

# ON THE EXISTENCE OF SOLUTIONS OF HAMILTONIAN ELLIPTIC SYSTEMS IN $\mathbb{R}^N$

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## 1 Introduction and Main Results

In this paper we study the following elliptic system of Hamiltonian type

$$\begin{cases} -\Delta u + b(x)u &= g(x, v) \\ -\Delta v + b(x)v &= f(x, u), \end{cases} \quad x \in \mathbb{R}^N, \quad (1)$$

where  $N \geq 3$  and  $f, g : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $b : \mathbb{R}^N \rightarrow \mathbb{R}$  are given continuous functions. We suppose that  $f$  and  $g$  are subcritical (see below).

Systems of this kind in a bounded domain, even with more general nonlinearities, were studied in a number of papers (see [CFM], [CM], [FF], [HV] and the references in these works ; supercritical systems were treated in [SZ], see the remarks at the end of this section). The case of the whole space was first considered in a recent paper by de Figueiredo and Yang [FY]. They studied the system

$$\begin{cases} -\Delta u + u &= g(x, v) \\ -\Delta v + v &= f(x, u), \end{cases} \quad x \in \mathbb{R}^N, \quad (2)$$

proving under some conditions the existence of a strong radial solution (Theorem 5.1 in [FY]). They also showed that strong solutions of (2) decay at infinity, together with their derivatives (Theorem 2.1 in [FY] ; see below for a definition of a strong solution).

Studying Theorems 2.1 and 5.1 in [FY], it appeared to us that they require a number of hypotheses which seem too strong, when compared to the well-known conditions under which existence and regularity of solutions of a scalar equation have been proved. One of the goals of this paper is to extend the results of de Figueiredo and Yang (Theorems 1 and 2 below), showing that these results hold under the hypotheses which represent the natural counterparts for (1) of the classical “mountain-pass” conditions, introduced in [AR] for a scalar equation.

We shall use the following assumptions.

(i) there exist positive constants  $C$  and  $\delta$ , such that

$$|f(x, t)| \leq C|t| \quad \text{and} \quad |g(x, t)| \leq C|t| \quad \text{for all } x \in \mathbb{R}^N \text{ and } |t| \leq \delta ;$$

(ii) for all  $x \in \mathbb{R}^N$

$$\begin{aligned} f(x, 0) &= g(x, 0) = 0, \\ \lim_{t \rightarrow 0} \frac{f(x, t)}{t} &= \lim_{t \rightarrow 0} \frac{g(x, t)}{t} = 0 ; \end{aligned}$$

(iii) there exist positive constants  $C, p$  and  $q$ , such that for all  $x \in \mathbb{R}^N$  and  $t \in \mathbb{R}$

$$\begin{aligned} |f(x, t)| &\leq C(1 + |t|^p), \\ |g(x, t)| &\leq C(1 + |t|^q). \end{aligned} \tag{3}$$

We suppose that  $p$  and  $q$  are both greater than one and satisfy the inequality

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N} ; \tag{4}$$

(iv) there exists a number  $\mu > 2$ , such that

$$\begin{aligned} tf(x, t) &\geq \mu F(x, t) > 0, \\ tg(x, t) &\geq \mu G(x, t) > 0, \end{aligned} \tag{5}$$

for all  $x \in \mathbb{R}^N$  and  $t \in \mathbb{R} \setminus \{0\}$ ; here, as usual,  $F(x, t) = \int_0^t f(x, s) ds$  and  $G(x, t) = \int_0^t g(x, s) ds$ .

In the literature, the following terminology is common. A pair  $(u, v)$  is said to be a strong solution of (2), provided it satisfies (2) almost everywhere and  $u \in W^{2, \frac{p+1}{p}}$ ,  $v \in W^{2, \frac{q+1}{q}}$  (all function spaces are understood to be taken over  $\mathbb{R}^N$ ). A pair  $(u, v)$  is called a weak solution of (2) if it satisfies the weak formulation of (2) and  $(u, v) \in H^s \times H^t$ , where  $s$  and  $t$  are suitably chosen positive numbers (see Section 3.1 for a precise definition). In particular,  $s$  and  $t$  are chosen so that  $H^s$  is embedded into  $L^{p+1}$  and  $H^t$  is embedded into  $L^{q+1}$ .

We say a function is radial (in  $x$ ), provided it depends only on the distance to a fixed point in  $\mathbb{R}^N$ .

In our first theorem we establish some regularity and decay properties of the solutions of (2).

**Theorem 1** (a) *Suppose (i) and (iii) hold. Then strong solutions of (2) belong to  $W^{2,s}(\mathbb{R}^N)^2$ , for all  $2 \leq s < \infty$ . In particular, the functions  $u$  and  $v$  decay at infinity, together with their derivatives.*

(b) *Suppose (i) and (iii) hold. Then weak solutions of (2) belong to  $W_{loc}^{2,s}(\mathbb{R}^N)^2$ , for all  $2 \leq s < \infty$ .*

(c) *Suppose (ii) and (iii) hold. Then weak radial solutions of (2) belong to  $C^2(\mathbb{R}^N)$ . Furthermore, together with their derivatives of first and second order, they converge exponentially to zero at infinity.*

The inequality (4) describes the region in the plane where the couple  $(p, q)$  should be situated, so that variational techniques can be used for solving (1). This region is bounded from above by the so-called “critical” hyperbola, a notion first introduced in [CFM] and [HV]. It expresses the fact that one of the numbers  $p$  and  $q$  can be greater than the critical exponent  $\frac{N+2}{N-2}$ , provided the other is small enough to compensate (note that, when  $p = q$ , (4) reduces to  $p < \frac{N+2}{N-2}$ ). It is natural to expect that in Theorem 2.1 of [FY] the condition that both  $p$  and  $q$  be smaller than  $\frac{N+2}{N-2}$  can be avoided, as was conjectured by de Figueiredo and Yang. This is the content of our Theorem 1 (a), where the condition (H1) of [FY] is also removed. We use a bootstrap technique, which gives a simple proof and enables us to show that strong solutions belong to  $W^{2,s}(\mathbb{R}^N)$  for arbitrarily large  $s$  – a result which was not proved in [FY].

Theorem 1 (b) is actually the regularity result in [HV]. We give a simple bootstrap proof of this result.

Theorem 1 (c), an extension of Theorem 2.3 in [FY], is inspired by the famous result by Berestycki and Lions ([BL]) on scalar equations.

In view of well known results for scalar equations one may expect that hypotheses (H3) and (H4) in Theorem 5.1 of [FY] are too strong and that the hypothesis relating the parameters  $p, q, \alpha, \beta$  can be removed. In our next theorem the existence of a solution of (2) is proved only under the standard “mountain pass” assumptions on the nonlinearities. Theorem 2 relies on a critical point theorem for strongly indefinite functionals, due to Li and Willem ([LW]), and used in [FY].

**Theorem 2** *Suppose (ii), (iii) and (iv) hold. If  $f$  and  $g$  are radial in  $x$ , then there exists a non-trivial weak radial solution of (2).*

It should be stressed that, even though Theorem 1 and Theorem 2 are extensions of the results of de Figueiredo and Yang, our proofs are different from theirs.

Next we discuss the existence of a ground state of (2), that is, a solution which minimizes the energy functional over the set of all nontrivial weak solutions (see Section 3.2). We have the following result.

**Theorem 3** *Under the conditions of Theorem 2*

(a) *if all weak solutions of the system (2) are radial, then this system has a ground state ;*

(b) *the system*

$$\begin{cases} -\Delta u + u = g(v) \\ -\Delta v + v = f(u) \end{cases} \quad (6)$$

*has a ground state, provided there exist positive constants  $c$  and  $C$  such that, for all  $t$  in the range of  $u$  and  $v$ ,*

$$\begin{cases} c|t|^{p+1} \leq tf(t) \leq C|t|^{p+1}, \\ c|t|^{q+1} \leq tg(t) \leq C|t|^{q+1}, \end{cases} \quad (7)$$

*with  $p$  and  $q$  satisfying (4).*

In [FY] de Figueiredo and Yang showed that the system

$$\begin{cases} -\Delta u + u = (v^+)^q \\ -\Delta v + v = (u^+)^p \end{cases} \quad x \in \mathbb{R}^N \quad (8)$$

has a ground state for  $p$  and  $q$  both smaller than  $\frac{N+2}{N-2}$ . Their result depends on the particular form of the nonlinearities in (8). De Figueiredo and Yang used a restrictive symmetry theorem of Gidas-Ni-Nirenberg type, adapted to (8).

In Theorem 3 (a) we show that, any time we are able to prove that all solutions of (2) are radial, we can get as a consequence the existence of a ground state of the whole system (2). Recently a general symmetry result for decaying positive solutions of cooperative elliptic systems was established in [BS]. Together with Theorem 3 (a) it yields the existence of a ground state for a large class of systems (2), including (8). Let us note that the system (2) is cooperative when  $f$  and  $g$  are non-decreasing in  $u$  and  $v$  respectively.

Theorem 3 (b) is a result independent of the symmetry of solutions. We permit non-homogeneous nonlinearities and allow  $p$  and  $q$  to be in the whole region under the critical hyperbola (4).

It is possible to produce conditions on the more general system (1), under which it is solvable. It is known for single equations that compactness is regained when  $b(x) \rightarrow \infty$  or  $|f(x, t)| \leq h(x)|u|^p$ , with  $h(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . The case when  $b(x) \rightarrow \infty$  and both  $p$  and  $q$  are smaller than  $\frac{N+2}{N-2}$  has already been considered by Ding and Li ([DL]). It is not difficult to see that the proof of Theorem 2 can be adapted to the system (1), under the above ‘‘compactness’’ conditions on  $b$  and  $f$ . We can even show that, when  $b(x)$

is unbounded, we can admit nonlinearities which are unbounded in  $x$ . This may require a strengthening of the condition (4). For a single equation unbounded nonlinearities were recently considered in [S].

**Theorem 4** *Suppose (ii) and (iv) hold,*

$$\inf_{x \in \mathbb{R}^N} b(x) > 0, \quad (9)$$

$$|f(x, t)| \leq h_1(x)|u|^p \quad \text{and} \quad |g(x, t)| \leq h_2(x)|v|^q, \quad (10)$$

where  $h_1$  and  $h_2$  are continuous functions, with  $p, q$  satisfying (4).

(a) *If  $h_i(x) \rightarrow 0$  as  $x \rightarrow \infty$ , then (1) has a non-trivial (weak) solution.*

(b) *If*

$$b(x) \rightarrow \infty \quad \text{as} \quad |x| \rightarrow \infty \quad (11)$$

*and there exist numbers  $1 < \alpha \leq \infty$  and  $R_0 > 0$ , such that*

$$h_i(x) \leq Cb(x)^{\frac{1}{\alpha}} \quad \text{for} \quad |x| \geq R_0, \quad i = 1, 2, \quad (12)$$

*and*

$$\frac{\alpha - 1}{\alpha(p + 1) - 2} + \frac{\alpha - 1}{\alpha(q + 1) - 2} > 1 - \frac{2}{N}, \quad (13)$$

*then (1) has a non-trivial solution.*

The condition (13) is in general stronger than (4). It reduces to (4) when (12) holds for all  $\alpha > 1$  with  $R_0 = R_0(\alpha)$ . The simplest case in which (13) is equivalent to (4) is when  $h_1$  and  $h_2$  are bounded.

The hypotheses (10), (11) and (12) can be weakened as in [S]. We shall omit the details here.

We finish this section with several remarks concerning supercritical systems. In [SZ] Serrin and Zou proved the existence of positive radial solutions of the system

$$\begin{cases} -\Delta u = H_v(u, v) \\ -\Delta v = H_u(u, v) \end{cases} \quad (14)$$

in the supercritical case, for example, if  $H_v(u, v) = v^q$  and  $H_u(u, v) = u^p$ , where the numbers  $p$  and  $q$  are required to satisfy

$$\frac{1}{p + 1} + \frac{1}{q + 1} \leq 1 - \frac{2}{N}.$$

System (14) has a more general Hamiltonian structure than (1). In this situation conditions (i) and (ii) are not needed. Furthermore, Serrin and Zou

obtained their existence result under a weaker (compared to (5)) hypothesis on the growth of  $H_u$  and  $H_v$ .

A basic hypothesis in [SZ] is that the derivatives of  $H$  are non-negative – a case in which (14) turns out to have solutions for supercritical  $p$  and  $q$ . In contrast, in our work, these derivatives are *negative* for small values of  $u$  and  $v$ , and we obtain solutions for  $p$  and  $q$  in the subcritical range. The remarkable difference between these two cases has long been known for a single equation.

## 2 Proof of Theorem 1

First of all we remark that the theorem is trivial if  $p \leq \frac{2}{N-2}$  (or  $q \leq \frac{2}{N-2}$ ), since then  $\frac{p+1}{p} \geq \frac{N}{2}$  and by the Sobolev embedding theorem  $W^{2, \frac{p+1}{p}}$  embeds into  $L^s$ , for all  $s \in \left[ \frac{p+1}{p}, \infty \right)$ . Then standard elliptic regularity theory (see [GT]) applied to (2) gives the conclusion of Theorem 1. Therefore we may suppose that  $p$  and  $q$  are both strictly larger than  $\frac{2}{N-2}$ . Another way of writing hypothesis (4) is  $p \in (1, p^0)$ , with

$$p^0 = \frac{2q + N + 2}{(N-2)q - 2}$$

and, respectively,  $q \in (1, q^0)$ , with  $q^0 = \frac{2p + N + 2}{(N-2)p - 2}$ .

Since  $(u, v)$  is a strong solution, the Sobolev embedding theorem implies that  $u \in L^s$  for all  $s \in \left[ \frac{p+1}{p}, p_1 \right]$ , where

$$p_1 = \frac{N(p+1)}{Np - 2(p+1)},$$

and, respectively,  $v \in L^s$  for all  $s \in \left[ \frac{q+1}{q}, q_1 \right]$ ,  $q_1 = \frac{N(q+1)}{Nq - 2(q+1)}$ .

Next, we observe that either  $p_1 > p+1$  or  $q_1 > q+1$ . Indeed,  $p_1 > p+1$  is equivalent to  $p < \frac{N+2}{N-2}$ , whereas (4) prevents  $p$  and  $q$  from being both greater or equal to  $\frac{N+2}{N-2}$ .

Let for example  $q_1 > q+1$ . Then, since (i) and (iii) imply

$$|f(x, t)| \leq C(|t| + |t|^p) \quad \text{and} \quad |g(x, t)| \leq C(|t| + |t|^q), \quad (15)$$

we have  $g(x, v) \in L^{\frac{q_1}{q}}$ . Then standard elliptic regularity theory implies that  $u \in W^{2, \frac{q_1}{q}}$ . By the Sobolev embedding theorem  $u \in L^{p_2}$ , with

$$p_2 = \frac{Nq_1}{Nq - 2q_1}.$$

We are going to set up an iteration (bootstrap) process which will finally give Theorem 1. We remark that if at any stage of this process a denominator should be non-positive, then either  $u$  or  $v$  belongs to  $L^s$  for arbitrarily large  $s$ , and we can then conclude the proof with the help of elliptic regularity theory.

We have  $p_2 > p + 1$ . Indeed, this is equivalent to

$$p < \frac{(N+2)q_1 - Nq}{Nq - 2q_1} =: P,$$

and will hold if  $p^0 < P$ . After a computation, this last inequality turns out to be equivalent to  $q_1 > q + 1$ . Then  $f(x, u) \in L^{\frac{p_2}{p}}$  and again by standard elliptic theory  $v \in W^{2, \frac{p_2}{p}}$ , a space which is embedded into  $L^{q_2}$ , with

$$q_2 = \frac{Np_2}{Np - 2p_2}.$$

Next we claim that  $q_2 > q_1$ . This is equivalent to

$$q_1 > \frac{N(pq - 1)}{2p + 2} = h_q(p).$$

Then, considering the function  $h_q(p)$ , we see that it is increasing on  $[1, p^0]$ , so  $h_q(p)$  achieves its maximum at  $p^0$ . It is easy to compute that  $h_q(p^0) = q + 1$ , which proves the claim, since we have supposed  $q_1 > q + 1$ .

Now we can carry out our bootstrap argument. We have  $g(x, v) \in L^{\frac{q_2}{q}}$  and therefore  $u \in L^{p_3}$ , with

$$p_3 = \frac{Nq_2}{Nq - 2q_2}.$$

Here  $p_3 > p_2$  is equivalent to  $q_2 > q_1$ . Then  $v \in L^{q_3}$ , with  $q_3 > q_2$  equivalent to  $p_3 > p_2$ .

In such a way we construct two sequences  $\{p_n\}_{n=1}^{\infty}$  and  $\{q_n\}_{n=1}^{\infty}$ , such that

$$\begin{aligned} p_{n+1} &= \frac{Nq_n}{Nq - 2q_n} & \text{and} & \quad p_{n+1} > p_n \text{ for } n \geq 2, \\ q_{n+1} &= \frac{Np_{n+1}}{Np - 2p_{n+1}} & \text{and} & \quad q_{n+1} > q_n \text{ for } n \geq 1. \end{aligned}$$

If one of those two sequences has a limit, then the other sequence will have a limit too. Suppose that  $p_n \rightarrow l_1$  and  $q_n \rightarrow l_2$  as  $n \rightarrow \infty$ . Then we have

$$l_1 = \frac{Nl_2}{Nq - 2l_2} \quad \text{and} \quad l_2 = \frac{Nl_1}{Np - 2l_1},$$

from which we get

$$l_2 = \frac{N(pq - 1)}{2p + 2}.$$

Above we showed that this implies  $l_2 \leq q + 1$  – a contradiction with the fact that  $q_{n+1} > q_n$  for all  $n$ .

We conclude that both sequences tend to infinity. Standard elliptic theory yields  $u, v \in W^{2,s}$  for all  $s < \infty$ . The decay at infinity is a consequence of the Morrey theorem. Let us remark that, in case we suppose (ii) instead of (i), this decay is exponential, as was proved by de Figueiredo and Yang (Theorem 3.1 in [FY]). Part (a) is proved.

Next, we prove part (b). Take a weak solution  $(u, v)$ . We recall that  $u \in L^{p+1}$  and  $v \in L^{q+1}$ . From the equations (2) and the hypotheses (ii) and (iii) we have  $u \in W_{loc}^{2, \frac{q+1}{q}}$  and  $v \in W_{loc}^{2, \frac{p+1}{p}}$ . We need to be able to carry out a bootstrap argument for such  $u$  and  $v$ . As in part (a) we get  $u \in L_{loc}^{q_1}$  and  $v \in L_{loc}^{p_1}$ . If we set  $p_0 = q + 1$  and  $q_0 = p + 1$ , it is elementary to see that each one of the inequalities  $p_1 > p_0$  and  $q_1 > q_0$  is equivalent to (4). Then through a procedure similar to the one we employed in the proof of part (a) we obtain  $u \in L_{loc}^{q_n}$  and  $v \in L_{loc}^{p_n}$ , where

$$p_{n+1} = \frac{Nq_n}{Np - 2q_n} \quad \text{and} \quad q_{n+1} = \frac{Np_n}{Nq - 2p_n} \quad (16)$$

It is easy to see that the sequences  $p_n$  and  $q_n$  are increasing.

If we suppose that  $(u, v)$  is not a strong solution then at least one of these two sequences will be bounded. Then it will have a limit and by (16) the other sequence will have a limit too. Setting  $l_1 = \lim p_n$  we get, as in the proof of part (a)  $l_1 \leq q + 1$ , which is a contradiction.

Finally, let us prove part (c). Without loss of generality we suppose that  $s \geq 1$  (if  $s < 1$  then necessarily  $t > 1$ ). Then, by the Strauss radial lemma (see for ex [BL]),  $u \in L^s$ , for  $p + 1 \leq s \leq \infty$ . By elliptic theory  $v \in W^{2,s}$ , for  $p + 1 \leq s < \infty$ . Then  $v \in L^s$ , for  $q + 1 \leq s \leq \infty$ , so  $u \in W^{2,s}$ , for  $q + 1 \leq s < \infty$ . By embedding theory the solutions belong to  $C^{1,\alpha}$ , for  $0 < \alpha < 1$ , and tend to zero at infinity.

To prove that  $u, v \in C^2$  we can use the argument of Berestycki and Lions ([BL]). Let us sketch it here, applied to the function  $u$ . Since  $u$  is radial ( $u = u(r), r = |x|$ ), the first equation in (2) can be written as

$$\frac{d}{dr}(r^{N-1}u'(r)) = r^{N-1}g_1(r), \quad (17)$$

where  $g_1(r) = g(v(r)) - u(r)$ . Integrating (17) from 0 to  $r$ , after a change of variables we obtain

$$\frac{u'(r)}{r} \rightarrow -\frac{g_1(0)}{N} \quad \text{as } r \rightarrow 0.$$

By (17),  $u \in C^2$ . The exponential decay of  $u$  and its first derivatives follows from Theorem 3.1 in [FY], whence the exponential decay of the second derivatives is a consequence of (17). Theorem 1 is proved.  $\square$

### 3 Proofs of the Existence Results

#### 3.1 Proof of Theorem 2

In the proof of Theorem 2 we shall use the same variational setting as in [FY]. We shall search for critical points of the functional

$$\Phi(z) = \Phi(u, v) = \int_{\mathbb{R}^N} A^s u A^t v \, dx - \int_{\mathbb{R}^N} F(x, u) \, dx - \int_{\mathbb{R}^N} G(x, v) \, dx$$

defined on the space  $E = H_r^s(\mathbb{R}^N) \times H_r^t(\mathbb{R}^N)$ . Here  $H_r^s$  is the subspace of all radial functions in the fractional Sobolev space  $H^s$ ,  $s$  and  $t$  are positive real numbers, such that  $s + t = 2$ , and  $A^s : H^s \rightarrow L^2$  denotes a canonical isomorphism (see for example [FY] for a definition of  $A^s$ ). Let us note that  $\|u\|_{H^s} = \|A^s u\|_{L^2}$ . On  $E$  we consider the norm

$$\|(u, v)\|_E^2 = \|u\|_{H^s}^2 + \|v\|_{H^t}^2.$$

We choose  $0 < s < 2$ , such that

$$s < \frac{N}{2}, \quad 2 - s < \frac{N}{2}, \quad \frac{N(p-1)}{2(p+1)} < s < \frac{4(q+1) - N(q-1)}{2(q+1)},$$

and put  $t = 2 - s$ . Such a choice of  $s$  is possible because of (4). Then  $p+1 < \frac{2N}{N-2s}$  and  $q+1 < \frac{2N}{N-2t}$ , which implies, by a result of Lions ([L2]), that the embeddings  $H_r^s \hookrightarrow L^{p+1}$  and  $H_r^t \hookrightarrow L^{q+1}$  exist and are compact.

The derivative of  $\Phi$  is given by

$$\langle \Phi'(z), \eta \rangle = \int_{\mathbb{R}^N} A^s u A^t \psi + A^s \phi A^t v \, dx - \int_{\mathbb{R}^N} f(x, u) \phi \, dx - \int_{\mathbb{R}^N} g(x, v) \psi \, dx$$

for any  $z = (u, v) \in E$  and any  $\eta = (\phi, \psi) \in E$ . Taking  $\eta = (\phi, 0)$  and  $\eta = (0, \psi)$  we obtain the weak formulation of (2). In other words, critical points of  $\Phi$  are weak solutions of (2).

As in [FY], we shall use an infinite dimensional linking theorem due to Li and Willem (see [LW]). Let us recall this result for the reader's convenience.

Let  $E^1$  and  $E^2$  be Hilbert spaces with bases  $\{e_i^k\}_{i=1}^\infty, k = 1, 2$ . Let us note  $E = E^1 \times E^2$  and  $E_n = E_n^+ \oplus E_n^-$ , where  $E_n^+ = \text{span}\{(e_i^1, e_i^2)\}_{i=1}^n$  and  $E_n^- = \text{span}\{(e_i^1, -e_i^2)\}_{i=1}^n$ . Putting  $E^+ = \cup_{n=1}^\infty E_n^+$  (resp.  $E^-$ ), we have  $E = E^+ \oplus E^-$ .

**Theorem 5 (Li-Willem)** *Suppose that  $\Phi \in C^1(E, \mathbb{R})$  satisfies the following assumptions*

**(LW1)**  $\Phi$  has a local linking at zero, that is, for some  $r > 0$

$$\begin{aligned} \Phi(z) &\geq 0 && \text{if } z \in E^+ \cap \{z \in E : \|z\|_E \leq r\} && \text{and} \\ \Phi(z) &\leq 0 && \text{if } z \in E^- \cap \{z \in E : \|z\|_E \leq r\} ; \end{aligned}$$

**(LW2)**  $\Phi$  is bounded on bounded subsets of  $E$  ;

**(LW3)** for every  $n$

$$\Phi(z) \rightarrow -\infty \quad \text{as } z \text{ belongs to } E_n^+ \oplus E^- \quad \text{and } \|z\|_E \rightarrow \infty ;$$

**(LW4)** every sequence  $\{z_n\}_{n=1}^\infty \subset E$  such that

$$z_n \in E_n, \tag{18}$$

$$|\Phi(z_n)| \leq \text{const}, \tag{19}$$

$$|\langle \Phi'(z_n), \eta \rangle| \leq o(1)\|\eta\|_E \quad \text{for } \eta \in E_n \tag{20}$$

is precompact in the strong topology of  $E$ .

Then  $\Phi$  has a nontrivial critical point.

In our case we write  $E = E^+ \oplus E^-$  with

$$\begin{aligned} E^+ &= \{(u, A^{-t}A^s u) \mid u \in H_r^s\}, \\ E^- &= \{(u, -A^{-t}A^s u) \mid u \in H_r^s\}. \end{aligned}$$

Indeed, for any  $(u, v) \in E$  we have

$$(u, v) = (w_+, A^{-s}A^t w_+) + (w_-, -A^{-t}A^s w_-),$$

where

$$w_+ = \frac{u + A^{-s}A^t v}{2} \in H^s \quad \text{and} \quad w_- = \frac{u - A^{-s}A^t v}{2} \in H^s.$$

We now check that the hypotheses of the theorem of Li and Willem are satisfied.

**Proof of (LW1).** From (ii) and (iii) it follows that for every  $\varepsilon > 0$  there exists  $C_\varepsilon > 0$ , such that

$$\begin{aligned} |f(x, t)| &\leq \varepsilon|t| + C_\varepsilon|t|^p, \\ |F(x, t)| &\leq \varepsilon\frac{|t|^2}{2} + C_\varepsilon|t|^{p+1}, \end{aligned} \tag{21}$$

and the same for  $g$  and  $G$ . Let  $z = (u, A^{-t}A^s u) \in E^+$ . Then

$$\begin{aligned}
\Phi(z) &= \|A^s u\|_{L^2}^2 - \int_{\mathbb{R}^N} F(x, u) dx - \int_{\mathbb{R}^N} G(x, A^{-t}A^s u) dx \\
&\geq \|u\|_{H^s}^2 - \frac{1}{4}\|u\|_{L^2}^2 - C\|u\|_{L^{p+1}}^{p+1} - \frac{1}{4}\|u\|_{H^{s-t}}^2 - C\|A^{-t}A^s u\|_{L^{q+1}}^{q+1} \\
&\geq \frac{1}{2}\|u\|_{H^s}^2 - C\|u\|_{H^s}^{p+1} - C\|A^{-t}A^s u\|_{H^t}^{q+1} \\
&\geq \frac{1}{4}\|z\|_E^2 - C\|z\|_E^{p+1} - C\|z\|_E^{q+1} \\
&\geq C\|z\|_E^2
\end{aligned}$$

provided that  $\|z\|_E$  is sufficiently small. Here, and in the sequel,  $C$  denotes a constant whose value may change from line to line.

It is obvious that  $\Phi(z) \leq -\frac{1}{2}\|z\|_E^2$  for  $z \in E^-$ . This proves (LW1).

It is also straightforward to check (LW2) - we use (21) and the choice of  $s$  and  $t$ .

**Proof of (LW3).** By integrating (5) and by using (3) we see that

$$F(x, t) \geq d_1(x)|t|^\mu \quad \text{and} \quad G(x, t) \geq d_2(x)|t|^\mu,$$

where  $d_1(x)$  and  $d_2(x)$  are positive bounded functions.

Let  $z \in E_n^+ \oplus E^-$ . Then  $z = (u, v) = z^+ + z^- = (u^+, A^{-t}A^s u^+) + (u^-, -A^{-t}A^s u^-)$  for some  $u^+, u^- \in H^s$ . We have

$$\begin{aligned}
\Phi(z) &= \|u^+\|_{H^s}^2 - \|u^-\|_{H^s}^2 - \int_{\mathbb{R}^N} F(x, u) dx - \int_{\mathbb{R}^N} G(x, v) dx \\
&\leq -\frac{1}{2}\|z^-\|_E^2 + \frac{1}{2}\|z^+\|_E^2 - \int_{\mathbb{R}^N} d_1(x)|u|^\mu dx - \int_{\mathbb{R}^N} d_2(x)|v|^\mu dx.
\end{aligned}$$

For any  $z = (u, v) \in E$  we set

$$\varphi(z) = \left( \int_{\mathbb{R}^N} d_1(x)|u|^\mu dx \right)^{\frac{1}{\mu}} + \left( \int_{\mathbb{R}^N} d_2(x)|v|^\mu dx \right)^{\frac{1}{\mu}}$$

(note that  $2 < \mu < \min\{p+1, q+1\}$ ). It is easily checked that  $\varphi$  is a norm on  $E$ . We now use the following simple fact.

**Proposition 1** *Let  $E$  be a Hilbert space with  $E = E^+ \oplus E^-$ , and let  $\varphi$  be a norm on  $E$ . Then there exists a norm  $\psi$  on  $E^+$ , such that for any  $z^+ \in E^+$  we have*

$$\psi(z^+) \leq \varphi(z),$$

for every  $z = z^+ + e^-$ , with  $e^- \in E^-$ .

**Proof.** Indeed, one can take

$$\psi(z^+) = \inf \{ \varphi(z) : z = z^+ + e^-, e^- \in E^- \}$$

and check that  $\psi$  is a norm on  $E^+$ .  $\square$

Hence

$$\begin{aligned} \Phi(z) &\leq -\frac{1}{2}\|z^-\|_E^2 + \frac{1}{2}\|z^+\|_E^2 - 2^{1-\mu} [\varphi(z)]^\mu \\ &\leq -\frac{1}{2}\|z^-\|_E^2 + \frac{1}{2}\|z^+\|_E^2 - 2^{1-\mu} [\psi(z^+)]^\mu. \end{aligned} \quad (22)$$

We see that (LW3) follows from (22), since  $\psi$  and  $\|\cdot\|_E$  are equivalent on the finite dimensional space  $E_n^+$  (recall that  $\mu > 2$ ).

**Proof of (LW4).** Take a sequence  $\{z_n\}$ ,  $z_n = (u_n, v_n)$ , for which (18), (19) and (20) hold.

We recall that the derivative of  $\Phi$  is given by

$$\langle \Phi'(z), \eta \rangle = \int_{\mathbb{R}^N} A^s u A^t \psi + A^s \phi A^t v \, dx - \int_{\mathbb{R}^N} f(x, u) \phi \, dx - \int_{\mathbb{R}^N} g(x, v) \psi \, dx$$

for any  $z = (u, v) \in E$  and any  $\eta = (\phi, \psi) \in E$ . We shall make two special choices of  $\eta$  in (20), from which we shall derive the boundedness of the sequence  $\{z_n\}$ .

**Lemma 1** *If  $z = (u, v) \in E_n$  then  $\bar{z} = (A^{-s}A^t v, A^{-t}A^s u) \in E_n$ .*

**Proof.** Indeed

$$\begin{aligned} z &= (u, v) = (u^+, A^{-t}A^s u^+) + (u^-, -A^{-t}A^s u^-) \\ &= (u^+ + u^-, A^{-t}A^s(u^+ - u^-)), \end{aligned}$$

and hence

$$\begin{aligned} \bar{z} &= (u^+ - u^-, A^{-t}A^s(u^+ + u^-)) \\ &= (u^+, A^{-t}A^s u^+) + (-1) \cdot (u^-, -A^{-t}A^s u^-). \end{aligned} \quad \square$$

We now put  $\eta = (A^{-s}A^t v_n, A^{-t}A^s u_n)$  into (20). Then (20) becomes

$$\begin{aligned} \langle \Phi'(z_n), \eta \rangle &= \|u_n\|_{H^s}^2 + \|v_n\|_{H^t}^2 - \int_{\mathbb{R}^N} f(x, u_n) A^{-s} A^t v_n \, dx \\ &\quad - \int_{\mathbb{R}^N} g(x, v_n) A^{-t} A^s u_n \, dx \\ &\leq o(1)\|\eta\|_E = o(1)\|z_n\|_E. \end{aligned}$$

The hypothesis (ii) permits us to fix  $\delta > 0$  such that

$$|f(x, t)| \leq \frac{1}{4}|t| \quad \text{and} \quad |g(x, t)| \leq \frac{1}{4}|t|,$$

for  $|t| \leq \delta$ . By (iii) we have

$$|f(x, t)| \leq C|t|^p \quad \text{and} \quad |g(x, t)| \leq C|t|^q,$$

for  $|t| \geq \delta$ .

With our choice of  $s$  and  $t$ , we obtain from (3.1)

$$\begin{aligned} \|z_n\|_E^2 &\leq o(1)\|z_n\|_E + \int_{|u_n| \leq \delta} |f(x, u_n)| |A^{-s} A^t v_n| dx \\ &\quad + \int_{|v_n| \leq \delta} |g(x, v_n)| |A^{-t} A^s u_n| dx + \int_{|u_n| \geq \delta} |f(x, u_n)| |A^{-s} A^t v_n| dx \\ &\quad + \int_{|v_n| \geq \delta} |g(x, v_n)| |A^{-t} A^s u_n| dx \\ &\leq o(1)\|z_n\|_E + \frac{1}{4} (\|u_n\|_{L^2}^2 + \|v_n\|_{H^{t-s}}^2) + \frac{1}{4} (\|v_n\|_{L^2}^2 + \|u_n\|_{H^{s-t}}^2) \\ &\quad + \left( \int_{|u_n| \geq \delta} |f(x, u_n)|^{1+\frac{1}{p}} dx \right)^{\frac{p}{p+1}} \|A^{-s} A^t v_n\|_{L^{p+1}} \\ &\quad + \left( \int_{|v_n| \geq \delta} |g(x, v_n)|^{1+\frac{1}{q}} dx \right)^{\frac{q}{q+1}} \|A^{-t} A^s u_n\|_{L^{q+1}} \\ &\leq o(1)\|z_n\|_E + \frac{1}{2}\|z_n\|_E^2 + C \left( \int_{\mathbb{R}^N} u_n f(x, u_n) dx \right)^{\frac{p}{p+1}} \|v_n\|_{H^t} \\ &\quad + C \left( \int_{\mathbb{R}^N} v_n g(x, v_n) dx \right)^{\frac{q}{q+1}} \|u_n\|_{H^s} \\ &\leq o(1)\|z_n\|_E + \frac{1}{2}\|z_n\|_E^2 + C A_n^{\frac{p}{p+1}} \|z_n\|_E + C B_n^{\frac{q}{q+1}} \|z_n\|_E, \end{aligned}$$

where we have set

$$A_n = \int_{\mathbb{R}^N} u_n f(x, u_n) dx \quad \text{and} \quad B_n = \int_{\mathbb{R}^N} v_n g(x, v_n) dx.$$

Hence

$$\|z_n\|_E \leq o(1) + C A_n^{\frac{p}{p+1}} + C B_n^{\frac{q}{q+1}}. \quad (23)$$

The second choice of  $\eta$  we make in (20) is  $\eta = z_n$ . Then, by (19) and (20) we obtain

$$\begin{aligned} \left(1 - \frac{2}{\mu}\right) A_n + \left(1 - \frac{2}{\mu}\right) B_n &\leq 2\Phi(z_n) - \langle \Phi'(z_n), z_n \rangle \\ &\leq C + o(1)\|z_n\|_E. \end{aligned}$$

Combining this with (23), we get

$$A_n + B_n \leq C + o(1) + o(1) \left( A_n^{\frac{p}{p+1}} + B_n^{\frac{q}{q+1}} \right),$$

from which it follows that the sequences  $\{A_n\}$  and  $\{B_n\}$  are bounded. Then, by (23),  $\{z_n\}$  is bounded in  $E$ .

We can now extract a weakly convergent subsequence of  $\{z_n\}$  (from now on, any time we extract a subsequence we shall keep the notation of the original sequence). Let  $u_n \rightharpoonup u$  weakly in  $H^s$  and  $v_n \rightharpoonup v$  weakly in  $H^t$ . It follows that  $\{u_n\}$  (resp.  $\{v_n\}$ ) converges almost everywhere in  $\mathbb{R}^N$  and strongly in  $L^r_{loc}$ , for  $2 \leq r < \frac{2N}{N-2s}$  (resp. for  $2 \leq r < \frac{2N}{N-2t}$ ).

Now it is easy to see that  $\langle \Phi'(z_n), \bar{\eta} \rangle$  converges to  $\langle \Phi'(z), \bar{\eta} \rangle$ , for all  $\bar{\eta} \in C_c^\infty \times C_c^\infty$ . Then, by density,  $\langle \Phi'(z_n), \eta \rangle$  converges to  $\langle \Phi'(z), \eta \rangle$  for all  $\eta \in E$ . We fix  $m \geq 1$  and take  $\eta \in E_m$ . Then, passing to the limit in (20) (note that (20) holds for all  $n \geq m$ ), we obtain

$$\int_{\mathbb{R}^N} A^s u A^t \psi + A^s \phi A^t v \, dx - \int_{\mathbb{R}^N} f(x, u) \phi \, dx - \int_{\mathbb{R}^N} g(x, v) \psi \, dx = 0, \quad (24)$$

and this inequality holds for all  $\eta \in E_m$  and all  $m \geq 1$ . Since  $\cup_{m=1}^\infty E_m$  is dense in  $E$ , (24) holds for all  $\eta \in E$ . Putting  $\eta = (A^{-s} A^t v, A^{-t} A^s u)$  into (24), we get

$$\|z\|_E^2 - \int_{\mathbb{R}^N} f(x, u) A^{-s} A^t v \, dx - \int_{\mathbb{R}^N} g(x, v) A^{-t} A^s u \, dx = 0. \quad (25)$$

On the other hand, putting  $\eta = (A^{-s} A^t v_n, A^{-t} A^s u_n)$  into (20) yields

$$\|z_n\|_E^2 - \int_{\mathbb{R}^N} f(x, u_n) A^{-s} A^t v_n \, dx - \int_{\mathbb{R}^N} g(x, v_n) A^{-t} A^s u_n \, dx = o(1), \quad (26)$$

so (LW4) will follow from (25) and (26), provided we show that the second and the third terms of (26) converge to the second and the third terms of (25) respectively. For instance, let us prove the convergence of the second term.

**Claim.**

$$\int_{\mathbb{R}^N} f(x, u_n) A^{-s} A^t v_n \, dx \longrightarrow \int_{\mathbb{R}^N} f(x, u) A^{-s} A^t v \, dx \quad \text{as } n \rightarrow \infty.$$

**Proof.** First we note that

$$\int_{\mathbb{R}^N} |f(x, u)| |A^{-s} A^t (v_n - v)| \, dx \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This can be seen easily by using (21), the Holder inequality and the fact that  $v_n \rightharpoonup v$  weakly in  $H^t$  implies  $A^{-s}A^t v_n \rightharpoonup A^{-s}A^t v$  weakly in  $H^s$  and strongly in  $L^{p+1}$  (recall that we restricted ourselves to radial functions). Next, splitting the integral inside and outside a ball  $B_R = B(0, R)$ , we see that

$$\begin{aligned}
I_n &= \int_{\mathbb{R}^N} |f(x, u_n) - f(x, u)| |A^{-s}A^t v_n| dx \\
&\leq \|f(x, u_n) - f(x, u)\|_{L^{\frac{p+1}{p}}(B_R)}^{p+1} \|A^{-s}A^t v_n\|_{L^{p+1}} \\
&\quad + \varepsilon \int_{|x| \geq R} (|u_n| + |u|) |A^{-s}A^t v_n| dx \\
&\quad + C \int_{|x| \geq R} |u_n|^p |A^{-s}A^t v_n| dx + C \int_{|x| \geq R} |u|^p |A^{-s}A^t v_n| dx \\
&\leq o(1) + C\varepsilon + C\|u_n - u\|_{L^{p+1}}^p + C\|u\|_{L^{p+1}(\mathbb{R}^N \setminus B_R)}^p.
\end{aligned}$$

It follows that  $I_n \rightarrow \infty$ , since the last expression can be made arbitrarily small, by taking  $\varepsilon$  sufficiently small and  $n, R$  sufficiently large. Theorem 2 is proved.

**Remark 1.** It is clear from the above proof that the usual Palais-Smale sequences for  $\Phi$ , defined on the whole space  $H^s \times H^t$  (without restriction to radial functions), are bounded in  $E$ . We shall use this fact in the next section.

### 3.2 Proof of Theorem 3

We have to show that

$$\inf \{ \Phi(z) \mid z \in E, z \neq (0, 0) \text{ is a solution of (2)} \}$$

is attained.

Let us first show that the above infimum is finite (and actually non-negative). We use the weak formulation of (2), that is

$$\int_{\mathbb{R}^N} A^s u A^t \psi dx - \int_{\mathbb{R}^N} g(x, v) \psi dx = 0 \quad \text{for all } \psi \in H^t \quad (27)$$

$$\int_{\mathbb{R}^N} A^t v A^s \phi dx - \int_{\mathbb{R}^N} f(x, u) \phi dx = 0 \quad \text{for all } \phi \in H^s. \quad (28)$$

If  $z = (u, v)$  is a nontrivial solution of (2), by taking  $\psi = v$  and  $\phi = u$  in (27) and (28), we immediately see that

$$\Phi(z) = \int_{\mathbb{R}^N} \left( \frac{1}{2} u f(x, u) - F(x, u) + \frac{1}{2} v g(x, v) - G(x, v) \right) dx \quad (29)$$

$$\geq \left(\frac{\mu}{2} - 1\right) \int_{\mathbb{R}^N} (F(x, u) + G(x, v)) \, dx > 0,$$

for any  $(u, v) \neq (0, 0)$ .

The next step in the proof is to derive a positive lower bound for the norms of the nontrivial solutions of (2). Let  $z = (u, v)$  be one such solution. Putting  $\phi = A^{-s}A^t v$  into (28) and  $\psi = A^{-t}A^s u$  into (27), we obtain

$$\begin{aligned} \|z\|_E^2 &= \int_{\mathbb{R}^N} f(x, u)A^{-s}A^t v \, dx + \int_{\mathbb{R}^N} g(x, v)A^{-t}A^s u \, dx \\ &\leq \frac{1}{4} \int_{\mathbb{R}^N} (|u||A^{-s}A^t v| + |v||A^{-t}A^s u|) \, dx \\ &\quad + C \int_{\mathbb{R}^N} (|u|^p|A^{-s}A^t v| + |v|^q|A^{-t}A^s u|) \, dx \\ &\leq \frac{1}{4} (\|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|u\|_{H^{s-t}}^2 + \|v\|_{H^{t-s}}^2) \\ &\quad + C\|u\|_{L^{p+1}}^p \|A^{-s}A^t v\|_{L^{p+1}} + C\|v\|_{L^{q+1}}^q \|A^{-t}A^s u\|_{L^{q+1}} \\ &\leq \frac{1}{2} \|z\|_E^2 + C\|u\|_{H^s}^p \|v\|_{H^t} + C\|u\|_{H^s} \|v\|_{H^t}^q \\ &\leq \frac{1}{2} \|z\|_E^2 + C\|z\|_E^{p+1} + C\|z\|_E^{q+1}, \end{aligned}$$

which implies

$$\|z\|_E \geq \text{const} > 0. \quad (30)$$

Next we take a sequence of nontrivial solutions  $\{z_n\}$ , such that

$$\Phi(z_n) \longrightarrow \inf \{ \Phi(z) \mid z \in E, z \neq (0, 0) \text{ is a solution of (2)} \}.$$

Obviously  $\{z_n\}$  is a Palais-Smale sequence. Hence it is bounded in  $E$  (see Remark 1 at the end of the previous section).

In case (a) the sequence  $\{z_n\}$  is a Palais-Smale sequence in  $H_r^s \times H_r^t$ . Then the proof of Theorem 2 implies this sequence has a *strongly* convergent subsequence. Hence, by (30), its limit is a nontrivial solution and therefore a ground state.

In case (b) we extract a weakly convergent subsequence of  $\{z_n\}$ . Let  $\{u_n\}$  converges to a function  $u$  weakly in  $H^s$  and  $\{v_n\}$  converges to a function  $v$  weakly in  $H^t$ . Then  $\{u_n\}$  and  $\{v_n\}$  converge almost everywhere in  $\mathbb{R}^N$  and strongly in  $L_{loc}^r$  for  $2 \leq r < \frac{2N}{N-2s}$  (respectively for  $2 \leq r < \frac{2N}{N-2t}$ ). We use the fact that  $u_n$  and  $v_n$  are *exact* solutions, that is,

$$\begin{aligned} \int_{\mathbb{R}^N} A^s u_n A^t \psi \, dx - \int_{\mathbb{R}^N} g(v_n) \psi \, dx &= 0 & \text{for all } \psi \in H^t \\ \int_{\mathbb{R}^N} A^t v_n A^s \phi \, dx - \int_{\mathbb{R}^N} f(u_n) \phi \, dx &= 0 & \text{for all } \phi \in H^s. \end{aligned} \quad (31)$$

By taking  $\phi$  and  $\psi$  smooth with compact supports in (31), we see that  $z = (u, v)$  is also a solution of (6).

Since

$$\begin{aligned} \frac{1}{2}tf(t) &\geq \frac{1}{2}tf(t) - F(t) \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)tf(t), \\ \frac{1}{2}tg(t) &\geq \frac{1}{2}tg(t) - G(t) \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)tg(t), \end{aligned} \quad (32)$$

by putting  $z = z_n$  in (29) we see that

$$\Phi(z_n) = |||u_n|||_{L^{p+1}}^{p+1} + |||v_n|||_{L^{q+1}}^{q+1}$$

where, by (7) and (32),  $|||\cdot|||_{L^{p+1}}$  and  $|||\cdot|||_{L^{q+1}}$  are equivalent to the standard norms in  $L^{p+1}$  and  $L^{q+1}$ . Hence

$$\Phi(z) \leq \liminf_{n \rightarrow \infty} \Phi(z_n) \quad (33)$$

since  $u_n \rightharpoonup u$  in  $L^{p+1}$  and  $v_n \rightharpoonup v$  in  $L^{q+1}$ .

We have not shown that  $z$  is a nontrivial solution. Assume that  $z = (0, 0)$ . Without loss of generality we suppose that  $s \geq 1$  (if  $s < 1$  then necessarily  $t > 1$ ). We claim that (30) prevents  $\{u_n\}$  from converging strongly to zero in  $L^{p+1}$ . Indeed, if  $u_n \rightarrow 0$  in  $L^{p+1}$ , then since  $(u_n, v_n)$  satisfy (6), the standard elliptic theory implies  $v_n \rightarrow 0$  in  $L^{p_1}$ , with  $p_1$  defined as in the proof of Theorem 1 (b). Then repeating the bootstrap argument we used in the proof of Theorem 1 (b), we obtain a contradiction with (30).

Next we apply a standard concentration-compactness lemma due to Lions (see [L]). Since  $u_n \not\rightarrow 0$  in  $L^{p+1}$  we can find a sequence of points  $x_n \in \mathbb{R}^N$  and constants  $R, \alpha > 0$ , such that

$$\int_{B_R(x_n)} |u_n|^2 dx \geq \alpha.$$

Set  $\tilde{z}_n = (\tilde{u}_n, \tilde{v}_n)$ , with  $\tilde{u}_n(\cdot) = u_n(\cdot + x_n)$  and  $\tilde{v}_n = v_n(\cdot + x_n)$ . Then  $\tilde{z}_n$  is itself a solution of (6), satisfying  $\Phi(\tilde{z}_n) = \Phi(z_n)$ . Hence  $\tilde{z}_n \rightharpoonup \tilde{z}$  in  $E$ . On the other hand

$$\int_{B_R(0)} |\tilde{u}_n|^2 dx \geq \alpha$$

prevents  $\tilde{u}_n \rightarrow 0$  strongly in  $L^2_{loc}$ , so  $\tilde{z} \neq (0, 0)$ . We can then repeat the proof of (33) for the sequence  $\{\tilde{z}_n\}$ , to conclude that  $\tilde{z}$  is a nontrivial solution and a ground state.

### 3.3 Proof of Theorem 4

We sketch the proof of this theorem.

First let us adjust the variational setting. Let  $H_{b(x)}^s, 0 < s < 1$ , be the space of functions  $u$ , such that

$$b^{\frac{1}{2}}(x)u \in L^2(\mathbb{R}^N) \quad \text{and} \quad \frac{u(x) - u(y)}{|x - y|^{s + \frac{N}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N).$$

One can also define  $H_{b(x)}^s$  by interpolation between the spaces

$$L_{b(x)}^2 = \left\{ u : \int_{\mathbb{R}^N} b(x)u^2 dx < \infty \right\}$$

and

$$H_{b(x)}^1 = \left\{ u : \int_{\mathbb{R}^N} |\nabla u|^2 + b(x)u^2 dx < \infty \right\}.$$

The space  $H_{b(x)}^1$  is the function space which was used for studying the case of a single equation in [R],[C], [BW] and [S]. The space  $H_{b(x)}^s$  is its fractional equivalent.

It is clear that  $H_{b(x)}^s$  is continuously embedded into  $H^s$ . It is not difficult to see that  $H_{b(x)}^s$  is compactly embedded into  $L^2$  and therefore into  $L^r$  for all  $2 \leq r < \frac{2N}{N-2s}$ . The proof of this fact is very much the same as in the case  $s = 1$  (see [C]; see also [S] for a more general result).

Let  $T = -\Delta + b(x) : H_{b(x)}^2 \subset L^2 \rightarrow L^2$  and put  $A^s = T^{\frac{s}{2}} : H_{b(x)}^s \rightarrow L^2$  (see [FY] for a detailed definition of  $A^s$ ). In this situation weak solutions of (1) will be critical points of  $\Phi$  defined on  $H_{b(x)}^s \times H_{b(x)}^t$ . It is possible to use the theorem of Li and Willem in this setting. We shall omit the details here since similar computations can be found in [S]. When proving statement (b), the point is that we have

$$\int_{\mathbb{R}^N} h_1(x)|u|^{p+1} dx \leq C \left( \int_{\mathbb{R}^N} b(x)u^2 dx \right)^{\frac{1}{\alpha}} \left( \int_{\mathbb{R}^N} |u|^{\frac{\alpha(p+1)-2}{\alpha-1}} dx \right)^{\frac{\alpha-1}{\alpha}},$$

$$\int_{\mathbb{R}^N} h_2(x)|v|^{q+1} dx \leq C \left( \int_{\mathbb{R}^N} b(x)v^2 dx \right)^{\frac{1}{\alpha}} \left( \int_{\mathbb{R}^N} |v|^{\frac{\alpha(q+1)-2}{\alpha-1}} dx \right)^{\frac{\alpha-1}{\alpha}}$$

and

$$\frac{\alpha - 1}{\alpha(p + 1) - 2} + \frac{\alpha - 1}{\alpha(q + 1) - 2} > 1 - \frac{2}{N},$$

by (13).

Once we have the proof in Section 3.1, the statement (a) can be obtained in the following way. We can repeat the proof of Theorem 2, the only differing point being the Claim at the end of Section 3.1. Here we use the fact that the sequences  $\{u_n\}$  and  $\{v_n\}$  are bounded which, combined with (10), implies the existence of  $R > 0$  such that the difference between the integrals in the Claim, over  $\mathbb{R}^N \setminus B_R$ , is smaller than any given number. We can then conclude by using the fact that  $H^s$  is compactly embedded into  $L^r_{loc}$  for  $2 \leq r < \frac{2N}{N-2s}$ .

**Remark 2.** If (12) holds with different  $\alpha_1$  and  $\alpha_2$ , one can make (13) more precise by replacing it with

$$\frac{\alpha_1 - 1}{\alpha_1(p + 1) - 2} + \frac{\alpha_2 - 1}{\alpha_2(q + 1) - 2} > 1 - \frac{2}{N}.$$

## 4 Some Open Problems

As we already mentioned, the theory of scalar elliptic equations is far more advanced than the theory of elliptic systems. Many interesting questions, which have been answered for scalar equations, are still open for systems. For example, the general radial case

$$\begin{cases} -\Delta u &= g(r, u, v) \\ -\Delta v &= f(r, u, v), \end{cases} \quad x \in \mathbb{R}^N,$$

has not been studied, nor has the case

$$\begin{cases} -\Delta u + u &= h_2(x)|v|^{q-1}v \\ -\Delta v + v &= h_1(x)|u|^{p-1}u, \end{cases} \quad x \in \mathbb{R}^N,$$

with  $h_i(x) \rightarrow 1$  as  $|x| \rightarrow \infty$  and  $h_i(x) \geq 1$ .

One may also search for solutions of the ‘‘perturbed system’’

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u &= |v|^{q-1}v \\ -\varepsilon^2 \Delta v + V(x)v &= |u|^{p-1}u, \end{cases} \quad x \in \mathbb{R}^N,$$

for small  $\varepsilon$ . In the case of a scalar equation this problem has been studied extensively during the last fifteen years (see for example [DF] and the references therein).

We intend to pursue these questions in the future.

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