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The action of the Frobenius map on rank 2 vector bundles over a supersingular genus 2 curve in characteristic 2

Laurent Ducrohet

July 11, 2005

1 Introduction

Let X be a smooth proper genus g curve over a field k of characteristic $p > 0$. Denote by $M_X(r)$ the moduli space of semistable vector bundles of rank r and trivial determinant. If X_1 is the curve $X \times_{k,\sigma} k$, where $\sigma : k \to k$ is the Frobenius of k, the k-linear relative Frobenius $F : X \to X_1$ induces by pull-back a (rational) map $V : M_{X_1}(r) \dashrightarrow M_X(r)$. We are interested in determining, e.g., the surjectivity and the degree of V, the density of Frobenius-stable bundles, and the loci of Frobenius-destabilized bundles, with the aim of studying the behavior of the sequence $n \mapsto F_{\text{abs}}^{(n)^*}E$ of pull-back by n-fold iterated (absolute) Frobenius of a fixed rank r vector bundle E. Our interest in those questions comes from a result (proved in [LS]) which claims that a semistable rank r vector bundle E corresponds to an (irreducible) continuous representation of the algebraic fundamental group $\pi_1(X)$ in $\mathrm{GL}_r(\bar{k})$ (endowed with the discrete topology) if and only if one can find an integer $n > 0$ such that $F_{\text{abs}}^{(n)^*}E \cong E$.

For general (g, r, p), not much seems to be known (see the introductions of [LP1] and [LP2] for an overview of this subject).

When $g = 2$, $r = 2$, the isomorphism $D : M_X \to |2\Theta| \cong \mathbb{P}^3$ (see [NR] for the complex case) remains valid for an algebraically closed field of positive characteristic (see [LP1], section 5 for a sketch of proof in the characteristic 2 case) and we have the commutative diagram

$$\begin{tikzcd} M_{X_1}(2) \arrow[r, "V"] \arrow[d, "D"'] & M_X(2) \arrow[d, "D"] \\ |2\Theta_1| \arrow[r, "\widetilde{V}"] & |2\Theta| \end{tikzcd}$$

Furthermore, the semistable boundary of the moduli space $M_X(2)$ identifies via D with the Kummer quartic surface Kum_X, which is canonically contained in the linear system

$|2\Theta|$, and $\widetilde{V}$ maps Kum_{X_1} to Kum_X. In [LP2], it is shown that $\widetilde{V}$ is given by degree p polynomials and always has base-points.

When $p = 2$ and X is an ordinary curve, [LP1] determined the quadric equations of $\widetilde{V}$ in terms of the generalized theta constants of the curve X and thus, they could answer the above mentioned questions. In [LP2], they could give the equations of $\widetilde{V}$ in case of a nonordinary curve X with Hasse-Witt invariant equal to 1 by specializing a family $\mathcal{X}$ of genus 2 curves parameterized by a discrete valuation ring with ordinary generic fiber and special fiber isomorphic to X. In particular, they determined the coefficients of the quadrics of $\widetilde{\mathcal{V}}_\eta$, which coincide with the Kummer quartic surface coefficients, in terms of the coefficients of an affine equation for a birational model of the ordinary curve $\mathcal{X}_\eta$.

In this paper, we complete the study of the $(g, r, p) = (2, 2, 2)$ case by giving the equations of $\widetilde{V}$ in case of a supersingular curve X (Theorem 4.2). We adapt the strategy of [LP2]. Namely, we consider a family $\mathcal{X}$ of genus 2 curves parameterized by a discrete valuation ring R with ordinary generic fiber and special fiber isomorphic to X. In order to find an R-basis of the free R-module $\mathcal{W} := H^0(\mathcal{J}\mathcal{X}, 2\Theta)$ and to express the canonical theta functions of order 2 in that basis, we pull back 2Θ on the product $\mathcal{X} \times \mathcal{X}$ via the Abel-Jacobi map $\mathcal{X} \times \mathcal{X} \to \mathcal{J}\mathcal{X}$ and make use of an explicit theorem of the square for hyperelliptic Jacobians (see [AG]). As for the two other cases, we can easily deduce a full description of the Verschiebung $V : M_{X_1}(2) \dashrightarrow M_X(2)$ (Proposition 5.1).

I would like to thank C. Pauly for helpful discussions and Y. Laszlo for having introduced me to this question, for his help and encouragements.

2 Deformation of genus 2 curves

2.1 Specializing an ordinary curve

Let k be an algebraically closed field of characteristic 2, let R be the discrete valuation ring $k[[s]]$, and let K be its fraction field. We consider a proper, smooth, and supersingular curve X of genus 2 over k. By [L], we know that there exists a unique $\mu \in k$ such that X is birationally equivalent to the plane curve given by the equation

$$y^2 + y \;=\; x^5 + \mu^2 x^3 \tag{2.1}$$

The projection $(x, y) \mapsto x$ is the restriction of the ramified double cover $\pi : X \to \mathbb{P}^1_k = |K_X|$ (where K_X is the canonical divisor of X) with a single Weierstrass point, namely ∞.

Let us choose an element ω of $R - \{0, 1\}$ and denote by $\mathcal{X}$ the R-scheme defined by the two affine charts

$$y^2 + (s^2x + 1)(s^2\omega^2 x + 1)y = x^5 + \mu^2 x^3 \tag{2.2}$$

and

$$\widetilde{y}^2 + \widetilde{x}(s^2 + \widetilde{x})(s^2\omega^2 + \widetilde{x})\widetilde{y} = \widetilde{x} + \mu^2\widetilde{x}^3$$

glued by the isomorphism given by $x \mapsto \widetilde{x}^{-1}$ and $y \mapsto \widetilde{y}(\widetilde{x}^{-3})$. The R-scheme $\mathcal{X}$ is proper and smooth, and again, the projection $(x,\, y) \mapsto x$ is the restriction of an R-morphism $\mathcal{X} \to \mathbb{P}^1_R$, still denoted by π. It is convenient to introduce the notation

$$\begin{cases} q(x) = (s^2x+1)(s^2\omega^2x+1) = s^4\omega^2x^2 + s^2(1+\omega^2)x + 1 \\ p(x) = x^5 + \mu^2x^3 \end{cases}$$

The special fiber $\mathcal{X}_0$ is isomorphic to X, the generic fiber $\mathcal{X}_\eta$ is a proper and smooth ordinary curve of genus 2 over K, and $\pi_\eta : \mathcal{X}_\eta \to \mathbb{P}^1_K$ is a ramified double cover with Weierstrass points 0_η (with coordinate $1/s^2$), 1_η (with coordinate $1/s^2\omega^2$) and ∞. Thus, we have $\mathcal{J}\mathcal{X}[2]_\eta \cong (\mathbb{Z}/2\mathbb{Z})^2 \times \mu_2^2/K$ and $\mathcal{J}\mathcal{X}[2]_0$ is a self dual local-local group scheme over k of dimension 2 and height 4.

One can define (over R) the Abel-Jacobi map

$$\begin{array}{rccc} AJ: & \mathcal{X}\times\mathcal{X} & \to & \mathcal{J}\mathcal{X} \\ & (x,\, y) & \mapsto & \mathcal{O}(x+y)\otimes K_{\mathcal{X}}^{-1} \end{array}$$

as the compositum of the quotient map $\mathcal{X}\times\mathcal{X} \to \mathrm{Sym}^2\mathcal{X} = \mathrm{Div}^2(\mathcal{X})$ under the natural action of $\mathfrak{S}_2$ by the natural map $\mathrm{Div}^2\mathcal{X} \to \mathcal{J}\mathcal{X}$. It follows from the Riemann-Roch theorem that it is surjective, separable of degree 2. Denote by

$$[0]_\eta = AJ(1_\eta + \infty) \qquad [1]_\eta = AJ(0_\eta + \infty) \qquad [\infty]_\eta = AJ(0_\eta + 1_\eta)$$

the three nonzero elements of $\mathcal{J}\mathcal{X}[2]^{\mathrm{et}}_\eta$ and notice that they specialize to 0.

2.2 Standard birational model

There exists a unique triplet $(a,\, b,\, c)$ of elements in the algebraic closure $\bar{K}$ of K such that $\mathcal{X}_\eta$ is birationally equivalent to the plane curve given by the equation

$$w^2 + z(z+1)w = z(z+1)(az^3 + (a+b)z^2 + cz + c) \tag{2.3}$$

and such that, with these coordinates, $0_\eta = 0$, $1_\eta = 1$ and $\infty = \infty$. These latter conditions amount to an ordering of the Weierstrass points of $\mathcal{X}_\eta$, and this is equivalent to an isomorphism $\mathbb{F}_2^2 \xrightarrow{\sim} \mathcal{J}\mathcal{X}[2]^{\mathrm{et}}_\eta$, i.e., a level-2-structure on X_η (see [DO] for the characteristic 0 case).

One can determine explicitly this coordinate change. As x and z are rational coordinates on $\mathbb{P}^1_K$, the function $z = z(x)$ has to induce a K-isomorphism of $\mathbb{P}^1_K$, which fixes ∞, sends $1/s^2$ to 0 and $1/s^2\omega^2$ to 1. Thus, we have

$$z(x) = \frac{\omega^2}{1+\omega^2}(s^2x+1) \quad\Leftrightarrow\quad x(z) = \frac{1+\omega^2}{s^2\omega^2}z + \frac{1}{s^2} \tag{2.4}$$

In particular, one has $q(x) = \dfrac{(1+\omega^2)^2}{\omega^2}z(z+1)$ and replacing in (2.2), one sees that the function $y(z,\, w)$ has to be of the form

$$y(z,\, w) = \frac{(1+\omega^2)^2}{\omega^2}[w + (\alpha z^2 + \beta z + \gamma)] \tag{2.5}$$

with α, β and γ in $\bar{K}$. Conversely, we have

$$w(x,\, y) = \frac{\omega^2}{(1+\omega^2)^2}\left[y + \alpha\, q(x) + (1+\omega^2)\left[(\alpha+\beta)s^2x + \left((\alpha+\beta+\gamma) + \frac{\gamma}{\omega^2}\right)\right]\right] \quad (2.6)$$

2.1. *Remark.* At the Weierstrass point 0_η, (2.2) gives $y(0_\eta)^2 = p\left(\frac{1}{s^2}\right)$ and (2.3) gives $w(0_\eta) = 0$. Therefore, one has, using (2.5),

$$\gamma^2 = \frac{\omega^4}{(1+\omega^2)^4}\; p\left(\frac{1}{s^2}\right) \quad = \quad \frac{\omega^4}{s^{10}}\frac{1+\mu^2 s^4}{(1+\omega^2)^4} \quad (2.7)$$

Similarly with the Weierstrass point 1_η, one gets

$$(\alpha+\beta+\gamma)^2 = \frac{\omega^4}{(1+\omega^2)^4}\; p\left(\frac{1}{s^2\omega^2}\right) \quad = \quad \frac{1}{s^{10}\omega^6}\frac{1+\mu^2 s^4\omega^4}{(1+\omega^2)^4}$$

For further use, we observe

$$(\alpha+\beta)^2 \quad = \quad \frac{1}{s^{10}\omega^6}\left[\frac{1+\omega^{10}}{(1+\omega^2)^4} + \mu^2 s^4\omega^4\frac{1+\omega^6}{(1+\omega^2)^4}\right] \quad (2.8)$$

$$(\alpha+\beta+\gamma)^2 + \frac{\gamma^2}{\omega^4} \quad = \quad \frac{1}{s^{10}\omega^6}\left[\frac{1+\omega^6}{(1+\omega^2)^4} + \mu^2 s^4\omega^4\frac{1}{(1+\omega^2)^3}\right] \quad (2.9)$$

and

$$(\alpha+\beta)^2 + (1+\omega^2)^2\left((\alpha+\beta+\gamma)^2 + \frac{\gamma^2}{\omega^4}\right) \quad = \quad \frac{1}{s^{10}\omega^2}\frac{1+\mu^2 s^4\omega^2}{(1+\omega^2)^3} \quad (2.10)$$

Replacing x and y by their expressions (2.4) and (2.5), we deduce from (2.2) the equation

$$w^2 + z(z+1)w = \frac{\omega^4}{(1+\omega^2)^4}\left[[(\alpha+\alpha^2)z^4 + (\alpha+\beta)z^3 + (\beta^2+\beta+\gamma)z^2 + \gamma z + \gamma^2] + p(x(z))\right]$$

Identifying the right-hand term of this equality and the right-hand term of (2.3) (both are degree 5 polynomials), we get six equalities, the first five of which are

$$\begin{cases} a = \dfrac{1+\omega^2}{s^{10}\omega^6} \\ b = \alpha^2 + \alpha + \dfrac{1}{s^{10}\omega^4} \\ a+b+c = \alpha+\beta + \dfrac{\mu^2}{s^6\omega^2(1+\omega^2)} \\ 0 = \beta^2+\beta+\gamma + \dfrac{\mu^2}{s^6(1+\omega^2)^2} \\ c = \gamma + \dfrac{\omega^2(1+\mu^2 s^4)}{s^{10}(1+\omega^2)^3} \end{cases} \quad (2.11)$$

Equation (2.7) gives γ and then, the last equality above gives

$$c = \frac{1}{s^{10}} \frac{\omega^2}{(1+\omega)^6}(1+\nu_0) \text{ with } \nu_0 = \mu^2 s^4 + s^5(1+\mu s^2)(1+\omega^2). \tag{2.12}$$

Using the expressions of a and c, as well as the expression (2.9) of $\alpha+\beta$, the third equality gives

$$b = \frac{1}{s^{10}} \frac{1}{\omega^6(1+\omega)^6}(1+\nu_1) \text{ with } \nu_1 = \mu^2 s^4 \omega^4 + s^5 \omega^3 (1+\omega^2)(1+\mu s^2 \omega^2). \tag{2.13}$$

2.2. *Remark.* Notice that the scalars a, b and c lie in fact in a finite purely inseparable degree 2 extension of the subfield of K generated by the coefficients in the equation (2.2). That is why, for sake of simplicity, we have chosen Weierstrass points $1/s^2$ and $1/s^2\omega^2$ (which are squares in K) and the parameter μ^2 in equation (2.1).

2.3. *Remark.* The second and the fourth equations of (2.11) show that the coefficients α and β lie in the finite extension $K^{1/2}$ of K and that they are not uniquely defined (but their sum $\alpha+\beta$ is, and belongs to K). This indeterminacy follows from the fact that the conditions we impose only determine an automorphism of the projective line $\mathbb{P}^1_K$ and thus, an automorphism of $\mathcal{X}_\eta$, defined up to the exchange of the two sheets of the ramified cover $\pi : \mathcal{X}_\eta \to \mathbb{P}^1_K$, i.e., up to composition by the hyperelliptic involution of $\mathcal{X}_\eta$ (which is defined by the transformation $(z,\, w) \mapsto (z,\, w + z(z+1))$). More precisely, if α is a root of the second equation of (2.11), $\alpha+1$ is the other one and, changing α (resp. β) into $\alpha+1$ (resp. $\beta+1$) in (2.5), we find that

$$\begin{aligned} y'(z,\, w) &:= \frac{(1+\omega^2)^2}{\omega^2}[w + ((\alpha+1)z^2 + (\beta+1)z + \gamma)] \\ &= \frac{(1+\omega^2)^2}{\omega^2}[(w + z(z+1)) + (\alpha z^2 + \beta z + \gamma)] \\ &= y(z,\, w + z(z+1)). \end{aligned}$$

3 The space $H^0(\mathcal{J}\mathcal{X},\, \mathcal{O}(2\Theta))$ for a genus 2 curve $\mathcal{X}/R$

Let L be a field and C a smooth proper curve of genus 2 over L. If JC is its associated Jacobian variety, the Abel-Jacobi map $AJ : C \times C \to JC$ is a surjective, separable morphism of degree 2 of L-varieties. The canonical divisor K_C gives a two-sheeted ramified cover $\pi : C \to \mathbb{P}^1_L$, and we denote by $\bar{p}$ the involutive conjugate of any point p in C.

It is classical that the natural map $\mathrm{Sym}^2 C = \mathrm{Div}^2(C) \to JC$ can be identified with the blowing up $\widetilde{JC} \to JC$ of JC at the origin: $\mathrm{Div}^2(C) \to JC$ is a birational morphism of (nonsingular, projective) surfaces and, using the Riemann-Roch theorem for C, we see that $\bar{\Delta}$ is the only irreducible curve in $\mathrm{Div}^2(C)$ contracted to a point in JC, namely the

origin. Using [H] (Chapter V, Corollary 5.4.), we find that $\mathrm{Div}^2(C) \to JC$ is a monoidal transformation, which is necessarily the monoidal transformation $\widetilde{JC} \to JC$ with center the origin since $\mathrm{Div}^2(C) \to JC$ factors through it (*ibid.* Proposition 5.3.).

Suppose there exists an L-rational ramification point ∞. Then, one can embed (non-canonically) the curve C in JC by associating to each point $p \in C$ the degree 0 line bundle $\mathcal{O}(p - \infty)$. Denote by Θ the corresponding divisor on JC. Its support is the set $\{j \in JC \mid H^0(j \otimes \mathcal{O}_C(\infty)) \neq 0\}$ and it is well-known that $\mathcal{O}(\Theta)$ is a principal polarization on the jacobian JC. Consequently, we have $\dim H^0(JC, \mathcal{O}(2\Theta)) = 4$.

When considering the genus 2 curve $\mathcal{X} \to \operatorname{Spec} R$ constructed in the previous section, we can extend the construction and therefore, $\mathcal{W} := H^0(\mathcal{JX}, \mathcal{O}(2\Theta))$ is a rank 4 free R-module.

3.1 Canonical theta functions

In the case of an ordinary genus 2 curve over an algebraically closed field of characteristic 2, π has three ramification points. Upon composing with an automorphism of $\mathbb{P}^1_L$, one can assume that they are 0, 1 and ∞, and that C is birational to a plane curve of the standard form (3), namely

$$w^2 + z(z+1)w = z(z+1)(az^3 + (a+b)z^2 + cz + c)$$

with a, b, c in L. Thus, the three nonzero 2-torsion points of $JC[2]$ are

$$[0] = AJ(1+\infty) \qquad [1] = AJ(0+\infty) \qquad [\infty] = AJ(0+1)$$

Furthermore, JC has a canonical polarization Θ_B defined (see [R]) by means of the canonical theta-characteristic $B \cong \mathcal{O}_C(0+1+\infty) \otimes K_C^{-1}$.

The 4-dimensional L-vector space $W := H^0(JC, \mathcal{O}(2\Theta_B))$ is the unique irreducible representation of weight 1 of the Heisenberg group $G(\mathcal{O}(2\Theta_B))$, obtained as a central extension

$$0 \to \mathbb{G}_m \to G(\mathcal{O}(2\Theta_B)) \to JC[2] \to 0$$

(*cf.* [Mu2] for the general theory of Heisenberg groups and [Sek] for the characteristic 2 case).

Taking a nonzero section θ of $H^0(JC, \mathcal{O}(\Theta_B))$ and setting

$$X_B = \theta^2 \qquad X_0 = [0]^* X_B \qquad X_1 = [1]^* X_B \qquad X_\infty = [\infty]^* X_B$$

one obtains a basis (unique up to scalar if we ask for these conditions) of W (*cf.* [LP1], section 2). As $\mathrm{Supp}(X_B) = \{j \in JC \mid H^0(j \otimes B) \neq 0\}$, we have $\mathrm{Supp}(X_\infty) = \{j \in JC \mid H^0(j \otimes [\infty] \otimes B) \neq 0\}$. But $[\infty] \otimes B = \mathcal{O}_C(0+1) \otimes \mathcal{O}_C(0+1+\infty) \otimes K_C^{-2}$ and using the fact that 0 and 1 are Weierstrass points of C, one finds $[\infty] \otimes B = \mathcal{O}_C(\infty)$. Thus, $\mathrm{Supp}(X_\infty) = \mathrm{Supp}(\Theta)$ and finally $2\Theta = [\infty]^*(2\Theta_B)$.

Following [LP2] (Lemma 3.3), one can express, using the Abel-Jacobi map and an explicit theorem of the square ([AG]), the rational functions $\frac{X_B}{X_\infty}, \frac{X_0}{X_\infty}, \frac{X_1}{X_\infty} \in \bar{k}(JC) \subseteq \bar{k}(C \times C)$. We have the following equalities in $\bar{k}(C \times C)$:

$$F_B := AJ^*\left(\frac{X_B}{X_\infty}\right) = \alpha_B \frac{(W_1+W_2)^2}{P(z_1,\, z_2)} \text{ where } \begin{cases} W_i = \dfrac{w_i}{z_i(z_i+1)} \\ P(z_1,\, z_2) = \dfrac{(z_1+z_2)^2}{z_1 z_2 (z_1+1)(z_2+1)} \end{cases}$$

$$F_0 := AJ^*\left(\frac{X_0}{X_\infty}\right) = \alpha_0 z_1 z_2, \quad F_1 := AJ^*\left(\frac{X_1}{X_\infty}\right) = \alpha_1 (z_1+1)(z_2+1)$$

with nonzero scalars α_B, α_0 and α_1 explicitly determined in terms of the scalars a, b, c appearing in the equation of the standard birational model of C, namely

$$\alpha_B = \frac{1}{\sqrt{bc}} \qquad \alpha_0 = \sqrt{\frac{a}{c}} \qquad \alpha_1 = \sqrt{\frac{a}{b}}$$

3.2 Case of the genus 2 curve $\mathcal{X} \to \operatorname{Spec} R$

The latter results only apply over the generic fiber $\mathcal{X}_\eta$ but we can use the coordinates change formulae (2.4) and (2.6) to express the rational functions F_B, F_0 and F_1 in terms of the R-coordinates $(x_i,\, y_i)$ instead of the K-coordinates $(z_i,\, w_i)$. First, using the expressions (2.11), (2.12) and (2.13) of coefficients a, b and c, and putting τ_0 and τ_1 for $\sqrt{\frac{1}{1+\nu_0}}$ and $\sqrt{\frac{1}{1+\nu_1}}$ respectively (which belong to $R[\sqrt{s}]$), we have

$$\sqrt{\frac{a}{c}} = \frac{1+\omega^4}{\omega^4}\tau_0 \qquad \sqrt{\frac{a}{b}} = (1+\omega^4)\tau_1 \qquad \sqrt{\frac{1}{bc}} = s^{10}\omega^2(1+\omega^2)\tau_0\tau_1 \tag{3.1}$$

Then, straightforward computations give

$$F_0 = \tau_0(1+s^2(x_1+x_2)+s^4x_1x_2) \qquad F_1 = \tau_1(1+s^2\omega^2(x_1+x_2)+s^4\omega^4x_1x_2) \tag{3.2}$$

and

$$\begin{aligned} F_B \;=\; \tau_0\tau_1 \Bigg[& s^6\omega^2(1+\omega^2)\frac{1}{q(x_1)q(x_2)}\left(\frac{q(x_2)y_1+q(x_1)y_2}{x_1+x_2}\right)^2 \\ & +s^{18}\omega^6(1+\omega^2)^3(\alpha+\beta)^2\frac{(x_1x_2)^2}{q(x_1)q(x_2)} \\ & +\left(s^{14}\omega^6(1+\omega^2)^3\left((\alpha+\beta+\gamma)^2+\frac{\gamma^2}{\omega^4}\right)\right)\frac{(x_1+x_2)^2}{q(x_1)q(x_2)} \\ & +s^{10}\omega^2(1+\omega^2)^3\left((\alpha+\beta)^2+(1+\omega^2)^2\left((\alpha+\beta+\gamma)^2+\frac{\gamma^2}{\omega^4}\right)\right)\frac{1}{q(x_1)q(x_2)}\Bigg] \end{aligned}$$

We compute the coefficients of the last three terms, using equalities (2.8), (2.9) and (2.10) and finally obtain

$$\begin{aligned} F_B \;=\; \tau_0\tau_1 \Bigg[& s^6\omega^2(1+\omega^2)\frac{1}{q(x_1)q(x_2)}\left(\frac{q(x_2)y_1+q(x_1)y_2}{x_1+x_2}\right)^2 \\ & +s^8\left(\frac{1+\omega^{10}}{1+\omega^2}+\mu^2s^4\omega^4\frac{1+\omega^6}{1+\omega^2}\right)\frac{(x_1x_2)^2}{q(x_1)q(x_2)} \\ & +s^4\left(\frac{1+\omega^6}{1+\omega^2}+\mu^2s^4\omega^4\right)\frac{(x_1+x_2)^2}{q(x_1)q(x_2)}+(1+\mu^2s^4\omega^2)\frac{1}{q(x_1)q(x_2)}\Bigg] \end{aligned} \tag{3.3}$$

3.1. *Remark.* Notice that, as the product $q(x_1)q(x_2)$ specializes to 1 over the special fiber, the three rational functions $\frac{X_0}{X_\infty}$, $\frac{X_1}{X_\infty}$ and $\frac{X_B}{X_\infty}$ specialize to 1 over the special fiber. This corresponds to the specialization of the 2-torsion points $[0]_\eta$, $[1]_\eta$ and $[\infty]_\eta$ to 0.

3.3 Finding an R-basis

Let us consider again a smooth proper curve C of genus 2 over an arbitrary field L. Denote by $\imath$ the hyperelliptic involution that permutes the two sheets of the ramified cover $\pi : C \to \mathbb{P}^1_L$ given by the canonical map and assume that π has a L-rational ramification point ∞. In the sequel, we will use the Abel-Jacobi map $AJ : C\times C \to JC$ and the divisor Θ on JC defined as the image of C in JC via the embedding $p \mapsto \mathcal{O}_C(p-\infty)$.

Lemma 3.2 *The pull-back $AJ^*(\Theta)$ is the divisor $(\{\infty\}\times C + C\times\{\infty\}) + \bar\Delta$ on $C\times C$, where $\bar\Delta = (Id\times\imath)^*\Delta$ and Δ is the diagonal in the product $C\times C$.*

Proof. Let M (resp. N) be the prime divisor of $\mathrm{Sym}^2 C$ with support the set $\{(p+\infty),\, p\in C\}$ (resp. the set $\{(p+\bar p),\, p\in C\}$). From [AG], we know that $b^*(\Theta) = M + nN$, where n is a nonnegative integer, and that $\sigma^*(M) = C\times\{\infty\}+\{\infty\}\times C$. Furthermore, as $\sigma^{-1}(N) = \bar\Delta$, $\sigma^*(N) = k\bar\Delta$, where k is an nonnegative integer satisfying the equation $\deg(\sigma)(N)^2 = k^2(\bar\Delta)^2$.

On the one hand, since N is the exceptional curve of the blowing up $b : \mathrm{Sym}^2 C \to JC$, we have $(N)^2 = -1$. On the other hand, the self-intersection number $(\bar\Delta)^2$ is coincides with $\deg_{\bar\Delta}(\mathcal{O}(\bar\Delta)\otimes\mathcal{O}_{\bar\Delta})$. But $\mathcal{O}(\bar\Delta)\otimes\mathcal{O}_{\bar\Delta} = \mathcal{N}_{\bar\Delta/C\times C}$ and taking determinants in the following short exact sequence of $\mathcal{O}_{\bar\Delta}$-coherent sheaves

$$0 \to \mathcal{N}^{-1}_{\bar\Delta/C\times C} \to \Omega^1_{C\times C}\otimes\mathcal{O}_{\bar\Delta} \to \omega_{\bar\Delta} \to 0$$

we obtain $\omega_C\otimes\imath^*\omega_C \cong \omega_C^2 \cong \mathcal{N}^{-1}_{\bar\Delta/C\times C}\otimes\omega_{\bar\Delta}$ hence

$$\omega_{\bar\Delta}^{-1} \cong \mathcal{N}_{\bar\Delta/C\times C} \tag{3.4}$$

Therefore, $(\bar\Delta)^2 = 2-2g_C = -2$, hence $k=1$ and $AJ^*(\Theta) = C\times\{\infty\}+\{\infty\}\times C + n\bar\Delta$. One can compute self-intersection again to determine n. We write

$$\begin{aligned} \deg(AJ)(\Theta)^2 = \;& (C\times\{\infty\})^2 + (\{\infty\}\times C)^2 + n^2(\bar\Delta)^2 \\ & +2[(C\times\{\infty\}).(\{\infty\}\times C) + n((C\times\{\infty\}).\bar\Delta + (\{\infty\}\times C).\bar\Delta)] \end{aligned}$$

It is clear that $(C\times\{\infty\})^2$ (resp. $(\{\infty\}\times C)^2$) equals to zero for $C\times\{\infty\}$ (resp. $\{\infty\}\times C$) being algebraically equivalent to a divisor that does not meet $C\times\{\infty\}$ (resp. $\{\infty\}\times C$). It is clear as well that the intersection products $(C\times\{\infty\}).(\{\infty\}\times C)$, $(C\times\{\infty\}).\bar{\Delta}$ et $(\{\infty\}\times C).\bar{\Delta}$ equal to 1 since the two divisors in each pairs obviously meet transversally in a single point, namely (∞,∞). Furthermore, using the Riemann-Roch theorem for an abelian variety of dimension 2 and a principal (hence ample) divisor ([Mu1]), we have $(\Theta)^2=2$. Replacing, we obtain $n=1$. $\square$

Denote by p_i the canonical projection $C\times C\to C$ on the i-th factor ($i=1,2$). Because the canonical divisor of C is 2∞ and because $\Omega^1_{C\times C}\cong p_1^*(\omega_C)\oplus p_2^*(\omega_C)$, the canonical divisor $K_{C\times C}$ equals $2(C\times\{\infty\})+2(\{\infty\}\times C)$. Therefore, the previous lemma allows us to see the L-vector space $H^0(JC,\mathcal{O}(2\Theta))$ as the linear subspace of $H^0(C\times C,\mathcal{O}(K_{C\times C}+2\bar{\Delta}))$ consisting of symmetric sections (under the action of $\mathfrak{S}_2$) that take constant value along $\bar{\Delta}$.

Lemma 3.3 *The natural inclusion $H^0(C\times C,K_{C\times C})\hookrightarrow H^0(C\times C,\mathcal{O}(K_{C\times C}+2\bar{\Delta}))$ induces three linearly independent sections 1, x_1+x_2 and x_1x_2 of $H^0(JC,\mathcal{O}(2\Theta))$.*

Proof. Consider the following short exact sequence of $\mathcal{O}_{C\times C}$-coherent sheaves

$$0\to\mathcal{O}(-\bar{\Delta})\to\mathcal{O}_{C\times C}\to\mathcal{O}_{\bar{\Delta}}\to 0 \tag{3.5}$$

Tensoring with $\mathcal{O}(K_{C\times C}+2\bar{\Delta})$ and taking cohomology groups, one gets the following exact sequence of L-vector spaces

$$0\to H^0(\mathcal{O}(K_{C\times C}+\bar{\Delta}))\to H^0(\mathcal{O}(K_{C\times C}+2\bar{\Delta}))\to H^0(\mathcal{O}(K_{C\times C}+2\bar{\Delta})\otimes\mathcal{O}_{\bar{\Delta}})$$

But $\mathcal{O}(K_{C\times C}+2\bar{\Delta})\otimes\mathcal{O}_{\bar{\Delta}}\cong\big(\mathcal{O}(K_{C\times C})\otimes\mathcal{O}(\bar{\Delta})\otimes\mathcal{O}_{\bar{\Delta}}\big)\otimes_{\mathcal{O}_{\bar{\Delta}}}\big(\mathcal{O}(\bar{\Delta})\otimes\mathcal{O}_{\bar{\Delta}}\big)$. The first term of this tensor product is isomorphic to $\omega_{\bar{\Delta}}$ and the second to $\mathcal{N}_{\bar{\Delta}/C\times C}$. Thus, using the isomorphism (3.4), we obtain the structural sheaf $\mathcal{O}_{\bar{\Delta}}$ and we have an exact sequence

$$0\to H^0(\mathcal{O}(K_{C\times C}+\bar{\Delta}))\to H^0(\mathcal{O}(K_{C\times C}+2\bar{\Delta}))\to L$$

Tensoring (3.5) with $\mathcal{O}(K_{C\times C}+\bar{\Delta})$ and taking cohomology groups, one gets an exact sequence of L-vector spaces

$$0\to H^0(\mathcal{O}(K_{C\times C}))\to H^0(\mathcal{O}(K_{C\times C}+\bar{\Delta}))\to H^0(\mathcal{O}(K_{C\times C}+\bar{\Delta})\otimes\mathcal{O}_{\bar{\Delta}})\xrightarrow{\delta}H^1(\mathcal{O}(K_{C\times C}))$$

with $\mathcal{O}(K_{C\times C}+\bar{\Delta})\otimes\mathcal{O}_{\bar{\Delta}}\cong\omega_{\bar{\Delta}}$. By Serre duality, the dual exact sequence corresponds to the long exact sequence of cohomology groups associated to the short exact sequence (3.5). Therefore, using Serre duality again and Künneth isomorphism, one has the following commutative diagram of morphisms of L-vector spaces

$$\begin{tikzcd}
H^1(\mathcal{O}(K_{C\times C}))^\vee \arrow[r, "\delta^\vee"] & H^0(\omega_{\bar\Delta})^\vee \\
H^1(\mathcal{O}_{C\times C}) \arrow[u, "\wr"'] \arrow[r] & H^1(\mathcal{O}_{\bar\Delta}) \arrow[u, "\wr"'] \\
H^1(\mathcal{O}_C)\oplus H^1(\mathcal{O}_C) \arrow[u, "\wr"'] \arrow[r, "Id+H^1(\imath)"] & H^1(\mathcal{O}_C) \arrow[u, "\wr"']
\end{tikzcd}$$

As the bottom horizontal arrow is surjective, $\delta^\vee$ is surjective and δ is injective. Thus, the morphism $H^0(\mathcal{O}(K_{C\times C}))\to H^0(\mathcal{O}(K_{C\times C}+\bar\Delta))$ is an isomorphism and using the Künneth isomorphism again, one gets the following exact sequence of L-vector spaces

$$0\to H^0(\omega_C)\otimes H^0(\omega_C)\to H^0(\mathcal{O}(K_{C\times C}+2\bar\Delta))\to L$$

Denote by x the rational coordinate function of $\mathbb{P}^1_L$ with pole at ∞. Considering $H^0(\omega_C)$ as an L-vector subspace of the function field $\bar k(C)$ by means of the differential dx, it has a basis $\{1,\, x\}$. Thus, considering $H^0(\omega_C)\otimes H^0(\omega_C)$ as an L-vector subspace of the function field $\bar k(C\times C)$, we find that $(H^0(\omega_C)\otimes H^0(\omega_C))^{\mathfrak{S}_2}$ has a basis $\{1,\, x_1+x_2,\, x_1x_2\}$. Note that the three corresponding sections of $H^0(\mathcal{O}(K_{C\times C}+2\bar\Delta))^{\mathfrak{S}_2}$ are in the kernel of the evaluation $H^0(\mathcal{O}(K_{C\times C}+2\bar\Delta))\to L\cong H^0(\mathcal{O}_{\bar\Delta})$ so they have constant value (equal to zero) along $\bar\Delta$, hence define sections of $H^0(JC,\,\mathcal{O}(2\Theta))$. $\square$

3.4. *Remark.* Consider the exact sequence (3.5), tensored by $\mathcal{O}(K_{C\times C}+2\bar\Delta)$ and localized at the generic point $\bar\Delta$ of the irreducible subscheme $\bar\Delta$ of $C\times C$. If A denotes the discrete valuation ring $\mathcal{O}_{C\times C,\,\bar\Delta}$, if t is a element of A that generates the maximal ideal, we find the exact sequence

$$0\to t^{-1}.A\to t^{-2}.A\to \bar k(\bar\Delta)\to 0$$

Now, given $g\in H^0(\mathcal{O}(K_{C\times C}+2\bar\Delta))$, we can look at it as an element of $t^{-2}.A$ and write it as $a_{-2}t^{-2}+a_{-1}t^{-1}+\sum_{n\geq 0}a_nt^n$, where the a_i are *a priori* elements of the residue field $\bar k(\bar\Delta)$. Its class in $\bar k(\bar\Delta)$ is a_{-2}, which lies in fact in L since the following diagram (where the vertical arrows are localization) commutes

$$\begin{tikzcd}
H^0(\mathcal{O}(K_{C\times C}+2\bar\Delta)) \arrow[r] \arrow[d] & L \arrow[d] \\
t^{-2}.A \arrow[r] & \bar k(\bar\Delta)
\end{tikzcd}$$

Therefore, the morphism $H^0(\mathcal{O}(K_{C\times C}+2\bar\Delta))\to L$ amounts, in some sense, to the computation of a residue.

3.5. *Remark.* As $H^0(JC,\,\mathcal{O}(2\Theta))$ is of dimension 4, the morphism $H^0(\mathcal{O}(K_{C\times C}+2\bar\Delta))\to L$ cannot be zero and there must exist a symmetric rational function f on $C\times C$ such that

(1) $\operatorname{div}(f) + K_{C\times C} + 2\bar{\Delta}$ is effective;

(2) f has a pole of order 2 along $\bar{\Delta}$.

Thus, we will have a basis $\{1,\, x_1+x_2,\, x_1x_2,\, f\}$ of the vector space $H^0(JC,\, \mathcal{O}(2\Theta))$.

3.6. *Remark.* These two lemmas extend to the relative case of the genus 2 curve $\mathcal{X}$ over $\operatorname{Spec} R$, and we find three linearly independent sections 1, x_1+x_2 and x_1x_2 of $\mathcal{W}$.

We now restrict to the case of the genus 2 curve $\mathcal{X} \to \operatorname{Spec} R$. Note that F_B certainly satisfies condition (1) in the remark above: over the generic point, it is the pull-back by AJ of the rational function $\dfrac{X_B}{X_\infty}$, where X_B and X_∞ are two sections of the line bundle $\mathcal{O}(2\Theta)$, which corresponds, by Lemma 3.2, to the divisor $K_{\mathcal{X}_\eta\times\mathcal{X}_\eta} + 2\bar{\Delta}$. Furthermore, the origin of $\mathcal{J}\mathcal{X}_\eta$ does not belong to $\operatorname{Supp}(\Theta_B)$: if it did, we would have $h^0(\mathcal{X}_\eta, B) \geq 1$ and there would exist a point p in $\mathcal{X}_\eta$ such that the divisors $0_\eta + 1_\eta$ and $p+\infty$ were linearly equivalent, which is impossible since the three Weierstrass points of $\mathcal{X}_\eta$ are pairwise different. On the other hand, $H^0(\mathcal{X}_\eta, \mathcal{O}(\infty)) \geq 1$ so $\dfrac{X_B}{X_\infty}$ has a pole at the origin, and by pull-back, F_B satisfies condition (2) over the generic fiber.

Of course, using Lemma 3.3 and Remark 3.5, any linear combination

$$\mu\left(\frac{1}{\tau_0\tau_1}F_B + (\lambda + \lambda_\Sigma(x_1+x_2) + \lambda_\Pi(x_1x_2))\right)$$

with μ, λ, λ_Σ and λ_Π in K such that it belongs to the ring

$$A = R[x_1,\, x_2,\, y_1,\, y_2,\, (x_1+x_2)^{-1},\, (q(x_1)q(x_2))^{-1}]$$

will fulfill condition (1) over both generic and special fibers and condition (2) over the generic fiber.

Using (3.3), recall that

$$\begin{aligned}\frac{1}{\tau_0\tau_1}F_B \;=\;& s^6\omega^2(1+\omega^2)\frac{1}{q(x_1)q(x_2)}\left(\frac{q(x_2)y_1+q(x_1)y_2}{x_1+x_2}\right)^2\\ &+s^8\left(\frac{1+\omega^{10}}{1+\omega^2}+\mu^2s^4\omega^4\frac{1+\omega^6}{1+\omega^2}\right)\frac{(x_1x_2)^2}{q(x_1)q(x_2)}\\ &+s^4\left(\frac{1+\omega^6}{1+\omega^2}+\mu^2s^4\omega^4\right)\frac{(x_1+x_2)^2}{q(x_1)q(x_2)}+(1+\mu^2s^4\omega^2)\frac{1}{q(x_1)q(x_2)}\end{aligned}$$

As $\dfrac{1}{q(x_1)q(x_2)}\left(\dfrac{q(x_2)y_1+q(x_1)y_2}{x_1+x_2}\right)^2$ obviously has a pole of order 2 along $\bar{\Delta}$ over both generic and special fibers, we look for λ, λ_Σ and λ_Π in K such that we can take $\mu = \dfrac{1}{s^6\omega^2(1+\omega^2)}$. Looking at the class of $\dfrac{1}{\tau_0\tau_1}F_B$ in A/s^6, we find

$$\frac{1}{q(x_1)q(x_2)}\left[s^4\frac{1+\omega^6}{1+\omega^2}(x_1+x_2)^2+(1+\mu^2s^4\omega^2)\right]$$

On the other hand, $(\lambda + \lambda_\Sigma(x_1+x_2) + \lambda_\Pi(x_1x_2))q(x_1)q(x_2)$ equals

$$\lambda+(\lambda_\Sigma+\lambda s^2(1+\omega^2))(x_1+x_2)+(\lambda_\Pi+\lambda s^4(1+\omega^4))(x_1x_2)+(\lambda s^4\omega^2+\lambda_\Sigma s^2(1+\omega^2))(x_1+x_2)^2$$

in A/s^6. Thus, we see that

$$\frac{1}{\tau_0\tau_1}F_B + (1+\mu^2s^4\omega^2) + s^2(1+\omega^2)(x_1+x_2) + s^4(1+\omega^4)(x_1x_2)$$

belongs to s^6A. A few more calculations show that it is in fact in $s^6\omega^2(1+\omega^2)A$ and we set

$$f = \frac{1}{s^6\omega^2(1+\omega^2)}\left[\frac{1}{\tau_0\tau_1}F_B + (1+\mu^2s^4\omega^2) + s^2(1+\omega^2)(x_1+x_2) + s^4(1+\omega^4)(x_1x_2)\right]$$

Therefore,

$$F_B = \tau_0\tau_1((1+\mu^2s^4\omega^2) + s^2(1+\omega^2)(x_1+x_2) + s^4(1+\omega^4)(x_1x_2) + s^6\omega^2(1+\omega^2)f) \quad (3.6)$$

Proposition 3.7 *The projective space $\mathbb{P}\mathcal{W} \cong \mathbb{P}^3_R$ has homogeneous coordinates $\{z_1,\, z_2,\, z_3,\, z_\infty\}$ in terms of which, over the generic point $\operatorname{Spec} K \to \operatorname{Spec} R$, the canonical theta coordinates $\{x_B,\, x_0,\, x_1,\, x_\infty\}$ are given, up to scalar, by the formulae*

$$\begin{cases} x_B = z_1 \\ x_0 = \tau_1(z_1 + s^2\omega^2 z_2 + s^4\omega^4 z_3) \\ x_1 = \tau_0(z_1 + s^2 z_2 + s^4 z_3) \\ x_\infty = \tau_0\tau_1((1+\mu^2s^4\omega^2)z_1 + s^2(1+\omega^2)z_2 + s^4(1+\omega^4)z_3 + s^6\omega^2(1+\omega^2)z_\infty) \end{cases}$$

Proof. The previous calculations show that one can form an R-basis $\{Z_\bullet\}$ of $\mathcal{W}$ by setting

$$\begin{array}{ll} Z_\infty = X_\infty & Z_1 = f\,X_\infty \\ Z_2 = x_1x_2X_\infty & Z_3 = (x_1+x_2)X_\infty \end{array}$$

The expressions of the $X_\bullet$ in terms of the $Z_\bullet$ are then given by the equations (3.2) and (3.6). By duality, we easily deduce the corresponding change for coordinates. □

4 Equations of $\widetilde{V}$ for a supersingular genus 2 curve in characteristic 2

In [LP1] (section 5), it is shown that the morphism $D : M_X \to \mathbb{P}^3 = |2\Theta|$ is, as in the complex case (see [NR]), an isomorphism when X is an ordinary genus 2 curve over an algebraically closed field of characteristic 2. As asserted in [LP2], this identification extends to the relative case $\mathcal{X} \to \operatorname{Spec} R$, so that the Frobenius morphism $\mathcal{X} \to \mathcal{X}_1$ induces, by pull-back, a rational map

$$\begin{tikzcd} \mathbb{P}\,\mathcal{W}_1 \arrow[rr, "\widetilde{\mathcal{V}}"] \arrow[dr] & & \mathbb{P}\,\mathcal{W} \arrow[dl] \\ & \operatorname{Spec} R & \end{tikzcd}$$

where $\mathcal{W}_1$ stands for the 2-twist $H^0(\mathcal{J}\mathcal{X}_1,\, \mathcal{O}(2\Theta_1))$ of $\mathcal{W}$.

Over the generic point η, [LP1] (Proposition 3.1) gives the form of the rational map

$$\begin{array}{rcl} \widetilde{\mathcal{V}}_\eta : \mathbb{P}\,(\mathcal{W}_1)_\eta & \to & \mathbb{P}\,(\mathcal{W})_\eta \\ x = (x_\bullet) & \mapsto & (\lambda_B P_B(x) :\, \lambda_0 P_0(x) :\, \lambda_1 P_1(x) :\, \lambda_\infty P_\infty(x)) \end{array}$$

where the $x_\bullet$ are the theta coordinates of the spaces $(\mathcal{W}_1)_\eta$ and $(\mathcal{W})_\eta$ (which correspond via the K-semilinear isomorphism $i^* : (\mathcal{W})_\eta \to (\mathcal{W}_1)_\eta$), where the $P_\bullet$ are the quadrics

$$\begin{array}{ll} P_B(x) = x_B^2 + x_0^2 + x_1^2 + x_\infty^2 & P_0(x) = x_B x_0 + x_1 x_\infty \\ P_1(x) = x_B x_1 + x_0 x_\infty & P_\infty(x) = x_B x_\infty + x_0 x_1. \end{array}$$

and where the $\lambda_\bullet$ are nonzero constants depending on the curve X. In [LP2] (section 3), we find an explicit determination (up to scalar) of these coefficients, namely

$$(\lambda_B :\, \lambda_0 :\, \lambda_1 :\, \lambda_\infty) = (\sqrt{abc} :\, \sqrt{c} :\, \sqrt{b} :\, \sqrt{a})$$

Using the formulae given in Proposition 3.7 and the expressions (2.11), (2.12) and (2.13) for the coefficients a, b and c, we can compute the rational map

$$\widetilde{\mathcal{V}} : z = (z_\bullet) \mapsto (R_1(z) :\, R_2(z) :\, R_3(z) :\, R_\infty(z))$$

First, we express the polynomials $Q_\bullet(z) = P_\bullet(x)$ (notice that the coefficients appearing in the formulae given in Proposition 3.7 are squared since we are dealing with elements of the 2-twist $(\mathcal{W}_1)_\eta$). We find the following:

$$\begin{array}{rcl} Q_B(\underline{z}) & = & (\tau_0\tau_1)^4\left((\nu_0\nu_1 + \mu^4 s^8\omega^4)^2 z_1^2 + s^8(\nu_0\omega^4 + \nu_1)^2 z_2^2 + s^{16}(\nu_0\omega^8 + \nu_1)^2 z_3^2 + s^{24}\omega^8(1+\omega^8) z_\infty^2\right) \\ \\ Q_0(\underline{z}) & = & \tau_0^4\tau_1^2\left((\nu_0 + \mu^2 s^4\omega^2)^2 z_1^2 + s^4\omega^4(\nu_0 + \mu^2 s^4)^2 z_1 z_2 + s^8\omega^4(\nu_0\omega^2 + \mu^2 s^4)^2 z_1 z_3\right. \\ & & \quad + s^8(1+\omega^4) z_2^2 + s^{12}\omega^4(1+\omega^4)(z_2 z_3 + z_1 z_\infty) + s^{16}(1+\omega^8) z_3^2 \\ & & \quad\quad \left. + s^{16}\omega^4(1+\omega^4)(z_2 z_\infty + s^4 z_3 z_\infty)\right) \\ \\ Q_1(\underline{z}) & = & \tau_0^2\tau_1^4\left((\nu_1 + \mu^2 s^4\omega^2)^2 z_1^2 + s^4(\nu_1 + \mu^2 s^4\omega^4)^2 z_1 z_2 + s^8(\nu_1 + \mu^2 s^4\omega^6)^2 z_1 z_3\right. \\ & & \quad + s^8\omega^4(1+\omega^4) z_2^2 + s^{12}\omega^4(1+\omega^4)(z_2 z_3 + z_1 z_\infty) + s^{16}\omega^8(1+\omega^8) z_3^2 \\ & & \quad\quad \left. + s^{16}\omega^8(1+\omega^4)(z_2 z_\infty + s^4\omega^4 z_3 z_\infty)\right) \\ Q_\infty(\underline{z}) & = & (\tau_0\tau_1)^2 s^8\omega^4\left((\mu^4 z_1^2 + z_2^2) + s^4(1+\omega^4)(z_2 z_3 + z_1 z_\infty) + s^8\omega^4 z_3^2\right) \end{array}$$

Secondly, using the coefficients $\lambda_\bullet$ and the formulae of Proposition 3.7 again (after inversion and up to scalar), we obtain

$$\begin{aligned}
R_1(z) &= \frac{1}{(\tau_0\tau_1)^4 s^{18}\omega^6(1+\omega)^6}\, Q_B(z) \\
R_2(z) &= \frac{1}{(\tau_0\tau_1)^4 s^2\omega^2(1+\omega^2)}\left[\frac{1}{s^{18}\omega^6(1+\omega^2)}\, Q_B(z) + \frac{1}{s^8(1+\omega^4)}(Q_0(z)+Q_1(z))\right] \\
R_3(z) &= \frac{1}{(\tau_0\tau_1)^4 s^4\omega^2(1+\omega^2)}\left[\frac{1}{s^{18}\omega^6(1+\omega^4)}\, Q_B(z) + \frac{1}{s^8\omega^2(1+\omega^4)}(\omega^2 Q_0(z)+Q_1(z))\right] \\
R_\infty(z) &= \frac{1}{(\tau_0\tau_1)^4 s^6\omega^2(1+\omega^2)}\left[\frac{1+\mu^2 s^4\omega^2}{s^{18}\omega^6(1+\omega)^6}\, Q_B(z)\right. \\
&\qquad\qquad \left. + \frac{1}{s^8\omega^4(1+\omega^4)}(\omega^4 Q_0(z)+Q_1(z)) + \frac{1}{s^8\omega^4} Q_\infty(z)\right]
\end{aligned}$$

Now, using the expansions of ν_0 and ν_1, we obtain

$$\begin{aligned}
R_1(z) &= \mu^4 z_1^2 + z_2^2 + s^2\bar{R}_1(z) \\
R_2(z) &= z_1^2 + s^2\bar{R}_2(z) \\
R_3(z) &= (z_2 z_3 + z_1 z_\infty) + s^2\bar{R}_3(z) \\
R_\infty(z) &= (\mu^4 z_3^2 + z_\infty^2 + \mu^2 z_1^2 + z_1 z_2) + s^2\bar{R}_\infty(z)
\end{aligned} \tag{4.1}$$

where the $\bar{R}_\bullet(z)$ are quadrics with coefficients in $R[\omega^{-1},\, (1+\omega)^{-1}]$.

4.1. *Remark.* A few more calculations show that the coefficients of the quadrics $\bar{R}_\bullet(z)$ lie in fact in R, so that we can choose any ω we want in $R \setminus \{0,\, 1\}$ (and not necessarily an element that specializes in $k \setminus \{0,\, 1\}$). It means that any deformation of X leads to the same equations.

Let us introduce the set of homogeneous coordinates $\{y_1,\, y_2,\, y_3,\, y_\infty\}$ of $|2\Theta|$ given by

$$\begin{aligned}
y_1 &= z_1, & y_2 &= z_2 + \mu z_1, \\
y_3 &= z_3, & y_\infty &= z_\infty + \mu z_3.
\end{aligned}$$

Thus, the corresponding coordinates change in $|2\Theta_1|$ is obtained by squaring the coefficients and we have proved the following.

Theorem 4.2 *Let X be a smooth, proper, and supersingular curve of genus 2 over an algebraically closed field of characteristic 2. There exist coordinates $\{z_\bullet\}$ (resp. $\{y_\bullet\}$) on $|2\Theta|$ (resp. $|2\Theta_1|$) such that the equations of $\widetilde{V}$ are given by*

$$\widetilde{V} : |2\Theta_1| \to |2\Theta|, \quad y = (y_\bullet) \mapsto z = (z_\bullet) = (Q_1(y) : Q_2(y) : Q_3(y) : Q_\infty(y))$$

with

$$\begin{aligned}
Q_1(y) &= y_2^2, & Q_2(y) &= y_1^2, \\
Q_3(y) &= y_2 y_3 + y_1 y_\infty, & Q_\infty(y) &= y_\infty^2 + y_1 y_2.
\end{aligned}$$

5 Frobenius action on M_X for a supersingular genus 2 curve in characteristic 2

We know ([LP2], section 3) that the Kummer surface $\mathrm{Kum}_{\mathcal{X}_\eta}$ is defined, in terms of the theta coordinates $\{x_\bullet\}$ on $|2\Theta|_\eta$, by the homogeneous quartic

$$c(x_B^2x_0^2+x_1^2x_\infty^2)+b(x_B^2x_1^2+x_0^2x\infty^2)+a(x_B^2x_\infty^2+x_0^2x_1^2)+x_Bx_0x_1x_\infty,$$

a, b, c being the scalars appearing in the standard birational model (2.3) of the curve $\mathcal{X}_\eta$.

The same kind of calculations as in section 4 give the equation of Kum_X in the coordinate system $\{z_\bullet\}$ of $|2\Theta|$, namely

$$\mu^2z_1^3z_2+z_1^3z_\infty+z_1^2z_2z_3+\mu^4z_1^2z_3^2+z_1z_2^3+z_2^2z_\infty^2+z_3^4. \tag{5.1}$$

As in [LP1], we easily deduce from Theorem 4.2 and from the latter calculations a complete description of the action of Frobenius on M_X, more precisely of its separable part $V : M_{X_1} \dashrightarrow M_X$, that we identify, using the isomorphism $D : M_X \xrightarrow{\sim} |2\Theta|$ (see the introduction), with $\widetilde{V} : |2\Theta_1| \dashrightarrow |2\Theta|$.

Proposition 5.1 *Let X be a smooth, proper, and supersingular curve of genus 2 over an algebraically closed field of characteristic 2.*

1. *The semistable boundary of M_X (resp. M_{X_1}) is isomorphic (via D) to the Kummer quartic surface Kum_X (resp. Kum_{X_1}), an equation of which is (5.1). In particular, V maps Kum_{X_1} onto Kum_X.*

2. *There is a unique stable bundle $E_{\mathrm{bad}} \in M_{X_1}$ which is destabilized by Frobenius (i.e., F^*E_{bad} is not semistable). We have $E_{\mathrm{bad}} = F_*B^{-1}$ and its projective coordinates are $(0:0:1:0)$.*

3. *Let H_1 be the hyperplane in $|2\Theta_1|$ defined by $y_2=0$. The map V contracts H_1 to the conic $\mathrm{Kum}_X \cap H$, where H is the hyperplane in $|2\Theta|$ defined by $z_1=0$.*

4. *The fiber of V over a point $[E] \in M_X$ is*

   - *a single point $[E_1] \in M_{X_1}$, if $[E] \notin H$,*
   - *empty, if $[E] \in H \setminus \mathrm{Kum}_X \cap H$,*
   - *a projective line passing through E_{bad}, if $[E] \in \mathrm{Kum}_X \cap H$.*

   In particular, V is dominant, nonsurjective.

5. *The total (resp. separable) degree of V is 4 (resp. 1).*

Proof. The fact that the Kummer quartic surface Kum_X is the semistable boundary of M_X comes from [NR]. On the one hand, the inverse image of Kum_X is a closed subspace in $|2\Theta_1|$ the ideal of which is generated by the homogeneous polynomial obtained by replacing the $\{z_\bullet\}$ in equation (5.1) by the $\{Q_\bullet(y)\}$ given in Theorem 4.2, namely

$$y_2^4[y_1^3y_\infty + \mu^2y_1^2y_2^2 + y_1^2y_2y_3 + \mu^4(y_1^2y_\infty^2 + y_2^2y_3^2) + y_1y_2^3 + y_2^2y_\infty^2 + y_3^4]$$

On the other hand, we obtain the equation

$$y_1^3y_\infty + \mu^2y_1^2y_2^2 + y_1^2y_2y_3 + \mu^4(y_1^2y_\infty^2 + y_2^2y_3^2) + y_1y_2^3 + y_2^2y_\infty^2 + y_3^4$$

of Kum_{X_1} in $|2\Theta_1|$ by squaring the coefficients in equation (5.1) and replacing the coordinates $\{z_\bullet\}$ on $|2\Theta_1|$ by their expressions

$$\begin{array}{ll} z_1 = y_1 & z_2 = y_2 + \mu^2y_1 \\ z_3 = y_3 & z_\infty = y_\infty + \mu^2y_3 \end{array}$$

in terms of the $\{y_\bullet\}$. Thus, $V^{-1}(\mathrm{Kum}_X) = \mathrm{Kum}_{X_1} \bigcup H_1$ (where $H_1 = (y_2 = 0)$), hence (1). Furthermore, H_1 is mapped onto the hyperplane $H = (z_1 = 0)$ of $|2\Theta|$, hence (3).

In [LP1] (Proposition 6.1), it is shown that there is exactly one base-point, namely F_*B^{-1}. Now, solving $Q_\bullet(y) = 0$, we find that $(0 : 0 : 1 : 0)$ is the unique base-point of V, hence (2).

Let $[E]$ be a k-point of M_X, with coordinates $(a^2 : b^2 : c^2 : d^2)$. Let us solve the system

$$y_2^2 = a^2, \quad y_1^2 = b^2, \quad y_2y_3 + y_1y_\infty = c^2, \quad y_\infty^2 + y_1y_2 = d^2. \tag{5.2}$$

We must have $y_1 = b$ and $y_2 = a$, so $y_\infty = d + \sqrt{ab}$ and finally $ay_3 = c^2 + bd + b\sqrt{ab}$. If $[E]$ belongs to H, i.e., if $a = 0$, it has solutions if and only if $c^2 = bd$, that is if and only if $[E]$ belongs to $H \cap \mathrm{Kum}_X$. In that case, there is no condition on y_3 and the inverse image of such a point is a projective line passing through E_{bad}. If $[E]$ does not belong to H, i.e., if $a \neq 0$), we see that the inverse image is a single point.

Finally, defining $u = y_1/y_2$, $v = y_3/y_2$ and $w = y_\infty/y_2$, the field extension

$$k(u^2,\, v + uw,\, w^2 + u) \subseteq k(u,\, v,\, w) \cong k(u^2,\, v + uw,\, w^2 + u)[t,\, s]/(t^2 - u^2,\, s^2 - w^2)$$

corresponds to the Verschiebung V, and the last assertion follows. $\square$

Denote by N_X (resp. N_{X_1}) the moduli of semistable bunbles with rank 2 and degree 0 over X (resp. X_1). In the case of an ordinary genus 2 curve, [LP1] (Proposition 6.4) showed the surjectivity of the (rational map) Verschiebung $N_{X_1} \dashrightarrow N_X$, $[E_1] \mapsto [F^*E_1]$. In the case of a supersingular curve, this result does not hold any more. Let E be a vector bundle in N_X. The Jacobian JX being divisible, we can assume that $\det E$ is trivial. Now, if $[F^*E_1] = [E]$, we have $F^*(\det E_1) = \det E$ and $\det E_1$ has to be a 2-torsion point of JX_1. Therefore, $\det E_1$ is trivial as well and the following is a corollary of Proposition 5.1.

Proposition 5.2 *Let X be a smooth, proper, and supersingular curve of genus 2 over an algebraically closed field of characteristic 2. The rational map $N_{X_1} \to N_X$ given by $[E_1] \mapsto [F^*E_1]$ is not surjective.*

We are now interested in those vector bundles over X that are destabilized by a finite number of iteration of the (absolute) Frobenius. Let Ω^{Frob} be the complementary set of classes of semistable rank 2 vector bundles E with trivial determinant over X such that ${F_{\mathrm{abs}}^{(n)}}^*E$ is semistable for all $n \geq 1$.

Proposition 5.3 *Let X be a smooth, proper, and supersingular curve of genus 2 over an algebraically closed field of characteristic 2. The open subset $M_X \setminus \{E_{\mathrm{bad}}\}$ is stable under the action of the (absolute) Frobenius. In particular, Ω^{Frob} is the Zariski open (dense) subset $M_X \setminus \{E_{\mathrm{bad}}\}$ of M_X.*

Proof. Pulling back a semistable bundle E over X by the semilinear isomorphism $i : X_1 \to X$, we obtain a semistable bundle i^*E over X_1. If E has trivial determinant, i^*E has trivial determinant as well. If E ($\neq E_{\mathrm{bad}}$) has coordinates $(a : b : c : d)$ in the system $\{z_\bullet\}$, i^*E has squared coordinates in the corresponding system $\{z_\bullet\}$ of M_{X_1}. Using Theorem 4.2 (actually, the quadrics given by equations (4.1) from which we deduce Theorem 4.2 after the indicated coordinates change), we find that $F_{\mathrm{abs}}^*E = V^*(i^*E)$ has coordinates

$$(\mu^4a^4 + b^4 : a^4 : a^2d^2 + b^2c^2 : \mu^4c^4 + d^4 + \mu^2a^4 + a^2b^2)$$

in the system $\{z_\bullet\}$, hence cannot be E_{bad}. $\square$

Corollary 5.4 *Let k be a finite field of characteristic 2, let X be a smooth, proper, and supersingular curve of genus 2 over k, and let E be a semistable rank 2 vector bundle with trivial determinant, defined over k and different from E_{bad}. Then some twist ${F^{(n_0)}}^*E$ comes from a continuous representation of the algebraic fundamental group $\pi_1(X)$ in $\mathrm{GL}_2(\bar{k})$.*

Proof. For such a vector bundle, the sequence ${F_{\mathrm{abs}}^{(n)}}^*E$ ($n \geq 1$) takes its values in the finite set $M_X(k)$ of S-equivalence classes of k-rational rank 2 semistable vector bundles with trivial determinant. Therefore, the sequence ${F_{\mathrm{abs}}^{(n+n_0)}}^*E$ ($n \geq 1$) is periodic for suitable n_0 and we use [LS] (Theorem 1.4) to conclude. $\square$

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Laurent Ducrohet
Université Pierre et Marie Curie
Analyse algébrique, UMR 7586
4, place Jussieu
75252 Paris Cedex 05 France
e-mail: ducrohet@math.jussieu.fr