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POLYMER PINNING AT AN INTERFACE

NICOLAS PETRELIS

ABSTRACT. We consider a model of hydrophobic homopolymer in interaction with an interface between oil and water. The configurations of the polymer are given by the trajectories of a simple symmetric random walk $(S_i)_{i \geq 0}$. On the one hand the hydrophobicity of each monomer tends to delocalize the polymer in the upper half plane, that is why we define h , a non negative energetic factor that the chain gains for every monomer in the oil (above the origin). On the other hand the chain receives a random price (or penalty) on crossing the interface. At site i this price is given by $\beta(1 + s\zeta_i)$, where $(\zeta_i)_{i \geq 1}$ is a sequence of i.i.d. centered random variables, and (s, β) are two non negative parameters. Since the price is positive on the average, the interface attracts the polymer and a localization effect may arise. We transform the measure of each trajectory with the hamiltonian $\beta \sum_{i=1}^N (1 + s\zeta_i) \mathbf{1}_{\{S_i=0\}} + h \sum_{i=1}^N \text{sign}(S_i)$, and study the critical curve $h_c^s(\beta)$ that divides the phase spaces in a localized and a delocalized area.

It is not difficult to show that $h_c^0(\beta) \leq h_c^s(\beta)$ for every $s \geq 0$, but in this article we give a method to improve in a quantitative way this lower bound. To that aim, we transform the strategy developed by Bolthausen and Den Hollander in [4] on taking into account the fact that the chain can target the sites where it comes back to the origin. Then we deduce from this last result a corollary in terms of pure pinning model, namely with the hamiltonian $\sum_{i=1}^N (-u + s\zeta_i) \mathbf{1}_{\{S_i=0\}}$ we find a lower bound of the critical curve $u_c(s)$ for small s . In this situation, we improve the existing lower bound of Alexander and Sidoravicius [1].

Keywords: Polymers, Localization-Delocalization Transition, Pinning, Random Walk, wetting.

AMS subject classification: 82B41, 60K35, 60K37

1. INTRODUCTION AND RESULTS

1.1. the model. We consider a simple random walk $(S_n)_{n \geq 0}$, defined as $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$ where $(X_i)_{i \geq 1}$ is a sequence of iid bernouilli trials verifying $P(X_1 = \pm 1) = 1/2$. We denote by $\Lambda_i = \text{sign}(S_i)$ if $S_i \neq 0$, $\Lambda_i = \Lambda_{i-1}$ otherwise. We also define $(\zeta_i)_{i \geq 1}$ a sequence of iid random variables non a.s. equal to 0, verifying $\mathbb{E}(e^{\lambda|\zeta_1|}) < \infty$ for every $\lambda > 0$ and $\mathbb{E}(\zeta_1) = 0$.

Now let $h \geq 0$, $s \geq 0$ and for each trajectory of the random walk we define the following hamiltonian

$$H_{N,\beta,h}^{\zeta,s}(S) = \beta \sum_{i=1}^N (1 + s\zeta_i) \mathbf{1}_{\{S_i=0\}} + h \sum_{i=1}^N \Lambda_i$$

With this hamiltonian we perturb the law of the random walk as follow

$$\frac{dP_{N,\beta,h}^{\zeta,s}}{dP}(S) = \frac{\exp\left(H_{N,\beta,h}^{\zeta,s}(S)\right)}{Z_{N,\beta,h}^{\zeta,s}}$$

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This new measure $P_{N,\beta,h}^{\zeta,s}$ is called polymer measure of size N . Under this measure two sorts of trajectories are "a priori" favored. On the one hand the localized trajectories, often coming back to the origin to receive some positive pinning rewards along the x axis. On the other hand, trajectories called delocalized, spending most time in the upper half plane and both favored by the second term of the hamiltonian and the fact that they are much more numerous than those staying close to the origin. So a competition between these two possible behaviors arises.

1.2. previous results and physical motivations. Systems of random walk attracted by a potential at an interface are closely studied at this moment (see [7], [12]). One of the major issue about that subject consists in understanding better the influence of a random potential compared to a constant one. Namely, if it seems intuitively clear that a random potential has a stronger power of attraction than a constant one of same expectation, it is much more complicated to quantify this difference.

In the present work, we consider a potential at the interface and also the fact that the polymer prefers lying in the upper half plan than in the lower one. That type of system has been studied numerically in [11], and can describe for example the situation of an hydrophobic homopolymer at an interface between oil and water. Close to this horizontal separation between the two solvents, some very small droplets of a third solvent (microemulsions) are put and have a big power of attraction on the monomers composing our chain. So the pinning prices our chain can receive when it comes back to the origin represent the attractive emulsions our polymer can touch close to the interface.

We expose here precise theoretical results about the critical curve arising from this system. We investigate new strategies of localization for the polymer consisting in targeting the sites where it comes back to the interface, and we find an explicit lower bound of the critical curve strictly above the non random one.

Our result covers, as a limit case as h goes to infinity the wetting transition model. Effectively in the last ten years the wetting problem, namely the case of a polymer interacting with an (impenetrable) interface has attracted a lot of interest since it can be regarded as a Polland Sheraga model of the DNA strand (see [7]). The localization transition with a constant disorder occurs for the pinning reward $\log 2$, and a lot of questions arising from this first result are linked with the effect of a small random perturbation add to the price $\log 2$. Moreover, with the constant pinning reward $\log 2$ the simple random walk conditioned to stay positive has the same law than the reflected random walk (see [10]). That is why, to study the wetting model around the pinning price $\log(2)$, it suffices to consider the pure pinning model, namely a reflected random walk pinned at the origin by small random variables.

This last model has therefore been closely studied, for example in [12] a particular type of positive potential has been considered and a criterium has been given to decide for every disorder realization if it localizes the polymer or not. But a very difficult question consists in estimating, for small s , the critical delocalization average $u_c(s)$ of an iid disorder of type $-u + s\zeta_i$ with ζ_i centered of variance 1 (namely $\text{Var}(-u + s\zeta_i) = s^2$). The annealed critical curve is given by $u_a(s) = \log E(\exp(s\zeta_i)) \stackrel{s \rightarrow 0}{\sim} s^2/2$ (even $= s^2/2$ when $\zeta_i \stackrel{D}{=} N(0, 1)$) and verifies as usual $u_c(s) \leq u_a(s)$. In the last 20 years there has been a lot of activity on this question, mostly from the physicists side and it is now widely believed that $u_c(s)$ behaves as $s^2/2$ but it is still an open question whether $u_c(s) = s^2/2$ (see [6]) for s small or $u_c(s) < s^2/2$ for every s (see [5] or [13]).

However up to now the only rigorous thing that has been proved is in [1], where Sidravicious and Alexander have studied a general class of random walk pinned either by an interface between two solvents or by an impenetrable wall. If we apply their results in our case it gives that for iid centered $(\zeta_i)_{i \geq 0}$ of fixed positive variance, the quenched quantity $u_c(s)$ is strictly larger than the non disordered one $u_c(0)$. In this paper, the new localization strategies we develop allows us to go further on giving an upper bound of $u_c(s)$ of type $-cs^2$, which has the same scale than the annealed lower bound.

1.3. the free energy. To decide for fixed parameters if our system is localized or not we introduce the free energy called $\Psi^s(\beta, h)$ and defined as

$$\Psi^s(\beta, h) = \lim_{N \rightarrow \infty} \frac{1}{N} \log Z_{N, \beta, h}^{\zeta, s}$$

This limit $\Psi^s(\beta, h)$ is not random any more and occurs \mathbb{P} almost surely in ζ and \mathbb{L}^1 . The proof of that sort of convergence is well known (see [8] or [4]). This free energy can easily be bounded from below on computing it on a restriction of the trajectories set. That way we denote by D_N the set $\{S : S_i > 0 \forall i \in \{1, \dots, N\}\}$. For each trajectory of D_N the hamiltonian is equal to hN since the chain stays in the upper half plane and never comes back to the origin. Moreover $P(D_N) \sim c/N^{1/2}$ as N goes to ∞ . Hence

$$\Psi^s(\beta, h) \geq \liminf_{N \rightarrow \infty} \frac{1}{N} \log E \left(e^{hN} \mathbf{1}_{\{D_N\}} \right) \geq h + \liminf_{N \rightarrow \infty} \frac{\log(P(D_N))}{N} \geq h$$

so the free energy is always larger than h , and from now on we will say that the polymer is delocalized if $\Psi^s(\beta, h) = h$, because the utterly delocalized trajectories of D_N give us the whole free energy whereas it will be delocalized if $\Psi^s(\beta, h) > h$.

This separation between localized and delocalized regime seems a bit raw, because many trajectories come back only a few times to the origin and should also be called delocalized since they spend almost all their time in the upper half plane. So taking only into account the utterly delocalized trajectories could be not sufficient. But it is in fact because for convexity reasons, in all the localized phase the chains come back to the origin a positive density of times. Another result can help us to understand the localization effect. It is due to Sinai in [16] and with the same technics we can control the vertical expansion of the chain in the localized area. That way we transform a bit the hamiltonian which becomes $\left(\beta \sum_{i=1}^N (1 + s\zeta_{N-i}) \mathbf{1}_{\{S_i=0\}} + h \sum_{i=1}^N (\Lambda_i) \right)$ so that the disorder is fixed in the neighborhood of S_N . Notice that the free energy is not modified by this transformation and allows us to say for $\Psi^s(\beta, h) > 0$, $\epsilon > 0$ and every realization of the disorder ζ that there exists a constant $C_\zeta^\epsilon > 0$, \mathbb{P} almost surely finite verifying for every $L \geq 0$ and $N \geq 0$

$$P_{N, \beta, h}^{\zeta, s}(|S_N| > L) \leq C_\zeta^\epsilon \exp(-(\Psi^s(\beta, h) - \epsilon)L)$$

This result can not occur if we keep the original hamiltonian because the disorder is not fixed close to S_N . As a consequence we meet almost surely arbitrary long stretches of negative rewards that push rarely but sometimes S_N far away from the interface.

Some pathwise results have also been proved in the delocalized area for polymer systems. In our case we can use the method developed in the last part of [3] to prove that \mathbb{P} almost surely in ζ and for every $K > 0$, $\lim_{N \rightarrow \infty} E_{N, \beta, h}^{\zeta, s}(\#\{i \in \{1, \dots, N\} : S_i > K\}/N) = 1$. These results allow us to understand more deeply what localization and delocalization mean.

Now we want to transform the hamiltonian, in order to simplify the localization condition. In that way notice that

$$\Psi^s(\beta, h) - h = \lim_{N \rightarrow \infty} \frac{1}{N} \log \left(E \left(\exp \left(\beta \sum_{i=1}^N (1 + s\zeta_i) \mathbf{1}_{\{S_i=0\}} + h \sum_{i=1}^N (\Lambda_i - 1) \right) \right) \right)$$

so we put $\Phi^s(\beta, h) = \Psi^s(\beta, h) - h$, the delocalization condition becomes $\Phi^s(\beta, h) = 0$ and the localization one $\Phi^s(\beta, h) > 0$. To finish with these new notations we denote $\Delta_i = 1$ if $\Lambda_i = -1$ and $\Delta_i = 0$ if $\Lambda_i = 1$. The hamiltonian becomes

$$H_{N,\beta,h}^{\zeta,s}(S) = \beta \sum_{i=1}^N (1 + s\zeta_i) \mathbf{1}_{\{S_i=0\}} - 2h \sum_{i=1}^N \Delta_i$$

and we keep $Z_{N,\beta,h}^{\zeta,s} = E \left(e^{H_{N,\beta,h}^{\zeta,s}} \right)$, so we have

$$\Phi^s(\beta, h) = \lim_{N \rightarrow \infty} \frac{1}{N} \log Z_{N,\beta,h}^{\zeta,s}$$

This function Φ^s is convex and continue in both variables, non decreasing in β and non increasing in h . In this paper we are particularly interested in the critical curve of the system, namely the curve that divides the phases space (h, β) in a delocalized zone, and a localized one. But before defining this curve precisely, it is helpful to consider the non disordered case ($s = 0$), which is much simpler to perform and provides intuitions about what happens in the disordered case ($s \neq 0$).

1.4. the critical curve. Above the critical curve the system will be delocalized and localized below, in appendix C) we compute the equation of this curve when $s = 0$, we obtain

$$h_c^0 : [0, \log(2)) \rightarrow \mathbb{R}$$

$$\beta \longrightarrow h_c^0(\beta) = \frac{1}{4} \log \left(1 - 4 \left(1 - e^{-\beta} \right)^2 \right) \quad (1.1)$$

So the curve is increasing, convex and goes to ∞ when β goes to $\log(2)$ from the left. But when $\beta \geq \log(2)$ the system is always localized, in fact as large as h is chosen the free energy remains strictly positive, that is why this critical curve is only defined on $[0, \log(2))$ (see Fig 1).

This lets us think that when $s \neq 0$, the critical curve should have a form of the same type as (1.1). Notice also that $h_c^0(\beta) \sim \beta^2$ as β goes to 0.

Proposition 1. *For every $s \geq 0$ and $\beta \geq 0$ there exists $h_c^s(\beta) \in [0, +\infty]$ such that for every $h < h_c^s(\beta)$ the free energy $\Phi^s(\beta, h)$ is strictly positive, whereas $\Phi^s(\beta, h) = 0$ if $h \geq h_c^s(\beta)$. This function $h_c^s(\beta)$ is convex, increasing in β , hence for every $s \geq 0$ there exists $\beta_0(s) \in [0, \infty]$ verifying $h_c^s(\beta) < +\infty$ if $\beta < \beta_0(s)$ and $h_c^s(\beta) = +\infty$ if $\beta > \beta_0(s)$.*

We will prove also that for every $s \geq 0$ the non disordered critical curve $h_c^0(\beta)$ is a lower bound of $h_c^s(\beta)$. As a consequence $\beta_0(s) \leq \beta_0(0) = \log(2)$

Remark 1. *The case $\beta = \beta_0(s)$ remains open, more precisely two different behavior of the curve may occur. Either $\lim_{\beta \rightarrow \beta_0^-(s)} h_c(\beta) = +\infty$, or there exists $h_0^s < \infty$ such that $\lim_{\beta \rightarrow \beta_0^-(s)} h_c(\beta) = h_0^s$ and by continuity of Φ^s in β we have $\Phi(\beta_0(s), h_0^s) = 0$ and $h_c(\beta_0(s)) = h_0^s$.*

We Find an upper bound of $h_c^s(\beta)$ as usual, on computing the annealed free energy, which is by the Jensen inequality an upper bound of the quenched free energy. The annealed system gives birth to a critical curve ($h_{ann.c}^s(\beta)$) which is an upper bound of the quenched critical curve. The annealed free energy is given by

$$\Phi_{ann.}^s(h, \beta) = \lim_{N \rightarrow \infty} \frac{1}{N} \log E \mathbb{E} \left(\exp \left(\beta \sum_{i=1}^N (1 + s\zeta_i) \mathbf{1}_{\{S_i=0\}} - 2h \sum_{i=1}^N \Delta_i \right) \right)$$

hence if we integrate over \mathbb{P} we obtain

$$\Phi_{ann.}^s(h, \beta) = \lim_{N \rightarrow \infty} \frac{1}{N} \log E \left(\exp \left(\left(\beta + \log \mathbb{E}(e^{\beta s \zeta_1}) \right) \sum_{i=1}^N \mathbf{1}_{\{S_i=0\}} - 2h \sum_{i=1}^N \Delta_i \right) \right)$$

Finally $\Phi_{ann.}^s(h, \beta) = \Phi^0(h, \beta + \log \mathbb{E}(e^{\beta s \zeta_1}))$ and the annealed critical curve can be expressed with the help of the non disordered one, namely if we call β_{ann}^s the only solution of $\beta + \log \mathbb{E}(e^{\beta s \zeta_1}) = \log 2$, for every $\beta \in [0, \beta_{ann}^s)$ the value of the annealed critical curve is $h_{ann.c}^s(\beta) = h_c^0(\beta + \log \mathbb{E}(e^{\beta s \zeta_1}))$ (see Fig 1).

Once again notice that the annealed critical curve verifies $h_{ann.c}^s(\beta) \sim \beta^2$ as β goes to 0.

1.5. The disordered model. Here comes the main part of the paper, we develop a new strategy to find a lower bound on the quenched critical curve. A strategy to find that kind of lower bound consists in computing the free energy on a particular restriction of the trajectories, namely, in the localized area, trajectories that often come back to the origin ([2]). Here we are going to develop another method, that consists in transforming (using radon Nikodym densities) the law of the excursions out of the origin. First (as done in [4]) we constrain the chain to come back to the origin a positive density of times, but without targeting the sites of the x axes it will touch. Then we make the chain choose at each excursion a trajectory law adapted to the local environment.

Notice first that proposition 1 tells us that for every $s \geq 0$ and $\beta \geq \log(2)$ we have $h_c^s(\beta) = \infty$ hence in any case the critical curve is not defined after $\log 2$, that is why, from now on we only consider the case $\beta \leq \log(2)$.

Theorem 2. *If $Var(\zeta_1) \in (0, \infty)$, there exists two strictly positive constant c_1 and c_2 such that for every $s \leq c_1$ and $\beta \in [0, \log 2 - c_2 s^2 \beta^2)$, we can bound from below the critical curve as follow*

$$h_c^s(\beta) \geq -\frac{1}{4} \log \left(1 - 4 \left(1 - e^{-\beta - c_2 s^2 \beta^2} \right)^2 \right) = m^s(\beta)$$

Remark 2. *This lower bound is strictly above the non disordered one (see proposition 1 and Fig 1) when $s > 0$.*

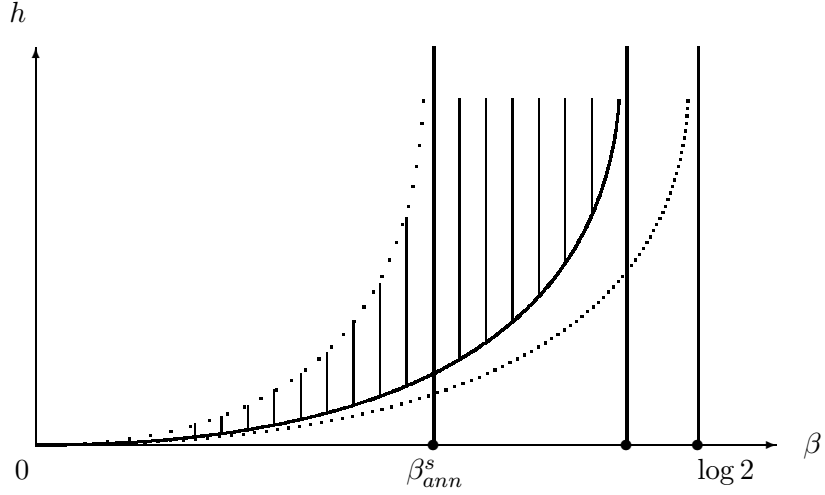



Fig. 1: $h_{ann.c}^s(\beta)$
 ————— $m^s(\beta)$  possible location of $h_c^s(\beta)$
 $h_c^0(\beta)$

Remark 3. *The possible values of c_1 and c_2 depend on the law of ζ_1 . For example, as showed in the proof if $\mathbb{P}(\zeta_1 > 0) = 1/2$ and $\mathbb{E}(\zeta_1 \mathbf{1}_{\{\zeta_1 > 0\}}) = 1$, the values $c_1 = 1$ and $c_2 = 1/(5 \times 2^{14})$ suit. But with other conditions the strategy to obtain the lower bound remains the same.*

Remark 4. *The precise value of c_2 ($1/(5 \times 2^{14})$) could certainly be improved, on building more complicated law of return to the origin. For example on building a law of return to the origin that depends more deeply on the environment (taking into account ζ_{i+2}, ζ_{i+4} etc...). The computations would be quite more complicated and our aim here is not to optimize the value of c but to expose a simple strategy that improves the non disordered lower bound of a term $cs^2\beta^2$ with $c > 0$.*

1.6. The pure pinning model. The pure pinning model is a bit different from our previous one, the h term of entropic repulsion vanishes and we consider pinning rewards at the origin of the form $-u + s\zeta_i$ with $u \geq 0$. The corresponding hamiltonian is

$$H_{N,s}^{\zeta,u}((i, S_i)_{i \in \{0, \dots, N\}}) = \sum_{i=1}^N (-u + s\zeta_i) \mathbf{1}_{\{S_i=0\}}$$

In that case, the condition of localization and delocalization in term of free energy remains the same and we have a critical u called $u_c(s)$ such that for $u \geq u_c(s)$ the system is delocalized, whereas for $u < u_c(s)$ it is localized. Recall also that if $\text{Var}(\zeta_1) = 1$ the annealed case tells us that $u_c(s) \leq u_c^{ann}(s) \sim_{s \rightarrow 0} s^2/2$. Now, a corollary of our theorem gives us a lower bound on $u_c(s)$ which has the good scale.

Corollary 3. *If $\text{Var}(\zeta_1) \in (0, \infty)$, there exists two strictly positive constant c_3 and c_4 such that for every $s \leq c_3$*

$$u_c(s) \geq c_4 s^2$$

Remark 5. *Once again the values of c_3 and c_4 depend on the law of ζ_1 . We will keep in our proof the conditions of remark 3 concerning ζ_1 . The values $c_3 = \log 2$ and $c_4 = 1/(5 \times 2^{16})$ suit.*

2. PROOF OF THEOREM AND PROPOSITION

2.1. Proof of Proposition 1. First define for every $\beta \geq 0$ and $s \geq 0$ the set $J_\beta^s = \{h \geq 0 \text{ such that } \Phi^s(\beta, h) = 0\}$. We put $h_c^s(\beta)$ the lower bound of J_β^s . Then recall that Φ is continuous, not increasing in h , and positive hence the set J_β^s can be written $[h_c^s(\beta), +\infty)$ (when it is not empty). Moreover Φ is not decreasing in β because $\Phi^s(0, h) = 0$ for every $h \geq 0$, $\Phi(\beta, h) \geq 0$ for every β and Φ is convex in β . So if $\beta_1 \geq \beta_2$ we have $J_{\beta_1}^s \subset J_{\beta_2}^s$. This gives us the fact that $h_c(\beta)$ is not decreasing, and we put $\beta_0(s) = \sup\{\beta \geq 0 : J_\beta^s \neq \emptyset\}$. The annealed computation shows us that $\beta_0(s) > 0$ because $\Phi^s(h, \beta) \leq \Phi_{ann.}^s(h, \beta)$. Thus $J_{ann.} \subset J_\beta$ and $\beta_0(s) \geq \beta_{ann.}^s > 0$. Now we want to prove that $h_c(\beta)$ is convex. That way it is continuous on the interval $[0, \beta_0(s)]$.

To prove this convexity we put $0 < a < b$ and $\lambda \in [0, 1]$. So remark that

$$H_{N, \lambda a + (1-\lambda)b, \lambda h_c^s(a) + (1-\lambda)h_c^s(b)}^{\zeta, s} = H_{N, \lambda a, \lambda h_c^s(a)}^{\zeta, s} + H_{N, (1-\lambda)b, (1-\lambda)h_c^s(b)}^{\zeta, s}$$

hence by holder inequality

$$\begin{aligned} \frac{1}{N} \log E \left(\exp \left(Z_{N, \lambda(a, h_c^s(a)) + (1-\lambda)(b, h_c^s(b))}^{\zeta, s} \right) \right) &\leq \frac{\lambda}{N} \log E \left(\exp \left(Z_{N, a, h_c^s(a)}^{\zeta, s} \right) \right) \\ &\quad + \frac{1-\lambda}{N} \log E \left(\exp \left(Z_{N, b, h_c^s(b)}^{\zeta, s} \right) \right) \end{aligned}$$

so as N goes to infinity the two terms of the rhs goes to zero because by continuity of Φ in h we have $\Phi(a, h_c^s(a)) = \Phi(b, h_c^s(b)) = 0$. Hence $\Phi^s(\lambda a + (1-\lambda)b, \lambda h_c^s(a) + (1-\lambda)h_c^s(b)) = 0$ and $h_c^s(\lambda a + (1-\lambda)b) \leq \lambda h_c^s(a) + (1-\lambda)h_c^s(b)$. This completes the proof.

Now it remains to give a short proof of the fact that $h_c^s(\beta) \geq h_c^0(\beta)$ for every $s \geq 0$. We will in fact prove that for $s \geq 0$, $\beta \geq 0$ and $h < h_c^0(\beta)$ the free energy $\Phi^s(\beta, h) > 0$. This will be sufficient to complete the proof. Hence notice that for fixed (β, h) the function $\Phi^s(\beta, h)$ is convex in s since it is the limit as N goes to infinity of the function sequence $\Phi_N^s(\beta, h) = \mathbb{E} \left(\frac{1}{N} \log E \left(\exp \left(H_{N, \beta, h}^{\zeta, s} \right) \right) \right)$ which are convex in s . Moreover for every $N > 0$, $\Phi_N^s(\beta, h)$ can be derived in s and this gives

$$\frac{\partial \Phi_N^s(\beta, h)}{\partial s} = \frac{1}{N} \mathbb{E} \left(\frac{E \left(\beta \sum_{i=1}^N \zeta_i \mathbf{1}_{\{S_i=0\}} \exp \left(H_{N, \beta, h}^{\zeta, s} \right) \right)}{E \left(\exp \left(H_{N, \beta, h}^{\zeta, s} \right) \right)} \right)$$

But when $s = 0$ the hamiltonian does not depend on the disorder (ζ) any more, so by Fubini Tonelli and the fact that the ζ_i are centered we can write

$$\left. \frac{\partial \Phi_N^s(\beta, h)}{\partial s} \right|_{s=0} = \frac{1}{N} \frac{E \left(\beta \sum_{i=1}^N \mathbb{E}(\zeta_i) \mathbf{1}_{\{S_i=0\}} \exp \left(H_{N, \beta, h}^{\zeta, 0} \right) \right)}{E \left(\exp \left(H_{N, \beta, h}^{\zeta, 0} \right) \right)} = 0$$

hence the convergence of Φ_N to Φ and their convexity allow us to say

$$\left. \frac{\partial_{right} \Phi^s(\beta, h)}{\partial s} \right|_{s=0} \geq \lim_{N \rightarrow \infty} \left. \frac{\partial_{right} \Phi_N^0(\beta, h)}{\partial s} \right|_{s=0} = 0$$

so since $\Phi^s(\beta, h)$ is convex in s we can conclude that it is not decreasing on $[0, \infty)$. Hence for every $s \geq 0$, $\Phi^s(\beta, h) \geq \Phi^0(\beta, h) > 0$. That is why $h_c^s(\beta) \geq h_c^0(\beta)$.

To finish with this proof, we show that $h_c^s(\beta)$ is increasing in β . In fact since $h_c^s(0) = 0$ and $h_c^s(\beta) \geq h_c^0(\beta) > 0$ for $\beta > 0$ the convexity of $h_c^s(\beta)$ gives us the result. \square

2.2. Proof of Theorem 2. In the following we consider $h > 0$, $\beta \leq \log(2)$, $\mathbb{P}(\zeta_1 > 0) = 1/2$, $\mathbb{E}(\zeta_1 \mathbf{1}_{\{\zeta_1 > 0\}}) = 1$ and $s \leq 1$.

STEP 1: transformation of the excursions law.

Definition 4. From now on we will call i_j the site where the j^{th} return to the origin takes place, so $i_0 = 0$ and $i_j = \inf\{i > i_{j-1} : S_i = 0\}$ and $\tau_j = i_j - i_{j-1}$ is the length of the j^{th} excursion out of the origin. We also call l_N the number of return to the origin before time N .

Thus by independence of the excursions signs we can rewrite the partition function as

$$H_N = E \left(\exp \left(\beta s \sum_{j=1}^{l_N} \zeta_{i_j} \right) \exp(\beta l_N) \prod_{j=1}^{l_N} \left(\frac{1 + \exp(-2h\tau_j)}{2} \right) \left(\frac{1 + \exp(-2h(N - i_{l_N}))}{2} \right) \right) \quad (2.1)$$

Now we want to transform the law of excursions out of the origin to constrain the chain to come back to zero a positive density of times. That way we introduce $P_{\alpha, h}^\beta$ the law of an homogeneous positive recurrent markov process, whose excursion law are given by

$$\forall n \in \mathbb{N} - \{0\} \quad P_{\alpha, h}^\beta(\tau_1 = 2n) = \left(\frac{1 + \exp(-4hn)}{2} \right) \alpha^{2n} \frac{P(\tau = 2n)}{H_{\alpha, h}^\beta} \exp(\beta) \quad (2.2)$$

where $H_{\alpha, h}^\beta$ can be computed as follow

$$H_{\alpha, h}^\beta = \sum_{i=1}^{\infty} \frac{\exp(-4hi) + 1}{2} e^\beta \alpha^{2i} P(\tau = 2i) = e^\beta \left(1 - \frac{\sqrt{1 - \alpha^2} + \sqrt{1 - e^{-4h}\alpha^2}}{2} \right) \quad (2.3)$$

Notice also that the function we are considering in the expectation of (2.1) only depends on l_N and the position of the return to the origin, namely i_1, \dots, i_{l_N} . Hence we can rewrite H_N as an expectation over $P_{\alpha, h}^\beta$ since we know the Radon Nikodym density $dP/dP_{\alpha, h}^\beta(\{i_1, \dots, i_{l_N}\})$. Hence H_N becomes

$$H_N = E_{\alpha, h}^\beta \left(\exp \left(\beta s \sum_{j=1}^{l_N} \zeta_{i_j} \right) \prod_{j=1}^{l_N} \frac{H_{\alpha, h}^\beta}{\alpha^{\tau_j}} \left(\frac{1 + e^{-2h(N - i_{l_N})}}{2} \right) \frac{P(\tau \geq N - i_{l_N})}{P_{\alpha, h}^\beta(\tau \geq N - i_{l_N})} \right)$$

Now we aim at transforming the excursions law again, so that the chain comes back more often in sites where the pinning reward is large. In fact we want the chain to take into account its local environment. So we define $P_{\alpha, h}^{\beta, \zeta, \alpha_1}$ the law of a non homogenous Markov process which depends on the environment. Its excursion laws are defined as follow. We set: $\alpha_1 < \left(1 - P_{\alpha, h}^\beta(\tau = 2) \right) / P_{\alpha, h}^\beta(\tau = 2)$, such that $\mu_1 = 1 - \left(\alpha_1 P_{\alpha, h}^\beta(\tau = 2) \right) / \left(1 - P_{\alpha, h}^\beta(\tau = 2) \right) > 0$ and

$$\begin{aligned}
P_{\alpha,h}^{\beta,\zeta,\alpha_1}(\tau=2) &= P_{\alpha,h}^{\beta}(\tau=2)(1+\alpha_1)\mathbf{1}_{\{\zeta_2>0\}} \\
P_{\alpha,h}^{\beta,\zeta,\alpha_1}(\tau=2r) &= P_{\alpha,h}^{\beta}(\tau=2r)\mu_1^{\mathbf{1}_{\{\zeta_2>0\}}} \quad \text{for } r \geq 2
\end{aligned} \tag{2.4}$$

So, under the law of this process, if the chain comes back to the origin at time i , the law of the following excursion is $P_{\alpha,h}^{\beta,\zeta_{i+},\alpha_1}$. Thus the chain checks whether the reward at time $i+2$ is positive or negative. If $\zeta_{i+2} \geq 0$ the probability to come back to zero at time $i+2$ increases. Else it remains the same.

With this new process we can write

$$\begin{aligned}
H_N &= E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\exp \left(\beta s \sum_{j=1}^{l_N} \zeta_{i_j} \right) \prod_{j=1}^{l_N} \left(\frac{H_{\alpha,h}^{\beta}}{\alpha^{\tau_j}} \right) \left(\frac{1}{2} + \frac{e^{-2h(N-i_{l_N})}}{2} \right) \right. \\
&\quad \left. \prod_{j=1}^{l_N} \left(\frac{P_{\alpha,h}^{\beta}(\tau_j)}{P_{\alpha,h}^{\beta,\zeta_{i_{j-1}+},\alpha_1}(\tau_j)} \right) \frac{P(\tau \geq N - i_{l_N})}{P_{\alpha,h}^{\beta,\zeta_{i_{l_N}+},\alpha_1}(\tau \geq N - i_{l_N})} \right) \\
H_N &\geq E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\exp \left(\beta s \sum_{j=1}^{l_N} \zeta_{i_j} \right) (H_{\alpha,h}^{\beta})^{l_N} \frac{1}{2} \prod_{j=1}^{l_N} \left(\frac{P_{\alpha,h}^{\beta}(\tau_j)}{P_{\alpha,h}^{\beta,\zeta_{i_{j-1}+},\alpha_1}(\tau_j)} \right) P(\tau \geq N - i_{l_N}) \right)
\end{aligned}$$

Now we apply the Jensen formula and

$$\begin{aligned}
\mathbb{E} \left(\frac{1}{N} \log H_N \right) &\geq \frac{\beta s}{N} \mathbb{E} E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\sum_{j=1}^{l_N} \zeta_{i_j} \right) + \log(H_{\alpha,h}^{\beta}) \mathbb{E} E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_N}{N} \right) + \frac{1}{N} \log \left(\frac{1}{2} \right) \\
&\quad + \frac{1}{N} \mathbb{E} E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\sum_{j=1}^{l_N} \log \left(\frac{P_{\alpha,h}^{\beta}(\tau_j)}{P_{\alpha,h}^{\beta,\zeta_{i_{j-1}+},\alpha_1}(\tau_j)} \right) \right) + \frac{1}{N} \log(P(\tau \geq N))
\end{aligned} \tag{2.5}$$

At this point, we can divide in two parts the lower bound of (2.5). The first one (called $E_1(N)$) is a positive energetic term corresponding to the additional reward the chain can expect on coming back often in "high reward" sites. Namely

$$E_1(N) = \frac{\beta s}{N} \mathbb{E} E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\sum_{j=1}^{l_N} \zeta_{i_j} \right)$$

the second one ($E_2(N)$) is a negative entropic term, because the measures transformations we did have an entropic cost, namely

$$\begin{aligned}
E_2(N) &= \log(H_{\alpha,h}^{\beta}) \mathbb{E} E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_N}{N} \right) + \frac{1}{N} \log \left(\frac{1}{2} \right) \\
&\quad + \frac{1}{N} \mathbb{E} E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\sum_{j=1}^{l_N} \log \left(\frac{P_{\alpha,h}^{\beta}(\tau_j)}{P_{\alpha,h}^{\beta,\zeta_{i_{j-1}+},\alpha_1}(\tau_j)} \right) \right) + \frac{1}{N} \log(P(\tau \geq N))
\end{aligned}$$

STEP2: energy term computation. First remark that

$$\sum_{j=1}^{l_N} \zeta_{i_j} = \sum_{i=0}^{N-2} \zeta_{i+2} \mathbf{1}_{\{S_i=0\}} \mathbf{1}_{\{S_{i+2}=0\}} + \sum_{k=3}^N \sum_{s=0}^{N-k} \zeta_{s+k} \mathbf{1}_{\{S_s=0\}} \mathbf{1}_{\{S_i \neq 0 \forall i \in \{s+1, \dots, s+k-1\} \text{ and } S_{s+k}=0\}}$$
(2.6)

So we put $A = \sum_{i=0}^{N-2} \zeta_{i+2} \mathbf{1}_{\{S_i=0\}} \mathbf{1}_{\{S_{i+2}=0\}}$

and $B = \sum_{k=3}^N \sum_{s=0}^{N-k} \zeta_{s+k} \mathbf{1}_{\{S_s=0\}} \mathbf{1}_{\{S_i \neq 0 \forall i \in \{s+1, \dots, s+k-1\} \text{ and } S_{s+k}=0\}}$

Hence we can compute separately the contributions of A and B

$$\mathbb{E} E_{\alpha, h}^{\beta, \zeta, \alpha_1} (B) = \sum_{k=3}^N \sum_{s=0}^{N-k} \mathbb{E} E_{\alpha, h}^{\beta, \zeta, \alpha_1} (\zeta_{s+k} \mathbf{1}_{\{S_s=0\}} \mathbf{1}_{\{S_i \neq 0 \forall i \in \{s+1, \dots, s+k-1\} \text{ and } S_{s+k}=0\}})$$

By Markov property

$$\begin{aligned} \mathbb{E} E_{\alpha, h}^{\beta, \zeta, \alpha_1} (B) &= \sum_{k=3}^N \sum_{s=0}^{N-k} \mathbb{E} \left(\mathbf{1}_{\{\zeta_{s+2} > 0\}} E_{\alpha, h}^{\beta, \zeta, \alpha_1} (\mathbf{1}_{\{S_s=0\}}) P_{\alpha, h}^{\beta} (k) \mu_1 \zeta_{s+k} \right) \\ &\quad + \mathbb{E} \left(\mathbf{1}_{\{\zeta_{s+2} \leq 0\}} E_{\alpha, h}^{\beta, \zeta, \alpha_1} (\mathbf{1}_{\{S_s=0\}}) P_{\alpha, h}^{\beta} (k) \zeta_{s+k} \right) \end{aligned}$$

But we notice that $E_{\alpha, h}^{\beta, \zeta, \alpha_1} (\mathbf{1}_{\{S_s=0\}})$ only depends on $\{\zeta_1, \zeta_2, \dots, \zeta_s\}$, hence by independence of the $\{\zeta_i\}_{i \geq 1}$ and since they are centered and $k \geq 3$ we have: $\mathbb{E} E_{\alpha, h}^{\beta, \zeta, \alpha_1} (B) = 0$.

Now let's consider the contribution of part A in (2.6)

$$\begin{aligned} \mathbb{E} E_{\alpha, h}^{\beta, \zeta, \alpha_1} (A) &= \sum_{i=0}^{N-2} \mathbb{E} \left(E_{\alpha, h}^{\beta, \zeta, \alpha_1} (\mathbf{1}_{\{S_i=0\}}) P_{\alpha, h}^{\beta} (2) (1 + \alpha_1) \zeta_{i+2} \mathbf{1}_{\{\zeta_{i+2} > 0\}} \right) \\ &\quad + \sum_{i=0}^{N-2} \mathbb{E} \left(E_{\alpha, h}^{\beta, \zeta, \alpha_1} (\mathbf{1}_{\{S_i=0\}}) P_{\alpha, h}^{\beta} (2) \zeta_{i+2} \mathbf{1}_{\{\zeta_{i+2} \leq 0\}} \right) \\ &= \alpha_1 P_{\alpha, h}^{\beta} (2) \mathbb{E} (\zeta_1 \mathbf{1}_{\{\zeta_1 > 0\}}) \mathbb{E} E_{\alpha, h}^{\beta, \zeta, \alpha_1} (\#\{i \in \{0, \dots, N-2\} : S_i = 0\}) \end{aligned}$$

So the contribution of this energy term is

$$E_1(N) = \beta s \alpha_1 P_{\alpha, h}^{\beta} (2) \frac{\mathbb{E} E_{\alpha, h}^{\beta, \zeta, \alpha_1} (\#\{i \in \{0, \dots, N-2\} : S_i = 0\})}{N} \geq \beta s \alpha_1 P_{\alpha, h}^{\beta} (2) \frac{\mathbb{E} E_{\alpha, h}^{\beta, \zeta, \alpha_1} (l_N)}{N}$$
(2.7)

STEP3: computation of entropic term. First notice that the terms $1/N \log(P(\tau \geq N))$ and $1/N \log(1/2)$ go to 0 as N goes to ∞ independently of all the other parameters. So we put $R_N = 1/N \log(P(\tau \geq N)) + 1/N \log(1/2)$ and we can write

$$E_2(N) = \frac{S_N}{N} + \log \left(H_{\alpha, h}^{\beta} \right) \mathbb{E} E_{\alpha, h}^{\beta, \zeta, \alpha_1} \left(\frac{l_N}{N} \right) + R_N$$

where we have put

$$S_N = \mathbb{E} E_{\alpha, h}^{\beta, \zeta, \alpha_1} \left(\sum_{j=1}^{l_N} \log \left(\frac{P_{\alpha, h}^{\beta} (\tau_j)}{P_{\alpha, h}^{\beta, \zeta_{i_{j-1} + \dots + \alpha_1}} (\tau_j)} \right) \right)$$
(2.8)

The definitions (2.2) and (2.4) of $P_{\alpha,h}^{\beta,\zeta_{i_{j-1}+\dots+\alpha_1}}$ and $P_{\alpha,h}^\beta$ give us immediately

$$\begin{aligned} S_N &= -\mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\sum_{j=1}^{l_N} \mathbf{1}_{\{\zeta_{i_{j-1}+2}>0\}} \left(\mathbf{1}_{\{\tau_j=2\}} \log(1+\alpha_1) + \mathbf{1}_{\{\tau_j>2\}} \log(\mu_1) \right) \right) \\ &= -\sum_{i=0}^{N-2} \mathbb{E} \left(E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\mathbf{1}_{\{S_i=0\}} \mathbf{1}_{\{S_{i+2}=0\}} \right) \mathbf{1}_{\{\zeta_{i+2}>0\}} \log(1+\alpha_1) \right) \\ &\quad - \sum_{k=3}^N \sum_{s=0}^{N-k} \mathbb{E} \left(E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\mathbf{1}_{\{S_s=0\}} \mathbf{1}_{\{S_{s+k}=0\}} \mathbf{1}_{\{S_i \neq 0 \ \forall i \in \{s+1, \dots, s+k-1\}\}} \right) \mathbf{1}_{\{\zeta_{s+2}>0\}} \log(\mu_1) \right) \end{aligned}$$

And once again, by Markov property we have

$$\mathbf{1}_{\{\zeta_{i+2}>0\}} E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\mathbf{1}_{\{S_i=0\}} \mathbf{1}_{\{S_{i+2}=0\}} \right) = \mathbf{1}_{\{\zeta_{i+2}>0\}} E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\mathbf{1}_{\{S_i=0\}} \right) (1+\alpha_1) P_{\alpha,h}^\beta \quad (2)$$

We notice that $E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\mathbf{1}_{\{S_i=0\}} \right)$ is independant of ζ_{i+2} and $\mathbb{P}(\zeta_{i+2} > 0) = 1/2$ hence

$$\begin{aligned} S_N &= -\frac{P_{\alpha,h}^\beta(2)}{2} (1+\alpha_1) \log(1+\alpha_1) \mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}(l_{N-2}) \\ &\quad - \sum_{k=3}^N \frac{\mu_1 \log(\mu_1)}{2} P_{\alpha,h}^\beta(k) \mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1}(l_{N-k}) \end{aligned}$$

Finally the entropic contribution is

$$\begin{aligned} E_2(N) &= \log \left(H_{\alpha,h}^\beta \right) \mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_N}{N} \right) - \frac{1}{2} P_{\alpha,h}^\beta(2) (1+\alpha_1) \log(1+\alpha_1) \mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_{N-2}}{N} \right) \\ &\quad - \sum_{k=3}^N \frac{\mu_1 \log(\mu_1)}{2} P_{\alpha,h}^\beta(k) \mathbb{E}E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_{N-k}}{N} \right) + R_N \quad (2.9) \end{aligned}$$

So (2.7) and (2.9) give us a precise lower bound of formula (2.5) of the form

$$\mathbb{E} \left(\frac{1}{N} \log(H_N) \right) \geq E_1(N) + E_2(N) \quad (2.10)$$

STEP4: estimation of $H_{\alpha,h}^\beta$ and choice of α and α_1 . Now we want to evaluate $H_{\alpha,h}^\beta$ with its expression of (2.2)

$$H_{\alpha,h}^\beta = e^\beta \left(1 - \frac{\sqrt{1-\alpha^2} + \sqrt{1-e^{-4h}\alpha^2}}{2} \right)$$

In order to compare $\log(H_{\alpha,h}^\beta)$ with the other terms of (2.10), we put $\alpha^2 = 1 - c\alpha_1^2$, with $c > 0$ and $\sqrt{c}\alpha_1 \leq 1$. That way we obtain

$$H_{\alpha,h}^\beta = e^\beta \left(1 - \frac{\sqrt{1-e^{-4h}}}{2} + \frac{\sqrt{1-e^{-4h}} - \sqrt{1-e^{-4h}(1-c\alpha_1^2)} - \sqrt{c}\alpha_1}{2} \right)$$

$$H_{\alpha,h}^\beta = e^\beta \left(1 - \frac{\sqrt{1-e^{-4h}}}{2} \right) \left(1 + \frac{\sqrt{1-e^{-4h}} \left(1 - \sqrt{1 + \frac{ce^{-4h}\alpha_1^2}{1-e^{-4h}}} \right) - \sqrt{c}\alpha_1}{2 - \sqrt{1-e^{-4h}}} \right)$$

But $\sqrt{1+x} \leq 1 + x/2$ for $x \in (-1, +\infty)$ and $2 - \sqrt{1-e^{-4h}} \geq 1$ hence:

$$\log(H_{\alpha,h}^\beta) \geq \log \left(e^\beta \left(1 - \frac{\sqrt{1-e^{-4h}}}{2} \right) \right) + \log \left(1 - \sqrt{c}\alpha_1 - \frac{c\alpha_1^2 e^{-4h}}{2\sqrt{1-e^{-4h}}} \right)$$

But as $\sqrt{c}\alpha_1 \leq 1$ we can bound by above the term

$$\sqrt{c}\alpha_1 + \frac{c\alpha_1^2 e^{-4h}}{2\sqrt{1-e^{-4h}}} = \sqrt{c}\alpha_1 \left(1 + \frac{\sqrt{c}\alpha_1 e^{-4h}}{2\sqrt{1-e^{-4h}}} \right) \leq \sqrt{c}\alpha_1 \left(1 + \frac{1}{2\sqrt{1-e^{-4h}}} \right) \quad (2.11)$$

To continue our computation we need to choose precise values for α_1 and c . That is why recalling that ($\alpha^2 = 1 - c\alpha_1^2$) we put

$$\alpha_1 = \beta s / (5 \times 2^8) \quad \sqrt{c} = \beta s / \left(3 \times 2^4 \left(1 + \frac{1}{2\sqrt{1-e^{-4h}}} \right) \right) \quad (2.12)$$

Notice that $\log(1-x) \geq -3x/2$ if $x \in [0, 1/3]$, and since $\beta s \leq \log(2)$ the rhs of (2.11) verify $\sqrt{c}\alpha_1 \left(1 + 1 / \left(2\sqrt{1-e^{-4h}} \right) \right) \leq \beta^2 s^2 / (15 \times 2^{12}) \leq \frac{1}{3}$ hence $\log(H_{\alpha,h}^\beta)$ becomes

$$\begin{aligned} \log(H_{\alpha,h}^\beta) &\geq \log \left(e^\beta \left(1 - \frac{\sqrt{1-e^{-4h}}}{2} \right) \right) - \frac{3}{2} \sqrt{c}\alpha_1 \left(1 + \frac{1}{2\sqrt{1-e^{-4h}}} \right) \\ &\geq \log \left(e^\beta \left(1 - \frac{\sqrt{1-e^{-4h}}}{2} \right) \right) - \frac{\beta^2 s^2}{5 \times 2^{13}} \end{aligned}$$

Hence as $\log(1+\alpha_1) \leq \alpha_1$ we can rewrite equation (2.5)

$$\begin{aligned} \mathbb{E} \left(\frac{1}{N} \log(H_N) \right) &\geq \left[\beta s \alpha_1 P_{\alpha,h}^\beta(2) - \frac{1}{2} P_{\alpha,h}^\beta(2) (1 + \alpha_1) \alpha_1 \right. \\ &\quad \left. + \log \left(e^\beta \left(1 - \frac{\sqrt{1-e^{-4h}}}{2} \right) \right) - \frac{\beta^2 s^2}{5 \times 2^{13}} \right] \mathbb{E} \left(E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_N}{N} \right) \right) \\ &\quad - \sum_{k=3}^N P_{\alpha,h}^\beta(k) \frac{\mu_1 \log(\mu_1)}{2} \mathbb{E} \left(E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_{N-k}}{N} \right) \right) + R_N \quad (2.13) \end{aligned}$$

STEP5: intermediate computation. To conclude this computation we need some inequalities on $P_{\alpha,h}^\beta$ and $H_{\alpha,h}^\beta$. As $\beta s \leq \log(2)$ equations (2.12) show that $\alpha_1 \sqrt{c} \in [0, 1/4]$, hence $\alpha^2 = 1 - c\alpha_1^2 \geq 1 - 1/2^4 \geq 3/4$. So we can bound from above and below the quantity $H_{\alpha,h}^\beta$ (introduced in (2.3))

$$e^\beta \geq H_{\alpha,h}^\beta \geq e^\beta \left(1 - \frac{\sqrt{c}\alpha_1}{2} - \frac{1}{2} \right) \geq \frac{3e^\beta}{8}$$

At this point we need to bound from above and below the quantity $P_{\alpha,h}^\beta(2)$, which has been defined in (2.2). With the previous inequalities we have $e^\beta/H_{\alpha,h}^\beta \geq 1$ and $\sqrt{1-\alpha^2} \leq 1/4$ so

$$P_{\alpha,h}^\beta(2) = 1 - \sum_{i=2}^{\infty} P_{\alpha,h}^\beta(2i) \leq 1 - \sum_{i=2}^{\infty} \frac{1}{2} \alpha^{2i} P(\tau = 2i) = 1 - \frac{1}{2} \left(1 - \sqrt{1-\alpha^2} - \frac{\alpha^2}{2} \right) \leq \frac{7}{8} \quad (2.14)$$

and

$$\frac{1}{8} = \frac{1}{4} \times \frac{e^\beta}{2e^\beta} \leq P_{\alpha,h}^\beta(2) \quad (2.15)$$

And to finish with these preliminary inequalities, we notice with (2.14) and (2.15) that

$$\frac{1}{8} \leq 1 - P_{\alpha,h}^\beta(2) \quad \text{and} \quad \frac{1}{7} \leq \frac{P_{\alpha,h}^\beta(2)}{1 - P_{\alpha,h}^\beta(2)} \leq 7 \quad (2.16)$$

Hence the condition $\alpha_1 < P_{\alpha,h}^\beta(\tau = 2) / (1 - P_{\alpha,h}^\beta(\tau = 2))$ is obviously verified.

STEP 6: conclusion. In the equation (2.13) we still have to evaluate the term

$$\sum_{k=3}^N P_{\alpha,h}^\beta(k) \mathbb{E} \left(E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_{N-k}}{N} \right) \right)$$

So if $N \geq N_0$

$$\begin{aligned} \sum_{k=3}^N P_{\alpha,h}^\beta(k) \mathbb{E} \left(E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_{N-k}}{N} \right) \right) &\geq P_{\alpha,h}^\beta(\{3, \dots, N_0\}) \mathbb{E} E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_{N-N_0}}{N} \right) \\ &\geq \left(1 - P_{\alpha,h}^\beta(2) \right) \mathbb{E} E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_N}{N} \right) - \frac{N_0}{N} \\ &\quad - P_{\alpha,h}^\beta(\{N_0 + 1, \dots, \infty\}) \mathbb{E} E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_N}{N} \right) \end{aligned}$$

Hence equation (2.13) becomes

$$\begin{aligned}
\mathbb{E} \left(\frac{1}{N} \log (H_N) \right) &\geq \left[\beta s \alpha_1 P_{\alpha, h}^\beta (2) - \frac{1}{2} P_{\alpha, h}^\beta (2) (1 + \alpha_1) \alpha_1 - \frac{\beta^2 s^2}{5 \times 2^{13}} \right. \\
&\quad \left. + \log \left(e^\beta \left(1 - \frac{\sqrt{1 - e^{-4h}}}{2} \right) \right) - (1 - P_{\alpha, h}^\beta (2)) \frac{\mu_1 \log (\mu_1)}{2} \right. \\
&\quad \left. + P_{\alpha, h}^\beta (\{N_0 + 1, \dots, \infty\}) \frac{\mu_1 \log (\mu_1)}{2} \right] \mathbb{E} \left(E_{\alpha, h}^{\beta, \zeta, \alpha_1} \left(\frac{l_N}{N} \right) \right) \\
&\quad + \frac{N_0 \mu_1 \log (\mu_1)}{N} + R_N
\end{aligned} \tag{2.17}$$

We can now bound from below with (2.12) and (2.15)

$$\beta s \alpha_1 P_{\alpha, h}^\beta (2) \geq \frac{\beta s}{2^3} \frac{\beta s}{5 \times 2^8} = \frac{\beta^2 s^2}{5 \times 2^{11}}$$

Moreover $\mu_1 = 1 - \frac{\alpha_1 P_{\alpha, h}^\beta (2)}{1 - P_{\alpha, h}^\beta (2)}$ and $-\log(1 - x) \geq x$ for $x \in [0, 1)$ so we have

$$-\frac{1 - P_{\alpha, h}^\beta (2)}{2} \mu_1 \log (\mu_1) \geq \frac{\alpha_1 P_{\alpha, h}^\beta (2)}{2} - \frac{\alpha_1^2 P_{\alpha, h}^\beta (2)^2}{2(1 - P_{\alpha, h}^\beta (2))}$$

We noticed before in (2.15) and (2.16) that $P_{\alpha, h}^\beta (2) \leq 7/8$ and $P_{\alpha, h}^\beta (2) / (2(1 - P_{\alpha, h}^\beta (2))) \leq \frac{7}{2}$, hence

$$-\frac{1 - P_{\alpha, h}^\beta (2)}{2} \mu_1 \log (\mu_1) \geq \frac{\alpha_1 P_{\alpha, h}^\beta (2)}{2} - \frac{7^2 \alpha_1^2}{2^4} \geq \frac{\alpha_1 P_{\alpha, h}^\beta (2)}{2} - 4\alpha_1^2$$

That way the inequality (2.17) must now be written

$$\begin{aligned}
\mathbb{E} \left(\frac{1}{N} \log (H_N) \right) &\geq \left[\frac{\beta^2 s^2}{5 \times 2^{12}} - \frac{1}{2} P_{\alpha, h}^\beta (2) (1 + \alpha_1) \alpha_1 + \frac{\alpha_1 P_{\alpha, h}^\beta (2)}{2} - 4\alpha_1^2 \right. \\
&\quad \left. + \log \left(e^\beta \left(1 - \frac{\sqrt{1 - e^{-4h}}}{2} \right) \right) \right. \\
&\quad \left. + P_{\alpha, h}^\beta (\{N_0 + 1, \dots, \infty\}) \frac{\mu_1 \log (\mu_1)}{2} \right] \mathbb{E} \left(E_{\alpha, h}^{\beta, \zeta, \alpha_1} \left(\frac{l_N}{N} \right) \right) \\
&\quad + \frac{N_0}{N} \mu_1 \log (\mu_1) + R_N
\end{aligned} \tag{2.18}$$

By (2.16) and (2.15) we know that $P_{\alpha, h}^\beta (2) \leq 7/8$ and $P_{\alpha, h}^\beta (2) / (1 - P_{\alpha, h}^\beta (2)) \leq 7$. Hence we have the inequalities

$$-\frac{1}{2} P_{\alpha, h}^\beta (2) (1 + \alpha_1) \alpha_1 + \frac{\alpha_1 P_{\alpha, h}^\beta (2)}{2} - 4\alpha_1^2 \geq -5\alpha_1^2 \geq -\frac{\beta^2 s^2}{5 \times 2^{16}} \tag{2.19}$$

$$\text{and } \frac{\alpha_1 P_{\alpha, h}^\beta (2)}{1 - P_{\alpha, h}^\beta (2)} \leq 7\alpha_1 = \frac{7\beta s}{5 \times 2^8} < \frac{1}{3} \tag{2.20}$$

Now since $\mu_1 \leq 1$ and $\log(1-x) \geq -3x/2$ for $x \in [0, 1/3]$ equation 2.20 allows us to bound by below

$$\mu_1 \log(\mu_1) \geq -\frac{3}{2} \frac{P_{\alpha,h}^\beta(2)}{1 - P_{\alpha,h}^\beta(2)} \alpha_1 \geq -\frac{21\beta s}{5 \times 2^9} \geq -1$$

So equation (2.18) becomes

$$\begin{aligned} \mathbb{E} \left(\frac{1}{N} \log(H_N) \right) &\geq \left[\frac{\beta^2 s^2}{5 \times 2^{13}} + \log \left(e^\beta \left(1 - \frac{\sqrt{1 - e^{-4h}}}{2} \right) \right) \right. \\ &\quad \left. - P_{\alpha,h}^\beta(\{N_0 + 1, \dots, \infty\}) \right] \mathbb{E} \left(E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_N}{N} \right) \right) - \frac{N_0}{N} + R_N \end{aligned} \quad (2.21)$$

But as proved in appendix A.1), $P_{\alpha,h}^\beta(\{N_0 + 1, \dots, \infty\})$ goes to zero as N_0 goes to ∞ independently of $h \geq 0$, hence for N_0 large enough and for all $h > 0$

$$P_{\alpha,h}^\beta(\{N_0 + 1, \dots, \infty\}) \leq \frac{\beta^2 s^2}{5 \times 2^{14}}$$

So if we put $q(s) = \frac{\beta^2 s^2}{5 \times 2^{14}}$ the equation (2.21) gives us for all $N \geq N_0$ and $h > 0$

$$\mathbb{E} \left(\frac{1}{N} \log(H_N) \right) \geq \left[q(s) + \log \left(e^\beta \left(1 - \frac{\sqrt{1 - e^{-4h}}}{2} \right) \right) \right] \mathbb{E} \left(E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_N}{N} \right) \right) + R_N^{N_0} \quad (2.22)$$

with $R_N^{N_0} = R_N - N_0/N$.

As proved in appendix A.2) for every $N \geq 1$ $\mathbb{E} \left(E_{\alpha,h}^{\beta,\zeta,\alpha_1} (l_N/N) \right) \geq \mathbb{E} \left(E_{\alpha,h}^\beta (l_N/N) \right)$. So if we note $h_0(\beta)$ the quantity verifying $\log \left(e^\beta \left(1 - \sqrt{1 - e^{-4h_0(\beta)}/2} \right) \right) = -q(s)$ we have for every $h < h_0(\beta)$ and $N \geq N_0$ that

$$\mathbb{E} \left(\frac{1}{N} \log(H_N) \right) \geq \left[q(s) + \log \left(e^\beta \left(1 - \frac{\sqrt{1 - e^{-4h}}}{2} \right) \right) \right] \mathbb{E} \left(E_{\alpha,h}^\beta \left(\frac{l_N}{N} \right) \right) + R_N^{N_0}$$

and consequently

$$\Phi^s(\beta, h) \geq \left[q(s) + \log \left(e^\beta \left(1 - \frac{\sqrt{1 - e^{-4h}}}{2} \right) \right) \right] \liminf_{N \rightarrow \infty} \mathbb{E} \left(E_{\alpha,h}^\beta \left(\frac{l_N}{N} \right) \right)$$

Notice also that $\liminf_{N \rightarrow \infty} \mathbb{E} \left(E_{\alpha,h}^\beta \left(\frac{l_N}{N} \right) \right) > 0$ (because $\alpha \in (0, 1)$). Hence for every β in $[0, \log(2) - q_s)$, $h_0(\beta)$ is a lower bound of $h_c(\beta)$

$$h_c(\beta) \geq h_0(\beta) = -\frac{1}{4} \log \left(1 - 4 \left(1 - e^{-\beta - q(s)} \right)^2 \right)$$

□

2.3. Proof of Corollary 3. As showed just before in (2.22) we have a rank $N_0 \in \mathbb{N} - \{0\}$ such that for all $h > 0$ and $N \geq N_0$

$$\mathbb{E} \left(\frac{1}{N} \log E \left(\exp \left(\beta \sum_{i=1}^N \mathbf{1}_{\{S_i=0\}} (s\zeta_i + 1) - 2h \sum_{i=1}^N \Delta_i \right) \right) \right) \geq \left[\frac{\beta^2 s^2}{5 \times 2^{14}} + \log \left(e^\beta \left(1 - \frac{\sqrt{1 - e^{-4h}}}{2} \right) \right) \right] \mathbb{E} \left(E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_N}{N} \right) \right) + R_N^{N_0}$$

but in appendix A.2) we prove the following inequalities

$$\mathbb{E} \left(E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_N}{N} \right) \right) \geq \mathbb{E} \left(E_{\alpha,h}^\beta \left(\frac{l_N}{N} \right) \right) \geq \mathbb{E} \left(E_{\alpha,\infty}^0 \left(\frac{l_N}{N} \right) \right) > 0 \quad (2.23)$$

and for fixed β, s, N let h go to ∞

$$\mathbb{E} \left(\frac{1}{N} \log E \left(\exp \left(\beta \sum_{i=1}^N \mathbf{1}_{\{S_i=0\}} (s\zeta_i + 1) \right) \mathbf{1}_{\{S_i \geq 0, \forall i \in \{1, \dots, N\}\}} \right) \right) \geq \left[\frac{\beta^2 s^2}{5 \times 2^{14}} + \log \left(e^\beta \frac{1}{2} \right) \right] \mathbb{E} \left(E_{\alpha,\infty}^0 \left(\frac{l_N}{N} \right) \right) + R_N^{N_0}$$

Now recall that $P(\{S_i \geq 0, \forall i \in \{1, \dots, N\}\}) \sim c/N^{1/2}$, the lower bound becomes

$$\mathbb{E} \left(\frac{1}{N} \log E \left(\exp \left(\beta \sum_{i=1}^N \mathbf{1}_{\{S_i=0\}} (s\zeta_i + 1) \right) \middle| \{S_i \geq 0, \forall i \in \{1, \dots, N\}\} \right) \right) \geq \left[\frac{\beta^2 s^2}{5 \times 2^{14}} + \log \left(e^\beta \frac{1}{2} \right) \right] \mathbb{E} \left(E_{\alpha,\infty}^0 \left(\frac{l_N}{N} \right) \right) + K_N^{N_0}$$

With $K_N^{N_0} = R_N^{N_0} - 1/N \log(P(\{S_i \geq 0, \forall i \in \{1, \dots, N\}\}))$, so that it goes to 0 as N goes to ∞ independently of all the other parameters. Now by [10] we can apply the fact that for an odd number of steps the RW conditioned to stay positive becomes the reflected RW if it is pinned by $\log 2$, that is to say

$$\frac{P_{refl.RW}}{P_{RW \text{ cond. to be } \geq 0}}(S) = \frac{e^{\log(2) \sum_{i=1}^{2N+1} \mathbf{1}_{\{S_i=0\}}} \mathbf{1}_{\{S_i \geq 0 \forall i \in \{0, 2N+1\}\}}}{V_{2N+1}}$$

With $\frac{1}{N} \log(V_N)$ goes to 0 as N goes to ∞ . Hence we put $\beta = \log(2) - u$

$$\mathbb{E} \left(\frac{1}{2N+1} \log E \left(\exp \left(\log(2) \sum_{i=1}^{2N+1} \mathbf{1}_{\{S_i=0\}} + \sum_{i=1}^{2N+1} \mathbf{1}_{\{S_i=0\}} (-u + \beta s \zeta_i) \right) \middle| \{S_i \geq 0, \forall i \leq 2N+1\} \right) \right) \geq \left[\frac{\beta^2 s^2}{5 \times 2^{14}} - u \right] \mathbb{E} \left(E_{\alpha,\infty}^0 \left(\frac{l_{2N+1}}{2N+1} \right) \right) + K_{2N+1}^{N_0}$$

$$\mathbb{E} \left(\frac{1}{2N+1} \log E \left(\exp \left(\sum_{i=1}^{2N+1} \mathbf{1}_{\{S_i=0\}} (-u + \beta s \zeta_i) \right) \right) \right) \geq \left[\frac{\beta^2 s^2}{5 \times 2^{14}} - u \right] \mathbb{E} \left(E_{\alpha, \infty}^0 \left(\frac{l_{2N+1}}{2N+1} \right) \right) + K_{2N+1}^{N_0} + \frac{1}{2N+1} \log(V_{2N+1})$$

Now let $N \rightarrow \infty$, and recall $\beta = \log(2) - u$

$$\lim_{N \rightarrow \infty} \mathbb{E} \left(\frac{1}{N} \log E \left(\exp \left(\sum_{i=1}^N \mathbf{1}_{\{S_i=0\}} (-u + \beta s \zeta_i) \right) \right) \right) \geq \left[\frac{\beta^2 s^2}{5 \times 2^{14}} - u \right] \lim_{N \rightarrow \infty} E_{\alpha, \infty}^0 \left(\frac{l_N}{N} \right)$$

so, for $u \leq \log(2)/2$, ($\beta \geq \log(2)/2$)

$$\lim_{N \rightarrow \infty} \mathbb{E} \left(\frac{1}{N} \log E \left(\exp \left(\sum_{i=1}^N \mathbf{1}_{\{S_i=0\}} (-u + \beta s \zeta_i) \right) \right) \right) \geq \left[\frac{\log(2)^2 s^2}{5 \times 2^{16}} - u \right] \lim_{N \rightarrow \infty} E_{\alpha, \infty}^0 \left(\frac{l_N}{N} \right)$$

By convexity, the free energy Φ , defined by

$$\Phi(u, v) = \lim_{N \rightarrow \infty} \mathbb{E} \left(\frac{1}{N} \log E \left(\exp \left(\sum_{i=1}^N \mathbf{1}_{\{S_i=0\}} (-u + v \zeta_i) \right) \right) \right)$$

is not decreasing in v hence

$$\Phi(u, \log(2)s) \geq \left[\frac{\log(2)^2 s^2}{5 \times 2^{16}} - u \right] \lim_{N \rightarrow \infty} E_{\alpha, \infty}^0 \left(\frac{l_N}{N} \right)$$

and for $s \in [0, \log(2)]$

$$u_c(s) \geq \frac{s^2}{5 \times 2^{16}}$$

□

APPENDIX A.

A.1. First we have to prove the first point, namely $P_{\alpha, h}^\beta(\{N_0, \dots, +\infty\})$ goes to 0 as N_0 goes to infinity independently of $h \geq 0$. That way we bound by above the quantity (2.2)

$$\begin{aligned} P_{\alpha, h}^\beta(\tau_1 = 2n) &= \left(\frac{1 + \exp(-4hn)}{2} \right) \alpha^{2n} \frac{P(\tau = 2n)}{H_{\alpha, h}^\beta} \exp(\beta) \\ &\leq \frac{\alpha^{2n} P(\tau = 2n)}{\sum_{j=1}^{+\infty} \frac{1}{2} \alpha^{2j} P(\tau = 2j)} \end{aligned}$$

So the rhs of this inequality does not depend on h any more and is the general term of a convergent serie hence we have the uniform convergence in h .

A.2. Now we want to prove the inequalities of (2.23), that is to say

$$\mathbb{E} \left(E_{\alpha, h}^{\beta, \zeta, \alpha_1} \left(\frac{l_N}{N} \right) \right) \geq \mathbb{E} \left(E_{\alpha, h}^\beta \left(\frac{l_N}{N} \right) \right) \geq \mathbb{E} \left(E_{\alpha, \infty}^0 \left(\frac{l_N}{N} \right) \right) \quad (\text{A.1})$$

That way we recall a coupling theorem (see [14] or [15])

Theorem 5. μ_1 and μ_2 are two probability measures on $2\mathbb{N} - \{0\}$. If for every bounded and non decreasing function f defined on $2\mathbb{N} - \{0\}$ we have $\mu_1(f) \leq \mu_2(f)$ we can define on the same probability space (Ω, P) two random variables (T_1, T_2) of law (μ_1, μ_2) such that $T_1 \leq T_2$ P almost surely.

Remark 6. We notice that to satisfy the hypothesis of the theorem it is enough to show that there exists an integer i_0 such that $\mu_1(2i) \geq \mu_2(2i)$ for every $i \in \{1, \dots, i_0\}$ and $\mu_1(2i) \leq \mu_2(2i)$ for every $i \geq i_0 + 1$. We can prove it easily on writing

$$\mu_2(f) - \mu_1(f) = \sum_{i=1}^{i_0} (\mu_2(2i) - \mu_1(2i))f(2i) + \sum_{i=i_0+1}^{\infty} (\mu_2(2i) - \mu_1(2i))f(2i)$$

But as f is not decreasing $f(2i) \geq f(2i_0)$ for every $i \geq i_0 + 1$ and $f(2i) \leq f(2i_0)$ for every $i \leq i_0$. Moreover since $\mu_2(2i) - \mu_1(2i)$ is positive when $i \geq i_0 + 1$ and negative else we have the inequality

$$\begin{aligned} \mu_2(f) - \mu_1(f) &\geq f(2i_0) \sum_{i=1}^{i_0} \mu_2(2i) - \mu_1(2i) + f(2i_0) \sum_{i=i_0+1}^{\infty} \mu_2(2i) - \mu_1(2i) \\ &\geq -f(2i_0) (\mu_1 - \mu_2) (\{2, \dots, 2i_0\}) + f(2i_0) (\mu_2 - \mu_1) (\{2(i_0 + 1), \dots, \infty\}) \end{aligned}$$

But $(\mu_2 - \mu_1) (\{2(i_0 + 1), \dots, \infty\}) = -(\mu_2 - \mu_1) (\{2, \dots, 2i_0\})$ hence

$$\mu_2(f) - \mu_1(f) \geq -f(2i_0)(\mu_1 - \mu_2) (\{2, \dots, 2i_0\}) + f(2i_0)(\mu_1 - \mu_2) (\{2, \dots, 2i_0\}) \geq 0$$

That is why we can use theorem 5 in this situation.

We now want to apply this remark to the following probability measures on $2\mathbb{N} - \{0\}$: $P_{\alpha, \infty}^0$, $P_{\alpha, h}^\beta$ and $P_{\alpha, h}^{\beta, +, \alpha_1}$ which is the law defined in (2.4) when $\zeta_2 \geq 0$.

First we compare $P_{\alpha, h}^\beta$ and $P_{\alpha, h}^{\beta, +, \alpha_1}$ which is in fact very easy since

$$\begin{aligned} P_{\alpha, h}^{\beta, +, \alpha_1}(\tau = 2) &= P_{\alpha, h}^\beta(\tau = 2)(1 + \alpha_1) \\ P_{\alpha, h}^{\beta, +, \alpha_1}(\tau = 2r) &= P_{\alpha, h}^\beta(\tau = 2r)\mu_1 \text{ for } r > 2 \end{aligned}$$

But $\alpha_1 > 0$ and $\mu_1 < 1$ hence $P_{\alpha, h}^{\beta, +, \alpha_1}(\tau = 2) > P_{\alpha, h}^\beta(\tau = 2)$ and $P_{\alpha, h}^{\beta, +, \alpha_1}(\tau = 2r) < P_{\alpha, h}^\beta(\tau = 2r)$ for $r \geq 2$. Thus remark 6 tells us that we can use theorem 5 and define on a probability space (Ω, P) a sequence of iid random variables $(T_i^1, T_i^2)_{i \geq 1}$ such that

- $P_{\alpha, h}^{\beta, +, \alpha_1}$ is the law of T_i^1 for every $i \geq 1$
- $P_{\alpha, h}^\beta$ the law of T_i^2 for every $i \geq 1$
- P almost surely $T_i^1 \leq T_i^2$ for every $i \geq 1$

At this point for every fixed disorder ζ we define by recurrence another process $(T_i^3)_{i \geq 1}$ with

$$\begin{aligned} T_i^3 &= T_i^2 \text{ if } \zeta_{T_1^3 + \dots + T_{i-1}^3 + 2} \geq 0 \\ &= T_i^1 \text{ if } \zeta_{T_1^3 + \dots + T_{i-1}^3 + 2} < 0 \end{aligned}$$

Hence with these notations $(T_i^2)_{i \geq 1}$ is the sequence of the excursion length of a random walk under the law $P_{\alpha, h}^\beta$ and $(T_i^3)_{i \geq 1}$ the one of a random walk under the law $P_{\alpha, h}^{\beta, \zeta, \alpha_1}$.

But by construction $T_i^3 \leq T_i^2$ for every $i \geq 1$, so for $j = 2$ or 3 if we put $l_N^j = \max\{s \geq 1 | T_1^j + \dots + T_s^j \leq N\}$ we have immediately that P almost surely $l_N^3 \geq l_N^2$. Thus for every ζ we have

$$E_{\alpha,h}^{\beta,\zeta,\alpha_1} \left(\frac{l_N}{N} \right) = E_P \left(\frac{l_N^3}{N} \right) \geq E_P \left(\frac{l_N^2}{N} \right) = E_{\alpha,h}^{\beta} \left(\frac{l_N}{N} \right)$$

and integrating over ζ we obtain the left hand side of inequality (A.1).

To finish with these inequalities we must show that the same argument allow us to compare $\mathbb{E} \left(E_{\alpha,h}^{\beta} \left(\frac{l_N}{N} \right) \right)$ and $\mathbb{E} \left(E_{\alpha,\infty}^0 \left(\frac{l_N}{N} \right) \right)$. Namely we want to prove that remark 6 also occur. So recall

$$P_{\alpha,h}^{\beta} (\tau_1 = 2n) = \left(\frac{1 + \exp(-4hn)}{2} \right) \alpha^{2n} \frac{P(\tau = 2n)}{H_{\alpha,h}^{\beta}} \exp(\beta)$$

$$P_{\alpha,\infty}^0 (\tau_1 = 2n) = \frac{\alpha^{2n} P(\tau = 2n)}{2H_{\alpha,\infty}^0}$$

So if we note

$$L_n = \frac{P_{\alpha,h}^{\beta} (\tau_1 = 2n)}{P_{\alpha,\infty}^0 (\tau_1 = 2n)} = (1 + \exp(-4hn)) \frac{H_{\alpha,\infty}^0}{H_{\alpha,h}^{\beta}} \exp(\beta)$$

we immediately notice that L_n decreases with n , but we have also

$$\sum_{i=1}^{\infty} P_{\alpha,h}^{\beta} (\tau_1 = 2i) = \sum_{i=1}^{\infty} P_{\alpha,\infty}^0 (\tau_1 = 2i) = 1$$

hence necessarily there exists i_0 in $\mathbb{N} - \{0\}$ such that $P_{\alpha,h}^{\beta} (\tau_1 = 2i) \geq P_{\alpha,\infty}^0 (\tau_1 = 2i)$ for $i \leq i_0$ and $P_{\alpha,h}^{\beta} (\tau_1 = 2i) \leq P_{\alpha,\infty}^0 (\tau_1 = 2i)$ for $i > i_0$. And the proof is complete. \square

APPENDIX B.

B.1. Proof of Proposition 1.1. First of all, we recall a classical property which tells us that we do not transform the free energy if we oblige the last monomer of the chain to touch the 0 axis. It is proved for example in a different case in [4] but the same technic works with our hamiltonian. So we can write

$$\Phi^0(h, \beta) = \lim_{N \rightarrow \infty} \mathbb{E} \frac{1}{2N} \log E \left(\exp \left(\beta \sum_{i=1}^{2N} \mathbf{1}_{\{S_i=0\}} - 2h \sum_{i=1}^{2N} \Delta_i \right) \mathbf{1}_{\{S_{2N}=0\}} \right)$$

In the following we note $Z_{2N,\beta,h} = E \left(\exp \left(\beta \sum_{i=1}^{2N} \mathbf{1}_{\{S_i=0\}} - 2h \sum_{i=1}^{2N} \Delta_i \right) \mathbf{1}_{\{S_{2N}=0\}} \right)$. Remark that $Z_{2N,\beta,h}$ can be rewrite as follow

$$\begin{aligned} Z_{2N,\beta,h} &= \sum_{j=1}^N E \left(e^{\beta j} e^{-2h \sum_{i=1}^{2N} \Delta_i} \mathbf{1}_{\{l_{2N}=j\}} \mathbf{1}_{\{S_{2N}=0\}} \right) \\ &= \sum_{j=1}^N \sum_{\substack{\vec{l} \in \mathbb{N}^{*j} \\ |\vec{l}|=N}} \prod_{i=1}^j \left(e^{\beta l_i} V_{h,l_i} \right) \end{aligned}$$

with $V_{h,l} = P(\tau = 2l) (e^{-4hl} + 1) / 2$. We aim at computing the generating function of $Z_{2N,\beta,h}$ called $\theta_h(z)$

$$\begin{aligned} \theta_h(z) &= \sum_{N=1}^{\infty} Z_{2N,\beta,h} z^{2N} = \sum_{N=1}^{\infty} z^{2N} \sum_{j=1}^N e^{\beta j} \sum_{\substack{\vec{l} \in \mathbb{N}^{*j} \\ |\vec{l}|=N}} \prod_{i=1}^j V_{h,l_i} \\ &= \sum_{j=1}^{\infty} \sum_{N=j}^{\infty} \sum_{\substack{\vec{l} \in \mathbb{N}^{*j} \\ |\vec{l}|=N}} \prod_{i=1}^j \left(e^{\beta z^{2l_i}} V_{h,l_i} \right) \\ &= \sum_{j=1}^{\infty} \left(\sum_{l=1}^{\infty} e^{\beta z^{2l}} V_{h,l} \right)^j = \sum_{j=1}^{\infty} \left(\sum_{l=1}^{\infty} \frac{P(\tau = 2l)}{2} (1 + e^{-4hl}) e^{\beta z^{2l}} \right)^j \end{aligned}$$

Now, recall that

$$\sum_{l=1}^{\infty} P(\tau = 2l) z^{2l} = 1 - \sqrt{1 - z^2}$$

hence the computation finally gives

$$\theta_h(z) = \sum_{j=1}^{\infty} \left(\frac{e^{\beta}}{2} \left(2 - \sqrt{1 - z^2} - \sqrt{1 - z^2 e^{-4h}} \right) \right)^j$$

So, this serie converges when $e^{\beta} \left(2 - \sqrt{1 - z^2} - \sqrt{1 - z^2 e^{-4h}} \right) < 2$, and if we note R its convergence radius, we have $\Phi(\beta, h) = -\log(R)$. That is why $\Phi(\beta, h) > 0$ if and only if $R < 1$. So, we can say that (h, β) is on the critical curve if and only if for $z = 1$: $e^{\beta} \left(2 - \sqrt{1 - z^2} - \sqrt{1 - z^2 e^{-4h}} \right) = 2$, which can be write $\sqrt{1 - e^{-4h}} = 2(1 - e^{-\beta})$. It gives us the critical curve equation

$$h_c^0(\beta) = \frac{1}{4} \log \left(1 - 4 \left(1 - e^{-\beta} \right)^2 \right)$$

□

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