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A simple background-independent hamiltonian quantum model

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Abstract

We study formulation and probabilistic interpretation of a simple general-relativistic hamiltonian quantum system. The system has no unitary evolution in background time. The quantum theory yields transition probabilities between measurable quantities (partial observables). These converge to the classical predictions in the $\hbar \to 0$ limit. Our main tool is the kernel of the projector on the solutions of Wheeler-deWitt equation, which we analyze in detail. It is a real quantity, which can be seen as a propagator that propagates "forward" as well as "backward" in a local parameter time. Individual quantum states, on the other hand, may contain only "forward propagating" components. The analysis sheds some light on the interpretation of background independent transition amplitudes in quantum gravity.

1 Introduction

Conventional mechanics describes evolution in "background time": the independent time variable t is assumed to represent a measurable but non-dynamical physical quantity. This is true in the classical as well as in the quantum theory, where evolution in background time is given by unitary transformations. With general relativity, however, we have learned that there is no non-dynamical measurable time in nature: there is no background spacetime, and no background time in particular. Therefore, at the fundamental level physics cannot describe evolution in time. It can only describe relations, or correlations, between measurable quantities. It is therefore necessary to extend the formalism of classical and quantum mechanics to such a background independent context.

A strategy in this direction has been explored in a number of recent works [1, 2, 3].[1] Here we study a simple system in order to illustrate and test ideas and techniques discussed in these works. We do not restate the general theory here, for which we refer the reader to the references above, but this paper is nevertheless self-contained. The system we consider has one degree of freedom. It has a well defined classical dynamics. It could describe a simple cosmological model – or a simple mechanical system studied without using an external clock. The system has two variables, which we call a and b. Locally, we can view b as the independent variable and a as the dependent one. The model describes then the evolution $a(b)$ of a as a function of b. In other words, locally we can view b as the time and a as the physical degree of freedom. But globally this picture breaks down: the dynamics predicts relations betweent the two variables a and b which are on an equal footing. The quantum evolution in b fails to be unitary. This is the difficulty that generates the much debated "problem of time" in quantum general relativity [6, 7].

The system we consider has been investigated in [5, 8]. The difficulty is not to define the quantum states of the theory – a rather easy task using standard hamiltonian methods. The difficulty is to define a consistent probabilistic interpretation of the formalism. Due to the lack of unitary time evolution, the

[1] For alternative systematic attempts to formulate general covariant quantum theory, see [4, 5].

conventional probabilistic interpretation of the wave function at fixed time does not apply. A good discussion of the difficulties of the "naive" interpretation of the wave function as a probability density, and of tentative strategies to solve the problem can be found in [5]. In [8], one of us has studied the physical interpretation of this system using operators well defined on the physical Hilbert space. These describe "complete" observables, (or "perennials") and include, in particular the "evolving constants of the motion" [7, 8]. This way of interpreting general relativistic quantum systems is correct in principle, but may be cumbersome in practice.

In [1], it was pointed out that the conventional notion of "observable" may not be the most suitable one in a background independent context. The weaker notion of "partial observable" may be more useful in such a context. A partial observable is any quantity that can be measured – even if it cannot be predicted by the theory. Using this notion, general relativistic systems admit a simple reading: they predict correlations between partial observables. This is the point of view developed in [1, 2, 3]. In this philosophy the key object that yields the interpretation of the quantum formalism is the "propagator" $K(a, b, a', b')$. This is defined as the kernel of the projector from the kinematical Hilbert space $\mathcal{K}$ (the space representing all possible outcomes of measurements of partial observables) to the physical Hilbert space $\mathcal{H}$ (the space representing the sole outcomes of measurements of partial observables that are allowed by dynamics). The propagator plays a role similar to the propagator of the Schrödinger equation in nonrelativistic quantum mechanics, which, as emphasized by Feynman, expresses the quantum dynamics. The propagator $K(a, b, a', b')$ can be given a direct physical interpretation [2]. This point of view might prove useful for the interpretation of the formalisms of non perturbative quantum gravity – in particular in loop quantum gravity [9] and its sum over path version, the spinfoam formalism [10], which gives us the analog of $K(a, b, a', b')$ in quantum general relativity [11].

Here we study the propagator $K(a, b, a', b')$ of our simple model and we discuss physical interpretation and classical limit. We show that in a regime in which b can be taken as an independent time parameter the probabilistic interpretation proposed in [2] reduces to the conventional interpretation of the modulus square of the wave function as a spacial probability density at fixed time. We illustrate the semiclassical limit and some semiclassical states. The propagator turns out to be real. It can be seen as propagating "forward" as well as "backward", in a local parameter time, in spite of the fact that the difference between forward and backward propagation is not manifest in the classical theory. The quantum states, on the other hand, split among forward and backward propagating ones. This might be the same situation that we find in quantum general relativity.

In the section 2 we discuss the classical theory. We formulate the theory in a fully covariant language, that illustrate the structure of general-relativistic hamiltonian mechanics. We study the quantum theory in Section 3. In the appendices, we recover an approximate Schrödinger equation, we recall the probabilistic interpretation based on the flux of the conserved current, and we illustrate a derivation of the propagator using functional integral techniques. A complete formulation and a general discussion of the classical and quantum formalism that we utilize here can be found in chapters 3 and 5 of reference [3].

2 Classical theory

2.1 Definition of the model

A simple harmonic oscillator can be viewed as system with two partial observables, q and t. A motion of the system defines a relation between q and t. A given motion is characterized by the two constants $A \in [0, \infty]$ and $\phi \in [0, 2\pi]$, and is given by the equation

$$f(q, t) = q - A \sin(\omega t + \phi) = 0, \tag{1}$$

where ω is a constant characterizing the system. This motion is a sinusoid of amplitude A and phase ϕ in the (q, t) plane, the extended configurations space $\mathcal{C}$. This dynamics is the classical limit of a quantum dynamics.

Here, we consider a system with two partial observables a and b. A given motion of the system is characterized by two constants $A \in [0, \sqrt{2M}]$ and $\phi \in [0, 2\pi]$, and is given by the equation

$$f(a,b) = \left(\frac{a}{A}\right)^2 + \left(\frac{b}{B}\right)^2 - 2\cos\phi\,\frac{a}{A}\frac{b}{B} = \sin^2\phi, \tag{2}$$

where M is a constant characterizing the system and $B^2 \equiv 2M - A^2$. This motion is an ellipse of radii A and B and inclination ϕ in the (a,b) plane, the extended configuration space $\mathcal{C}$. We want to understand if this system can be seen as the classical limit of a quantum system, with a well defined probabilistic interpretation.

2.2 Classical dynamics

A relativistic hamiltonian formulation of the system above can be given on the extended configuration space $\mathcal{C} = R^2$ coordinatized by the partial observables (a,b). The dynamics (2) is governed by the relativistic hamiltonian

$$H(a,b,p_a,p_b) = \frac{1}{2}\left(a^2 + b^2 + p_a^2 + p_b^2\right) - M \tag{3}$$

defined on the cotangent space $T^*\mathcal{C} = R^4$ coordinatized by (a,b,p_a,p_b), where p_a and p_b are the momenta conjugate to a and b. In general, in a general-relativistic hamiltonian system the dynamics is defined by the surface $H(q^a,p_a) = 0$, in $T^*\mathcal{C}$ (see chapter 3 of [3].) The hamiltonian (3) yields the Hamilton-Jacobi equation

$$\left(\frac{\partial S(a,b)}{\partial a}\right)^2 + \left(\frac{\partial S(a,b)}{\partial b}\right)^2 + a^2 + b^2 - 2M = 0. \tag{4}$$

Given a one parameter family $S(a,b;A)$ of solutions of this equation, the physical motions are determined –given two constants A and p_A– by the equation

$$\frac{\partial S(a,b;A)}{\partial A} - p_A = 0. \tag{5}$$

A family of solutions of the Hamilton-Jacobi equation (4) is given by

$$S(a,b;A) = \frac{a}{2}\sqrt{A^2 - a^2} + \frac{A^2}{2}\arctan\left(\frac{a}{\sqrt{A^2 - a^2}}\right) + \frac{b}{2}\sqrt{B^2 - b^2} + \frac{B^2}{2}\arctan\left(\frac{b}{\sqrt{B^2 - b^2}}\right). \tag{6}$$

inserting this in (5) gives (2), where $p_A = A\phi$, with simple algebra.

2.3 Action

Consider a path $\lambda : [0,1] \to T^*\mathcal{C}$ in the space of the coordinates a, b and their momenta p_a, p_b, and let $\tilde{\lambda} : [0,1] \to \mathcal{C}$ be its projection on the extended configuration space $\mathcal{C}$ (just the coordinates a, b). The action $S[\lambda]$ is defined by

$$S[\lambda] = \int_\lambda (p_a da + p_b db). \tag{7}$$

The physical motions $\tilde{\lambda}$ are the ones such that λ extremizes this action in the class of paths that satisfy

$$H(a,b,p_a,p_b) = 0 \tag{8}$$

and such that $\tilde{\lambda}$ has fixed extrema $\tilde{\lambda}(0) = (a,b)$ and $\tilde{\lambda}(1) = (a',b')$. These motions follow the ellipse (2). This is a very general structure, common to all non relativistic as well as general relativistic systems [3].

2.4 Geometry

Since the space $T^*\mathcal{C}$ is a cotangent space, it is naturally equipped with the Poincaré one-form

$$\theta = p_a da + p_b db \tag{9}$$

and the symplectic two-form $\omega = d\theta$. Notice that the action (7) is simply the line integral along the path of the Poincaré one-form

$$S[\lambda] = \int_\lambda \theta. \tag{10}$$

Equation (8) defines a surface Σ in the space $T^*\mathcal{C}$. In the model we are considering this surface is a three-sphere of radius $\sqrt{2M}$. The restriction of ω to Σ is degenerate: it has a null direction X. That is, there is a vector field X on Σ that satisfies $\omega(X) = 0$. It can be easily verified that this is

$$X = p_a \frac{\partial}{\partial a} + p_b \frac{\partial}{\partial b} - a \frac{\partial}{\partial p_a} - b \frac{\partial}{\partial p_b}. \tag{11}$$

The integral lines of X on Σ define the motions. It is easy to see that these integral lines are circles. In fact, they define a standard Hopf fibration of the three-sphere $S_3 \sim S_2 \times S_1$. The motions are precisely the ones given by (2), and the corresponding momenta are

$$\begin{aligned} p_a &= \sqrt{A^2 - a^2}, \\ p_b &= \sqrt{B^2 - b^2}. \end{aligned} \tag{12}$$

The quotient of Σ by the motions $S_2 \sim S_3/S_1$ is the physical phase space of the system, coordinatized by the two constants A and ϕ

$$2A^2 = 2M + p_a^2 - p_b^2 + a^2 - b^2 \tag{13}$$

$$\tan\phi = \frac{p_a b - p_b a}{p_a p_b + ab}, \tag{14}$$

which are constant along the motions.

2.5 Parametrization and the associate nonrelativistic system

The dynamics of the system becomes easier to deal with if we introduce a non-physical parameter τ along the physical motions (2). We can parametrize the ellipse (2) as follows

$$\begin{aligned} a(\tau) &= A\ \sin(\tau + \phi), \\ b(\tau) &= B\ \sin(\tau). \end{aligned} \tag{15}$$

The momenta are then given by

$$\begin{aligned} p_a(\tau) &= \sqrt{A^2 - a^2} = A\ \cos(\tau + \phi) = \frac{da(\tau)}{d\tau}, \\ p_b(\tau) &= \sqrt{B^2 - b^2} = B\ \cos(\tau) = \frac{db(\tau)}{d\tau}. \end{aligned} \tag{16}$$

The parameter τ is not connected with observability; it is only introduced for convenience. In particular the physical predictions of our system regard the relation between a and b, not the dependence of a and b on τ. The parameter τ corresponds to the time coordinate of general relativity: physically meaningful quantities are independent from the time coordinate.

It is useful to consider also a *distinct* dynamical system with *two* degrees of freedom $a(\tau)$ and $b(\tau)$ evolving according to equations (15) in a time parameter τ. We call this system the *associate* nonrelativistic system. It is important not to confuse the two systems. The associate system has two degrees of freedom, while our original general-relativistic system has one degree of freedom. The associate system is governed by the function H given in (3), seen now as a conventional non-general-relativistic hamiltonian. Indeed, it is immediate to see that equations (15-16) are the solutions of the Hamilton equations of H. This is the hamiltonian of two harmonic oscillators with unit mass and unit angular frequency, minus a constant energy M.[2]

The original system can be derived from the associate nonrelativistic systems in two steps. First, by restricting the motions to the ones with total energy equal to zero (namely the ones in which the energy $E = E_a + E_b$ of the two oscillators is $E = M$). Second, by gauging away the time evolution in τ: that is, by restricting the physical observables to the τ independent ones, namely to the sole relations between a and b. This reduces by one the number of degrees freedom. Intuitively, we "throw away the clock that measures τ".

2.6 The Hamilton function

As beautifully emphasized by Hamilton [12], the solution of any dynamical system is entirely coded in its Hamilton function. This is true for background independent systems as well. Furthermore, the Hamilton function plays a key role in relating the classical system to the quantum system, and we shall use it extensively. It is essentially the classical limit of the quantum propagator.

The Hamilton function $S(a, b, a', b')$ is a function on $\mathcal{C} \times \mathcal{C}$ that satisfies the Hamilton-Jacobi equation in both sets of variables. It is defined as the value of the action of the physical motion that goes from (a', b') to (a, b). If $S(a, b, a', b')$ is known, all physical motions can be obtained just taking derivatives, as follows. Defining

$$p'_a(a, b, a', b') = \frac{\partial S(a, b, a', b')}{\partial a'}, \quad p'_b(a, b, a', b') = \frac{\partial S(a, b, a', b')}{\partial b'}, \tag{17}$$

the algebraic equations

$$p'_a(a, b, a', b') = p'_a, \quad p'_b(a, b, a', b') = p'_b \tag{18}$$

are equivalent to (2) and give the physical motions. Here a motion is characterized by the "initial data" (a', b', p'_a, p'_b) instead than by A and ϕ.

Let us study the Hamilton functions of our system. Given two points (a', b') and (a, b) in $\mathcal{C}$, we first need to find the physical motion that goes from one to the other. That is, we need to find the value of the two constants A and ϕ of this motion. To this aim, it is convenient to parametrize the motion as in (15) and search the value of the three constants A, ϕ and τ such that

$$\begin{cases} a' = a(\tau') = A\ \sin(\tau' + \phi), \\ b' = b(\tau') = B\ \sin(\tau'); \end{cases} \qquad \begin{cases} a = a(\tau' + \tau) = A\ \sin(\tau' + \tau + \phi), \\ b = b(\tau' + \tau) = B\ \sin(\tau' + \tau). \end{cases} \tag{19}$$

With some algebra we find

$$A^2 = \frac{a^2 + a'^2 - 2aa' \cos\tau}{\sin^2\tau} \tag{20}$$

and

$$M = \frac{(a^2 + b^2 + a'^2 + b'^2) - 2(aa' + bb') \cos\tau}{\sin^2\tau} \tag{21}$$

[2] In turn, the associate system can be cast in covariant form as well. Its extended phase space has coordinates (a, b, τ) and its relativistic hamiltonian is $H_{associate} = p_\tau + H$.

The last equation can solved for τ, giving

$$\tau(a,b,a',b') = \arccos \frac{aa' + bb' \pm \sqrt{(aa'+bb')^2 + M(M - a^2 - b^2 - a'^2 - b'^2)}}{M} \tag{22}$$

Inserting this value in (20) gives $A = A(a,b,a',b')$.

The value of the Hamilton function is obtained by performing the integration that defines the action (7) of the motion that goes from (a',b') to (a,b). The result is

$$S(a,b,a',b') = S\Big(a,b,a',b';\ A(a,b,a',b')\Big) \tag{23}$$

where

$$S(a,b,a',b';\ A) = S(a,b,A) - S(a',b',A). \tag{24}$$

Here $S(a,b,A)$ is given in (6) and $A(a,b,a',b')$ is the value of A of the ellipse (2) that crosses (a,b) and (a',b'), determined by the equations (20) and (22).

It easy to show that

$$\left.\frac{\partial S(a,b,a',b';\ A)}{\partial A}\right|_{A=A(a,b,a',b')} = 0 \tag{25}$$

We can also write the hamilton function in another form, that turns out to be useful in the following. Using (15) and (21) we can write

$$S(a,b,a',b') = S(a,b,a',b';\ \tau(a,b,a',b')) \tag{26}$$

where

$$S(a,b,a',b';\ \tau) = \frac{1}{2}\frac{(a^2+b^2+a'^2+b'^2)\cos\tau - 2(aa'+bb')}{\sin\tau} + M\tau. \tag{27}$$

This result is not surprising. The Hamilton function of a single harmonic oscillator is

$$S(a,a';\tau) = \frac{1}{2}\frac{(a^2+a'^2)\cos\tau - 2aa'}{\sin\tau}; \tag{28}$$

equation (27) shows that the action of the relativistic system is equal to the action of the associate system of two harmonic oscillators $a(\tau)$ and $b(\tau)$ plus the constant M term, calculated for the time τ determined by the requirement that the motion reproduces the constrained one of the covariant system. That is

$$S(a,b,a',b') = S(a,a';\tau(a,b,a',b')) + S(b,b';\tau(a,b,a',b')) + M\tau(a,b,a',b'). \tag{29}$$

Finally, it is easy to check that

$$\left.\frac{\partial S(a,b,a',b';\ \tau)}{\partial \tau}\right|_{\tau=\tau(a,b,a',b')} = 0 \tag{30}$$

An equation that plays an important role below.

3 Quantum theory

The classical system described above can be viewed as the classical limit of a quantum system. The quantum system is defined on the kinematical Hilbert space $\mathcal{K} = L_2[R^2, da\, db]$ formed by the wave functions $\psi(a,b)$. The partial observables a and b act as (self-adjoint) multiplicative operators. The dynamics is defined by the Wheeler-deWitt equation

$$\left(-\hbar^2 \frac{\partial^2}{\partial a^2} - \hbar^2 \frac{\partial^2}{\partial b^2} + a^2 + b^2 - 2M\right) \psi(a,b) = 0. \tag{31}$$

The operator in parenthesis is the Wheeler-deWitt operator H. The space of the solutions of this equation form the Hilbert space $\mathcal{H}$ of the theory. It is easy to find the general solution of this equation. The equation can be rewritten as

$$H(a,b,p_a,p_b)\ \psi(a,b) = (H_a + H_b - M)\ \psi(a,b) = 0. \tag{32}$$

where H_a (and H_b) is the Shrödinger Hamiltonian operator (with unit mass and unit frequency) in the a (resp b) variable of a single harmonic oscillator. Since the eigenvalues of this operator are $E_a = \hbar(n_a + 1/2)$ (and $E_b = \hbar(n_b + 1/2)$) with non negative integer n_a (and n_b), in the basis of the energy eigenstates, $|n_a, n_b\rangle$, equation (31) takes the form

$$H(a,b,p_a,p_b)\ |n_a,n_b\rangle = (\hbar(n_a + n_b + 1) - M)\ |n_a,n_b\rangle = 0. \tag{33}$$

This equation has solution only if

$$M = \hbar(n_a + n_b + 1) \tag{34}$$

namely if $N = M/\hbar - 1$ is a nonnegative integer. We shall assume so from now on. The general non-normalized solution of (31) is then easily

$$\psi(a,b) = \sum_{n_a+n_b+1=N} c_{n_a,n_b} \psi_{n_a}(a)\ \psi_{n_b}(b) \tag{35}$$

where ψ_n is the normalized n-th eigenfunction of the Harmonic oscillator, and the coefficients c_{n_a,n_b} satisfy $\sum_{n_a+n_b+1=N} |c_{n_a,n_b}|^2 = 1$. The state space $\mathcal{K}$ is the Hilbert space of the associate nonrelativistic system. A basis in $\mathcal{K}$ is formed by the states $|n_a, n_b\rangle$ with n_a a-quanta and n_b b-quanta:

$$\langle a,b|n_a,n_b\rangle = \psi_{n_a}(a)\ \psi_{n_b}(b) \tag{36}$$

The physical Hilbert space $\mathcal{H}$ is the one spanned by the states with N quanta. It is convenient to write $j = N/2$ and to introduce the quantum number $m = \frac{1}{2}(n_a - n_b)$ that runs from $m = -j$ to $m = j$. So that $\mathcal{H}$ is spanned by the $(2j+1)$ states

$$|m\rangle \equiv |j-m, j+m\rangle. \tag{37}$$

That is

$$\langle a,b|m\rangle = \psi_{j-m}(a)\ \psi_{j+m}(b) \equiv \psi_m(a,b). \tag{38}$$

Explicitly,

$$\psi_m(a,b) = \frac{1}{\sqrt{2^{2j}\hbar\pi(j+m)!(j-m)!}}\ H_{j+m}(a/\sqrt{\hbar}) H_{j-m}(b/\sqrt{\hbar})\ e^{-\frac{a^2+b^2}{2\hbar}} \tag{39}$$

where $H_n(q)$ is the n-th Hermite polynomial. (As pointed out by Schwinger we can define angular momentum operators on the Hilbert space of two oscillators; the total Hamiltonian is proportional to the total

angular momentum. $\mathcal{H}$ is then the Hilbert space of the irreducible spin-j component of $\mathcal{K}$. m is the quantum number associated to one of the components of the angular momentum:

$$L_z = \frac{1}{4}(p_a^2 - p_b^2 + a^2 - b^2) \tag{40}$$

see [8].) Alternatively, we can diagonalize the angular momentum

$$L = \frac{1}{2}(bp_a - ap_b). \tag{41}$$

Its eigenstates are easily written in polar coordinates $a = r\sin\varphi$, $b = r\cos\varphi$ as

$$\psi_{m'}(r,\varphi) = e^{\frac{i}{\hbar}m'\varphi}\psi_{m'}(r), \tag{42}$$

where m' is the eigenvalue of L and $\psi_{m'}(r)$ is the (unique) solution of the radial equations with the correct energy

$$\psi_{m'}(r) = e^{-r^2/2}\, r^{|m'|}\, {}_1F_1(-n, |m'|+1, r^2) \tag{43}$$

where ${}_1F_1(-n, |m'|+1, r^2)$ is the confluent hypergeometric function and $n = 0, 1, ..., j$, $|m'| = 0, 1, ..., 2j$, with the condition

$$2j = 2n + |m'|. \tag{44}$$

$\mathcal{H}$ is a proper subspace of $\mathcal{K}$, therefore the scalar product of $\mathcal{K}$ is well defined in $\mathcal{H}$. As well known, in several general relativistic systems the space of solutions of the Wheeler-deWitt equation is a space of generalized states, namely a subspace of a suitable completion of $\mathcal{K}$. This is sometimes described as serious conceptual difficulty, but it is not: there are many equivalent techniques for deriving a scalar product of $\mathcal{H}$ from the scalar product of $\mathcal{K}$. Here we are not concerned with this technicality.

3.1 The propagator

Since $\mathcal{H}$ is a proper subspace of $\mathcal{K}$, there is an orthogonal projection operator

$$P : \mathcal{K} \to \mathcal{H}. \tag{45}$$

The relativistic propagator $K(a, b, a', b')$ is defined as the integral kernel of P. Easily, this projector is given by

$$P = \sum_{n_a+n_b+1=N} |n_a, n_b\rangle\langle n_a, n_b| \tag{46}$$

Therefore

$$K(a, b, a', b') = \sum_{n_a+n_b+1=N} \langle a, b|n_a, n_b\rangle\langle n_a, n_b|a', b'\rangle. \tag{47}$$

Explicitly, we can write this as

$$K(a, b, a', b') = \sum_{m=-j}^{j} \Psi_m^*(a', b')\Psi_m(a, b) \tag{48}$$

$$= \sum_{m=-j}^{j} e^{-\frac{1}{2\hbar}(a^2+b^2+a'^2+b'^2)}\, \frac{H_{j+m}(a/\sqrt{\hbar})H_{j-m}(b/\sqrt{\hbar})H_{j+m}(a'/\sqrt{\hbar})H_{j-m}(b'/\sqrt{\hbar})}{2^{2j}(j+m)!(j-m)!\hbar\pi} \tag{49}$$

using the properties of the Hermite polynomials, this can be written for integer j also as

$$K(a, b, a', b') = \frac{1}{\pi}\, e^{-\frac{a^2+b^2+a'^2+b'^2}{2\hbar}} \sum_{i=0}^{j} \frac{(2(aa'+bb')/\hbar)^{2j-2i}}{(2j-2i)!}\, L_i^{2j-2i}\left(\frac{a^2+b^2+a'^2+b'^2}{\hbar}\right) \tag{50}$$

where L_n^m is the n-th generalized Laguerre polynomial with parameter m.

3.2 Alternative expressions for the propagator

The expression (46) of the propagator can be rewritten as follows

$$\begin{aligned} P &= \sum_{n_a+n_b+1=N} |n_a,n_b\rangle\langle n_a,n_b| = \sum_{n_a,n_b} |n_a,n_b\rangle\langle n_a,n_b|\ \delta_{n_a+n_a+1-N} \\ &= \sum_{n_a,n_b} |n_a,n_b\rangle\langle n_a,n_b| \int_0^{2\pi} d\tau\ e^{-i(n_a+n_b+1-N)\tau} = \sum_{n_a,n_b} |n_a,n_b\rangle\langle n_a,n_b|\ \int_0^{2\pi} d\tau\ e^{-\frac{i}{\hbar}H\tau} \\ &= \int_0^{2\pi} d\tau\ e^{-\frac{i}{\hbar}H\tau}. \end{aligned} \tag{51}$$

That is

$$P\psi(a,b) = \int_0^{2\pi} d\tau\ \psi(a,b,\tau), \tag{52}$$

where $\psi(a,b,\tau)$ is the wave function of the associate system that evolves in a time τ from the initial state $\psi(a,b,0) = \psi(a,b)$. The evolution of the associate system is easy to determine. We have

$$\psi(a,b,\tau) = e^{-\frac{i}{\hbar}H\tau}\psi(a,b,0) = \int da' \int db' K(a,a',\tau)K(b,b'\tau)e^{-\frac{i}{\hbar}M\tau}\ \psi(a,b,0) \tag{53}$$

where $K(a,a'\tau)$ is the well known propagator of a single Harmonic oscillator

$$K(a,a',\tau) = \langle a|e^{-\frac{i}{\hbar}H_a\tau}|a'\rangle = \frac{1}{\sqrt{2\pi i\hbar\sin\tau}} \exp\left(\frac{i}{2\sin\tau}\left(\left[(a^2+a'^2)\cos\tau - 2aa'\right]\right)\right), \tag{54}$$

and the term $e^{-\frac{i}{\hbar}M\tau}$ is present because of the constant term in the hamiltonian. Inserting (54) in (53) we find a simple integral expression for the propagator

$$K(a,b,a',b') = \int_0^{2\pi} d\tau \frac{1}{2\pi i\hbar\sin\tau} \exp\left(\frac{i}{2\hbar\sin\tau}\left[(a^2+b^2+a'^2+b'^2)\cos\tau - 2(a'a+b'b)\right] + \frac{i}{\hbar}M\tau\right) \tag{55}$$

that is

$$K(a,b,a',b') = \int_0^{2\pi} d\tau \frac{1}{2\pi i\hbar\sin\tau} \exp\left(\frac{i}{\hbar}S(a,b,a',b';\tau)\right) \tag{56}$$

where $S(a,b,a',b';\tau)$ is the hamilton function given by (27).

3.3 Properties of the propagator

Let us summarize some of the properties of the propagator. The propagator

- satisfies the Wheeler-deWitt equation (31) in both sets of variables.

$$\left(-\hbar\frac{\partial^2}{\partial a^2} - \hbar\frac{\partial^2}{\partial b^2} + a^2 + b^2 - (2j+1)\hbar\right) K(a,b,a',b') = 0; \tag{57}$$

- satisfies the composition law

$$\int da' \int db'\ K(a,b,a',b')\ K(a',b',a'',b'') = K(a,b,a'',b''), \tag{58}$$

  this follows immediately from the fact that P is a projector;

- propagates a physical state from (a',b') to (a,b) in the sense

$$\int da'db'\ K(a,b,a',b')\ \psi(a',b') = \psi(a,b); \tag{59}$$

- projects an arbitrary function $\psi(a,b)$ (not a solution of (31)) to a solution of (31);
- is gauge invariant, in the sense that it is independent from the parameter τ;
- is real;
- is symmetric in the exchange $(a,b) \leftrightarrow (a',b')$.

4 Formal relations between quantum and classical theory

From now we assume that $M \gg \hbar$ and we study the relation between the classical and the quantum theory.

4.1 The semiclassical limit of the propagator

We begin by relating the propagator $K(a,b,a',b')$ and the Hamilton function $S(a,b,a',b')$. A semiclassical approximation of the propagator can be derived from the expression (55). In the limit of small $\hbar$ we can evaluate the integral using a saddle point approximation. The result is

$$K(a,b,a',b') = \sum_i \frac{1}{2\pi i\hbar \sin\tau_i} \exp\left(\frac{i}{2\hbar\sin\tau_i}\left[(a^2+b^2+a'^2+b'^2)\cos\tau_i - 2(a'a+b'b)\right] + \frac{i}{\hbar}M\tau_i\right) \tag{60}$$

where $\tau_i = \tau_i(a,b,a',b')$ are the values of τ for which the exponential has an extremum. That is, they are determined by

$$\frac{d}{d\tau}\left(\frac{i}{2\hbar\sin\tau}\left[(a^2+b^2+a'^2+b'^2)\cos\tau - 2(a'a+b'b)\right] + \frac{i}{\hbar}M\tau\right)\Big|_{\tau=\tau_i} = 0 \tag{61}$$

But notice that the exponent is precisely $S(a,b,a',b',\tau)$, and equation (61) is precisely equation (30) that determines the classical time τ along the classical trajectory from (a',b') to (a',b'). Therefore the exponent in (60) is *precisely* $(\frac{i}{\hbar})$ the Hamilton function !

There is however an essential subtlety, that we have disregarded during the classical discussion. In general, equation (61) does not has one, but rather two solutions (a smooth periodic function cannot have a single extremum). Why two? Because in general there are two paths that connect two points (a',b') and (a,b): a shorter one and a longer one obtained subtracting the shorter one from the entire ellipses. Indeed, it is clear that if τ is a solution, so is $2\pi-\tau$. Therefore the sum over τ_i contains two terms: $\tau_1 = \tau(a,b,a',b')$ where $\tau(a,b,a',b')$ is given by (22) and $\tau_2 = 2\pi - \tau(a,b,a',b')$. Inserting this in (62) we conclude that, to the first relevant order in $\hbar$,

$$K(a,b,a',b') = \frac{1}{2\pi i\hbar\sin\tau(a,b,a',b')}\ \left(e^{-\frac{i}{\hbar}S(a,b,a',b')} - e^{+\frac{i}{\hbar}S(a,b,a',b')}\right), \tag{62}$$

or

$$K(a,b,a',b') = \frac{1}{\pi\hbar\sin\tau(a,b,a',b')}\ \sin\left(\frac{1}{\hbar}S(a,b,a',b')\right). \tag{63}$$

4.2 Quasi classical states

Consider a coherent state of the associate nonrelativistic system. Using standard coherent state technology, the state determined by the classical position (a,b) and the classical momenta (p_a,p_b) can be written as

$$|a,b,p_a,p_b\rangle = \sum_{n_a=0}^{\infty}\sum_{n_b=0}^{\infty}\frac{\alpha_a^{n_a}\alpha_b^{n_b}}{\sqrt{n_a!n_b!}}\exp\left(-\frac{|\alpha_a|^2+|\alpha_b|^2}{2}\right)|n_a,n_b\rangle \tag{64}$$

where $\alpha_a = a + ip_a$ and $\alpha_b = b + ip_b$. This is a minimum spread wave packet centered in the position (a, b) and with minimum spread momentum (p_a, p_b). Recall that the coherent states of the harmonic oscillator have the property

$$|a, b, p_a, p_b, \tau\rangle \equiv e^{-\frac{i}{\hbar}H\tau} |a, b, p_a, p_b\rangle = |a(\tau), b(\tau), p_a(\tau), p_b(\tau)\rangle \tag{65}$$

where $(a(\tau), b(\tau), p_a(\tau), p_b(\tau))$ is the classical motion determined by the initial conditions a, b, p_a, p_b. That is, they follow classical trajectories exactly.

Let us now project this state on $\mathcal{H}$. We obtain

$$P\,|a, b, p_a, p_b\rangle = \sum_{n_a+n_b+1=N} \frac{\alpha_a^{n_a}\alpha_b^{n_b}}{\sqrt{n_a!n_b!}} \exp\left(-\frac{|\alpha_a|^2 + |\alpha_b|^2}{2}\right) |n_a, n_b\rangle \tag{66}$$

On the other hand, we have also

$$\psi_{a,b,p_a,p_b}(a', b') = \int_0^{2\pi} d\tau\ \psi_{a,b,p_a,p_b}(a', b', \tau) \tag{67}$$

where $\psi_{a,b,p_a,p_b}(a, b, \tau)$ is the time evolution of the coherent state of the associate system. In the small $\hbar$ limit the state $\psi(a, b, \tau)$ has essentially support only along a narrow strip around the classical motion determined by the initial conditions (a, b, p_a, p_b). Therefore the physical state $\psi_{a,b,p_a,p_b}(a', b')$ will have support on this same region, namely on one of the ellipses (2). The figure shows the square of the amplitude of a coherent state $\psi(a, b)$ computed numerically: it is picked around the classical solution.

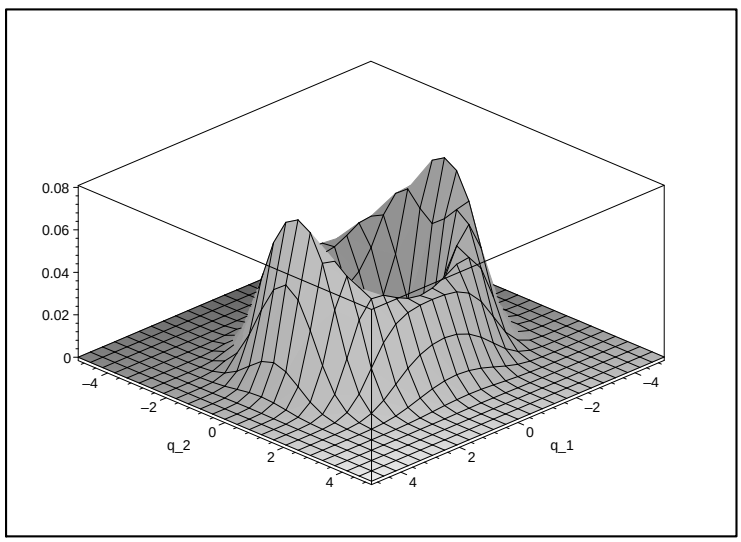

Figure 1: Support of a quasi classical state

4.3 Forward and backward propagation

In the classical theory, an ellipse defines a classical motion. We can assume that the versus of the ellipse, clockwise or anticlockwise, has no physical significance. In fact the equation (2) that fixes the motion does not contain any reference to the versus of the ellipse. However, in the quantum theory there are distinct "clockwise and anticlockwise propagating" states. To see this, consider, instead of the coherent state defined in the previous section, the one defined by the classical initial values $(a, b, -p_a, -p_b)$. In the associate system, the same ellipses is followed in the opposite direction. After integrating in $d\tau$, the resulting physical coherent state $\psi_{a,b,-p_a,-p_b}$ has the same support as ψ_{a,b,p_a,p_b}, but it is not the same state ! A moment of reflection shows that it is its complex conjugate

$$\psi_{a,b,-p_a,-p_b}(a', b') = \psi^*_{a,b,p_a,p_b}(a', b'). \tag{68}$$

In fact, the physical Hilbert space can be split into the forward and backward propagating subspaces according to the sign of the eigenvalues of the operator L.

A particularly simple semiclassical state, which illustrates well some general features of these states, can be obtained as follows. Consider the classical motion formed by a circle, namely the case $A = B = \sqrt{M}$. The two corresponding semiclassical states are easily obtained. They are the states (42) with $m' = \pm 2j$, that maximize and minimize the angular momentum L. In the large $M/\hbar$ limit, the radial function become a narrow gaussian around the classical radius $\sqrt{M}$, as can be seen from (43). Indeed, when $m' = \pm 2j$, $n = 0$, and since

$$ {}_1F_1(-n, |m'| + 1, r^2) = 1 \tag{69} $$

we have

$$ \psi_{\pm 2j}(r, \varphi) \;=\; e^{-r^2/2 + 2j \ln r}\; e^{\pm \frac{i}{\hbar} 2j\varphi}. \tag{70} $$

The maximum is in $r = \sqrt{2j} \approx \sqrt{M}$.

This observation sheds light on the fact that the propagator is real and contains the two terms in the right hand side of (62). The propagator propagates both sectors of the Hilbert space – the one propagating clockwise, $\psi_{-2j}(r, \varphi)$, and the one propagating anticlockwise, $\psi_{+2j}(r, \varphi)$. Each of the two terms of (62) propagates one of these sectors. Since the two are simply the complex conjugate of each other, the propagator is real.

On the other hand, there are quantum states that are in only one of the two sectors. They are "clockwise propagating" states. Restricted to those states, the propagator can be written as a single exponential of the Hamilton function.

Notice that the two terms of (62) can be interpreted in two equivalent manners. Either as related to the two branches of the ellipses that connect (a, b) and (a', b'). Or as one term evolving from (a, b) to (a', b') and other from (a', b') to (a, b). The math of section 4.1 can be interpreted either way. Thus, the second term of the propagator can be seen as the backward propagating term. We suspect that this term is therefore going to be present irrespectively of whether the classical orbits are closed.

Recall that in quantum general relativity the three-geometry to three-geometry transition amplitude has a surprising tendency of turning out to be real. For instance, the Ponzano-Regge transition amplitudes are real. Similarly, spin foam model tend to give real transition amplitudes. We suspect that the propagator of a covariant theory is naturally defined as a real propagation –as in the example studied here– where there is no a priori distinction between forward and backward propagation. On the other hand, a quantum state may contain one component only, and therefore be able to "select" the appropriate part of the propagator. Form this point of view, the attempt by Oriti and Livine to separate the two directions of propagation in the spinfoam sums [13] can be seen as attempts to separate locally the general relativistic analog of the two terms of (62).

5 Probabilistic interpretation

The idea developed in [1, 2, 3] is that the variables of the extended configuration space $\mathcal{C}$ can be interpreted as partial observables, namely quantities that can be measured. A point in $\mathcal{C}$, called an "event", represents a simultaneous measurement of partial observables ((a, b) in our case). The classical theory predicts which events can be measured: a classical state determines a relation between partial observables and therefore a subset of events. The quantum theory assigns a probability amplitude for the measurement of each event. According to the general prescription given in [2], the probability of observing an event in a small region $\mathcal{R}$ of $\mathcal{C}$ if an event has been observed in a small region $\mathcal{R}'$ is

$$ P_{\mathcal{R}\mathcal{R}'} = \frac{|K_{\mathcal{R}\mathcal{R}'}|^2}{K_{\mathcal{R}\mathcal{R}}\; K_{\mathcal{R}'\mathcal{R}'}} \tag{71} $$

where

$$K_{\mathcal{R}\mathcal{R}'} = \int_{\mathcal{R}} dadb \int_{\mathcal{R}'} da'db' \quad K(a,b,a',b'). \tag{72}$$

The propagator K contains then the full physically relevant information about the quantum theory. Alternatively, given a physical state ψ, the probability for detecting an event around a point (a,b) in a small region $\mathcal{R}$ of volume V, where $\psi(a,b)$ can be taken to constant, is

$$P_{\mathcal{R}} \;=\; \frac{|\int_R \psi(a,b)\; da\, db|^2}{K_{\mathcal{R}\mathcal{R}}} \;=\; \frac{V^2}{K_{\mathcal{R}\mathcal{R}}} \;\; |\psi(a,b)|^2. \tag{73}$$

To test the viability of this interpretation in the context of our system, we must relate it to the standard interpretation of quantum mechanics. According to the standard interpretation of the wave function $\psi(x,t)$, the modulus square of the wave function *at fixed time* is to be interpreted as the probability density *in space* for the position of the particle. A probability is therefore given by

$$dP = |\psi(x,t)|^2 dx. \tag{74}$$

Can we recover this expression from (73) in a limit in which our system behaves as a nonrelativistic system where, say, b is taken as the independent variable (as t) and a as a dynamical variable (as x) varying in time?

For this, we must look for a regime where the fact that the evolution "comes back" in the time b is negligible. Consider a region $\mathcal{R}$ such that $a^2, b^2 \ll M$. As in [2], imagine we have a detector active for very short "time" b. That is, consider a small detection region $\Delta b \ll (\Delta a)^2 \ll 1$. Let us then calculate the detection probability using (73). In the region, $\tau(a,b,a',b') \ll 1$. Assume also that $M \gg \hbar$ and we can use the saddle point approximation. The propagator takes the form

$$K(a,b,a',b') = \frac{1}{\pi\hbar\tau} \sin\left(\frac{(a-a')^2 + (b-b')^2}{2\hbar\tau} + \frac{M}{\hbar}\tau \right) \tag{75}$$

Using the expression of the parameter τ from (21)

$$K(a,b,a',b') = \frac{\sqrt{M}}{\pi\hbar} \frac{\sin\left(\frac{3}{2\hbar}\sqrt{M}\sqrt{(a-a')^2 + (b-b')^2}\right)}{\sqrt{(a-a')^2 + (b-b')^2}} \tag{76}$$

Since $M \gg \hbar$, this gives

$$K(a,b,a',b') \;\; \approx \;\; \frac{3\sqrt{M}}{2\pi\sqrt{\hbar}}\, \delta\left(\sqrt{(a-a')^2 + (b-b')^2}/\hbar \right) \tag{77}$$

Performing the integration in the region $\mathcal{R}$, and using our conditions on the shape of $\mathcal{R}$ (which are not symmetric in a and b) we obtain

$$K_{\mathcal{R}\mathcal{R}} = \int_{\mathcal{R}} dadb \int_{\mathcal{R}} da'db' \quad K(a,b,a',b') \approx \frac{3\sqrt{M}}{2\pi\hbar^2}\; \Delta a (\Delta b)^2 \tag{78}$$

so

$$P_{\mathcal{R}} = \frac{|\int_R \psi da\, db|^2}{K_{\mathcal{R}\mathcal{R}}} \approx \frac{|\psi(a,b)|^2 (\Delta a)^2 (\Delta b)^2}{\Delta a (\Delta b)^2} = |\psi(a,b)|^2 \Delta a \tag{79}$$

where we have absorbed a proportionality constant in the normalization of the wave function. The key result is that the probability is independent from Δb and proportional to Δa. The first fact allows us to use the notion of measurement *istantaneous* in b, because the "time" b needed by the apparatus to perform the measurmenet has no effect on the result. The second implies that $|\psi(a,b)|^2$ can be interpreted as a probability density in a at fixed b. Therefore we have recovered the standard probabilistic interpretation of the wave function in quantum mechanics: $|\psi|^2$ is the probability density in space, at fixed time.

6 Conclusion

We have studied a simple dynamical system that reproduces some key aspects of the background independence of general relativity. We have used this system to illustrate some general features of the structure of general covariant dynamical systems.

According to the point of view we have taken here, the dynamics of a system with n degrees of freedom does not describe the evolution in time of n observables. Rather, it describes the correlations between $n+1$ partial observables. The space of the partial observables is the extended configuration space $\mathcal{C}$, and the dynamics is governed by a (vanishing) relativistic hamiltonian H on $T^*\mathcal{C}$. In the quantum theory, the kinematical Hilbert space $\mathcal{K}$ expresses all potential outcomes of measuments of partial observables. Dynamics is a restriction on these states and expresses the existence of correlations among measurements of partial observables. Such restriction of the states in $\mathcal{K}$ is given by the Wheeler-deWitt equation $H\psi = 0$. Given a physical state, the probability that the system is detected in a small region of $\mathcal{C}$ is governed by (73). (A discussion of the precise meaning of probability in this context is in [2].) Here we have shown that in our simple system this probability prescription reduces to the conventional one in the appropriate regime.

This definition of probability yields positive probabilities, unlikely the definition of probability as a flux of a current (see Appendix B), often used in quantum cosmology [5]. It is reasonable to expect the two definitions to agree for states that propagate in only one direction. We also expect it to agree with the probabaility computed with histories techniques studied in [5] and we think that the precise relation between these two points of view deserves to be studied.

The kernel of the projector from the kinematical to the physical state space is the propagator, which codes the dynamical content of the theory and can be taken as the basis of the probabilistic interpretation. We have studied the propagator of our model in detail. We have shown that in the semiclassical limit it has a simple relation with the Hamilton function of the classical theory, but this relation is not a simple exponential, as one might have expexted. Instead, the propagator is real. It is the sum of two exponential terms complex conjugate to each other, that propagate backward and forward, respectively, along the motions. Accordingly, the physical Hilbert space splits between forward and backward propagating states. We expect this structure to be the same in quantum general relativity.

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A Appendix: Approximate Schrödinger picture

If we want to describe just the local evolution, we can cast the problem in a Schrödinger form by choosing an "internal time" variable as a function on the phase-space, before quantizing. The starting point for this is to find a function $T(a, b, p_a, p_b)$ that satisfies

$$\{T, H\} = 1, \tag{80}$$

We have then $dT/d\tau = 1$, here τ is the evolution parameter. We chose

$$T(b, p_b) = \arctan\left(\frac{b}{p_b}\right), \tag{81}$$

that satisfies (80). Next, we search for a canonnical transformation from the canonical pair b, p_b to a canonical pair T, P_T. This is given by

$$b = \pm\sqrt{2P_T}\sin T, \tag{82}$$

$$p_b = \pm\sqrt{2P_T}\cos T. \tag{83}$$

A generating function for this canonical transformation is

$$S(b, T) = \frac{b^2}{2\tan T}. \tag{84}$$

Indeed, we have

$$P = -\frac{\partial S}{\partial Q} \tag{85}$$

and

$$p_b = -\frac{\partial S}{\partial b}. \tag{86}$$

The Hamiltonian in the canonical variables (a, T, p_a, P_T) is then

$$H = P_T + H_{eff} \tag{87}$$

where

$$H_{eff} = \frac{1}{2}(p_a^2 + a^2) - M \tag{88}$$

from which we obtain the Schrödinger equation

$$-i\hbar\frac{\partial\psi}{\partial T} = H_{eff}\psi \tag{89}$$

It is important to notice that the range of T is contained in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, therefore this change of variables cannto be used around the points $b = \pm\frac{\pi}{2}$, where the "time" b "comes back". The propagator is, up to a phase, the one of the standard harmonic oscillator.

$$K(a, T, a', 0) = \frac{1}{\sqrt{2\pi i\hbar \sin T}} \exp\left(\frac{i}{2\hbar \sin T}\left[(a^2 + a'^2)\cos T - 2a'a\right] + \frac{i}{\hbar}MT\right). \tag{90}$$

B WKB

We recall here the interpretation of the semiclassical approximation of a quantum cosmological model in terms of the conserved current of the WKB approximation.

Let us search approximate solutions of the Wheeler-deWitt equation (31) in the form

$$\psi(a, b) = A(a, b)\ e^{\frac{i}{\hbar}S(a,b)}, \tag{91}$$

where $A(a, b)$ is a function that varies slowly, in a sense that we specify in a moment. Inserting this function in (31) we obtain for the real part

$$\frac{\partial^2 A}{\partial^2 a} + \frac{\partial^2 A}{\partial^2 b} - \frac{A}{\hbar^2}\left(\left(\frac{\partial S}{\partial a}\right)^2 + \left(\frac{\partial S}{\partial b}\right)^2\right) = \frac{-2(2j+1) + a^2 + b^2}{\hbar^2}A. \tag{92}$$

If $A(a, b)$ varies slowly in the sense

$$\frac{\frac{\partial^2 A}{\partial^2 a} + \frac{\partial^2 A}{\partial^2 b}}{A} \ll \frac{-2(2j+1) + a^2 + b^2}{\hbar^2}, \tag{93}$$

then $S(a, b)$ must satisfy the Hamilton-Jacobi equation

$$\frac{1}{2}\left(\frac{\partial S}{\partial a}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial b}\right)^2 + \frac{a^2}{2} + \frac{b^2}{2} - (2j+1) = 0 \tag{94}$$

For the imaginary part we have

$$2\frac{\partial A}{\partial a}\frac{\partial S}{\partial a} + 2\frac{\partial A}{\partial b}\frac{\partial S}{\partial b} + A\left(\frac{\partial^2 S}{\partial^2 a} + \frac{\partial^2 S}{\partial^2 b}\right) = 0 \tag{95}$$

which can be written as

$$\frac{\partial}{\partial a}\left(A^2\frac{\partial S}{\partial a}\right)+\frac{\partial}{\partial b}\left(A^2\frac{\partial S}{\partial b}\right)=0 \tag{96}$$

and interpreted as a continuity equation $\partial_a j^a + \partial_b j^b = 0$ for the current $\vec{j} = (j^a, j^b)$

$$\vec{j} \;=\; A^2\;\vec{\nabla}S. \tag{97}$$

where

$$\vec{\nabla}S=\left(\frac{\partial S}{\partial a},\frac{\partial S}{\partial b}\right) \tag{98}$$

The WKB approximation is a widely used technique in quantum cosmological models. The central question is how to extract physical predictions from a WKB solution. The Wheeler-DeWitt equation is typically a second-order equation like the Klein-Gordon equation, and the associated current

$$\vec{J}=i(\psi\vec{\nabla}\psi^*-\psi^*\vec{\nabla}\psi) \tag{99}$$

is conserved: $div\vec{J}=0$. This current J is intended to provide the interpretation of the WKB solution: the flux of $\vec{J}$ across a surface Σ defines the probability that the set of classical trajectories with momentum $\vec{p}=\vec{\nabla}S$ corresponding to the wave function intersect the surface Σ. In general, this current produces negative probabilities. In particular, we have a full basis of real solutions where $\vec{J}=0$.

C Path integral representation of the propagator

Finally, we illustrate a path integral derivation of the propagator. The action of the relativistic system can be formulated on an extended configuration space of the coordinates a, b and a lagrangian multiplier N that implements the constraint (8)

$$S[a,b,N]=\int_0^1 d\tau\left(p_a\dot{a}+p_b\dot{b}-N\frac{1}{2}(p_a^2+p_b^2+a^2+b^2-2M)\right) \tag{100}$$

where a dot denotes the derivative with respect to τ. The variation of the action with respect to the variables $a(\tau), p_a(\tau), b(\tau), p_b(\tau)$ and $N(\tau)$ gives the canonical equations of motion and the constraint equation. The action is invariant under reparametrization of the parameter τ labeling the motion. This invariance implies a relation between the velocities and the momenta, obtained varying the action with respect to p_a and p_b:$\dot{a}=Np_a$ and $\dot{b}=Np_b$. The action can be put in the form

$$S[a,b,N]=\frac{1}{2}\int_0^1 d\tau\;\left(\frac{\dot{a}^2+\dot{b}^2}{N}-N(a^2+b^2-2M)\right) \tag{101}$$

which has the same structure as the ADM action of general relativity. The path integral representation of the propagator of the quantized theory has the form

$$K(a',b',a,b)=\int dN\int_a^{a'}\mathcal{D}a\int_b^{b'}\mathcal{D}b\;e^{iS[a,b,N]} \tag{102}$$

In order to sum over only inequivalent trajectories we fix the gauge in the action choosing $N(\tau)=constant$. After gauge-fixing and after rescaling the parameter τ as $\tilde{\tau}=\tau N$, the propagator can be written

$$K(a',b',a,b)=\int dN\int_a^{a'}\mathcal{D}a\int_b^{b'}\mathcal{D}b\exp\left[i\int_0^N d\tilde{\tau}\frac{1}{2}\left(\dot{a}^2+\dot{b}^2-a^2-b^2+2M\right)\right]$$

The range of integration of the lagrangian multiplier will be fixed by the symmetry property of the integrand. the explicit expression of the propagator results to be

$$\begin{aligned} K(a',b',a,b) &= \int dN\mathcal{D}a\mathcal{D}b \exp\left[i\int_0^N d\tilde{\tau}\frac{1}{2}\left(\dot{a}^2+\dot{b}^2-a^2-b^2\right)+iMN\right] \\ &= \int dN\left(\int \mathcal{D}a \exp\left[i\int_0^N d\tilde{\tau}\frac{1}{2}\left(\dot{a}^2-a^2\right)\right]\right)\left(\int \mathcal{D}b \exp\left[i\int_0^N d\tilde{\tau}\frac{1}{2}\left(\dot{b}^2-b^2\right)\right]\right)e^{iMN} \end{aligned}$$

The expressions in parenthesis have the form of the path integral for the propagator of a free one-dimensional harmonic oscillator over the time N in the coordinates a and b respectively:

$$K(a',b',a,b) = \int dN K(a',a,N)K(b',b,N)e^{iMN} \tag{103}$$

The propagator of a free harmonic oscillator is a periodic function of the time:

$$K(a',a,N+2\pi) = K(a',a,N) \tag{104}$$

Consequently we are free to choose the interval $[0,2\pi]$ as the range of integration for N

$$K = \int_0^{2\pi} dN K(a',a,N)K(b',b,N)e^{iMN}. \tag{105}$$

This representation of the propagator is exactly the expression (55).