

EXISTENCE AND MULTIPLICITY OF SOLUTIONS OF SEMI-LINEAR ELLIPTIC EQUATIONS IN \mathbb{R}^N

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1 Introduction and Main Results

In this paper we study the equation

$$-\Delta u + b(x)u = a(x)|u|^{p-1}u, \quad x \in \Omega, \quad (1)$$

and its more general version

$$-\Delta u + b(x)u = f(x, u), \quad x \in \Omega, \quad (2)$$

with Dirichlet boundary conditions ; here Ω is a regular (unbounded) domain in \mathbb{R}^N , $a(x) \in C(\overline{\Omega}, \mathbb{R})$, $b(x) \in C^1(\overline{\Omega}, \mathbb{R})$, $f(x, u) \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, and

$$\begin{aligned} 1 < p < p^\# \leq \frac{N+2}{N-2} & \quad \text{if } N \geq 3 \\ 1 < p < \infty & \quad \text{if } N = 1, 2 \end{aligned} \quad (3)$$

(the definition of $p^\#$ is given below).

Equations of this kind stem from various physical problems, such as the existence of stationary states of nonlinear Schrödinger and Klein-Gordon equations (for a discussion see for example [BL]).

We take $\Omega = \mathbb{R}^N$ throughout the paper. It is easy to see that Theorems 1.1-1.3 carry over, with minor changes, to the case of an arbitrary regular domain.

In the classical paper [AR] Ambrosetti and Rabinowitz considered the case of a bounded domain Ω . A great number of papers deal with these equations in unbounded domains. We shall quote here [BL], where the autonomous case is studied, [Li2], [BWi1] and [BWi2], where the radial non-autonomous case is considered, [Ls], where a "concentration-compactness" argument is used, as well as [DW] and [Li1] (we refer also to the references in these papers). All these works present different ways of overcoming the

problem of "lack of compactness", typical for elliptic problems in unbounded domains.

In the first part of this paper we consider another situation in which this problem can be dealt with – the case when the potential $b(x)$ is "large" at infinity. The pioneering work in this direction was done by Rabinowitz ([R]), whose hypotheses on $b(x)$ were the following

$$(r1) \quad b_0 = \inf_{x \in \mathbb{R}^N} b(x) > 0 ;$$

$$(r2) \quad \lim_{|x| \rightarrow \infty} b(x) = \infty.$$

Theorem 1.7 of [R] states that (2) has a positive solution, under (r1), (r2) and the standard "mountain-pass" assumptions on $f(x, u)$. Rabinowitz showed that a subsequence of the sequence associated with the mountain pass value of the functional

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + b(x)u^2 dx - \int_{\mathbb{R}^N} F(x, u) dx,$$

defined on the space

$$H = \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} b(x)|u|^2 dx < \infty \right\},$$

$$\|u\|_H^2 := \int_{\mathbb{R}^N} |\nabla u|^2 + b(x)u^2 dx,$$

converges weakly to a nontrivial solution of (2). The question whether this functional satisfies the Palais-Smale condition on H remained open and was later answered in the affirmative by Costa ([C]) (see also [OW]).

Note that condition (r2) is equivalent to the following hypothesis.

(r2) For every $M > 0$ the set

$$\Omega_M = \{x \in \mathbb{R}^N \mid b(x) < M\} \tag{4}$$

is bounded.

In ([BWa]) Bartsch and Wang proved that the functional Φ satisfies the Palais-Smale condition on H , and thus obtained a positive solution of (2), provided that (r1) holds and

(bw2) for every $M > 0$ the set Ω_M has finite Lebesgue measure.

Bartsch and Wang used the same hypotheses on f as in [R]. They also remarked that (2) has infinitely many solutions when $f(x, u)$ is odd in u .

To our knowledge, up to now (bw2) is the best hypothesis under which Φ is known to satisfy the Palais-Smale condition. Note that (r1), (r2) and (bw2) do not cover even cases like $b(x) = x_1^2 x_2^2$, $N = 2$, for which this condition is clearly expected to hold.

Our goal is to extend the results of [BWa] in several directions. First, we investigate to what extent (bw2) can be weakened, so that the compactness in the functional is preserved. We obtain a necessary and sufficient condition, on the domain and the potential $b(x)$, for the functional to satisfy the Palais-Smale condition (see Theorem 3.1).

Second, we weaken condition (r1). For instance, $b(x)$ can be negative on a ball which is not too large, in a sense to be explained below. Actually, $b(x)$ does not even have to be positive in an exterior domain. We find a necessary and sufficient condition for the existence of a positive solution of (2) in H .

Third, we show that the right-hand side of (2) does not have to be bounded in x . We can obtain non-trivial solutions of (2), provided the growth of f is controlled by the growth of $b(x)$. Here appears the upper bound for p . More precisely, we suppose that there exist a function $A(x) \in L_{loc}^\infty(\mathbb{R}^N)$, with $A(x) \geq 1$, and constants $\alpha > 1$, $C_0 > 0$, $R_0 > 0$, $\mu_0 > 2$ such that

$$A(x) \leq C_0 \left(1 + (\max\{0, b(x)\})^{\frac{1}{\alpha}}\right), \quad \text{if } |x| \geq R_0; \quad (5)$$

$$|f(x, u)| \leq C_0 A(x) (1 + |u|^p), \quad \text{for all } x \in \mathbb{R}^N \text{ and } u \in \mathbb{R}; \quad (6)$$

$$\frac{f(x, u)}{A(x)} = o(|u|) \quad \text{as } u \rightarrow 0 \text{ uniformly in } x; \quad (7)$$

$$0 < \mu_0 F(x, u) = \mu_0 \int_0^u f(x, s) ds \leq u f(x, u), \quad \text{if } u \neq 0. \quad (8)$$

The preceding works dealt with the case $A(x) \equiv 1$. To our knowledge, this is the first time a result covering unbounded nonlinearities in the non-radial case appears in literature.

In the case of (1) we suppose (5) for $A(x) = \max\{1, |a(x)|\}$, but we do not suppose that $a(x)$ is everywhere positive.

We set

$$p^\# = \frac{N+2}{N-2} - \frac{4}{\alpha(N-2)}.$$

We see that if (5) holds for any $\alpha > 1$ and some $R_0 = R_0(\alpha)$, we may take $p^\# = \frac{N+2}{N-2}$ in (3). Let us stress that the case when $A(x)$ is bounded is the simplest one in which (5) holds for any $\alpha > 1$.

While studying equations (1) and (2) in the above framework, we came across an intriguing observation, related to Pohozaev's identity. In case $a(x)$

is bounded and $p \in [\frac{N+2}{N-2}, \infty]$, it can be inferred from this identity that (1) has only the trivial solution in the variational space, whereas variational methods ensure that (1) has non-trivial solutions for $p \in [1, \frac{N+2}{N-2})$. It turns out that unbounded nonlinearities lead to a gap between the “existence” range for p , given by variational methods, and the “non-existence” range, provided by Pohozaev’s identity (see Section 4 for details).

We use the following notation. If $G \subset \mathbb{R}^N$ is open and $2 \leq s < \frac{2N}{N-2}$, we set

$$\nu_s(G) = \inf_{u \in \mathcal{M}_s(G)} \int_G |\nabla u|^2 + b(x)u^2 \, dx,$$

where $\mathcal{M}_s(G) = \{u \in H_0^1(G) \mid \|u\|_{L^s(G)} = 1\}$. We put $\nu_s(\emptyset) = \infty$.

The assumptions on $b(x)$ that we use are the following.

(i) There exists a constant $B \geq 0$ such that

$$b(x) \geq -B \quad \text{for all } x \in \mathbb{R}^N;$$

(ii)_s For any $r > 0$ and any sequence $\{x_n\} \subset \mathbb{R}^N$ which goes to infinity

$$\lim_{n \rightarrow \infty} \nu_s(B_n) = \infty,$$

where $B_n = B(x_n, r)$ stands for the open ball with center x_n and radius r .

We assume (i) throughout the paper. We consider the following condition

$$\lambda_1 := \inf_{u \in H, \|u\|_{L^2} = 1} \int_{\mathbb{R}^N} |\nabla u|^2 + b(x)u^2 \, dx > 0. \quad (9)$$

Note that (9) allows for potentials which change sign.

We now state our theorems about the existence of solutions of equations (1) and (2).

Theorem 1.1 *We suppose (3), (5)–(9), and (ii)_{p+1}. Then (2) has a positive solution. If, in addition, $f(x, u)$ is odd in u then (2) has infinitely many solutions.*

Furthermore, if we suppose (ii)₂ instead of (ii)_{p+1} then (9) is a necessary and sufficient condition for the existence of a positive solution of (2).

Theorem 1.2 *We suppose (3), (ii)_{p+1}, (5), and (9). Then sufficient condition for the existence of a positive solution of (1) is that $a(x)$ is positive somewhere. If we suppose (ii)₂ instead of (ii)_{p+1}, this is also necessary. Furthermore, under (ii)₂, if $a(x) \geq 0$ and $a(x) \not\equiv 0$ then (9) is a necessary and sufficient condition for the existence of a positive solution of (1).*

Theorem 1.3 *We suppose (3), (ii)₂, (5), and (9). If $a(x)$ is positive somewhere then there exist infinitely many solutions of (1).*

It is useful to have geometric conditions on $b(x)$ under which (ii)_s holds for all $s \in [2, \frac{2N}{N-2})$, and the above theorems apply.

Theorem 1.4 *The condition (ii)_s holds for all $s \in [2, \frac{2N}{N-2})$, provided that for any $M > 0$, any $r > 0$ and any sequence $\{x_n\} \subset \mathbb{R}^N$ which goes to infinity, we have*

$$\lim_{n \rightarrow \infty} |\Omega_M \cap B_n| = 0, \quad (10)$$

where $|\cdot|$ denotes the Lebesgue measure in \mathbb{R}^N .

In a way, Theorem 1.4 implies that Theorems 1.1, 1.2 and 1.3 apply provided that for all $M > 0$ the set $\Omega_M \setminus \overline{B_R}$ becomes "narrower and narrower" when R goes to infinity. On the other hand, it is well-known that the Palais-Smale condition fails if a level set of $b(x)$ contains an unbounded sequence of balls with fixed radius.

In the second part of the paper we study the situation when the potential $b(x)$ is only "sufficiently large" at infinity. We consider the equation

$$-\Delta u + \lambda b_\lambda(x)u = f(x, u), \quad (11)$$

where $\lambda \geq 1$ is a parameter, and show, under suitable assumptions on $b_\lambda(x)$, it possesses a non-trivial solution for sufficiently large λ . More specifically, we suppose that

(bl1) $b_\lambda(x) \geq 0$, for $\lambda \geq 1$ and $x \in \mathbb{R}^N$;

(bl2) there exists $M > 0$ such that for all $\lambda \geq 1$

$$|\Omega_{M,\lambda}| < \infty,$$

where $\Omega_{M,\lambda} = \{x \in \mathbb{R}^N \mid b_\lambda(x) < M\}$.

(bl3) $\lim_{\lambda \rightarrow \infty} b_\lambda(0) = 0$.

In [BWa] Bartch and Wang studied the case $b_\lambda(x) = b(x) + \frac{1}{\lambda}$, where the function $b(x)$ was supposed to be non-negative, to have a set Ω_M of finite measure for some $M > 0$, and to take value zero on a non-empty open subset of \mathbb{R}^N . The necessity of the last condition was left as an open question. Applied to this $b_\lambda(x)$, our results give the existence only under the hypothesis that $b(x)$ vanishes at one point.

We set

$$\mu = \max\{\bar{\mu} > 0 \mid \text{there exists a positive continuous function } d(x) \text{ such that } F(x, u) \geq d(x)|u|^{\bar{\mu}} \text{ for } x \in \mathbb{R}^N, u \in \mathbb{R}\}.$$

It is clear that $\mu_0 \leq \mu \leq p + 1$.

Our first result about equation (11) concerns the most frequent case when $\mu = p + 1$.

Theorem 1.5 *Assume that (bl1)-(bl3) and (3), (5)-(9) hold (with b replaced by b_λ in (5)). Suppose also $\mu = p + 1$. Then there exists $\lambda_0 \geq 1$, depending only on the various constants involved in the assumptions, such that (11) has a non-trivial (positive) solution, for any $\lambda \geq \lambda_0$.*

If we want to remove the hypothesis $\mu = p + 1$, we have to strengthen (bl3), replacing it with a more precise condition about the behaviour of $b_\lambda(x)$ near the origin, for λ sufficiently large.

Theorem 1.6 *Assume that (bl1), (bl2), (3), (5)-(9) hold, and that*

(bl4) *there exist constants $C_1, \varepsilon_0, \beta > 0$, such that*

$$b_\lambda(x) \leq C_1 \left(|x|^\beta + \lambda^{-\frac{\beta}{\beta+2}} \right) \quad \text{for } |x| \leq \varepsilon_0 \lambda^{-\frac{1}{\beta+2}},$$

and

$$\frac{2}{\beta + 2} \left(\frac{\mu}{\mu - 2} - \frac{N}{2} \right) < \frac{p + 1}{p - 1} - \frac{N}{2}.$$

Then the conclusion of Theorem 1.5 holds.

It is clear that the result in [BWa] is a particular case of Theorem 1.6.

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2 Preliminary Results and Proof of Theorem 1.4

Our first lemma enables us to settle the variational setting.

Lemma 2.1 *Assume (9) holds. Then the space H , endowed with the scalar product*

$$\langle u, v \rangle = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v + b(x)uv \, dx, \quad (12)$$

is a Hilbert space. Moreover, H is continuously embedded into $H^1(\mathbb{R}^N)$.

Proof. We have only to show that there exists a positive constant a such that, for $u \in H$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 + b(x)u^2 dx \geq a \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

Suppose that there exists a sequence $\{u_n\} \subset H$, such that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla u_n|^2 + b(x)u_n^2 dx \leq \frac{1}{n}.$$

The last inequality, together with $\lambda_1 > 0$, implies $u_n \rightarrow 0$ in $L^2(\mathbb{R}^N)$. We obtain the contradiction

$$o(1) = -B \int_{\mathbb{R}^N} u_n^2 dx \leq \int_{\mathbb{R}^N} b(x)u_n^2 dx \leq \frac{1}{n} - 1. \quad \square$$

The following lemma shows the relation between the different $(ii)_s$.

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^N$ be open and let $2 \leq p < q < \frac{2N}{N-2}$. Then*

$$\nu_q(\Omega) \geq C(p, q, N, \lambda_1) (\nu_p(\Omega))^{\alpha_1},$$

where α_1 is a fixed number in $(0, 1)$. If $2 < q < p < \frac{2N}{N-2}$ then

$$\nu_q(\Omega) \geq C(p, q, N, \lambda_1) (\nu_p(\Omega))^{\alpha_2}.$$

Therefore, $(ii)_p$ implies $(ii)_q$, for $2 \leq p \leq q$, and for $2 < q \leq p$.

Proof. This is a consequence of the previous lemma and the Gagliardo–Nirenberg inequality. Indeed, there exist a number $\alpha \in (0, 1)$ and a positive constant C , depending only on p, q, N and λ_1 , such that

$$\begin{aligned} \nu_q(\Omega) &= \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + b(x)u^2 dx}{\left(\int_{\mathbb{R}^N} |u|^q dx\right)^{\frac{2}{q}}} \\ &\geq \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + b(x)u^2 dx}{C \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{2\alpha}{p}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx\right)^{1-\alpha}} \\ &\geq \frac{1}{C} \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \left(\frac{\int_{\mathbb{R}^N} |\nabla u|^2 + b(x)u^2 dx}{\left(\int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{2}{p}}} \right)^\alpha \\ &= \frac{1}{C} (\nu_p(\Omega))^\alpha. \end{aligned}$$

The proof of the second inequality goes the same way. \square

In the following proposition we establish an equivalent way of stating condition $(ii)_s$, which will be useful in the sequel.

Proposition 2.1 *Condition (ii)_s holds if and only if*

$$\lim_{R \rightarrow \infty} \nu_s(\mathbb{R}^N \setminus \overline{B_R}) = \infty. \quad (13)$$

Proof. First, it is clear that (13) implies (ii)_s, since $\nu_s(\Omega_2) \leq \nu_s(\Omega_1)$ for any $\Omega_1 \subset \Omega_2$.

Suppose that (13) is false. Then we can find sequences $R_n \rightarrow \infty$ and $\{u_n\} \subset H$ such that $u_n \in H^1(\mathbb{R}^N)$, $\text{supp } u_n \subset \mathbb{R}^N \setminus B_{R_n}$,

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + b(x)u_n^2 dx \leq C$$

and

$$\int_{\mathbb{R}^N} |u_n|^s dx = 1$$

(everywhere in the sequel C will denote a positive constant whose value is irrelevant and may change from line to line). Note that the sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ (by Lemma 2.1) and does not converge to zero in $L^s(\mathbb{R}^N)$. This permits us to use a classical concentration result of Lions (see [Ls] or Lemma 1.21 in [W]). This result states that there exist a sequence $\{x_n\} \subset \mathbb{R}^N$ and a number $r_0 > 0$, such that

$$\int_{B(x_n, r_0)} |u_n|^s dx \geq c_0 > 0$$

(up to a subsequence of $\{u_n\}$). Since the support of u_n is contained in $\mathbb{R}^N \setminus B_{R_n}$, we have $x_n \rightarrow \infty$. We take a cut-off function φ_n , such that $\varphi_n \in C_c^\infty(\mathbb{R}^N)$, $0 \leq \varphi_n \leq 1$, $\varphi_n \equiv 0$ in $\mathbb{R}^N \setminus B(x_n, 2r_0)$ and $\varphi_n \equiv 1$ in $B(x_n, r_0)$. Then the sequence $v_n = \varphi_n u_n$ is bounded in H , belongs to $H_0^1(B(x_n, 2r_0))$, and

$$\|v_n\|_{L^s(B(x_n, 2r_0))} \geq c_0 > 0.$$

It is not difficult to infer from this inequality that (ii)_s does not hold, for $r = 2r_0$. \square

Proof of Theorem 1.4. By Lemma 2.2, it suffices to prove that (10) implies (ii)₂. Suppose for contradiction that this is not true, that is, there exist a sequence $\{x_n\}$, a number $r > 0$, and a sequence $\{u_n\} \subset H_0^1(B_n)$, such that

$$\int_{B_n} |\nabla u_n|^2 + b(x)u_n^2 dx \leq C, \quad (14)$$

with

$$\int_{B_n} |u_n|^2 dx = 1. \quad (15)$$

Note that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, by Lemma 2.1.

For any $M > 0$ we set $\Omega_{M,n} = \{x \in B_n \mid b(x) < M\}$, so that

$$\lim_{n \rightarrow \infty} |\Omega_{M,n}| = 0.$$

We have, for a fixed M ,

$$\begin{aligned} C &\geq \int_{B_n \setminus \Omega_{M,n}} b(x) u_n^2 dx + \int_{\Omega_{M,n}} b(x) u_n^2 dx \\ &\geq M \int_{B_n} u_n^2 - (B + M) \int_{\Omega_{M,n}} u_n^2 dx. \end{aligned} \quad (16)$$

We use the following lemma.

Lemma 2.3 *Let $\omega_n \subset \mathbb{R}^N$ be open, with*

$$\lim_{n \rightarrow \infty} |\omega_n| = 0.$$

Then, for any $C > 0$ and any $2 \leq p < 2^ = \frac{2N}{N-2}$,*

$$\lim_{n \rightarrow \infty} \left(\sup_{\int_{\mathbb{R}^N} |\nabla u|^2 \leq C} \int_{\omega_n} |u|^p dx \right) = 0.$$

Proof. By Hölder and Sobolev inequalities

$$\begin{aligned} \int_{\omega_n} |u|^p dx &\leq |\omega_n|^{1-\frac{p}{2^*}} \|u\|_{L^{2^*}(\mathbb{R}^N)}^p \\ &\leq C |\omega_n|^{1-\frac{p}{2^*}} \|\nabla u\|_{L^2(\mathbb{R}^N)}^p, \end{aligned}$$

which proves the lemma. \square

Finally, (15), (16) and Lemma 2.3 yield, for n sufficiently large,

$$C \geq \frac{M}{2},$$

which is a contradiction, since M is arbitrary. Theorem 1.4 is proved. \square

The next proposition will be used to derive the necessary conditions stated in Theorems 1.1 and 1.2.

Proposition 2.2 *Suppose that (i) and (ii)₂ hold. Then λ_1 is the smallest eigenvalue of the operator $-\Delta + b(x)$ in \mathbb{R}^N , corresponding to a positive principal eigenfunction. In other words, the equation*

$$\begin{cases} -\Delta\varphi_1 + b(x)\varphi_1 = \lambda_1\varphi_1 \\ \varphi_1 \in H \cap C^2(\mathbb{R}^N) \\ \varphi_1 > 0 \end{cases} \quad (17)$$

has a solution which, normalised so that $\int_{\mathbb{R}^N} \varphi_1^2 dx = 1$, is a minimiser for the right-hand side of (9).

Proof. We shall prove that the infimum in the right-hand side of (9) is attained. First of all we remark that $\lambda_1 \geq -B > -\infty$.

Let $\{u_n\}$ be a sequence of non-negative functions (if necessary, replace u_n by $|u_n|$) in H such that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + b(x)u_n^2 dx \rightarrow \lambda_1 \quad \text{as } n \rightarrow \infty, \quad (18)$$

and

$$\int_{\mathbb{R}^N} u_n^2 dx = 1. \quad (19)$$

From (i), (18) and (19) we obtain, for sufficiently large n ,

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \leq \lambda_1 + 1 + B. \quad (20)$$

It follows that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and converges (up to a subsequence) to a function u weakly in $H^1(\mathbb{R}^N)$, almost everywhere in \mathbb{R}^N (which implies $u \geq 0$) and strongly in $L^2(B_R)$, for all $R > 0$.

We take a function $\varphi \in C^\infty(\mathbb{R}^N)$ such that $\varphi \equiv 0$ on B_R and $\varphi \equiv 1$ on $\mathbb{R}^N \setminus B_{R+1}$. We obtain

$$\begin{aligned} \|u_n - u\|_{L^2(\mathbb{R}^N)}^2 &\leq \|(1 - \varphi)(u_n - u)\|_{L^2(\mathbb{R}^N)}^2 + \|\varphi(u_n - u)\|_{L^2(\mathbb{R}^N)}^2 \\ &\leq \beta_n + \frac{1}{\nu_2(\mathbb{R}^N \setminus B_R)} \int_{\mathbb{R}^N} |\nabla(\varphi(u_n - u))|^2 \\ &\quad + b(x)(u_n - u)^2 dx \\ &\leq \beta_n + \gamma_R, \end{aligned}$$

where β_n and γ_R are quantities which tend to zero as n and R tend to infinity. It follows that $u_n \rightarrow u$ strongly in $L^2(\mathbb{R}^N)$. Hence

$$\int_{\mathbb{R}^N} u^2 dx = 1. \quad (21)$$

Furthermore, the weak H^1 -convergence yields

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx. \quad (22)$$

Since $u_n \rightarrow u$ strongly in $L^2(\mathbb{R}^N)$ implies $b^{\frac{1}{2}}(x)u_n \rightharpoonup b^{\frac{1}{2}}(x)u$ weakly in $L^2(G)$, with $G = \{x \in \mathbb{R}^N \mid b(x) > 1\}$, we get

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} b(x)u_n^2 dx &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus G} b(x)u_n^2 dx + \liminf_{n \rightarrow \infty} \int_G b(x)u_n^2 dx \\ &\geq \int_{\mathbb{R}^N} b(x)u^2 dx, \end{aligned} \quad (23)$$

where we have used the fact that $b(x) \in L^\infty(\mathbb{R}^N \setminus G)$.

Finally, it follows from (21), (22) and (23) that $u \in H$ is a minimiser for the right-hand side of (9). Since $u \in H^1(\mathbb{R}^N)$, a standard bootstrap argument yields $u \in C^2(\mathbb{R}^N)$. The strong maximum principle implies $u > 0$ in \mathbb{R}^N , so u is a solution of (17). \square

3 The Palais-Smale Condition. Existence Results

We consider the functional

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + b(x)u^2 dx - \int_{\mathbb{R}^N} F(x, u) dx.$$

Of course, we set $F(x, u) = \frac{a(x)|u|^{p+1}}{p+1}$ in the case of (1).

Lemma 3.1 *Suppose (3), (5)–(9) hold. Then the functional Φ is well defined and C^1 on H . Furthermore, for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$\int_{\mathbb{R}^N} |F(x, u)| dx \leq \varepsilon \|u\|_H^2 + C_\varepsilon \|u\|_H^{p+1}. \quad (24)$$

Proof. First we note that (6) and (7) imply

$$|F(x, u)| \leq C_0 A(x) (\varepsilon |u|^2 + C_\varepsilon |u|^{p+1}). \quad (25)$$

By using (5) we get

$$\int_{\mathbb{R}^N} A(x)|u|^{p+1} dx \leq \left(C_0 + \max_{|x| \leq R_0} A(x) \right) \|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}$$

$$\begin{aligned}
& + C_0 \int_{\{x : b(x) > 0\}} b(x)^{\frac{1}{\alpha}} |u|^{p+1} dx \\
\leq & C \left(\|u\|_{H^1(\mathbb{R}^N)}^{p+1} + \left(\int_{\{x : b(x) > 0\}} b(x) u^2 dx \right)^{\frac{1}{\alpha}} \right. \\
& \left. \times \left(\int_{\{x : b(x) > 0\}} |u|^{\frac{\alpha(p+1)-2}{\alpha-1}} dx \right)^{\frac{\alpha-1}{\alpha}} \right) \\
\leq & C \left(\|u\|_H^{p+1} + \left(\|u\|_H^2 + B \int_{\{x : b(x) \leq 0\}} u^2 dx \right)^{\frac{1}{\alpha}} \right. \\
& \left. \times \|u\|_H^{\frac{\alpha(p+1)-2}{\alpha}} \right) \\
\leq & C \left(1 + \left(1 + \frac{B}{\lambda_1} \right)^{\frac{1}{\alpha}} \right) \|u\|_H^{p+1},
\end{aligned}$$

where we have used Lemma 2.1, together with the fact that assumption (3) implies $\frac{\alpha(p+1)-2}{\alpha-1} < \frac{2N}{N-2}$. We get, in a similar way,

$$\int_{\mathbb{R}^N} A(x) u^2 dx \leq C \|u\|_H^2,$$

which proves (24). It is standard to see that Φ is C^1 on H . \square

It is easy to check that critical points of Φ on H are solutions of (2). The proof of the existence of solutions of (1) and (2) then follows the standard "Mountain Pass" procedure.

It is the Palais–Smale condition that creates difficulties in studying this type of elliptic problems in unbounded domains. In order to establish its validity in our case, we prove that H is compactly embedded into some weighted Lebesgue spaces.

Let $L_{A(x)}^s(\mathbb{R}^N)$ denote the set of measurable functions u on \mathbb{R}^N such that

$$\int_{\mathbb{R}^N} A(x) |u|^s dx < \infty.$$

We take the s -th square of this quantity to be the norm on $L_{A(x)}^s(\mathbb{R}^N)$.

Proposition 3.1 *Suppose (5) and (9) hold. Then H is continuously embedded into $L_{A(x)}^s(\mathbb{R}^N)$, for $2 \leq s < p^\# + 1$. Furthermore, $(ii)_s$ is a necessary and sufficient condition for this embedding to be compact.*

Proof. The continuous embedding was established in the proof of Lemma 3.1.

Let us show that $(ii)_s$ is sufficient for the embedding $H \hookrightarrow L^s_{A(x)}(\mathbb{R}^N)$ to be compact. Suppose that $u_n \rightharpoonup 0$ weakly in H . By Lemma 2.1 we know that $u_n \rightarrow 0$ in $L^s(B_R)$, for all $2 \leq s < \frac{2N}{N-2}$, and all $R > 0$.

We obtain, as in the proof of Proposition 2.2,

$$\begin{aligned} \|u_n\|_{L^s(\mathbb{R}^N)} &\leq \|(1-\varphi)u_n\|_{L^s(\mathbb{R}^N)} + \|\varphi u_n\|_{L^s(\mathbb{R}^N)} \\ &\leq \beta_n + \gamma_R. \end{aligned}$$

By using $(ii)_s$ and Proposition 2.1 we see that β_n and γ_R tend to zero as n and R tend to infinity. This implies $u_n \rightarrow 0$ in $L^s(\mathbb{R}^N)$. Then, proceeding as in the proof of Lemma 3.1, we obtain for any $u \in H$

$$\|u\|_{L^s_{A(x)}(\mathbb{R}^N)} \leq C \left(\|u\|_{L^s(\mathbb{R}^N)} + \left(1 + \frac{B}{\lambda_1}\right)^{\frac{1}{\alpha}} \|u\|_{\frac{2}{H}}^{\frac{2}{\alpha}} \|u\|_{L^{\frac{\alpha s-2}{\alpha-1}}(\mathbb{R}^N)}^{\frac{\alpha s-2}{\alpha}} \right), \quad (26)$$

from which we infer that $u_n \rightarrow 0$ in $L^s_{A(x)}(\mathbb{R}^N)$, since $s \leq \frac{\alpha s-2}{\alpha-1} < \frac{2N}{N-2}$.

We turn to the necessity of $(ii)_s$. Assume $(ii)_s$ does not hold, that is, by Proposition 2.1, there exists a sequence $R_n \rightarrow \infty$ such that $\nu_s(\mathbb{R}^N \setminus \overline{B_{R_n}}) \leq C$. This implies the existence of functions $u_n \in H^1(\mathbb{R}^N)$ such that

$$\text{supp } u_n \subset \mathbb{R}^N \setminus B_{R_n}, \quad (27)$$

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + b(x)u_n^2 \, dx \leq C + 1, \quad (28)$$

and

$$\int_{\mathbb{R}^N} |u_n|^s \, dx = 1. \quad (29)$$

If the embedding $H \hookrightarrow L^s_{A(x)}(\mathbb{R}^N)$ were compact, from (28) we could infer the existence of a subsequence of $\{u_n\}_{n=1}^\infty$ such that $u_n \rightarrow u$ strongly in $L^s_{A(x)}(\mathbb{R}^N)$, for some function u . On the other hand (27), (28) and $R_n \rightarrow \infty$ imply $u_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$, that is $u \equiv 0$. This is a contradiction with (29), since $u_n \rightarrow 0$ in $L^s_{A(x)}(\mathbb{R}^N)$ clearly implies $u_n \rightarrow 0$ in $L^s(\mathbb{R}^N)$. \square

We now state our main result about the Palais-Smale condition.

Theorem 3.1 *Suppose that (3) and (5)–(9) hold. Then $(ii)_{p+1}$ is a sufficient condition for the functional Φ to satisfy the Palais-Smale condition on H . If, in addition,*

$$\inf_{\substack{x \in \mathbb{R}^N \\ |u| \geq R_0}} F(x, u) > 0, \quad (30)$$

for some $R_0 > 0$, then $(ii)_{p+1}$ is also necessary.

Remark. For the necessity of $(ii)_{p+1}$ we need to add a condition like (30). For instance, in the case of (1), if $a(x) \rightarrow 0$ as $|x| \rightarrow \infty$ then Φ satisfies the Palais-Smale condition even for $b(x) = \text{const}$ (when $(ii)_{p+1}$ clearly fails). However, if we do not suppose (30), it is still possible to produce necessary and sufficient conditions for (PS), which involve the ratio between a and b . We leave the details to the reader.

Proof of Theorem 3.1. Suppose $(ii)_{p+1}$ holds. It is standard to check that the following claim implies the Palais-Smale condition for Φ .

Claim 1 *The map*

$$\begin{cases} u & \rightarrow \int_{\mathbb{R}^N} F(x, u) dx \\ (H, \text{weak}) & \rightarrow \mathbb{R} \end{cases}$$

is continuous.

Proof of Claim 1. Let $u_n \rightharpoonup u$ weakly in H . Then $u_n \rightarrow u$ strongly in $L^{p+1}_{A(x)}(\mathbb{R}^N)$. By the inverse Lebesgue theorem, we can find a subsequence of $\{u_n\}_{n=1}^\infty$ and a function $h \in L^{p+1}_{A(x)}(\mathbb{R}^N)$ such that $|u_n(x)| \leq h(x)$ and u_n converges to u almost everywhere in \mathbb{R}^N .

Then, if we note

$$v_n = |F(x, u_n) - F(x, u)|,$$

it follows that $v_n \rightarrow 0$ almost everywhere in \mathbb{R}^N . By using (25) we see that we can find functions $w_n, f \in L^1(\mathbb{R}^N)$ such that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$, for which

$$\|w_n\|_{L^1(\mathbb{R}^N)} \leq C,$$

and

$$v_n \leq \varepsilon w_n + C_\varepsilon f.$$

Then it is readily checked that for every $\delta > 0$ there exists $R > 0$ for which

$$\|v_n\|_{L^1(\mathbb{R}^N \setminus B_R)} < \delta \quad \text{for all } n. \quad (31)$$

It is also easily seen that for all $\delta > 0$ we can find $r > 0$ such that for any $E \subset \mathbb{R}^N$, with $|E| < r$, we have

$$\|v_n\|_{L^1(E)} < \delta \quad \text{for all } n. \quad (32)$$

We finish the proof of Claim 1 by noticing that (31) and (32) are exactly what we need in order to apply Vitali's theorem, which yields $v_n \rightarrow 0$ strongly in $L^1(\mathbb{R}^N)$. \square

Let us prove the necessity of $(ii)_{p+1}$. Suppose that there exist a number $r > 0$ and a sequence $\{x_n\} \subset \mathbb{R}^N$ which goes to infinity, such that

$$\nu_{p+1}(B_n) \leq C. \quad (33)$$

By standard existence results for bounded domains we can find a function $u_n \in H_0^1(B_n)$, such that

$$\Phi(u_n) = c_n \quad \text{and} \quad \Phi'(u_n) = 0,$$

where $c_n > 0$. Note that we can take, as in [W],

$$c_n = \inf_{\gamma \in \Gamma_n} \max_{t \in [0,1]} \Phi(\gamma(t)),$$

where

$$\Gamma_n = \{ \gamma \in C([0, 1], H_0^1(B_n)) \mid \gamma(0) = 0, \Phi(t\gamma(1)) < 0 \text{ for } t \geq 1 \}.$$

It is clear that $u_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$ (we suppose that the functions in $H_0^1(B_n)$ are extended as zero outside the ball B_n).

We first show that the sequence of positive numbers $\{c_n\}$ is bounded from above. Note that

$$c_n \leq \inf_{u \in H_0^1(B_n) \setminus \{0\}} \max_{t \geq 0} \Phi(tu),$$

since every ray in $H_0^1(B_n)$ can be parametrised in such a way that a part of it belongs to Γ_n . Furthermore, (8), (9) and (30) imply that for all $u \in H_0^1(B_n)$

$$\Phi(u) \leq \Psi(u) := \frac{C_1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + b(x)u^2 dx - C_2 \int_{\mathbb{R}^N} |u|^\mu dx.$$

Hence

$$c_n \leq \inf_{u \in H_0^1(B_n) \setminus \{0\}} \max_{t \geq 0} \Psi(tu). \quad (34)$$

It is easy to compute that for any $u \in H_0^1(B_n) \setminus \{0\}$

$$\max_{t \geq 0} \Psi(tu) = \frac{C(\mu - 2)}{2\mu^{\frac{\mu}{\mu-2}}} \left(\frac{\int_{\mathbb{R}^N} |\nabla u|^2 + b(x)u^2 dx}{\left(\int_{\mathbb{R}^N} |u|^\mu dx \right)^{\frac{2}{\mu}}} \right)^{\frac{\mu}{\mu-2}},$$

so

$$\begin{aligned} c_n &\leq \text{const.} \nu_\mu(B_n)^{\frac{\mu}{\mu-2}} \\ &\leq C, \end{aligned}$$

by (33) and Lemma 2.2. This proves that $\{u_n\}$ is a Palais-Smale sequence.

It follows from (25), (26), Lemmas 2.1 and 3.1 that

$$\begin{aligned}\Phi(u) &\geq \frac{1}{4}\|u\|_{H(\mathbb{R}^N)}^2 - C\|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} - C\|u\|_{L^{q+1}(\mathbb{R}^N)}^{q+1} \\ &\geq C_0\|u\|_{H^1(\mathbb{R}^N)}^2 - C\|u\|_{L^{q+1}(\mathbb{R}^N)}^{q+1},\end{aligned}$$

for all $u \in H$, with $q = \frac{\alpha p - 1}{\alpha - 1}$. Hence $\{c_n\}$ is bounded from below by a positive constant. It is clear that this prevents $u_n \rightarrow 0$ strongly in H , which leads to a contradiction with the Palais-Smale condition.

Proofs of Theorem 1.1 and 1.2. Let $\lambda_1 > 0$. To see that this is a sufficient condition for the existence of a positive solution of (1) and (2), we use the mountain pass theorem of Ambrosetti and Rabinowitz. Its hypotheses are satisfied, since, by Lemma 3.1,

$$\Phi(u) \geq \frac{1}{4}\|u\|_H^2 - C\|u\|_H^{p+1} \quad \text{for } u \in H \quad (35)$$

and

$$\Phi(tu_0) \leq \frac{t^2}{2} - Ct^{\mu_0} \rightarrow -\infty \quad \text{as } t \rightarrow \infty,$$

where u_0 is a fixed function with compact support, such that $\|u_0\|_H = 1$ (for Theorem 1.2 we also suppose $\text{supp } u_0 \subset \{x \mid a(x) > 0\}$).

To see that the mountain pass procedure yields a positive solution, we replace the function $f(x, u)$ by

$$f^+(x, u) = \begin{cases} f(x, u) & \text{if } u > 0 \\ 0 & \text{if } u \leq 0, \end{cases}$$

and carry out the same argument.

We see that $\lambda_1 > 0$ is a necessary condition for the existence of a positive solution of (2) (and (1), if $a(x) \geq 0$) by multiplying both sides of this equation by φ_1 and by integrating. In the same way we show, when $\lambda_1 > 0$, that (1) can have a positive solution only if $a(x)$ takes a positive value.

Finally, it is standard to check, in the case of an odd f , that the Symmetric Mountain Pass theorem implies the existence of an unbounded sequence of critical values of Φ . This concludes the proof of Theorem 1.1 and Theorem 1.2. \square

Proof of Theorem 1.3. In the case of a bounded domain this theorem was proved in [AT]. To see that their proof carries over to our case, it suffices to notice that Proposition 3.1 standardly implies the following result.

Lemma 3.2 *The spectrum of the operator $-\Delta + b(x)$ on H consists of a sequence $\{\lambda_n\}$ of eigenvalues such that $\lambda_n \rightarrow \infty$ when $n \rightarrow \infty$.*

Then, following [AT], we can prove that there exist a subspace $X \subset H$ of finite codimension and a number $r > 0$ such that

$$\inf_{u \in X, \|u\|_H = r} F(u) > F(0).$$

It is not difficult to see, as in [AT], that for all $k \geq 1$ there exist subspaces Y_k of H such that $\dim Y_k \geq k$ and

$$\sup_{u \in Y_k} F(u) < \infty.$$

Indeed, it suffices to take $X = E_j^\perp$, where E_j denotes the subspace of H spanned by the proper vectors of $-\Delta + b(x)$ corresponding to $\lambda_1 \dots \lambda_j$. Fixing j so that $\lambda_{j+1} > 0$, we get (35) on X . As for Y_k , one can take $Y_k = \text{span}\{\phi_1 \dots \phi_k\}$, where $\phi_1 \dots \phi_k$ are arbitrary $C_c^\infty(\mathbb{R}^N)$ -functions with disjoint supports, contained in the set $a(x) > 0$ (see [AT] for details). We finish the proof with the help of a Benci's pseudoindex theorem (see [AT] and [Be]). \square

Remark 3.1 Theorem 1.4 and the remark immediately after it give handy conditions to check whether we can or cannot apply the existence theorems. We can get plenty of examples of functions that satisfy neither (r1) nor (bw2) but for which equation (2) is still solvable – for example $x_1^2 \dots x_N^2 - A$, with any constant $A > 0$ chosen so that $\lambda_1 > 0$. It is clear, for most “reasonable” potentials, that Theorem 1.4 and the remark following it should give an answer to the question whether Theorems 1-3 apply.

Remark 3.2 It is not difficult to see that the same results hold for a general strictly elliptic operator in divergence form

$$\sum_{i,j=1}^{\infty} \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial}{\partial x_i} \right) + b(x),$$

under suitable conditions on the coefficients $a_{ij}(x)$.

Remark 3.3 It is worth noting that in the case of a bounded domain there are some results concerning the existence of a positive solution of (1) when $\lambda_1 \leq 0$ (see [AT] and [BCN]). For example, when $\lambda_1 = 0$, necessary and sufficient conditions are

$$a(x) \text{ changes sign in } \Omega$$

and

$$\int_{\Omega} a(x)e_1^{p+1} dx < 0,$$

where e_1 denotes the principal eigenfunction of $-\Delta + b(x)$ on the domain Ω . It is not difficult to see, under (i) and (ii)₂, that these results carry over to \mathbb{R}^N . We leave the details to the reader.

4 The Pohozaev Identity does not Give a Full Answer to the Existence Question

We consider the following example

$$-\Delta u + |x|^a u = |x|^b |u|^{p-1} u, \quad (36)$$

where $N \geq 3, 0 \leq b < a$. We are searching for solutions $u \in H$ such that $u \in L_{|x|^b}^{p+1}(\mathbb{R}^N)$. Theorem 1.2 states that (36) has a non-trivial solution, provided $p \in (1, p^\#)$, where

$$p^\# = \frac{N+2}{N-2} - \frac{4b}{a(N-2)}.$$

On the other hand, the generalised Pohozaev identity, established by Pucci and Serrin ([PS]), gives

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{N+a}{2} \int_{\mathbb{R}^N} |x|^a u^2 dx - \frac{N+b}{p+1} \int_{\mathbb{R}^N} |x|^b u^{p+1} = 0, \quad (37)$$

for any solution u of (36). By multiplying (36) by u and by integrating we see that

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |x|^a u^2 dx - \int_{\mathbb{R}^N} |x|^b u^{p+1} = 0. \quad (38)$$

By putting together (37) and (38) we obtain

$$\left(\frac{N-2}{2} - \frac{N+b}{p+1} \right) \int_{\mathbb{R}^N} |\nabla u|^2 dx + \left(\frac{N+a}{2} - \frac{N+b}{p+1} \right) \int_{\mathbb{R}^N} |x|^a u^2 dx = 0.$$

We see that this implies $u \equiv 0$, provided $p \in [p^*, \infty)$, with

$$p^* = \frac{N+2}{N-2} + \frac{2b}{N-2}.$$

As we see, $p^\# < \frac{N+2}{N-2} < p^*$ when $b > 0$. In other words, an unbounded nonlinearity diminishes both the range of p 's where variational methods ensure the existence of a non-trivial solution and the range of p 's where the Pohozaev identity says no such solutions exist. We do not know if non-trivial solutions exist for $p \in [p^\#, p^*)$.

5 Proof of Theorem 1.5 and Theorem 1.6

The proofs of Theorems 1.5 and 1.6 will be carried out jointly.

As before, we consider the functional corresponding to (11)

$$\Phi_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda b_\lambda(x) u^2 dx - \int_{\mathbb{R}^N} F(x, u) dx.$$

We introduce the space

$$E_\lambda = \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} b_\lambda(x) u^2 dx < \infty \right\}$$

and the quantity

$$\|u\|_\lambda^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda b_\lambda(x) u^2 dx,$$

for $u \in E_\lambda$.

The following lemma summarizes some easy preliminary facts.

Lemma 5.1 *For every $\lambda \geq 1$,*

(a) *there exists $C_\lambda > 0$ such that*

$$\|u\|_{H^1(\mathbb{R}^N)} \leq C_\lambda \|u\|_\lambda.$$

This implies that $(E_\lambda, \|\cdot\|_\lambda)$ is a Hilbert space, continuously embedded into $H^1(\mathbb{R}^N)$.

(b) *the functional Φ_λ is well-defined and C^1 on E_λ .*

(c) *the functional Φ_λ has a “mountain-pass” geometry on E_λ .*

(d) *there exists a number $c_\lambda > 0$ and a sequence $\{u_n^\lambda\}_{n=1}^\infty \subset E_\lambda$ such that*

$$\begin{aligned} \Phi_\lambda(u_n^\lambda) &\rightarrow c_\lambda \quad \text{as } n \rightarrow \infty \\ \Phi'_\lambda(u_n^\lambda) &\rightarrow 0 \quad \text{in } E'_\lambda \quad \text{as } n \rightarrow \infty. \end{aligned}$$

(e) *we have*

$$\limsup_{n \rightarrow \infty} \|u_n^\lambda\|_\lambda^2 \leq \rho c_\lambda,$$

where $\rho > 0$ is a constant independent of λ .

(f) *there exists a solution u_0^λ of (11) such that a subsequence of $\{u_n^\lambda\}$ converges to u_0^λ weakly in E_λ .*

Proof. We have (we set $G^c = \mathbb{R}^N \setminus G$, for any $G \subset \mathbb{R}^N$)

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u|^2 + b_\lambda(x)u^2 dx &\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2} \int_{\Omega_{M,\lambda}} |\nabla u|^2 + \int_{\Omega_{M,\lambda}^c} b_\lambda(x)u^2 \\ &\geq \min \left\{ \frac{1}{2}, \frac{C_{M,\lambda}}{2}, M \right\} \|u\|_{H^1(\mathbb{R}^N)}^2, \end{aligned}$$

where $C_{M,\lambda}$ is the constant in the Poincaré's inequality for the set $\Omega_{M,\lambda}$.

As in the proof of Lemma 3.1, we can show that

$$\int_{\mathbb{R}^N} A(x)|u|^s dx \leq C_\lambda \|u\|_\lambda^s,$$

for some $C_\lambda > 0$ and all $u \in E_\lambda$, $2 \leq s \leq p+1$. Statement (b) follows standardly.

To prove (c) it suffices to take a sufficiently small $\varepsilon = \varepsilon(\lambda)$ in (25). Statements (d), (e) and (f) are also trivial. \square

Our goal is to show that $u_0^\lambda \not\equiv 0$, for sufficiently large λ . Let us suppose the contrary, that is, there exists a sequence $\{\lambda_m\}_{m=1}^\infty$, with $\lambda_m \rightarrow \infty$, such that $u_0^{\lambda_m} \equiv 0$. Then we obtain (dropping the subscript m)

$$\begin{aligned} c_\lambda &= \lim_{n \rightarrow \infty} \left(\Phi_\lambda(u_n^\lambda) - \frac{1}{2} \langle \Phi'_\lambda(u_n^\lambda), u_n^\lambda \rangle \right) \\ &\leq \frac{1}{2} \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} u_n^\lambda f(x, u_n^\lambda) dx \\ &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \varepsilon A(x) |u_n^\lambda|^2 + C_\varepsilon A(x) |u_n^\lambda|^{p+1} dx \\ &\leq \liminf_{n \rightarrow \infty} \int_{B_R} + \limsup_{n \rightarrow \infty} \int_{B_R^c} \\ &\leq \varepsilon \limsup_{n \rightarrow \infty} \|u_n^\lambda\|_{L^2_{A(x)}(B_R^c)}^2 + C_\varepsilon \limsup_{n \rightarrow \infty} \|u_n^\lambda\|_{L^{p+1}_{A(x)}(B_R^c)}^{p+1}, \end{aligned} \quad (39)$$

where ε and R are arbitrary positive numbers.

We now need an estimate for the L^2 -norm of u_n^λ outside “large” balls.

Lemma 5.2 *For every $\lambda \geq 1$ there exists $R(\lambda) > 0$ such that*

$$\limsup_{n \rightarrow \infty} \|u_n^\lambda\|_{L^2(B_{R(\lambda)}^c)}^2 \leq \bar{\rho} \frac{C_\lambda}{\lambda},$$

where $\bar{\rho}$ is a constant independent of λ .

Proof. By Lemma 5.1 (e) we obtain for all $R > 0$

$$\begin{aligned}
\rho c_\lambda + o(1) &\geq \int_{\mathbb{R}^N} |\nabla u_n^\lambda|^2 + \lambda b_\lambda(x) |u_n^\lambda|^2 \\
&\geq \int_{\mathbb{R}^N} |\nabla u_n^\lambda|^2 + \lambda \int_{\Omega_{M,\lambda}} b_\lambda(x) |u_n^\lambda|^2 + \lambda M \int_{\Omega_{M,\lambda}^c} |u_n^\lambda|^2 \\
&= \int_{\mathbb{R}^N} |\nabla u_n^\lambda|^2 + \lambda M \int_{\mathbb{R}^N} |u_n^\lambda|^2 - \lambda \int_{\Omega_{M,\lambda}} (M - b_\lambda(x)) |u_n^\lambda|^2 \\
&\geq \int_{\mathbb{R}^N} |\nabla u_n^\lambda|^2 + \lambda M \int_{\mathbb{R}^N} |u_n^\lambda|^2 - \lambda M \int_{\Omega_{M,\lambda} \cap B_R} |u_n^\lambda|^2 \\
&\quad - \lambda M \int_{\Omega_{M,\lambda} \setminus B_R} |u_n^\lambda|^2 \\
&\geq \int_{\mathbb{R}^N} |\nabla u_n^\lambda|^2 + \lambda M \int_{B_R^c} |u_n^\lambda|^2 - \lambda M \int_{\Omega_{M,\lambda} \setminus B_R} |u_n^\lambda|^2.
\end{aligned}$$

Lemma 2.3 permits us to choose $R = R(\lambda)$ such that

$$\int_{\Omega_{M,\lambda} \setminus B_R} |u_n^\lambda|^2 \leq \frac{1}{2\lambda M} \int_{\mathbb{R}^N} |\nabla u_n^\lambda|^2.$$

Lemma 5.2 follows. \square

We set $G = \mathbb{R}^N \setminus B_{R(\lambda)}$ and fix $s \in \{2, p+1\}$. With the help of the Gagliardo-Nirenberg inequality we see that for all $u \in E_\lambda$

$$\begin{aligned}
\int_G A(x) |u|^s dx &\leq C \|u\|_{L^s(G)}^s + \int_G b_\lambda(x)^{\frac{1}{\alpha}} |u|^s dx \\
&\leq C \|\nabla u\|_{L^2(G)}^{\theta s} \|u\|_{L^2(G)}^{(1-\theta)s} + C \left(\int_G b_\lambda(x) u^2 dx \right)^{\frac{1}{\alpha}} \\
&\quad \times \left(\int_G |u|^{\frac{\alpha s - 2}{\alpha - 1}} dx \right)^{\frac{\alpha - 1}{\alpha}} \\
&\leq C \|\nabla u\|_{L^2(G)}^{\theta s} \|u\|_{L^2(G)}^{(1-\theta)s} + C \left(\int_G b_\lambda(x) u^2 dx \right)^{\frac{1}{\alpha}} \\
&\quad \times \|\nabla u\|_{L^2(G)}^{\frac{\bar{\theta} \alpha s - 2}{\alpha}} \|u\|_{L^2(G)}^{\frac{(1-\bar{\theta}) \alpha s - 2}{\alpha}}, \tag{40}
\end{aligned}$$

where $\theta = \frac{N(s-2)}{2s}$ and $\bar{\theta} = \frac{N\alpha(s-2)}{2(\alpha s - 2)}$. It follows from Lemma 5.1 (e) that

$$\limsup_{n \rightarrow \infty} \|\nabla u_n^\lambda\|_{L^2(\mathbb{R}^N)}^2 \leq \rho c_\lambda \quad \text{and} \quad \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} b_\lambda(x) |u_n^\lambda|^2 dx \leq \rho \frac{c_\lambda}{\lambda}.$$

We obtain, by (40) and Lemma 5.2,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_G A(x) |u_n^\lambda|^s dx &\leq C c_\lambda^{\frac{\theta s}{2}} \left(\frac{c_\lambda}{\lambda} \right)^{\frac{(1-\theta)s}{2}} \\ &\quad + C \left(\frac{c_\lambda}{\lambda} \right)^{\frac{1}{\alpha}} c_\lambda^{\frac{\bar{\theta} \alpha s - 2}{2\alpha}} \left(\frac{c_\lambda}{\lambda} \right)^{\frac{(1-\bar{\theta})(\alpha s - 2)}{2\alpha}} \\ &\leq C \lambda^{-\frac{(1-\theta)s}{2}} c_\lambda^{\frac{s}{2}}. \end{aligned}$$

By putting $R = R(\lambda)$ and by taking ε sufficiently small in (39) we get

$$c_{\lambda_m} \leq C \lambda_m^{\frac{N(p-1)-2(p+1)}{4}} c_{\lambda_m}^{\frac{p+1}{2}}.$$

Hence

$$\limsup_{\lambda \rightarrow \infty} \lambda^{\frac{N}{2} - \frac{p+1}{p-1}} c_\lambda > 0.$$

The following lemma provides a contradiction, both for Theorem 1.5 and Theorem 1.6. This is where we use assumptions (bl3) and (bl4).

Lemma 5.3 *Suppose the assumptions of either Theorem 1.5 or Theorem 1.6 hold. Then*

$$\lim_{\lambda \rightarrow \infty} \lambda^{\frac{N}{2} - \frac{p+1}{p-1}} c_\lambda = 0.$$

Proof. We develop an idea which already appeared in an earlier work by the author (see [S]). As in the proof of Theorem 3.1,

$$c_\lambda \leq \inf_{u \in E_\lambda \setminus \{0\}} \max_{t \geq 0} \Phi_\lambda(tu)$$

and

$$\Phi_\lambda(u) \leq \Psi_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda b_\lambda(x) u^2 dx - \int_{\mathbb{R}^N} d(x) |u|^\mu dx.$$

We obtain

$$\max_{t \geq 0} \Psi_\lambda(tu) = \frac{\mu - 2}{2\mu^{\frac{\mu}{\mu-2}}} \left(\frac{\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda b_\lambda(x) u^2 dx}{\left(\int_{\mathbb{R}^N} d(x) |u|^\mu dx \right)^{\frac{2}{\mu}}} \right)^{\frac{\mu}{\mu-2}}. \quad (41)$$

Set

$$g_\mu(\lambda) = \inf_{u \in E_\lambda \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda b_\lambda(x) u^2 dx}{\left(\int_{\mathbb{R}^N} d(x) |u|^\mu dx \right)^{\frac{2}{\mu}}},$$

so that

$$c_\lambda \leq C (g_\mu(\lambda))^{\frac{\mu}{\mu-2}}. \quad (42)$$

By making the change of variables $y = \lambda^{\frac{1}{2}}x$ we see that

$$\begin{aligned} g_\mu(\lambda) &= \lambda^{1-\frac{N}{2}+\frac{N}{\mu}} \inf_{u \in E_\lambda \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + b_\lambda(\lambda^{-\frac{1}{2}}y)u^2 dy}{\left(\int_{\mathbb{R}^N} d(\lambda^{-\frac{1}{2}}y)|u|^\mu dy\right)^{\frac{2}{\mu}}} \\ &=: \lambda^{1-\frac{N(\mu-2)}{2\mu}} f_\mu(\lambda). \end{aligned} \quad (43)$$

First we consider the case $\mu = p + 1$ (Theorem 1.5). From (42) and (43) we obtain

$$\begin{aligned} c_\lambda &\leq C (g_{p+1}(\lambda))^{\frac{p+1}{p-1}} \\ &\leq C \lambda^{\frac{p+1}{p-1}-\frac{N}{2}} (f_{p+1}(\lambda))^{\frac{p+1}{p-1}}. \end{aligned}$$

The statement of Lemma 5.3 follows from this, provided we show that

$$\lim_{\lambda \rightarrow \infty} f_{p+1}(\lambda) = 0. \quad (44)$$

Proof of (44). This was proved in [S], so we only outline the proof. We shall prove (44) for any f_μ , with $\mu \in (2, p^\# + 1)$.

Note that

$$f_\mu(\lambda) = \inf_{u \in \mathcal{N}_\lambda} \int_{\mathbb{R}^N} |\nabla u|^2 + b_\lambda(\lambda^{-\frac{1}{2}}y)u^2 dy,$$

where $\mathcal{N}_\lambda = \left\{ u \in E_\lambda \mid \int_{\mathbb{R}^N} d(\lambda^{-\frac{1}{2}}y)|u|^\mu dy = 1 \right\}$.

Let us assume, for some sequence $\lambda_m \rightarrow \infty$, that $f_\mu(\lambda_m) \geq a_0 > 0$. In what follows we drop the subscript m . We take a sequence $\{u_n\}_{n=1}^\infty$ in $C_c^\infty(\mathbb{R}^N)$ such that

$$\lim_{n \rightarrow \infty} \|\nabla u_n\|_{L^2(\mathbb{R}^N)} = 0 \quad \text{and} \quad \|u_n\|_{L^\mu(\mathbb{R}^N)} = 1.$$

Set

$$v_{n,\lambda} = \frac{u_n}{\left(\int_{\mathbb{R}^N} d(\lambda^{-\frac{1}{2}}y)|u_n|^\mu dy\right)^{\frac{1}{\mu}}},$$

so that $v_{n,\lambda} \in \mathcal{N}_\lambda$ for all n and all λ .

Next we note that for every n there exists $\lambda_n > 0$ such that for $\lambda > \lambda_n$

$$\int_{\mathbb{R}^N} d(\lambda^{-\frac{1}{2}}y)|u_n|^\mu dy > \frac{d(0)}{2}.$$

Hence we can find n_0 such that

$$\int_{\mathbb{R}^N} |\nabla v_{n_0, \lambda}|^2 dy < \frac{a_0}{2},$$

for $\lambda > \lambda_{n_0}$. However

$$\int_{\mathbb{R}^N} |\nabla v_{n_0, \lambda}|^2 + b_\lambda(\lambda^{-\frac{1}{2}}y)|v_{n_0, \lambda}|^2 dy \geq a_0,$$

so hypothesis (bl3) leads to a contradiction, for λ sufficiently large and greater than λ_{n_0} . \square

Finally, suppose that (bl4) holds. Then we have, by setting $\delta = \frac{\beta}{2(\beta+2)}$ and by making use of the change of variables $z = \lambda^{-\delta}y$,

$$\begin{aligned} f_\mu(\lambda) &\leq \inf_{u \in H_0^1(B_{\varepsilon_0 \lambda^\delta}) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + b_\lambda(\lambda^{-\frac{1}{2}}y)u^2 dy}{\left(\int_{\mathbb{R}^N} d(\lambda^{-\frac{1}{2}}y)|u|^\mu dy\right)^{\frac{2}{\mu}}} \\ &\leq C \inf_{u \in H_0^1(B_{\varepsilon_0 \lambda^\delta}) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + \lambda^{-\frac{\beta}{2}}|y|^\beta u^2 + \lambda^{-\frac{\beta}{\beta+2}}u^2 dy}{\left(\int_{\mathbb{R}^N} d(x)|u|^\mu dy\right)^{\frac{2}{\mu}}} \\ &\leq C \lambda^{-2\delta(1-\frac{N}{2}+\frac{N}{\mu})} \inf_{u \in H_0^1(B_{\varepsilon_0}) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + (1+\varepsilon_0^\beta)u^2 dz}{C \left(\int_{\mathbb{R}^N} |u|^\mu dz\right)^{\frac{2}{\mu}}} \\ &= \text{const.} \lambda^{-2\delta(1-\frac{N}{2}+\frac{N}{\mu})}. \end{aligned}$$

This implies that

$$g_\mu(\lambda) \leq C \lambda^{\frac{2}{\beta+2}(1-\frac{N(\mu-2)}{2\mu})}.$$

Hence, by (42),

$$c_\lambda \lambda^{\frac{N}{2}-\frac{p+1}{p-1}} \leq C \lambda^{\frac{2}{\beta+2}(\frac{\mu}{\mu-2}-\frac{N}{2})-(\frac{p+1}{p-1}-\frac{N}{2})}.$$

Lemma 5.3 follows, because of (bl4). \square

The proof of Theorems 1.5 and 1.6 is complete.

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