x) = x_B x_\infty + x_0 x_1. \end{array}$$

and where the $\lambda_\bullet$ are non zero constants depending on the curve X. In [LP2] (section 3), we find explicit determination (up to scalar) of these coefficients, namely

$$(\lambda_B :\, \lambda_0 :\, \lambda_1 :\, \lambda_\infty) = (\sqrt{abc} :\, \sqrt{c} :\, \sqrt{b} :\, \sqrt{a})$$

Using the base change formulae given in Proposition (3.7) and the expressions (2.11), (2.12) and (2.13) of coefficients a, b and c, we can compute the rational map

$$\widetilde{\mathcal{V}} : z = (z_\bullet) \mapsto (R_1(z) :\, R_2(z) :\, R_3(z) :\, R_\infty(z))$$

First, we express the polynomials $Q_\bullet(z) = P_\bullet(x)$ (notice that the coefficients appearing in the formulae given in Proposition (3.7) are squared since we are dealing with elements of the 2-twist $(\mathcal{W}_1)_\eta$). We find

$$\begin{array}{rcl} Q_B(\underline{z}) & = & (\tau_0\tau_1)^4 \left((\nu_0\nu_1 + \mu^4 s^8\omega^4)^2 z_1^2 + s^8(\nu_0\omega^4 + \nu_1)^2 z_2^2 + s^{16}(\nu_0\omega^8 + \nu_1)^2 z_3^2 + s^{24}\omega^8(1+\omega^8) z_\infty^2\right) \\ \\ Q_0(\underline{z}) & = & \tau_0^4\tau_1^2 \left((\nu_0 + \mu^2 s^4\omega^2)^2 z_1^2 + s^4\omega^4(\nu_0 + \mu^2 s^4)^2 z_1 z_2 + s^8\omega^4(\nu_0\omega^2 + \mu^2 s^4)^2 z_1 z_3\right. \\ & & \quad + s^8(1+\omega^4) z_2^2 + s^{12}\omega^4(1+\omega^4)(z_2 z_3 + z_1 z_\infty) + s^{16}(1+\omega^8) z_3^2 \\ & & \quad\quad \left. + s^{16}\omega^4(1+\omega^4)(z_2 z_\infty + s^4 z_3 z_\infty)\right) \\ \\ Q_1(\underline{z}) & = & \tau_0^2\tau_1^4 \left((\nu_1 + \mu^2 s^4\omega^2)^2 z_1^2 + s^4(\nu_1 + \mu^2 s^4\omega^4)^2 z_1 z_2 + s^8(\nu_1 + \mu^2 s^4\omega^6)^2 z_1 z_3\right. \\ & & \quad + s^8\omega^4(1+\omega^4) z_2^2 + s^{12}\omega^4(1+\omega^4)(z_2 z_3 + z_1 z_\infty) + s^{16}\omega^8(1+\omega^8) z_3^2 \\ & & \quad\quad \left. + s^{16}\omega^8(1+\omega^4)(z_2 z_\infty + s^4\omega^4 z_3 z_\infty)\right) \\ Q_\infty(\underline{z}) & = & (\tau_0\tau_1)^2 s^8\omega^4 \left((\mu^4 z_1^2 + z_2^2) + s^4(1+\omega^4)(z_2 z_3 + z_1 z_\infty) + s^8\omega^4 z_3^2\right) \end{array}$$

Secondly, using the coefficients $\lambda_\bullet$ and the formulae of the Proposition (3.7) again (after inversion and up to scalar), we obtain that

$$\begin{aligned}
R_1(z) &= \frac{1}{(\tau_0\tau_1)^4 s^{18}\omega^6(1+\omega)^6}\, Q_B(z) \\
R_2(z) &= \frac{1}{(\tau_0\tau_1)^4 s^{2}\omega^2(1+\omega^2)}\left[\frac{1}{s^{18}\omega^6(1+\omega^2)}\, Q_B(z) + \frac{1}{s^8(1+\omega^4)}(Q_0(z)+Q_1(z))\right] \\
R_3(z) &= \frac{1}{(\tau_0\tau_1)^4 s^{4}\omega^2(1+\omega^2)}\left[\frac{1}{s^{18}\omega^6(1+\omega^4)}\, Q_B(z) + \frac{1}{s^8\omega^2(1+\omega^4)}(\omega^2 Q_0(z)+Q_1(z))\right] \\
R_\infty(z) &= \frac{1}{(\tau_0\tau_1)^4 s^{6}\omega^2(1+\omega^2)}\left[\frac{1+\mu^2 s^4\omega^2}{s^{18}\omega^6(1+\omega)^6}\, Q_B(z)\right. \\
&\qquad \left. + \frac{1}{s^8\omega^4(1+\omega^4)}(\omega^4 Q_0(z)+Q_1(z)) + \frac{1}{s^8\omega^4} Q_\infty(z)\right]
\end{aligned}$$

Now, using the expansions of ν_0 and ν_1, we obtain that

$$\begin{aligned}
R_1(z) &= \mu^4 z_1^2 + z_2^2 + s^2\bar{R}_1(z) \\
R_2(z) &= z_1^2 + s^2\bar{R}_2(z) \\
R_3(z) &= (z_2 z_3 + z_1 z_\infty) + s^2\bar{R}_3(z) \\
R_\infty(z) &= (\mu^4 z_3^2 + z_\infty^2 + \mu^2 z_1^2 + z_1 z_2) + s^2\bar{R}_\infty(z)
\end{aligned} \tag{4.1}$$

where the $\bar{R}_\bullet(z)$ are quadrics with coefficients in $R[\omega^{-1}, (1+\omega)^{-1}]$.

4.1. *Remark.* A few more calculations show that the $\bar{R}_\bullet(z)$ coefficients lie in fact in R, so that we can choose any ω we want in $R - \{0, 1\}$ (and not necessarily an element that specializes in $k - \{0, 1\}$). It means that any deformation of X leads to the same equations.

Let's introduce the set of homogeneous coordinates $\{y_1, y_2, y_3, y_\infty\}$ of $|2\Theta|$ given by

$$\begin{aligned}
y_1 &= z_1, & y_2 &= z_2 + \mu z_1, \\
y_3 &= z_3, & y_\infty &= z_\infty + \mu z_3.
\end{aligned}$$

Thus, the corresponding coordinate change in $|2\Theta_1|$ is obtained by squaring the coefficients and we have proved the following :

Theorem 4.2 *Let X be a smooth supersingular genus 2 curve over a characteristic 2 algebraically closed field. There exist coordinates $\{z_\bullet\}$ (resp. $\{y_\bullet\}$) on $|2\Theta|$ (resp. $|2\Theta_1|$) such that the equations of $\widetilde{V}$ are given by*

$$\widetilde{V} : |2\Theta_1| \to |2\Theta|, \qquad y = (y_\bullet) \mapsto z = (z_\bullet) = (Q_1(y) : Q_2(y) : Q_3(y) : Q_\infty(y))$$

with

$$\begin{aligned}
Q_1(y) &= y_2^2, & Q_2(y) &= y_1^2, \\
Q_3(y) &= y_2 y_3 + y_1 y_\infty, & Q_\infty(y) &= y_\infty^2 + y_1 y_2.
\end{aligned}$$

5 Frobenius action on M_X for a supersingular genus 2 curve in characteristic 2

We know ([LP2], section 3) that the Kummer surface $\mathrm{Kum}_{\mathcal{X}_\eta}$ is given, in terms of the theta coordinates $\{x_\bullet\}$ on $|2\Theta|_\eta$, by the homogeneous quartic

$$c(x_B^2 x_0^2 + x_1^2 x_\infty^2) + b(x_B^2 x_1^2 + x_0^2 x_2 \infty^2) + a(x_B^2 x_\infty^2 + x_0^2 x_1^2) + x_B x_0 x_1 x_\infty$$

the a, b, c being the scalars appearing in the standard birational model (2.3) of the curve $\mathcal{X}_\eta$.

The same kind of calculations as in section 4 give the equation of Kum_X in the coordinate system $\{z_\bullet\}$ of $|2\Theta|$, namely

$$\mu^2 z_1^3 z_2 + z_1^3 z_\infty + z_1^2 z_2 z_3 + \mu^4 z_1^2 z_3^2 + z_1 z_2^3 + z_2^2 z_\infty^2 + z_3^4 \tag{5.1}$$

As in [LP1], we easily deduce from theorem (4.2) and from the latter calculations a complete description of the action of Frobenius on M_X, more precisely of its separable part $V : M_{X_1} \to M_X$.

Proposition 5.1 *Let X be a smooth supersingular genus 2 curve over a characteristic 2 algebraically closed field.*

1. *The semi-stable boundary of M_X (resp. M_{X_1}) is isomorphic to Kummer's quartic surface Kum_X (resp. Kum_{X_1}) the equation of which is (5.1). In particular, $\widetilde{V}$ maps Kum_{X_1} onto Kum_X.*
2. *There is a unique stable bundle $E_{BAD} \in M_{X_1}$ which is destabilized by Frobenius (ie F^*E_{BAD} is not semi-stable). We have $E_{BAD} = F_*B^{-1}$ and its projective coordinates are $(0:0:1:0)$.*
3. *Let H_1 be the hyperplane in $|2\Theta_1|$ defined by $y_2 = 0$. The map $\widetilde{V}$ contracts H_1 to the conic $\mathrm{Kum}_X \cap H$, where H is the hyperplane in $|2\Theta|$ defined by $z_1 = 0$.*
4. *The fiber of $\widetilde{V}$ over a point $[E] \in M_X$ is*
   - *a single point $[E_1] \in M_{X_1}$, if $[E] \notin H$,*
   - *empty, if $[E] \in H \setminus \mathrm{Kum}_X \cap H$,*
   - *a projective line passing through E_{BAD}, if $[E] \in \mathrm{Kum}_X \cap H$.*

   In particular, $\widetilde{V}$ is dominant, non surjective. The separable degree of $\widetilde{V}$ is 1.

Proof : The fact that the Kummer's quartic surface Kum_X is the semi-stable boundary of M_X comes from [NR]. On one hand, the inverse image of Kum_X is a closed subspace in M_{X_1} the ideal of which is generated by the homogeneous polynomial obtained by replacing the $\{z_\bullet\}$ in equation (5.1) by the $\{Q_\bullet(y)\}$ given in theorem (4.2), namely

$$y_2^4[y_1^3 y_\infty + \mu^2 y_1^2 y_2^2 + y_1^2 y_2 y_3 + \mu^4(y_1^2 y_\infty^2 + y_2^2 y_3^2) + y_1 y_2^3 + y_2^2 y_\infty^2 + y_3^4]$$

One the other hand, we obtain the equation

$$y_1^3 y_\infty + \mu^2 y_1^2 y_2^2 + y_1^2 y_2 y_3 + \mu^4(y_1^2 y_\infty^2 + y_2^2 y_3^2) + y_1 y_2^3 + y_2^2 y_\infty^2 + y_3^4$$

of Kum_{X_1} in $|2\Theta_1|$ by squaring the coefficients in equation (5.1) and replacing the coordinates $\{z_\bullet\}$ on $|2\Theta_1|$ by their expressions

$$\begin{aligned} z_1 &= y_1, & z_2 &= y_2 + \mu^2 y_1, \\ z_3 &= y_3, & z_\infty &= y_\infty + \mu^2 y_3; \end{aligned}$$

in terms of the $\{y_\bullet\}$. Thus, $\widetilde{V}^{-1}(\mathrm{Kum}_X) = \mathrm{Kum}_{X_1} \bigcup H_1$ (where $H_1 = (y_2 = 0)$), hence (1). Furthermore, H_1 is mapped onto the hyperplane $H = (z_1 = 0)$ of M_X, hence (3).
In [LP1] (Prop. 6.1), it is shown that there is exactly one base point, namely F_*B^{-1}. Now, solving $Q_\bullet(y) = 0$, we find that $(0:0:1:0)$ is the unique base point of $\widetilde{V}$, hence (2).
Let's $[E]$ be a k-point of M_X, with coordinates $(a^2 : b^2 : c^2 : d^2)$, and solve the system

$$y_2^2 = a^2; \quad y_1^2 = b^2; \quad y_2 y_3 + y_1 y_\infty = c^2; \quad y_\infty^2 + y_1 y_2 = d^2. \tag{5.2}$$

We must have $y_1 = b$ and $y_2 = a$, so $y_\infty = d + \sqrt{ab}$ and finally $ay_3 = c^2 + bd + b\sqrt{ab}$. If $[E]$ belongs to H (ie if $a = 0$), it has solutions iff $c^2 = bd$, that is iff $[E]$ belongs to $H \cap \mathrm{Kum}_X$. In that case, there is no condition on y_3 and the inverse image of such a point is a projective line passing through E_{BAD}. If $[E]$ does not belong to H (ie if $a \neq 0$), we see that the inverse image is a single point $\square$

Denote by N_X (resp. N_{X_1}) the moduli of semi-stable bunbles with rank 2 and degree 0 over X (resp. X_1). In the case of an ordinary genus 2 curve, [LP1] (prop. 6.4) show the surjectivity of the (rational map) Verschiebung $N_{X_1} \to N_X; [E_1] \mapsto [F^* E_1]$. In the case of a supersingular curve, this result doesn't hold any more. Let E be a vector bundle in N_X. JX being divisible, we can assume that $\det E$ is trivial. Now, if $[F^* E_1] = [E]$, we have $F^*(\det E_1) = \det E$ and $\det E_1$ has to be a 2-torsion point of JX_1. Therefore, $\det E_1$ is trivial as well and the following appears as a corollary of proposition (5.1) :

Proposition 5.2 *Let X be a supersingular genus 2 curve over a characteristic 2 algebraically closed field. The rational map $N_{X_1} \to N_X$ given by $[E_1] \mapsto [F^* E_1]$ is not surjective.*

Now, we are interested in those vector bundle over X that are destabilized by a finite number of iteration of the (absolute) Frobenius. Let Ω^{Frob} be the complementary set of classes of semi-stable bundles E with trivial determinant over X such that ${F_{abs}^{(n)}}^* E$ is semi-stable for all $n \geq 1$.

Proposition 5.3 *Let X be a supersingular genus 2 curve over a characteristic 2 algebraically closed field. The open subset $M_X - \{E_{BAD}\}$ is stable under the action of the (absolute) Frobenius. In particular, $\Omega^{Frob} = M_X - \{E_{BAD}\}$ and is Zariski dense in M_X.*

Proof : Pulling-back a semi-stable bundle E over X by the semi-linear isomorphism $i : X_1 \to X$, we obtain a semi-stable bundle $i^* E$ over X_1. If E has trivial determinant, $i^* E$ has trivial determinant as well. If E ($\neq E_{BAD}$) has coordinates $(a : b : c : d)$ in the system $\{z_\bullet\}$, $i^* E$ has squared coordinates in the corresponding system $\{z_\bullet\}$ of M_{X_1}. Using theorem (4.2) (actually, the quadrics given by equations (4.1) from which we deduce theorem (4.2) after the mentioned coordinates change), we find that $F_{abs}^* E = V^*(i^* E)$ has coordinates

$$(\mu^4 a^4 + b^4 : a^4 : a^2 d^2 + b^2 c^2 : \mu^4 c^4 + d^4 + \mu^2 a^4 + a^2 b^2)$$

in the system $\{z_\bullet\}$, hence cannot be E_{BAD} $\square$

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