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SYMMETRY FOR EXTERIOR ELLIPTIC PROBLEMS AND TWO CONJECTURES IN POTENTIAL THEORY

Boyan SIRAKOV
Laboratoire d'analyse numérique
Université Paris 6
Tour 55-65, 5e étage
75252 Paris Cedex 05, FRANCE

Abstract. In this paper we extend a classical result of Serrin to a class of elliptic problems $\Delta u + f(u, |\nabla u|) = 0$ in exterior domains $\mathbb{R}^N \setminus G$ (or $\Omega \setminus G$ with Ω and G bounded). In case G is an union of a finite number of disjoint C^2-domains G_i and $u = a_i > 0$, $\frac{\partial u}{\partial n} = \alpha_i \leq 0$ on ∂G_i, $u \to 0$ at infinity, we show that if a non-negative solution of such a problem exists, then G has only one component and it is a ball. As a consequence we establish two results in electrostatics and capillarity theory. We further obtain symmetry results for quasilinear elliptic equations in the exterior of a ball.

1 Introduction and Main Results

This paper is a contribution to the study of symmetry properties of non-negative solutions of elliptic partial differential equations, which started with the classical papers by Serrin and by Gidas, Ni and Nirenberg ([S] and [GNN]; see also [BN] for a more recent approach). We concentrate on exterior and annuli-like domains, possibly multiply-connected, with different boundary conditions on each connected component of the boundary.

Consider the model problem

$$\left\{ \begin{array}{rcll} \Delta u + f(u) & = & 0 & \text{in} \quad \mathbb{R}^N \setminus G \\ u & \geq & 0 & \text{in} \quad \mathbb{R}^N \setminus G \\ u & \to & 0 & \text{as} \quad |x| \to \infty, \end{array} \right. \tag{1}$$

where $f \in C^1([0, \infty))$, with $f'(0) < 0$. We suppose that

$$G = \bigcup_{i=1}^{k} G_i, \tag{2}$$

where $k \in \mathbb{N}$ and G_i are bounded C^2–domains such that $\overline{G_i} \cap \overline{G_j} = \emptyset$ for $i \neq j$.

The boundary conditions that we impose on ∂G are the following

$$u = a_i > 0 \quad \text{and} \quad \frac{\partial u}{\partial n} = \alpha_i \leq 0 \quad \text{on } \partial G_i \ , \ i = 1, \ldots, k, \tag{3}$$

where $a_i, \alpha_i, i = 1, \ldots, k$, are constants and n denotes the inward normal to the boundary of $\Omega \setminus G$.

Our main result (Theorem 2 below), applied to problem (1), gives the following statement.

Theorem 1 *Suppose $u \in C^2(\mathbb{R}^N \setminus G)$ is a solution of (1), satisfying (3). Then G has only one connected component (i.e. k=1). Moreover, G is a ball and the solution u is radial with respect to the center of this ball.*

Let us now describe the general setting that we consider. We study boundary-value problems of the type

$$\begin{cases} Qu + f(u, |\nabla u|) = 0 \ , \quad u \geq 0 \ , \quad u \in C^2 \quad \text{in } \overline{\Omega \setminus G} \\ \text{Boundary Conditions} \quad \text{(BC)} \end{cases} \tag{4}$$

where Q is a (nonlinear) strongly elliptic operator, $\Omega \subset \mathbb{R}^N, N \geq 2$, is a C^2–domain, G is as in (2), $\overline{G} \subset \Omega$ and $\Omega \setminus G$ is connected.

We distinguish two cases for Ω.

Case A. Ω is bounded. Here the boundary conditions on $\partial\Omega$ read

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial n} = \beta \quad \text{on } \partial\Omega, \tag{5}$$

where β is a constant.

Case B. $\Omega = \mathbb{R}^N$. Then we suppose that

$$\nabla u(x) \to 0 \ \text{ and } \ u(x) \to 0 \quad \text{as } \ |x| \to \infty. \tag{6}$$

In the particular case when f does not depend on $|\nabla u|$, instead of (6) we only suppose that

$$u(x) \to 0 \quad \text{as } \ |x| \to \infty. \tag{7}$$

A positive solution satisfying (7) is often called a ground state.

We consider the following assumptions on the elliptic operator and the function f.

(q) $Qu = \operatorname{div}(g(|\nabla u|)\nabla u)$, where $g \in C^2([0, \infty))$, $g(s) > 0$ and $(sg(s))' > 0$ for all $s \geq 0$. In other words, we suppose that Q is a regular strongly elliptic operator.

(f) $f(u,p)$ is a locally Lipschitz continuous function in $[0,\infty)^2$ and, in Case B, is a non-increasing function of u for small positive values of u and p.

These assumptions are satisfied in the applications we present below. Our results remain true for any generalisation of (q) and (f) which leads to a "good" equation for the comparison function in the "moving planes" method (see Step 1 in Section 2.2). In particular, a natural question is whether we can consider nonlinearities which are not Lipschitz continuous in u at the origin. See Remark 2 at the end of this section.

In the sequel we denote by (BC) either (3) and (5), or (3) and (6), or (3) and (7), depending on the case we consider. The following theorem contains our main result.

Theorem 2 *Suppose (q) and (f) hold. If u is a solution of (4) satisfying (BC), then $k = 1$, Ω and G are concentric balls centered at some point $x^0 \in \mathbb{R}^N$, u is radial, that is $u = u(|x - x^0|)$, and*

$$\frac{du}{dr} < 0 \quad \text{for } r = |x - x^0| \in (\rho_1, \rho_2),$$

where ρ_1 and ρ_2 denote the radii of G and Ω respectively $(0 < \rho_1 < \rho_2 \leq \infty)$.

The presence of assumptions on both the solution and its normal derivative on the boundary makes problem (4) overdetermined. It can be viewed as a free boundary type problem. This kind of problem was introduced in Serrin's classical paper [S], where he considered the case $u > 0, G = \emptyset$, Ω bounded.

In recent years there have been some partial results for non-empty G and unbounded Ω. In particular, when Ω is bounded, Alessandrini ([A]) obtained Theorem 2 when $f \equiv 0$. In the same case (Ω bounded), Willms, Gladwell and Siegel ([WGS]) obtained the result for $f \equiv 1$ and $Q = \Delta$, provided $N = 2$, Ω and G_i are convex and satisfy some additional curvature conditions. The most general previous results were derived by Reichel ([R1], [R2], [R3]). He proved Theorem 2 under the additional hypotheses

(r) $$a_1 = \ldots = a_k = a \quad \text{and} \quad 0 < u < a \quad \text{in} \quad \Omega \setminus G.$$

Notice that in our result (r) is not assumed a priori, but is rather derived as a consequence of Theorem 2.

The statement of Theorem 2 thus unifies and extends all the above results for problem (4).

Another related result was derived by Aftalion and Busca ([AB]). Using a method based on the Kelvin transform in exterior domains, they obtained

Theorem 1 ($\Omega = \mathbb{R}^N, Q = \Delta$) for a different class of functions f, including power nonlinearities like u^p, for $\frac{N}{N-2} \leq p \leq \frac{N+2}{N-2}$. Aftalion and Busca suppose $k = 1$ and $0 \leq u \leq a$. One can see, using their method together with our approach here, that their result holds if in their work the latter two hypotheses are replaced by (2) and (3).

As a consequence of Theorem 2, we obtain two results in electrostatics and capillarity theory. These problems have been open for some time.

Theorem 3 *Two or more $C^{2,\alpha}$– regular conducting bodies in $\mathbb{R}^N$ do not admit constant equilibrium charge distributions on their boundaries.*

Theorem 4 *Two or more $C^{2,\alpha}$– regular solid cylinders, dipped into an infinite plain liquid reservoir, cannot raise a capillary surface to constant heights on their walls.*

As far as Theorem 3 is concerned, we recall that if Ω is a domain in $\mathbb{R}^N$, a charge distribution $\rho \in C(\partial\Omega)$ is said to be in equilibrium if the induced single-layer potential

$$\psi(x) = \int_{\partial\Omega} \rho(y)\gamma(|x-y|)\, d\sigma_y$$

is constant in Ω; here $\gamma(t) = -\frac{1}{2\pi}\log t$ if $N = 2$ and $\gamma(t) = -\frac{1}{4\pi t}$ if $N = 3$. Then the function ψ satisfies an equation of type (4) with $Q = \Delta$ and $f = 0$. Note that when $N = 2$ we have $\psi \to -\infty$ as $|x| \to \infty$, but this creates no problems when we apply Theorem 2 (see a remark by Reichel in [R2]).

The physical setting described in Theorem 4 leads to an equation of type (4), with

$$g(s) = \frac{1}{\sqrt{1+s^2}} \quad \text{and} \quad f(u) = -\kappa u,\ \kappa > 0$$

(the original discussion on this problem can be found in [S]; see also [R2] and [R3], where the cases of one conducting body or one cylinder were studied).

Our next result concerns the case when Ω or G (or both) is a priori supposed to be a ball. In this situation, to obtain a symmetry result on the solution, we do not need to assume that its normal derivative is constant on the corresponding boundary. A previous result in this sense was obtained by Reichel, who considered the case when Ω is bounded, and assumed condition (r) (with weak instead of strict inequalities, see [R1]).

Theorem 5 *Suppose G is a ball. Then the conclusion of Theorem 2 holds true if (3) is replaced by*

$$u = a > 0 \quad \textit{and} \quad \frac{\partial u}{\partial n} \leq 0 \quad \textit{on } \partial G. \tag{8}$$

Analogously, if Ω is a ball, the same conclusion holds if (5) is replaced by

$$u = 0 \quad \text{on } \partial\Omega.$$

There is no difficulty in extending Theorem 5 (with the same proof) to the limiting case $G = \{x_0\}$ (i.e. $\rho_1 = 0$), where $x^0 \in \mathbb{R}^N$ is such that $u(x^0) = \max_{x\in\Omega} u(x) > 0$. This actually means that, when Ω is a ball and $G = \{x_0\}$, Theorem 5 reduces to the classical symmetry result of Gidas, Ni and Nirenberg ([GNN], see also [CS]). Furthermore, in the same situation ($G = \{x_0\}$), when Ω is arbitrary, we obtain an extension of Serrin's result to non-negative solutions.

Theorem 6 *Suppose (q) and (f) hold and let $u \in C^2(\overline{\Omega})$ be a solution of the following problem*

$$\begin{cases} Qu + f(u, |\nabla u|) = 0 & \text{in} \quad \Omega \\ u \geq 0\,, \quad u \not\equiv 0 & \text{in} \quad \Omega \\ u = 0\,, \quad \dfrac{\partial u}{\partial n} = const & \text{on} \quad \partial\Omega. \end{cases} \tag{9}$$

Then Ω is a ball with radius ρ_2, centered at x^0, u is radial, and

$$\frac{du}{dr} < 0 \quad \text{for } r = |x - x^0| \in (0, \rho_2).$$

The result which we get by putting $\Omega = \mathbb{R}^N$ and $G = \{x^0\}$ in Theorem 5 was recently obtained independently by Serrin and Zou in [SZ] (see also [L]). In this paper they study the symmetry properties of a larger class of elliptic equations on $\mathbb{R}^N$, which includes singular operators like the p-Laplacian.

Our theorems rely on the widely used method of "moving planes", introduced by Alexandrov and developed in this setting by Serrin.

Finally, we describe several possible extensions of our theorems.

Remark 1. We can weaken the hypothesis on the regularity of the solution by adding an extra assumption on the shape of the domain $\Omega \setminus G$. All our theorems remain true for weak solutions in $C^1(\overline{\Omega \setminus G})$ (as in [SZ]), provided Ω and G are such that the critical positions $\lambda_\star$ and λ_Ω in the moving planes method are always attained when internal tangency occurs (see Section 2 for definitions of these). In particular, this assumption is satisfied for symmetric domains.

If we want to consider C^1-weak solutions, we need also to suppose that the function f vanishes at $u = 0$. Note that this hypothesis is a consequence of the existence of a C^2-solution.

Remark 2. After this work was completed the author learned of a recent paper on the strong maximum principle by Pucci, Serrin and Zou ([PSZ]), where, extending earlier results by Vasquez ([V]), they establish essentially optimal conditions on the function f under which a non-negative solution of $Qu+f(u,|\nabla u|)=0$ is strictly positive everywhere. They consider differential inequalities and singular elliptic operators.

For instance, when f is independent of $|\nabla u|$, one of their results says the strong maximum principle holds provided

$$\int_0 \frac{du}{\sqrt{F(u)}} = \infty, \tag{10}$$

where $F(u) = \int_0^u f(s)\,ds$. It is elementary to check that (10) is satisfied if f is Lipschitz continuous at $u=0$.

It is not difficult to see that condition (f) can be replaced by the hypotheses in [PSZ] which ensure the validity of the strong maximum principle.

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2 Proofs

We apply the method of "moving planes", in order to show that for any direction $\gamma \in \mathbb{R}^N \setminus \{0\}$ there exists $\lambda = \lambda(\gamma) \in \mathbb{R}$, such that the domain and the solution are symmetric with respect to the hyperplane

$$T_\lambda = \{x \in \mathbb{R}^N \mid <x,\gamma> = \lambda\};$$

here $<\cdot,\cdot>$ denotes the scalar product in $\mathbb{R}^N$.

We fix γ, say $\gamma = e_1 = (1,0,\ldots,0)$. For any $x=(x_1,\ldots,x_N) \in \mathbb{R}^N$ we put $x=(x_1,x') \in \mathbb{R}\times\mathbb{R}^{N-1}$ and denote by $B_\delta(x)$ the open ball with radius δ centered at x. For every $A \subset \mathbb{R}^N$ and $\lambda, t \in \mathbb{R}$ we set

$$\begin{aligned} D_\lambda &= \{x \in \mathbb{R}^N \mid x_1 > \lambda\}; \\ A^\lambda &= \text{the reflection of } A \text{ with respect to } T_\lambda; \\ \Gamma_t(A) &= A - te_1 = \{x \in \mathbb{R}^N \mid (x_1+t, x') \in A\}, \quad \Gamma(A) = \bigcup_{t\in\mathbb{R}} \Gamma_t(A). \end{aligned}$$

For $i=1,\ldots,k$, we define the quantities

$$d_i = \inf\{\lambda \in \mathbb{R} \mid T_\mu \cap \overline{G_i} = \emptyset \text{ for all } \mu > \lambda\};$$

$$\begin{aligned}\lambda_i &= \inf\{\lambda \le d_i \mid (\overline{G_i} \cap D_\mu)^\mu \subset G_i \text{ and } <n(z), e_1> > 0 \\ &\quad \text{for all } \mu > \lambda \text{ and all } z \in T_\mu \cap \partial G_i\} ; \\ d &= \max_{1\le i\le k} d_i , \quad \lambda_\star = \max_{1\le i\le k} \lambda_i.\end{aligned}$$

In other words, d_i is the x_1-coordinate of the right-hand cap of G_i. Note that T_{d_i} is tangent to ∂G_i and that $\lambda_i < d_i$ (this is well-known, see [AF]). For a bounded domain Ω we denote by d_Ω and λ_Ω the corresponding values for Ω.

We call λ_i (resp. $\lambda_\star$) the critical position for G_i (resp. G). We say that λ_i is attained at a position of internal tangency if $(\overline{G_i} \cap D_{\lambda_i})^{\lambda_i} \not\subset G_i$ (see Fig. 1). When λ_i is not attained at a position of internal tangency, we say it is attained at a position of orthogonality.

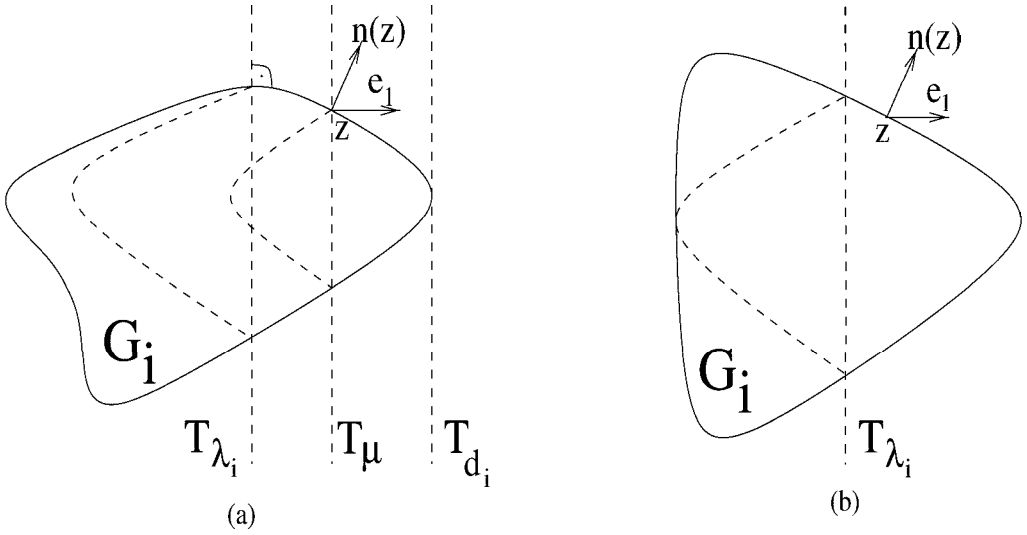

Figure 1: *Two types of domains G_i : the critical position λ_i is attained at a point of orthogonality (a), or at a point of tangency (b). For all $\mu > \lambda_i$ the part of $\overline{G_i}$ to the right of T_μ has its reflection inside G_i and the outward normal to ∂G_i at each point of the boundary of this part makes an acute angle with the direction e_1.*

We set $\Sigma_\lambda = (D_\lambda \cap \Omega) \setminus (\overline{G^\lambda} \cup \overline{G})$ and consider the function

$$w_\lambda(x) = u(x^\lambda) - u(x).$$

This function is well defined in $\overline{\Sigma_\lambda}$. Our goal is to show that $w_\lambda \equiv 0$ in Σ_λ for some $\lambda \in \mathbb{R}$ such that T_λ is a hyperplane of symmetry for Ω and G.

In the next section we establish some properties of the reflected set G^λ, which we use in Sections 2.2 and 2.3 to prove our theorems.

2.1 Some Reflection Properties of the Set G

The following easy property of G will permit us to treat this set in most cases like consisting of only one domain.

Lemma 2.1 *Let $\lambda \geq \lambda_\star$. For any $i \in \{1, \ldots, k\}$, we have*

$$\overline{G_i^\lambda} \cap \left(\bigcup_{j \neq i} (\overline{G_j^\lambda} \cup \overline{G_j}) \right) \cap D_\lambda = \emptyset.$$

Hence any point on $\partial G_i^\lambda \cap D_\lambda$ has a neighbourhood which does not intersect $\bigcup_{j \neq i} (\overline{G_j^\lambda} \cup \overline{G_j})$.

Proof. It is obvious that $\overline{G_i} \cap \overline{G_j} = \emptyset$ implies $\overline{G_i^\lambda} \cap \overline{G_j^\lambda} = \emptyset$. It is also clear that $\lambda \geq \lambda_\star$ implies, by the definition of $\lambda_\star$,

$$\overline{G_i} \cap D_\lambda \subseteq \overline{G_i^\lambda} \cap D_\lambda,$$

for all $i \in \{1, \ldots, k\}$. Hence, for any $i \neq j$,

$$\overline{G_i^\lambda} \cap (\overline{G_j^\lambda} \cup \overline{G_j}) \cap D_\lambda \subseteq (\overline{G_i^\lambda} \cap \overline{G_j^\lambda}) \cap D_\lambda = \emptyset. \qquad \square$$

We use Lemma 2.1 to obtain a characterisation of the boundary points of the reflected regions G_i^λ. It will be crucial in the subsequent discussion.

Let $z \in \partial G_i^\lambda \cap D_\lambda$ be such that $\Gamma_t(z)$ belongs to G_i^λ for small positive values of t. We define the quantities

$$\begin{aligned} \overline{t} = \overline{t}(z) &:= \min\{t > 0 \mid \Gamma_t(z) \in \partial G_i^\lambda\}; \\ \underline{t} = \underline{t}(z) &:= \min\{t > 0 \mid \Gamma_t(z) \in \partial G_i^\lambda \cup \partial G_i\}; \\ \rho = \rho(z) &:= \operatorname{dist}(z, T_\lambda) > 0. \end{aligned} \tag{11}$$

It is clear that $0 < \underline{t} \leq \overline{t} < \infty$, since G_i is smooth and bounded.

Lemma 2.2 *If $\lambda \geq \lambda_\star$, then any $z \in \partial G_i^\lambda \cap D_\lambda$, $i = 1, \ldots, k$, has one of the following properties (exclusively)*

(I) *$\Gamma_t(z) \in \Sigma_\lambda$ for small positive values of t or there exists a sequence $t_m \searrow 0$ such that $\Gamma_{t_m}(z) \in \partial G_i^\lambda \cap D_\lambda$;*

(II) *$0 < \underline{t}(z) \leq \rho_(z)$, the open segment $(\Gamma_{\underline{t}}(z), z)$ belongs to G_i^λ, and $\Gamma_{\underline{t}}(z) \in \partial G_i^\lambda$;*

(III) $0 < \underline{t}(z) < \rho_(z)$, *the open segment* $(\Gamma_{\underline{t}}(z), z)$ *belongs to* G_i^λ, *and* $\Gamma_{\underline{t}}(z) \in \partial G_i$;

(IV) $\lambda = \lambda_\star$ *and* $z \in \partial G_i^\lambda \cap \partial G_i$ *(the symmetry case).*

The four cases of Lemma 2.2 are shown on Fig. 2. In this way we obtain four types of points on $\partial G_i^\lambda \cap D_\lambda$.

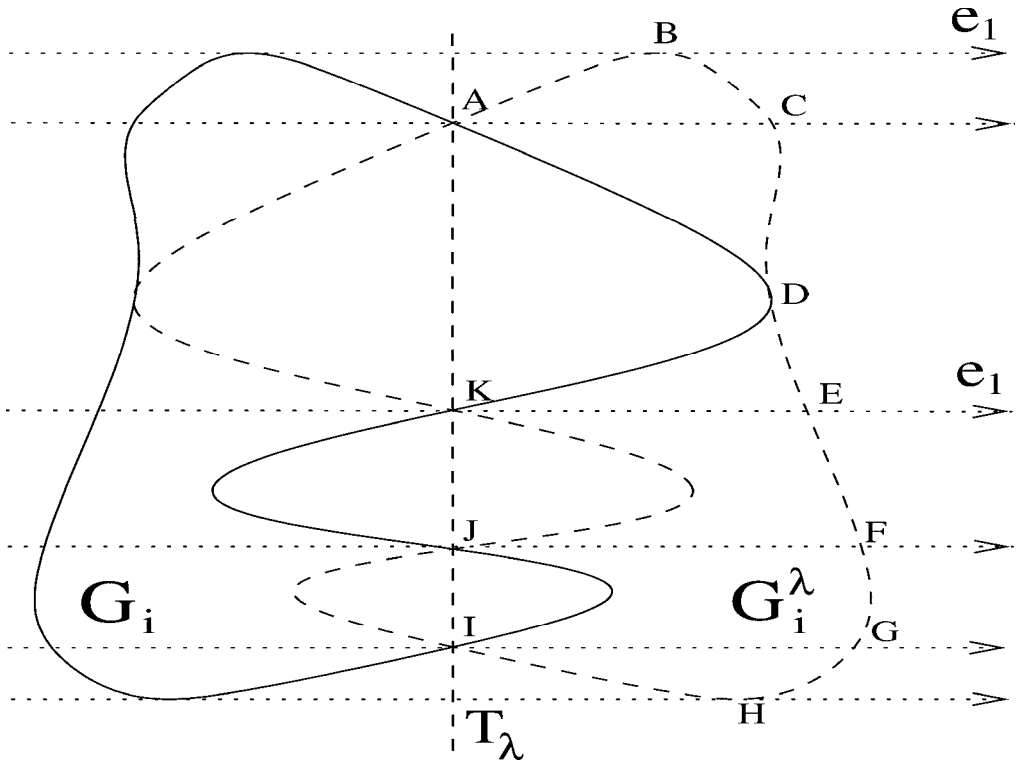

Figure 2: *Four types of points on* $\partial G_i^\lambda \cap D_\lambda$ *: the arcs (A,B], [H,I), (J,K) are of type (I), the arcs (B,C], [E,F], [G,H) are of type (II), the arcs (C,D), (D,E), (F,G) are of type (III), and the point D is of type (IV).*

Proof of Lemma 2.2. Fix a point $z \in \partial G_i^\lambda \cap D_\lambda$ and suppose (I) does not hold for z. We are going to show that z satisfies one of the other three alternatives.

Since (I) is false for z, Lemma 2.1 implies that $\Gamma_t(z) \in G_i^\lambda$ for small positive values of t, so that $\bar{t}(z)$ and $\underline{t}(z)$ are well-defined. First we observe that

$$\bar{t}(z) \leq 2\rho(z).$$

Indeed, if this is not true, we obtain a contradiction with the fact that $\lambda \geq \lambda_\star$. In order to write this rigorously we note that

$$\left\{\Gamma_{\overline{t}}(z)^\lambda\right\}^{\lambda+\frac{1}{2}(\overline{t}-2\rho)} = z^\lambda.$$

Since z^λ and $\Gamma_{\overline{t}}(z)^\lambda$ are two points on ∂G_i, this implies that we have internal tangency for G_i to the right of or at position $\lambda + \frac{1}{2}(\overline{t} - 2\rho)$. Consequently, in case $\overline{t} > 2\rho$,

$$\lambda_\star \geq \lambda_i \geq \lambda + \frac{1}{2}(\overline{t} - 2\rho) > \lambda,$$

which is a contradiction.

If $\overline{t}(z) = 2\rho(z)$, then it is obvious that $z^\lambda = \Gamma_{\overline{t}}(z) \in \partial G_i^\lambda$, so

$$z \in \partial G_i^\lambda \cap \partial G_i.$$

Since in this situation we have tangency to the right of or at position λ, we deduce $\lambda = \lambda_\star$, that is, we are in case (IV).

The last case to consider is $\overline{t}(z) < 2\rho(z)$. We claim that in this case $\underline{t}(z) \leq \rho(z)$, that is, $\Gamma_{\underline{t}}(z) \in \overline{D_\lambda}$. If $\overline{t}(z) \leq \rho(z)$, this is obvious. If

$$\rho(z) < \overline{t}(z) < 2\rho(z),$$

then the point $\Gamma_{\overline{t}}(z)^\lambda \in \partial G_i$ is to the left of z and to the right of T_λ; we infer

$$\underline{t}(z) \leq \text{dist}\left(z, \Gamma_{\overline{t}}(z)^\lambda\right) < \rho(z).$$

It follows from Lemma 2.1 that if $\underline{t}(z) \leq \rho(z)$, then $\underline{t}(z)$ does not change in case we replace G_i by G in (11). We infer from this fact that the open segment $(\Gamma_{\underline{t}}(z), z)$ belongs to G_i^λ.

Finally, it is clear that we have either case (II) or case (III), depending on whether the point at which we reach $\underline{t}(z)$ is on ∂G_i^λ or on ∂G_i.

2.2 The Case $\Omega = \mathbb{R}^N$.

We first give the general plan of the proof of Theorem 2. We use a hyperplane perpendicular to e_1 and say it has reached a position λ provided $w_\mu \geq 0$ in Σ_μ for all $\mu \geq \lambda$. The hyperplane "starts" at $\lambda = +\infty$ and "moves" as λ decreases. In the initializing step of our proof we show that this process can begin, that is, $w_\lambda \geq 0$ in Σ_λ for sufficiently large λ. Next we observe that if the moving plane has reached a position λ, then the solution u is strictly decreasing – in the x_1-direction – in the region to the right of the plane

T_λ. By using this fact we show that the moving plane reaches position d, which permits us to prove that, in a neighbourhood of ∂G, the solution u is strictly decreasing in the direction of the outward normal to ∂G. Then, having already all the necessary information on the solution, we can show that the moving plane reaches the critical position $\lambda_\star$. A device due to Serrin permits us to prove that $w_{\lambda_\star} \equiv 0$ in a connected component Z of $\Sigma_{\lambda_\star}$. We conclude by showing that all points on $\partial Z \setminus T_{\lambda_\star}$ are of the symmetry type (IV) (see Lemma 2.2), and by using a topological argument due to Fraenkel.

We divide the proof of Theorem 2 into ten steps.

Step 1 *There exists $\overline{\lambda} \in \mathbb{R}$ such that $w_\lambda \geq 0$ in Σ_λ, for all $\lambda \geq \overline{\lambda}$.*

Proof. In order to simplify the presentation, we first suppose that we are in the situation of Theorem 1, that is, hypotheses (q) and (f) are replaced by

(q_1) $\quad Q = \Delta\,;$

(f_1) $\quad f(u,p)$ does not depend on p, $f \in C^1([0,\infty))$ and $f'(0) < 0$.

Since u tends to zero at infinity we can take $\widetilde{\lambda} \in \mathbb{R}$ such that $\widetilde{\lambda} > d$ and

$$u(x) < \frac{1}{2} \min_{1 \leq i \leq k} a_i \qquad \text{for} \quad |x| > \widetilde{\lambda},$$

so that $w_\lambda > \dfrac{a_i}{2} > 0$ on ∂G_i^λ, for all i and all $\lambda \geq \widetilde{\lambda}$.

Now we can proceed in a standard way. Suppose the claim in Step 1 is false, that is, there exists a sequence $\{\lambda_m\}_{m=1}^\infty$ such that

$$\lim_{m \to \infty} \lambda_m = \infty\,, \ \lambda_m \geq \widetilde{\lambda},$$

and w_{λ_m} takes negative values in Σ_{λ_m}. Since w_λ is zero on T_λ and tends to zero at infinity for a fixed λ, we see that w_{λ_m} attains its negative minimum inside Σ_{λ_m}, say at a point $x^{(m)}$. Then

$$\nabla w_{\lambda_m}(x^{(m)}) = 0 \quad \text{and} \quad \Delta w_{\lambda_m}(x^{(m)}) \geq 0.$$

On the other hand, it is clear that the function $v_\lambda(x) = u(x^\lambda)$ satisfies in Σ_λ the same equation as u. By substracting the two equations we see that w_λ satisfies a linear equation of the type

$$\Delta w_\lambda + b_\lambda(x) w_\lambda = 0 \quad \text{in} \ \ \Sigma_\lambda \tag{12}$$

for all $\lambda \in \mathbb{R}$, where $b_\lambda(x) = f'(c(\lambda, x))$, with

$$c(\lambda, x) \in [\min\{u(x^\lambda), u(x)\}, \max\{u(x^\lambda), u(x)\}].$$

Since $w_{\lambda_m}(x^{(m)}) < 0$, we see that $0 \leq c(\lambda_m, x^{(m)}) \leq u(x^{(m)})$ and therefore $\lim_{m\to\infty} c(\lambda_m, x^{(m)}) = 0$. It follows that $b_{\lambda_m}(x^{(m)})$ is strictly negative for large m. Hence

$$0 \leq \Delta w_{\lambda_m}(x^{(m)}) = -b_{\lambda_m}(x^{(m)}) w_{\lambda_m}(x^{(m)}) < 0,$$

which is a contradiction.

The way to extend this argument to the general case is well-known. We sketch it here, for the sake of completeness. After substracting the two equations for u, and v_λ and doing some standard computations, we obtain a linear strictly elliptic equation with bounded coefficients, in the form

$$-\partial_i(a_{ij}(x)\partial_j w_\lambda) + b_i(x)\partial_i w_\lambda + c(x) w_\lambda = 0, \tag{13}$$

where $c(x) \leq 0$, provided the functions $u(x)$ and $v_\lambda(x)$ are sufficiently small. Then we can use the weak maximum principle, as in [GT], to conclude that $w_\lambda \geq 0$ in Σ_λ, for λ sufficiently large. □

Step 1 shows that the number

$$\lambda_0 = \inf\{\lambda \in \mathbb{R} \mid w_\mu \geq 0 \text{ in } \Sigma_\mu \text{ for all } \mu > \lambda\}$$

is well defined. It is obvious that λ_0 is finite. Notice that, by continuity, $w_{\lambda_0} \geq 0$ in Σ_{λ_0}.

Step 2 *We have*

$$\frac{\partial u}{\partial x_1} < 0$$

in the set $\{x \in \mathbb{R}^N \mid x_1 > \max\{\lambda_0, d\}\}$.

Proof. Fix $\mu > \max\{\lambda_0, d\}$. Notice that for all $\lambda > d$ the set Σ_λ is connected and regular. It follows from the strong maximum principle and Hopf's lemma, applied to (13), that for any $\lambda \in [\lambda_0, \infty) \cap (d, \infty)$, either $w_\lambda \equiv 0$ in Σ_λ or $w_\lambda > 0$ in Σ_λ, with $\dfrac{\partial w_\lambda}{\partial \nu} < 0$ on points of $\partial\Sigma_\lambda$ at which w_λ takes value zero (here and in the sequel ν denotes the outward normal to $\partial\Sigma_\lambda$).

If $w_\mu > 0$ in Σ_μ we obtain, using the fact that $w_\mu = 0$ on T_μ,

$$0 > \left.\frac{\partial w_\mu}{\partial \nu}\right|_{T_\mu} = -\left.\frac{\partial w_\mu}{\partial x_1}\right|_{T_\mu} = 2\left.\frac{\partial u}{\partial x_1}\right|_{T_\mu}. \tag{14}$$

Supose for contradiction that $w_\mu \equiv 0$ in Σ_μ. We distinguish two cases. First, if $w_\lambda > 0$ in Σ_λ for all $\lambda > \mu$ then, using (14) with μ replaced by λ, we see

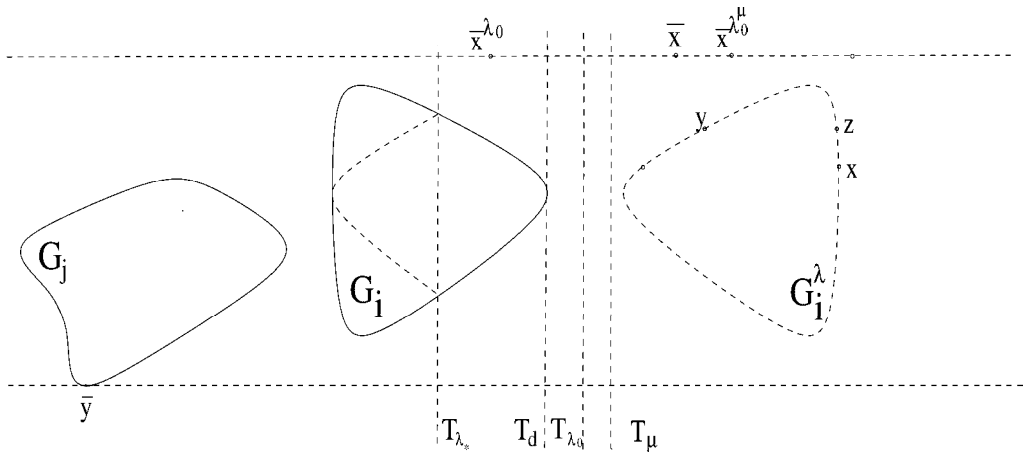

Figure 3: *The contradictions in Steps 2 and 3.*

that $\dfrac{\partial u}{\partial x_1} < 0$ in D_μ. Then we fix a point $\overline{x} \in \Sigma_\mu$ such that $\Gamma(\overline{x}) \cap \overline{G} = \emptyset$, and obtain the contradiction

$$u(\overline{x}) \leq u(\overline{x}^{\lambda_0}) = u\left((\overline{x}^{\lambda_0})^\mu\right) = u(\overline{x}_1 + 2(\mu - \lambda_0), \overline{x}') < u(\overline{x})$$

(see Fig. 3).

Second, if $w_{\widetilde{\mu}} \equiv 0$ in $\Sigma_{\widetilde{\mu}}$ for some $\widetilde{\mu} > \mu$, we take a point $\overline{y} \in \partial G$ such that $\Gamma(\overline{y}) \cap G = \emptyset$ and, by using consecutive reflexions with respect to T_μ and $T_{\widetilde{\mu}}$, obtain an unbounded sequence of points at which u takes a fixed positive value – contradiction with (BC). □

Step 3 $\lambda_0 \leq d$.

Proof. Suppose for contradiction that $\lambda_0 > d$. As explained above, either $w_{\lambda_0} \equiv 0$ in Σ_{λ_0} or $w_{\lambda_0} > 0$ in Σ_{λ_0}. First assume $w_{\lambda_0} \equiv 0$ in Σ_{λ_0}. We can always take two points $y, z \in \partial G_1^{\lambda_0}$ such that $y_1 < z_1$ and $y' = z'$. Then, by Step 2, $u(y) > u(z)$. However, we get

$$u(y) = u(y^{\lambda_0}) = a_1 = u(z^{\lambda_0}) = u(z),$$

because $w_{\lambda_0} \equiv 0$.

Next, suppose $w_{\lambda_0} > 0$ in Σ_{λ_0}, with $\dfrac{\partial w_{\lambda_0}}{\partial \nu} < 0$ on points of $\partial\Sigma_{\lambda_0}$ where $w_{\lambda_0} = 0$. By using the minimal choice of λ_0 we can find a sequence $\{\lambda_m\}_{m=1}^\infty$ such that $\lim\limits_{m\to\infty} \lambda_m = \lambda_0$, $\lambda_m < \lambda_0$, and w_{λ_m} takes negative values in Σ_{λ_m}. Let $x^{(m)} \in \overline{\Sigma_{\lambda_m}}$ be such that

$$w_{\lambda_m}(x^{(m)}) = \min_{x \in \overline{\Sigma_{\lambda_m}}} w_{\lambda_m}(x) < 0.$$

Using the assumption $\lambda_0 > d$, we fix m_0 such that

$$\text{dist}\left(\overline{G^{\lambda_m}},\, T_{\lambda_0}\right) \geq \frac{1}{2}\text{dist}\left(\overline{G^{\lambda_0}},\, T_{\lambda_0}\right) > 0, \tag{15}$$

for $m \geq m_0$.

We distinguish two cases.

Case 1 $x^{(m)} \in \text{int}\Sigma_{\lambda_m}$, for all $m \geq m_0$.

In this case a contradiction is obtained in a standard way. As in Step 1 we see that $\Delta w_{\lambda_m}(x^{(m)}) \geq 0$ and $\nabla w_{\lambda_m}(x^{(m)}) = 0$. If $\{x^{(m)}\}$ tends to infinity we obtain a contradiction exactly as in Step 1. If a subsequence of $\{x^{(m)}\}$ converges to a point x^0, then by passing to the limit we obtain $x^0 \in \overline{\Sigma_{\lambda_0}}$, $w_{\lambda_0}(x^0) \leq 0$ and $\nabla w_{\lambda_0}(x^0) = 0$. This means that $x^0 \in \partial\Sigma_{\lambda_0}$ and $w_{\lambda_0}(x^0) = 0$, so Hopf's lemma implies $\nabla u(x^0) \neq 0$, a contradiction.

Case 2 $x^{(m)} \in \partial\Sigma_{\lambda_m}$, for some $m \geq m_0$.

We drop the super(sub-)script m, for simplicity. Since $w_\lambda = 0$ on T_λ, we see that $x \in \partial G_i^\lambda$, for some $i \in \{1, \ldots, k\}$. We infer from (15) that $\Gamma_t(x)$ belongs to D_{λ_0}, for small $t > 0$.

We apply Lemma 2.2, which says x has one of the four properties described in the statement of this lemma. Here $\lambda > d > \lambda_\star$, hence x cannot be of type (IV). It cannot be of type (III) either, since $\overline{G} \subset \mathbb{R}^N \setminus \overline{D_\lambda}$. We are going to obtain a contradiction in the remaining two cases.

First, suppose x is of type (I). If $\Gamma_t(x)$ belongs to Σ_λ for small $t > 0$ we consider the directional derivative $\dfrac{\partial w_\lambda}{\partial(-e_1)}(x)$. By the minimal choice of x this derivative has to be non-negative. On the other hand, it is easy to see that

$$\frac{\partial w_\lambda}{\partial(-e_1)}(x) = \frac{\partial u}{\partial x_1}(x) + \frac{\partial u}{\partial x_1}(x^\lambda) < \frac{\partial u}{\partial x_1}(x^\lambda),$$

by Step 2.

We are going to show that $\dfrac{\partial u}{\partial x_1}(x^\lambda)$ is non-positive. The fact that $\Gamma_t(x) \in \Sigma_\lambda$ for small $t > 0$ implies $< n(x^\lambda), e_1 > \geq 0$ (it is a standard fact that a direction which makes an acute angle with the inward normal to the boundary of a smooth domain enters the domain). It is clear that $u \equiv \text{const}$ on ∂G_i implies

$$\left.\frac{\partial u}{\partial \xi_j}\right|_{\partial G_i} \equiv 0\,, \quad j = 1, \ldots, N-1,$$

for any orthonormal basis $\xi_1, \ldots, \xi_{N-1}$ in the tangent plane to ∂G_i. Hence, by (BC),

$$\frac{\partial u}{\partial x_1}(x^\lambda) = < n(x^\lambda), e_1 > \frac{\partial u}{\partial n}(x^\lambda) \leq 0,$$

which leads to a contradiction.

Next, still in case (I), suppose that there exists a sequence of positive numbers $t_m \to 0$, such that $x^{(m)} := \Gamma_{t_m}(x) \in \partial G_i^\lambda$. We have

$$w_\lambda(x^{(m)}) \geq w_\lambda(x),$$

by the minimal choice of x. On the other hand, Step 2 together with $t_m > 0$ implies $u(x^{(m)}) > u(x)$. Hence

$$w_\lambda(x^{(m)}) = a_i - u(x^{(m)}) < a_i - u(x) = w_\lambda(x),$$

a contradiction.

Finally, if x is of type (II), we see that $\Gamma_{\underline{t}}(x) \in \partial G_i^\lambda \cap D_{\lambda_0}$ (since $\overline{G^\lambda} \subset D_{\lambda_0}$), where $\underline{t} = \underline{t}(x)$ is the number defined in Section 2.1. Hence, by Step 2,

$$w_\lambda\left(\Gamma_{\underline{t}}(x)\right) = a_i - u\left(\Gamma_{\underline{t}}(x)\right) < a_i - u(x) = w_\lambda(x),$$

which contradicts the minimal choice of x. □

Remark 3. We can now prove Theorem 5 for $G = \{x^0\}$, where $x^0 \in \mathbb{R}^N$ is such that $u(x^0) = \max_{x \in \mathbb{R}^N} u(x) > 0$. By Step 3, $\lambda_0 \leq x_1^0$ and $w_{\lambda_0} \geq 0$ in Σ_{λ_0}. By taking a moving plane coming from the left we get $w_{x_1^0} \equiv 0$ in $\Sigma_{x_1^0}$. To be more precise, by "a plane coming from the left" we mean the same process of moving a hyperplane T_λ which starts at $\lambda = -\infty$ and moves as λ increases. The region Σ_λ would then be defined to the left of T_λ.

Step 4 *For any $z \in \partial G$ and any unit vector η, for which $<\eta, n(z)> > 0$, we can find a sufficiently small ball $B_\delta(z)$ such that*

$$\frac{\partial u}{\partial \eta}(\zeta) < 0 \quad \text{for all } \zeta \in B_\delta(z) \setminus \overline{G}.$$

This statement was already used by Reichel, who established it under the condition (r). Since we do not assume this stronger condition, we have to provide a different proof.

The proof of Step 4 is the only place in Section 2.2 where we have to be careful about the fact that there might be more than one domain G_i. We postpone this proof for the time being.

Step 5 *$w_\lambda > 0$ in Σ_λ, for any $\lambda \in [\lambda_0, \infty) \cap (\lambda_\star, \infty)$.*

Proof. In view of steps 2 and 3 we can restrict ourselves to the case $\lambda \leq d$. By the strong maximum principle, all we have to exclude is $w_\lambda \equiv 0$ in a

connected component Z of Σ_λ. Suppose for contradiction we are in this situation.

First we observe that any connected component Y of Σ_λ satisfies

$$\operatorname{dist}\left(\partial Y \cap D_\lambda\,,\, T_\lambda\right) = 0 \tag{16}$$

(see Fig. 4). Indeed, Y is connected, so Y^λ is also connected and hence Y^λ is a connected component of $(\mathbb{R}^N \setminus \overline{G}) \setminus \overline{D_\lambda}$. Since $\mathbb{R}^N \setminus \overline{G}$ is itself connected, we see that either Y^λ contains a left neighbourhood of T_λ or

$$\operatorname{dist}(\partial Y^\lambda \setminus \overline{D_\lambda}, T_\lambda) = 0. \tag{17}$$

The first alternative is impossible because of the fact that T_λ touches at least one of the domains G_i (recall that $\lambda \leq d$). By reflection (17) implies (16).

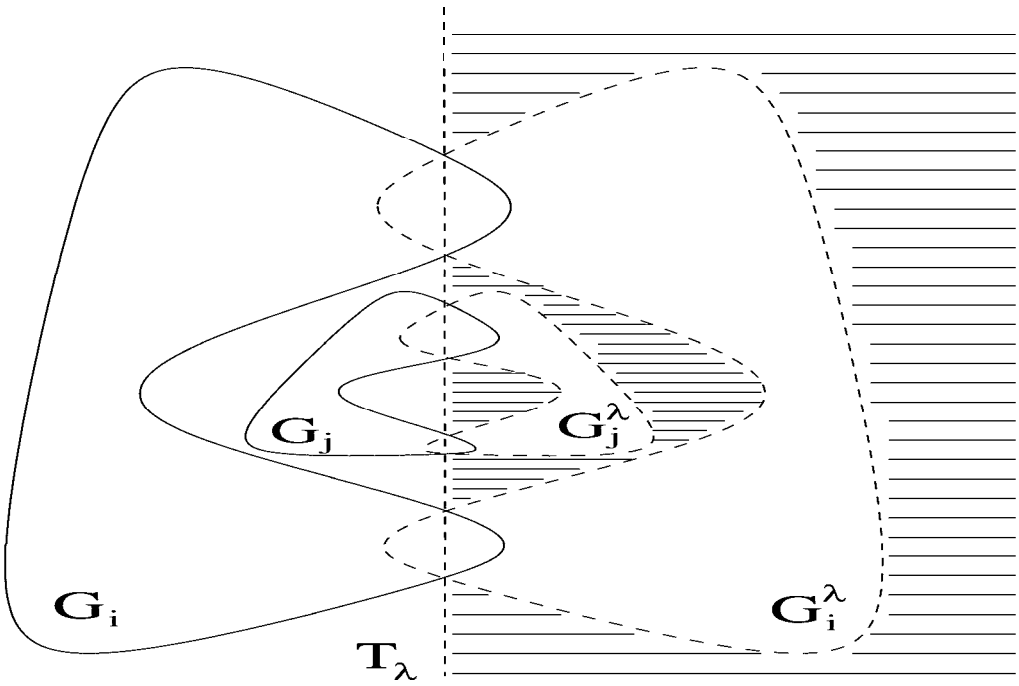

Figure 4: *The shaded regions are the connected components of* Σ_λ.

Now (16) permits us to take a sequence $\{z^{(m)}\}_{m=1}^\infty \subset \partial Z \cap D_\lambda$, which converges to a point $z^0 \in T_\lambda \cap \partial Z \cap \partial G$. Since $\lambda > \lambda_*$, we have $<n(z^{(m)^\lambda}), e_1> > 0$ for sufficiently large m, and the open segment between $z^{(m)^\lambda}$ and $z^{(m)}$ belongs to $B_\delta(z^0) \setminus \overline{G}$, where δ is chosen as in Step 4. Then, by Step 4, u decreases strictly from $z^{(m)^\lambda}$ to $z^{(m)}$, which yields $w_\lambda(z^{(m)}) > 0$, a contradiction with the fact that $w_\lambda \equiv 0$ in Z. This last argument was used by Reichel in [R1]. $\square$

Step 6 *We have*

$$\frac{\partial u}{\partial x_1} < 0 \quad \text{in } D_{\lambda^\star} \setminus \overline{G},$$

where $\lambda^\star = \max\{\lambda_0, \lambda_\star\}$.

Proof. By proceeding as in Step 2, we see that this is a direct consequence of Step 5 and Hopf's lemma. □

Step 7 $\lambda_0 \leq \lambda_\star$.

Proof. Suppose $\lambda_0 > \lambda_\star$. By Step 5 we know that $w_{\lambda_0} > 0$ in Σ_{λ_0}. Proceeding as in Step 3, we find two sequences $\{\lambda_m\}_{m=1}^{\infty}$, $\{x^{(m)}\}_{m=1}^{\infty}$, such that

$$\lim_{m\to\infty} \lambda_m = \lambda_0\,,\ \lambda_\star < \lambda_m < \lambda_0\,,\ x^{(m)} \in \overline{\Sigma_{\lambda_m}} \setminus T_{\lambda_m}$$

and w_{λ_m} attains its negative minimum in $\overline{\Sigma_{\lambda_m}}$ at $x^{(m)}$.

A number of different situations may arise. We are going to obtain a contradiction in each of them.

Case 1 There is a subsequence of $\{x^{(m)}\}$, such that $x^{(m)} \in \text{int}\Sigma_{\lambda_m}$.

If $\lim_{m\to\infty} |x^{(m)}| = \infty$ we obtain a contradiction as in Step 1. If a subsequence of $\{x^{(m)}\}$ converges to a point x^0 which belongs to the regular part of $\partial\Sigma_{\lambda_0}$, we obtain a contradiction as in Case 1 of Step 3. If x^0 belongs to the singular part of $\partial\Sigma_{\lambda_0}$, we have $x^0 \in T_{\lambda_0} \cap \partial G$, so that the argument we used at the end of the proof of Step 5 implies $w_{\lambda_m}(x^{(m)}) > 0$, for m sufficiently large.

Case 2 There is a subsequence of $\{x^{(m)}\}$, such that $x^{(m)} \in \partial\Sigma_{\lambda_m}$.

Notice that in this case $\{x^{(m)}\}$ is bounded, since $\partial\Sigma_{\lambda_m} \setminus T_{\lambda_m}$ is a bounded set. Fix i such that (along a subsequence) $x^{(m)} \in \partial G_i^{\lambda_m}$. We shall apply Lemma 2.2 to $x^{(m)}$ and reach a contradiction in all its four cases, with the help of Step 6. However, Step 6 gives information on the behaviour of the solution only to the right of T_{λ_0}, so we have first to exclude the case when $\{x^{(m)}\} \subset \partial\Sigma_{\lambda_m} \setminus D_{\lambda_0}$ (see Fig. 5). If such a subsequence exists, then $\lambda_m \to \lambda_0$ implies (along a subsequence) $x^{(m)} \to x^0 \in T_{\lambda_0} \cap \partial G_i$. Then the argument we used at the end of the proof of Step 5 leads to a contradiction.

Therefore we can assume that $x^{(m)} \in \partial G_i^{\lambda_m} \cap D_{\lambda_0}$. We apply Lemma 2.2 to $x^{(m)}$. If, for some m, the point $x^{(m)}$ is of type (I) we obtain a contradiction exactly as in Step 3. Points of type (IV) are also excluded, since $\lambda_m > \lambda_\star$.

It remains to reach a contradiction in case $x^{(m)}$ is of type (II) or (III) for all m. Let $\underline{t} = \underline{t}(m)$ be the quantity defined in Section 2.1 and let $\underline{x}^{(m)} := \Gamma_{\underline{t}}(x^{(m)})$. We treat separately the cases when $\underline{x}^{(m)}$ belongs or not to D_{λ_0} (note that by Lemma 2.2 we only know that $\underline{x}^{(m)} \in \overline{D_{\lambda_m}}$, see Fig. 5).

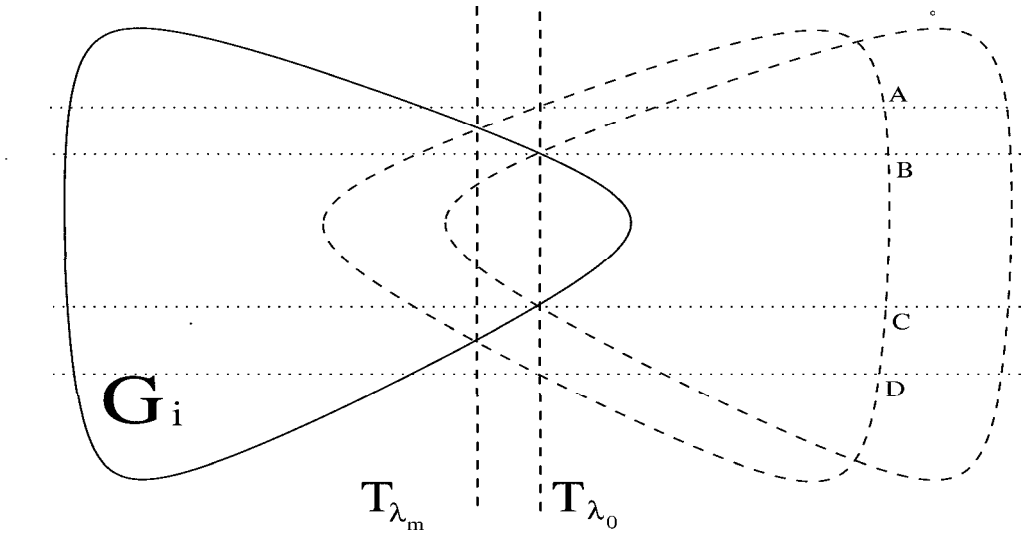

Figure 5: *The different situations in Case 2. For instance, the points on the arcs [A,B] and [C,D] give rise to Case 2.2.*

Case 2.1 $\underline{x}^{(m)} \in D_{\lambda_0}$ for some $m \in \mathbb{N}$.

First suppose $x^{(m)}$ is of type (II), that is, $\underline{x}^{(m)} \in \partial G_i^{\lambda_m} \cap D_{\lambda_0}$. Then we obtain a contradiction with the minimal choice of $x^{(m)}$, exactly as in Step 3.

Second suppose $x^{(m)}$ is of type (III), that is, $\underline{x}^{(m)} \in \partial G_i \cap D_{\lambda_0}$. Then $<n(\underline{x}^{(m)}), e_1>> 0$ (since $\lambda_0 > \lambda_\star$). Therefore u decreases strictly from $\underline{x}^{(m)}$ to $x^{(m)}$ (indeed, by Step 4 it starts from $\underline{x}^{(m)}$ by decreasing and continues decreasing, by Step 6). Hence $u(x^{(m)}) < u(\underline{x}^{(m)}) = a_i$, and

$$w_{\lambda_m}(x^{(m)}) = a_i - u(x^{(m)}) > 0,$$

a contradiction (recall that w_{λ_m} is negative at $x^{(m)}$).

Case 2.2 $\underline{x}^{(m)} \in \overline{D_{\lambda_m}} \setminus D_{\lambda_0}$ for all $m \in \mathbb{N}$.

Since $\underline{x}^{(m)} \in \partial G_i \cup \partial G_i^{\lambda_m}$, we see that $\{x^{(m)}\}$ is bounded. So we can extract a subsequence $\underline{x}^{(m)} \to \underline{x}^0 \in \partial G_i \cap T_{\lambda_0} = \partial G_i^{\lambda_0} \cap T_{\lambda_0}$. Then for m sufficiently large the segment between $\underline{x}^m$ and its orthogonal projection on T_{λ_0} belongs to $B_\delta(\underline{x}^0)$. Step 4 implies that u decreases on that segment. By using Step 6 to the right of T_{λ_0}, we see that u decreases on the whole segment $(\underline{x}^{(m)}, x^{(m)})$. Then we obtain a contradiction exactly as in Case 2.1, for both types of points. □

Step 8 $w_{\lambda_\star} \equiv 0$ *in at least one connected component of* $\Sigma_{\lambda_\star}$.

Proof. By Step 7, $w_{\lambda_\star} \geq 0$ in $\Sigma_{\lambda_\star}$. The following argument was carried out in [R2]. We outline the proof for completeness.

If there exists $z^0 \in \partial G \cap T_{\lambda_\star}$ such that $<n(z^0), e_1> = 0$, one can show that all first and second order derivatives of $w_{\lambda_\star}$ vanish at z^0 (see [R2], pp. 389-391 for detailed computations). Then Serrin's corner lemma (see [S]) immediately implies that $w_{\lambda_\star} \equiv 0$ in the connected component of $\Sigma_{\lambda_\star}$ whose boundary contains z^0.

If no such z^0 exists, by the definition of $\lambda_\star$ we can find $z_1 \in \partial G \cap D_{\lambda_\star}$ such that $z_1^{\lambda_\star} \in \partial G$ and $(\partial(G \cap D_{\lambda_\star}))^{\lambda_\star}$ is internally tangent to ∂G at $z_1^{\lambda_\star}$. By (BC), $w_{\lambda_\star}(z_1) = 0$ and $\dfrac{\partial w_{\lambda_\star}}{\partial \nu}(z_1) = 0$, so the claim of Step 8 follows from Hopf's lemma. □

Step 9 *Let Z be a connected component of $\Sigma_{\lambda_\star}$ such that $w_{\lambda_\star} \equiv 0$ in Z. Then*

$$\partial Z \setminus T_{\lambda_\star} \subset \partial G.$$

Proof. We shall once more make use of Lemma 2.2. Another way of stating Step 9 is to say that all points on $\partial Z \setminus T_{\lambda_\star}$ are of the symmetry type (IV). Suppose for contradiction that there exists a point $z \in \partial Z \setminus T_{\lambda_\star}$ (say $z \in \partial G_i^{\lambda_\star}$), which is not of type (IV). We are going to obtain a contradiction in all of the remaining three cases.

The point z is not of type (I), since if it were, $w_{\lambda_\star} \equiv 0$ in Z would lead to a contradiction, exactly as in Step 3. If z is of type (II) or (III), we set, as before, $\underline{z} = \Gamma_{\underline{t}(z)}(z) \in \overline{D_{\lambda_\star}}$. Then Step 6 implies that u decreases strictly on the segment $(\underline{z}, z)$. In case (II) we get

$$a_i = u(\underline{z}^{\lambda_\star}) \geq u(\underline{z}) > u(z) = u(z^{\lambda_\star}) = a_i,$$

since $w_{\lambda_\star} \geq 0$ in $\Sigma_{\lambda_\star}$ and $w_{\lambda_\star}(z) = 0$. Analogously, in case (III) we have

$$a_i = u(\underline{z}) > u(z) = u(z^{\lambda_\star}) = a_i.$$

Step 9 follows.

Step 10 *Conclusion.*

Proof. Once we have proved Step 9, the conclusion is obtained via a topological argument, due to Fraenkel and used in this setting by Reichel. Note that under condition (r) the previous Step 9 is obvious and is independent of all other steps.

We sketch the argument for completeness. Set $G^c = \mathbb{R}^N \setminus G$ and

$$X = Z \cup Z^\lambda \cup (\partial Z \cap G^c) \cup (\partial Z^\lambda \cap G^c).$$

The set X is symmetric with respect to T_λ, since Step 9 implies

$$\partial Z \cap G^c \subset T_\lambda \cap G^c.$$

One may check that X is open in G^c and hence $\partial X \subset \partial G$. Then $G^c \setminus X = G^c \setminus \overline{X}$, which implies that $X = G^c$, since G^c is connected. □

Finally we go back to Step 4.

Proof of Step 4. We use induction with respect to k. First assume $k = 1$. If $\alpha_1 < 0$, Step 4 is obvious, by continuity. Hence we can assume $\alpha_1 = 0$, or equivalently $\nabla u \equiv 0$ on ∂G. Notice that this implies $|D^2 u| = \left|\dfrac{\partial^2 u}{\partial n^2}\right|$ on ∂G.

Fix a point $z^0 \in T_d \cap \partial G$, so that

$$\frac{\partial u}{\partial x_1}(z^0) = \frac{\partial u}{\partial n}(z^0) = 0.$$

Steps 2 and 3, together with the assumption $\lambda_0 \leq d$, imply

$$\frac{\partial u}{\partial x_1}\left(\Gamma_t(z^0)\right) < 0,$$

for negative t. We conclude that

$$\frac{\partial^2 u}{\partial n^2}(z^0) = \frac{\partial^2 u}{\partial x_1^2}(z^0) \leq 0. \tag{18}$$

On the other hand, it is easy to compute that $u = \text{const}$ and $\nabla u = 0$ on ∂G imply

$$\begin{aligned} \operatorname{div}\left(Q(|\nabla u|)\nabla u\right)|_{\partial G} &= Q(|\nabla u|)\Delta u|_{\partial G} + <\nabla\left[Q(|\nabla u|)\right], \nabla u>|_{\partial G} \\ &= Q(0)\Delta u|_{\partial G} \end{aligned}$$

and

$$\Delta u|_{\partial G} = \left.\frac{\partial^2 u}{\partial n^2}\right|_{\partial G}.$$

Hence

$$\frac{\partial^2 u}{\partial n^2} \equiv -\frac{f(a_1, 0)}{Q(0)} = \text{const} \quad \text{on } \partial G.$$

By (18), $f(a_1, 0) \geq 0$. If $f(a_1, 0) > 0$, Step 4 follows easily, since

$$\frac{\partial^2 u}{\partial \eta^2} = <\eta, n>^2 \frac{\partial^2 u}{\partial n^2} \quad \text{on } \partial G.$$

If $f(a_1, 0) = 0$, we see that all first and second order derivatives of u vanish on ∂G. This implies that the function

$$\overline{u}(x) = \begin{cases} u(x) & \text{for} \quad x \in \mathbb{R}^N \setminus G \\ a_1 & \text{for} \quad x \in \overline{G} \end{cases}$$

belongs to $C^2(\mathbb{R}^N)$ and solves the equation

$$\begin{cases} Q\overline{u} + f(\overline{u}, |\nabla \overline{u}|) = 0 & \text{in} \quad \mathbb{R}^N \\ \overline{u} \geq 0\,, \quad \overline{u} \not\equiv 0 & \text{in} \quad \mathbb{R}^N \\ \overline{u}(x) \to 0 & \text{as} \quad x \to \infty. \end{cases} \tag{19}$$

However, the shape of $\overline{u}$ contradicts the result for equations on $\mathbb{R}^N$ that we have already proved (see Remark 3), which says a solution of (19) has a non-zero gradient everywhere, except at one point.

Suppose next that Step 4 is proved for $k-1$ domains G_i, and consider a problem with k domains. By Steps 5-10, Theorem 2 holds for problems with $k-1$ domains G_i. Set

$$I = \{i \mid \alpha_i < 0 \ \text{ or } \ \frac{\partial^2 u}{\partial n^2} < 0 \ \text{ on } \ \partial G_i\}$$

and $J = \{1, \ldots, k\} \setminus I$. Note that the statement of Step 4 is true for G_i with $i \in I$.

We claim that J is empty. Suppose this is not the case, and set

$$\underline{d} = \max_{j \in J} d_j.$$

It is clear that Step 5 can be proved for values of λ such that $\lambda \geq \lambda_0$ and $\lambda > \max\{\lambda_\star, \underline{d}\}$. Hence, as in Step 6,

$$\frac{\partial u}{\partial x_1} < 0 \quad \text{ in } \ D_{\max\{\lambda_\star, \underline{d}\}} \setminus \overline{G}.$$

It follows that, if $\lambda_\star > \underline{d}$, Steps 7-10 hold with the same proofs, yielding a contradiction. If $\lambda_\star \leq \underline{d}$ the moving plane reaches at least one domain G_{j_0}, with $j_0 \in J$. As in the case $k = 1$, this implies $f(a_{j_0}, 0) \geq 0$ and $f(a_{j_0}, 0) = 0$, by the definition of J. Then, defining the function $\overline{u}$ with a_1 replaced by a_{j_0} and $\overline{G}$ replaced by $\overline{G_{j_0}}$, we obtain a contradiction with Theorem 2, in the case when there are $k-1$ domains.

This concludes the proof of Theorem 2 for $\Omega = \mathbb{R}^N$.

Proof of Theorem 5. We now suppose that G is a ball and (8) holds. Observe that the hypotheses of Theorem 5 are sufficient to carry out the

first three steps in the above proof. It is easy to see that Step 3 implies $u < a$ in $\mathbb{R}^N \setminus G$. Indeed, if there exists $x \in \mathbb{R}^N \setminus G$ such that $u(x) \geq a$, we take moving planes $T_{\lambda(\gamma)}$, for $\gamma = x$, and see that u has to increase strictly on the ray from x to the boundary ∂G, which is a contradiction, since $u = a$ on ∂G.

The rest of the proof is elementary. We know that $w_\lambda > 0$ in Σ_λ for $\lambda > \max\{\lambda_0, \lambda_\star\}$. To prove $\lambda_0 \leq \lambda_\star$, we have only to consider Case 1 in Step 7. It remains the same, since $u < a$ in $\mathbb{R}^N \setminus G$ implies Step 4 (see for example [R2]). Finally, $w_{\lambda_\star} \equiv 0$ is obtained with the help of a moving plane coming from the left.

2.3 Bounded Domains Ω

We begin with a simple but basic observation, which permits us to treat this case in essentially the same way as the case $\Omega = \mathbb{R}^N$.

Lemma 2.3 *If $w_\lambda \equiv 0$ in a connected component Z of Σ_λ then*

$$\overline{Z} \cap \partial\Omega \cap \partial G^\lambda = \emptyset.$$

Proof. At any point $z \in \partial\Omega \cap \partial {G_i}^\lambda$ we have

$$w_\lambda(z) = a_i - 0 > 0.$$

Corollary 2.1 *If $w_\lambda \equiv 0$ in a connected component Z of Σ_λ, then either $dist(\overline{Z}, \partial\Omega) > 0$ or Z contains a neighbourhood of $\partial\Omega \cap \partial A(Z)$ in Ω, where $A(Z)$ denotes the connected component of $\Omega \cap D_\lambda$ containing Z.*

The next three lemmas ensure the symmetry of Ω, provided the moving plane reaches the critical position λ_Ω.

Lemma 2.4 *Either $\beta > 0$ or $\beta = 0$ and $\dfrac{\partial^2 u}{\partial n^2} > 0$ on $\partial\Omega$. Hence the statement of Step 4 in Section 2.2 is true for Ω and we can find $\varepsilon_0 > 0$ such that $u > 0$ in $\{x \in \Omega \mid dist(x, \partial\Omega) < 2\varepsilon_0\}$.*

Proof. Since $u \geq 0$ in Ω, we immediately see that either $\beta > 0$, or $\beta = 0$ and $\dfrac{\partial^2 u}{\partial n^2} \geq 0$ on $\partial\Omega$. If Lemma 2.4 were false, following the proof of Step 4 of Section 2.2 we obtain $|\nabla u| = \beta \equiv 0$ and $|D^2 u| = \dfrac{\partial^2 u}{\partial n^2} \equiv 0$ on $\partial\Omega$. This implies that the function

$$\overline{u}(x) = \begin{cases} u(x) & \text{for} \quad x \in \Omega \setminus G \\ 0 & \text{for} \quad x \in \mathbb{R}^N \setminus \overline{\Omega}. \end{cases}$$

belongs to $C^2(\mathbb{R}^N \setminus G)$ and solves (4) in $\mathbb{R}^N \setminus G$. On the other hand, the result we already proved in Section 2.2 says $\overline{u}$ has to be strictly positive.

Lemma 2.5 *If $\lambda \in [\lambda_\Omega\,,\, d_\Omega)$ and $w_\lambda \equiv 0$ in a connected component Z of Σ_λ which contains a neighbourhood of $\partial\Omega \cap \partial A(Z)$ in Ω, then $\lambda = \lambda_\Omega$ and Ω is symmetric with respect to T_λ.*

Proof. Note that for $\lambda \in [\lambda_\Omega\,,\, d_\Omega)$ the set $S := \partial\Omega \cap \partial A(Z)$ is connected and its reflexion with respect to T_λ is contained in $\overline{\Omega}$. We shall prove that $S^\lambda \subset \partial\Omega$, which is the desired symmetry property. Set

$$H = \{z \in S \mid z^\lambda \in \partial\Omega\}.$$

We are going to show that $H = S$. First, H is clearly not empty, since it contains the set $T_\lambda \cap S$. Second, the set H is open in S. To prove this, fix $z^0 \in H$ and $\varepsilon \in (0, \varepsilon_0)$, such that $B_\varepsilon(z^0) \cap A(Z) \subset Z$; here ε_0 is the number from the previous lemma. By Lemma 2.4, if $z \in (B_\varepsilon(z^0) \cap S) \setminus H$, then $w_\lambda(z) > 0$, which implies $z \notin \overline{Z}$ – a contradiction (see Fig 6). We infer that $B_\varepsilon(z) \cap S \subset H$, that is, H is open. Finally, it is clear that $S \setminus H$ is open in S, and we conclude that $H = S$.

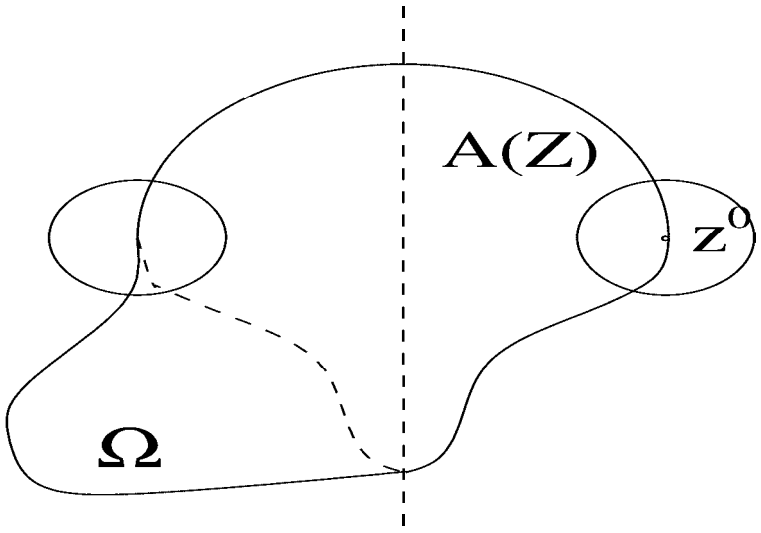

Figure 6: *The contradiction in Lemma 2.5.*

Lemma 2.6 *If $\lambda_0 \leq \lambda_\Omega$, then $w_{\lambda_\Omega} \equiv 0$ in some connected component Z of Σ_{λ_Ω}, which contains a neighbourhood of $\partial\Omega \cap \partial A(Z)$ in Ω.*

Proof. To construct Z such that $w_{\lambda_\Omega} \equiv 0$ in Z we repeat the proof of Step 8, Section 2.2. Then it is obvious from the construction that $\partial Z \cap \partial\Omega \neq \emptyset$, and Corollary 2.1 yields Lemma 2.6.

Proof of Theorem 2 (bounded domains). We shall only sketch this proof, as all its ingredients have already appeared above. We adapt the proof in Section 2.2 to the case of a bounded domain Ω. Step 1 is again the initialising step, in which we now prove that $w_\lambda \geq 0$ in Σ_λ for λ smaller, but sufficiently close to d_Ω (this is easy, by using the fact that the maximum principle holds for any linear strictly elliptic operator with bounded coefficients, provided the measure of the domain is sufficiently small). Step 2 is the same, since Corollary 2.1 and Lemma 2.5 prevent $w_\lambda \equiv 0$ in a connected component of Σ_λ and, in particular, in a right neighbourhood of T_λ, for $\lambda > \max\{d, \lambda_\Omega, \lambda_\star\}$. By using this fact, we can prove (as in Step 3) that $\lambda_0 \leq \max\{d, \lambda_\Omega\}$, replacing the convergence to infinity of $\{x^{(m)}\}$ by convergence to $\partial\Omega$. It is important to note that the points $x^{(m)}$ can neither lie on $\partial\Omega$ (since $w_\lambda \geq 0$ there), nor tend to a point on $\partial\Omega \cap \partial G_i$ (since $w_\lambda = a_i > 0$ at such points). The only difference with Step 3 is that here $\{x^{(m)}\}$ may tend to a point on $\partial\Omega \cap T_\lambda$. In this case we obtain a contradiction by using Lemma 2.4 and the argument at the end of Step 5.

Theorem 6 is already proved, since we can always suppose that we have $\lambda_\Omega \geq x_1^0 = d$ (if necessary, take a moving plane coming from the left). This means that the moving plane reaches the critical position λ_Ω, so Lemmas 2.5 and 2.6 permit us to obtain Theorem 6.

There are no other modifications in Steps 4-7, from which we infer that $\lambda_0 \leq \max\{\lambda_\Omega, \lambda_\star\}$. If $\lambda_\Omega \geq \lambda_\star$, then it follows from Lemmas 2.5 and 2.6 that Ω is symmetric with respect to T_{λ_Ω}. By applying the argument in Steps 9-10 to the set Z given by Lemma 2.6, we see that G is also symmetric (note that the symmetry of Ω implies $G^\lambda \subset \Omega$). If $\lambda_\star \geq \lambda_\Omega$, as in Steps 8-9 we obtain $w_{\lambda_\star} \equiv 0$ in a connected component Z of $\Sigma_{\lambda_\star}$, such that $\partial Z \setminus T_{\lambda_\star} \subset \partial G$. Since $\Omega \setminus G$ is connected, we can repeat the topological argument in Step 10 to conclude that $\Omega \setminus G$ is symmetric.

Proof of Theorem 5. If G is a ball, as in Section 2.2 we obtain $u < a$ in $\Omega \setminus G$. If Ω is a ball and $\lambda_\star > \lambda_\Omega$, we obtain a contradiction as in the last proof. If $\lambda_\star < \lambda_\Omega$, we consider a moving plane coming from the left.

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