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**INDEX THEORY AND
NON-COMMUTATIVE GEOMETRY
II. DIRAC OPERATORS AND INDEX BUNDLES
April 19, 2005**

MOULAY-TAHAR BENAMEUR AND JAMES L. HEITSCH

ABSTRACT. When the index bundle of a longitudinal Dirac type operator is transversely smooth, we define its Chern character in Haefliger cohomology and relate it to the Chern character of the K -theory index. This result gives a concrete connection between the topology of the foliation and the longitudinal index formula. Moreover, the usual spectral assumption on the Novikov-Shubin invariants of the operator is improved.

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INTRODUCTION

In this paper, we continue our systematic study of the index theorem in Haefliger cohomology of foliations. In [BH-I], we defined a Chern character for leafwise elliptic pseudodifferential operators on foliations. By using Connes' extension in [Con86], we then translated the Connes-Skandalis K -theory index theorem [CS84] into Haefliger cohomology, thus proving scalar index theorems in the presence of holonomy invariant currents.

In order to get more insight into topological invariants of foliations, we extend here the results of [He95] and [HL99], which tie the indices of a leafwise operator on a foliation of a compact manifold to the so-called index bundle of the operator. In particular, we show that for a generalized Dirac operator D along the leaves of a foliation with Hausdorff graph, the Chern character of the analytic index of D coincides with the Chern character of the index bundle of D . As in [He95] and [HL99], we assume that the projection onto the kernel of D is transversely smooth, and that the spectral projections of D^2 for the intervals $(0, \epsilon)$ are transversely smooth, for ϵ sufficiently small. In those two papers, we assumed that the Novikov-Shubin invariants of D were greater than three times the codimension of F . Here we use the K -theory index and we need only assume that they are greater than half the codimension of F . More precisely, the pairings of these Chern characters with a given Haefliger $2k$ -current agree whenever the Novikov-Shubin invariants of D are greater than k . We conjecture that this theorem is still true provided only that the Novikov-Shubin invariants are positive. Note that in the heat equation proof of the classical Atiyah-Singer families index theorem, [B86], it is assumed that there is a uniform gap about zero in the spectrum of the operator, which implies the conditions we assume on the spectral projections.

In [Con79, Con81], Connes extended the classical construction of Atiyah [A75] of the L^2 covering index theorem to leafwise elliptic operators on compact foliated manifolds. To do so he replaced the lifting and

deck transformations used by Atiyah by a lifting to the holonomy covers of the leaves invariant under the natural action of the holonomy groupoid. Moreover, he defined an analytic index map from the K-theory of the tangent bundle of the foliation to the K-theory of the C^* algebra of the foliation, which plays the role of the K-theory of the space of leaves. In [CS84], Connes and Skandalis defined a push forward map in K-theory for any K-oriented map from a manifold to the space of leaves of a foliation of a compact manifold. This allowed them to define a topological index map from the K-theory of the tangent bundle of the foliation to the K-theory of its C^* algebra. Their main result is that the analytic and topological index maps are equal, an extension of the classical Atiyah-Singer families index theorem. This theorem does not lead in general to a relation between the index of the operator and its index bundle, by which we mean the (graded) projection onto the kernel of the operator, even when this latter is transversely smooth so that its Chern character is well defined. This index bundle, which lives in a von Neumann algebra of the foliation, carries important information about the foliation, e.g. its Euler class, its higher signatures, etc.

In this paper, we extend the Chern character to the index bundle of D , provided the projection onto the kernel of D is transversely smooth. Our main result is that, with the conditions given in the first paragraph, the Chern character of D equals the Chern character of the index bundle of D . Since the Chern character of the index bundle equals the superconnection index defined in [He95], we obtain as a corollary the coincidence of the superconnection index with the Chern character of the analytic and topological indices. This Chern character is readily computable and directly relates the index of D with the topology of the foliation.

We point out the papers [GL03, GL05] where Gorokhovskiy and Lott prove, by a different method, an index theorem for longitudinal Dirac operators.

Here is a brief outline of the paper. In Section 1., we fix notation and briefly review some necessary material. In Section 2., we extend our Chern character to the K -theory of the space of super-exponentially decaying operators on the leaves of a foliation, recall the construction of Dirac operators and the heat index idempotent. In Section 3., we review the construction of the Chern character we use, and extend it to the index bundle of a leafwise Dirac operator. In Section 4., we prove our main theorem, Theorem 4.1. In Section 5., we show that the Chern character of the index bundle for D defined here is the same as that defined in [He95] using Bismut superconnections.

Some of our results are valid for all foliations, not just those with Hausdorff groupoid. We will alert the reader when we need to assume that the graph \mathcal{G} of F is Hausdorff. It is also worth pointing out that our results are valid if we replace the holonomy groupoid \mathcal{G} by any smooth groupoid between the monodromy and holonomy groupoids.

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1. NOTATION AND REVIEW

Throughout this paper M denotes a smooth compact Riemannian manifold of dimension n , and F denotes an oriented foliation of M of dimension p and codimension q . So $n = p + q$. The tangent bundle of F will be denoted TF . If $E \rightarrow N$ is a vector bundle over a manifold N , we denote the space of smooth sections by $C^\infty(E)$ or by $C^\infty(N; E)$ if we want to emphasize the base space of the bundle. The compactly supported sections are denoted by $C_c^\infty(E)$ or $C_c^\infty(N; E)$. The space of differential k -forms on N is denoted $\mathcal{A}^k(N)$, and we set $\mathcal{A}(N) = \bigoplus_{k \geq 0} \mathcal{A}^k(N)$. The space of compactly supported k -forms is denoted $\mathcal{A}_c^k(N)$, and $\mathcal{A}_c(N) = \bigoplus_{k \geq 0} \mathcal{A}_c^k(N)$.

The holonomy groupoid \mathcal{G} of F consists of equivalence classes of paths $\gamma : [0, 1] \rightarrow M$ such that the image of γ is contained in a leaf of F . Two such paths γ_1 and γ_2 are equivalent if $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$, and the holonomy germ along them is the same. Two classes may be composed if the first ends where the second begins, and the composition is just the juxtaposition of the two paths. This makes \mathcal{G} a groupoid. The space $\mathcal{G}^{(0)}$ of units of \mathcal{G} consists of the equivalence classes of the constant paths, and we identify $\mathcal{G}^{(0)}$ with M .

\mathcal{G} is a (in general non-Hausdorff) dimension $2p + q$ manifold. The basic open sets defining its manifold structure are given as follows. Let \mathcal{U} be a finite good cover of M by foliation charts as defined in [HL90]. Given U and V in this cover and a leafwise path γ starting in U and ending in V , define (U, γ, V) to be

the set of equivalence classes of leafwise paths starting in U and ending in V which are homotopic to γ through a homotopy of leafwise paths whose end points remain in U and V respectively. It is easy to see, using the holonomy defined by γ from a transversal in U to a transversal in V , that if $U, V \simeq \mathbb{R}^p \times \mathbb{R}^q$, then $(U, \gamma, V) \simeq \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^q$. If \mathcal{G} is non-Hausdorff, it is not true that compact sets are always closed, nor that the closure of a compact set is compact. Because of this, we define the notion of having compact support as follows. Given a bundle E over \mathcal{G} and any set (U, γ, V) as above, consider $E|_{(U, \gamma, V)}$, the restriction of E to (U, γ, V) . The space $C_c^\infty(E|_{(U, \gamma, V)})$ has a natural inclusion into the space of sections of E over \mathcal{G} by extending any element of $C_c^\infty(E|_{(U, \gamma, V)})$ to all of \mathcal{G} by defining it to be zero outside (U, γ, V) . We define the space $C_c^\infty(E) = C_c^\infty(\mathcal{G}; E)$ of smooth sections of E over \mathcal{G} with compact support to be all finite sums $\sum s_i$ where each s_i is the image of an element in some $C_c^\infty(E|_{(U, \gamma, V)})$. The space of smooth functions with compact support on \mathcal{G} , namely $C_c^\infty(\mathcal{G}; \mathbb{R})$, will be denoted $C_c^\infty(\mathcal{G})$. The metric on M induces a canonical metric on \mathcal{G} , denoted g_0 . See [He95] for the construction.

The source and range maps of the groupoid \mathcal{G} are the two natural maps $s, r : \mathcal{G} \rightarrow M$ given by $s([\gamma]) = \gamma(0)$, $r([\gamma]) = \gamma(1)$. \mathcal{G} has two natural transverse foliations F_s and F_r whose leaves are respectively $\tilde{L}_x = s^{-1}(x)$, $\tilde{L}^x = r^{-1}(x)$ for $x \in M$. Note that $r : \tilde{L}_x \rightarrow L$ is the holonomy covering of L .

The Haefliger cohomology of F , [H80], is given as follows. For each $U_i \in \mathcal{U}$, let $T_i \subset U_i$ be a transversal and set $T = \bigcup T_i$. We may assume that the closures of the T_i are disjoint. Let \mathcal{H} be the holonomy pseudogroup induced by F on T . Denote by $\mathcal{A}_c^k(M/F)$ the quotient of $\mathcal{A}_c^k(T)$ by the vector subspace generated by elements of the form $\alpha - h^*\alpha$ where $h \in \mathcal{H}$ and $\alpha \in \mathcal{A}_c^k(T)$ has support contained in the range of h . Give $\mathcal{A}_c^k(M/F)$ the quotient topology of the usual C^∞ topology on $\mathcal{A}_c^k(T)$, so this is not a Hausdorff space in general. The exterior derivative $d_T : \mathcal{A}_c^k(T) \rightarrow \mathcal{A}_c^{k+1}(T)$ induces a continuous differential $d_H : \mathcal{A}_c^k(M/F) \rightarrow \mathcal{A}_c^{k+1}(M/F)$. Note that $\mathcal{A}_c^k(M/F)$ and d_H are independent of the choice of cover \mathcal{U} . The complex $\{\mathcal{A}_c^*(M/F), d_H\}$ and its cohomology $H_c^*(M/F)$ are, respectively, the Haefliger forms and Haefliger cohomology of F .

As the bundle TF is oriented, there is a continuous open surjective linear map, called integration over the leaves,

$$\int_F : \mathcal{A}_c^{p+k}(M) \longrightarrow \mathcal{A}_c^k(M/F)$$

which commutes with the exterior derivatives d_M and d_H . Given $\omega \in \mathcal{A}_c^{p+k}(M)$, write $\omega = \sum \omega_i$ where $\omega_i \in \mathcal{A}_c^{p+k}(U_i)$. Integrate ω_i along the fibers of the submersion $\pi_i : U_i \rightarrow T_i$ to obtain $\int_{U_i} \omega_i \in \mathcal{A}_c^k(T_i)$.

Define $\int_F \omega \in \mathcal{A}_c^k(M/F)$ to be the class of $\sum_i \int_{U_i} \omega_i$. It is independent of the choice of the ω_i and of the cover \mathcal{U} .

As \int_F commutes with d_M and d_H , it induces the map $\int_F : H^{p+k}(M; \mathbb{R}) \rightarrow H_c^k(M/F)$.

2. THE K -THEORY INDEX

In this section, we recall the definition of the analytic index of a Dirac operator defined along the leaves of a foliation. We begin with some general remarks about operators along the leaves of foliations.

Let E_1 and E'_1 be two complex vector bundles over M with Hermitian metrics and connections, and set $E = r^*E_1$ and $E' = r^*E'_1$ with the pulled back metrics and connections. A pseudo-differential \mathcal{G} -operator with uniform support acting from E to E' is a smooth family $(P_x)_{x \in M}$ of \mathcal{G} -invariant pseudo-differential operators, where for each x , P_x is an operator acting from $E|_{\tilde{L}_x}$ to $E'|_{\tilde{L}_x}$. The \mathcal{G} -invariance property means that for any $\gamma \in \tilde{L}_x^y = \tilde{L}_x \cap \tilde{L}^y$, we have

$$(\gamma \cdot P)_y = U_\gamma \circ P_x \circ U_\gamma^{-1} = P_y,$$

where U_γ denotes the operator on sections of any bundle induced by the isomorphism $\gamma : \tilde{L}_y \rightarrow \tilde{L}_x$ given by composition with γ ; for instance

$$U_\gamma : C_c^\infty(\tilde{L}_x, E) \longrightarrow C_c^\infty(\tilde{L}_y, E).$$

The smoothness assumption is rigorously defined in [NWX96]. If we denote by K_x the Schwartz kernel of P_x , then the \mathcal{G} -invariance assumption implies that the family $(K_x)_{x \in M}$ induces a distributional section

K of $\text{Hom}(E, \widehat{E}')$ over \mathcal{G} which is smooth outside $\mathcal{G}^{(0)} = M$. Here $\widehat{E}' = s^*E'_1$, which is also the pullback bundle of E' under the diffeomorphism $\gamma \mapsto \gamma^{-1}$. Since M is compact, the uniform support condition becomes the assumption that the support of K is compact in \mathcal{G} . The space of uniformly supported pseudo-differential \mathcal{G} -operators from E to E' is denoted $\Psi^\infty(\mathcal{G}; E, E')$, and the space of uniformly supported regularizing \mathcal{G} -operators is denoted by $\Psi^{-\infty}(\mathcal{G}; E, E')$. When $E' = E$ we simply denote the corresponding spaces by $\Psi^\infty(\mathcal{G}; E)$ and $\Psi^{-\infty}(\mathcal{G}; E)$. The Schwartz Kernel Theorem identifies $\Psi^{-\infty}(\mathcal{G}; E, E')$ with $C_c^\infty(\mathcal{G}, \text{Hom}(E, \widehat{E}'))$, see [Con79, NWX96].

An element of $\Psi^\infty(\mathcal{G}; E, E')$ is elliptic if it is elliptic when restricted to each leaf of F_s . The parametrix theorem can be extended to the foliated case and we have

Proposition 2.1. [Con79] *Let P be a uniformly supported elliptic pseudo-differential \mathcal{G} -operator acting from E to E' . Then there exists a uniformly supported pseudo-differential \mathcal{G} -operator Q acting from E' to E such that*

$$I_E - Q \circ P \in \Psi^{-\infty}(\mathcal{G}; E) \text{ and } I_{E'} - P \circ Q \in \Psi^{-\infty}(\mathcal{G}; E').$$

Here I_E and $I_{E'}$ denote the identity operators of E and E' respectively.

A classical K -theory construction assigns to any uniformly supported elliptic pseudo-differential \mathcal{G} -operator P from E to E' , a K -theory class

$$\text{Ind}_a(P) \in K_0(\Psi^{-\infty}(\mathcal{G}; E \oplus E')) = K_0(C_c^\infty(\mathcal{G}, \text{Hom}(E \oplus E')))$$

called the analytic index of P [CM91, BH-I]. It will be useful to define this index class using functional calculus in a wider space of smoothing operators, so we now relax the uniform support condition and extend the above pseudodifferential calculus.

A super-exponentially decaying \mathcal{G} -operator from E to E' is a family $P = (P_x)_{x \in M}$ of smoothing \mathcal{G} -operators so that its Schwartz kernel $P_x(y, z)$ is smooth in x, y , and z , and satisfies

2.2. *Given non-negative integer multi indices α, β , and γ , there are positive constants ϵ, C_1 , and C_2 , such that for all $x \in M, y, z \in \tilde{L}_x$,*

$$\left\| \frac{\partial^{|\alpha|+|\beta|+|\gamma|} P_x(y, z)}{\partial x^\alpha \partial y^\beta \partial z^\gamma} \right\| \leq C_1 \exp \left[\frac{-d_x(y, z)^{1+\epsilon}}{C_2} \right].$$

Here $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ come from coordinates obtained from a finite good cover \mathcal{U} of M and $d_x(\cdot, \cdot)$ is the distance on \tilde{L}_x . The space of all such operators is denoted $\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; E, E')$ or $C_{\mathfrak{S}}^\infty(\mathcal{G}; \text{Hom}(E, \widehat{E}'))$. Again when $E' = E$ we denote the corresponding spaces by $\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; E)$ and $C_{\mathfrak{S}}^\infty(\mathcal{G}; \text{Hom}(E))$ for simplicity. When E and E' are trivial line bundles, we omit them and denote the corresponding spaces by $\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G})$ and $C_{\mathfrak{S}}^\infty(\mathcal{G})$.

Lemma 2.3. *When $E' = E$, the space $\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; E)$ is an algebra.*

Proof. Let P and $Q \in \Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; E)$, with constants ϵ_1, C_1, C_2 and ϵ_2, D_1, D_2 respectively, for the estimate given by Equation 2.2. We may replace ϵ_1 and ϵ_2 by $\epsilon = \min(\epsilon_1, \epsilon_2)$. Set $\alpha = 1 + \epsilon$, $C = C_1 D_1$ and $D = C_2 + D_2$. Then for $y, z \in \tilde{L}_x$,

$$\begin{aligned} |P_x \circ Q_x(y, z)| &= \left| \int_{\tilde{L}_x} P_x(y, w) Q_x(w, z) dw \right| \leq \int_{\tilde{L}_x} C_1 e^{-d(y, w)^\alpha / C_2} D_1 e^{-d(w, z)^\alpha / D_2} dw \leq \\ &\int_{\tilde{L}_x} C e^{-d(y, w)^\alpha / D} e^{-d(w, z)^\alpha / D} dw = C e^{-(d(y, z)^\alpha / 2^\alpha D)} \int_{\tilde{L}_x} e^{-(d(y, w)^\alpha + d(w, z)^\alpha - (d(y, z)^\alpha / 2)^\alpha) / D} dw \leq \\ &C e^{-(d(y, z)^\alpha / 2^\alpha D)} \left[\int_{S_z} e^{-d(y, w)^\alpha / D} dw + \int_{S_y} e^{-d(w, z)^\alpha / D} dw \right] \leq \\ &C e^{-(d(y, z)^\alpha / 2^\alpha D)} \left[\int_{\tilde{L}_x} e^{-d(y, w)^\alpha / D} dw + \int_{\tilde{L}_x} e^{-d(w, z)^\alpha / D} dw \right], \end{aligned}$$

where

$$S_z = \{w \in \tilde{L}_x \mid d(w, z) \geq d(y, z)/2\} \quad \text{and} \quad S_y = \{w \in \tilde{L}_x \mid d(y, w) \geq d(y, z)/2\}.$$

Now each of the integrals $\int_{\tilde{L}_x} e^{-d(y,w)^\alpha/D} dw$ and $\int_{\tilde{L}_x} e^{-d(w,z)^\alpha/D} dw$ is bounded independently of x, y , and z . This is a standard argument for foliations of compact manifolds. Since M is compact, the leaves \tilde{L}_x have at most (uniformly bounded) exponential growth, and the integrands are super-exponentially decaying with uniform super-exponential bounds. This gives us the estimate in 2.2 for $P \circ Q$.

To get the estimate for the derivatives $\partial^{|\alpha_1|+|\beta|+|\gamma|}(P \circ Q)_x(y, z)/\partial x^{\alpha_1} \partial y^\beta \partial z^\gamma$ we need only note that these are finite sums of the form

$$\sum_{\alpha_1+\alpha_2=\alpha} \int_{\tilde{L}_x} \left(\frac{\partial^{|\alpha_1|+|\beta|} P_x(y, w)}{\partial x^{\alpha_1} \partial y^\beta} \right) \left(\frac{\partial^{|\alpha_2|+|\gamma|} Q_x(w, z)}{\partial x^{\alpha_2} \partial z^\gamma} \right) dw.$$

We can then repeat the argument above, using the estimates for the individual integrands. \square

There is a continuous embedding of algebras

$$j_{\mathfrak{S}} : C_c^\infty(\mathcal{G}; \text{Hom}(E \oplus E')) \hookrightarrow C_{\mathfrak{S}}^\infty(\mathcal{G}; \text{Hom}(E \oplus E')),$$

and we define the Schwartz analytic index $\text{Ind}_a^{\mathfrak{S}}$ as the composition of the analytic index Ind_a and the induced morphism $j_{\mathfrak{S}*} : K_*(\mathcal{G}; \text{Hom}(E \oplus E')) \rightarrow K_0(C_{\mathfrak{S}}^\infty(\mathcal{G}; \text{Hom}(E \oplus E')))$. So if P is a uniformly supported elliptic pseudo-differential \mathcal{G} -operator,

$$\text{Ind}_a^{\mathfrak{S}}(P) = j_{\mathfrak{S}*}(\text{Ind}_a(P)) \in K_0(C_{\mathfrak{S}}^\infty(\mathcal{G}; \text{Hom}(E \oplus E'))).$$

By classical arguments, see for instance [MN96], it is easy to check that $\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; E, E')$ is a right module over the algebra $\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G})$. The extended pseudodifferential calculus is defined by:

$$\Psi_{\mathfrak{S}}^\infty(\mathcal{G}; E, E') := \Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; E, E') \otimes_{\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G})} \Psi^\infty(\mathcal{G}; E, E').$$

It is generated by $\Psi^\infty(\mathcal{G}; E, E')$ and $\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; E, E')$. When $E' = E$, we obtain in this way an algebra of pseudodifferential operators. The subspace $\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; E)$ is then an ideal in the algebra $\Psi_{\mathfrak{S}}^\infty(\mathcal{G}; E)$. This is due to the estimate given in 2.2. In particular, we may define $\text{Ind}_a^{\mathfrak{S}}(P)$ directly using a parametrix $Q \in \Psi_{\mathfrak{S}}^\infty(\mathcal{G}; E', E)$ and the classical construction, and it is obvious that the two definitions agree.

The construction of the Chern character $\text{ch}_a : K_0(C_c^\infty(\mathcal{G}; \text{Hom}(E \oplus E'))) \rightarrow H_c^*(M/F)$ in [BH-I], reviewed in Section 3 below, also extends to this case thanks to Lemma 2.5, p. 443 of [HL02]. Note that this lemma requires one of the elements to be uniformly exponentially decaying while the other must have uniformly bounded coefficients. But if an operator is uniformly exponentially decaying it does have uniformly bounded coefficients. Thus we have

$$\text{ch}_a^{\mathfrak{S}} : K_0(C_{\mathfrak{S}}^\infty(\mathcal{G}; \text{Hom}(E \oplus E'))) \longrightarrow H_c^*(M/F)$$

and

$$\text{ch}_a^{\mathfrak{S}} \circ j_{\mathfrak{S}*} = \text{ch}_a.$$

Finally, the formula for ch_a in Definition 3.2 below also holds for $\text{ch}_a^{\mathfrak{S}}$.

Now assume that the dimension p of F is even and denote by D a generalized Dirac operator for the foliation F . One of the most important examples of such an operator is given by the longitudinal Dirac operator with coefficients in a vector bundle over M . It is defined as follows. As above, let E_1 be a complex vector bundle over M with Hermitian metric and connection, and set $E = r^*(E_1)$ with the pulled back metric and connection. Assume that the tangent bundle TF of F is spin with a fixed spin structure. Then TF_s is also spin, and we endow it with the pulled back spin structure from TF . Denote by $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ the bundle of spinors along the leaves of F_s . Denote by ∇^0 the connection on TF_s given by the orthogonal projection of the Levi-Civita connection for g_0 on $T\mathcal{G}$. ∇^0 is then the Levi-Civita connection on each leaf of F_s for the induced metric. For all $x \in M$, ∇^0 induces a connection ∇^0 on $\mathcal{S}|_{\tilde{L}_x}$ and we denote also by ∇^0 the tensor product connection on $\mathcal{S} \otimes E|_{\tilde{L}_x}$. These data determine a smooth family $D = \{D_x\}$ of Dirac

operators, where D_x acts on sections of $\mathcal{S} \otimes E|_{\tilde{L}_x}$ as follows. Let X_1, \dots, X_p be a local oriented orthonormal basis of $T\tilde{L}_x$, and set

$$D_x = \sum_{i=1}^p \rho(X_i) \nabla_{X_i}^0$$

where $\rho(X_i)$ is the Clifford action of X_i on the bundle $\mathcal{S} \otimes E|_{\tilde{L}_x}$. Then D_x does not depend on the choice of the X_i , and it is an odd operator for the \mathbb{Z}_2 grading of $\mathcal{S} \otimes E = (\mathcal{S}^+ \otimes E) \oplus (\mathcal{S}^- \otimes E)$. Set $D^+ = D : C_c^\infty(\mathcal{S}^+ \otimes E) \rightarrow C_c^\infty(\mathcal{S}^- \otimes E)$ and $D^- = D : C_c^\infty(\mathcal{S}^- \otimes E) \rightarrow C_c^\infty(\mathcal{S}^+ \otimes E)$. For more on generalized Dirac operators, see [LM89].

A super-exponentially decaying \mathcal{G} -operator on $\mathcal{S} \otimes E$ is defined to be an operator of the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where each A_{ij} is a smoothing operator whose Schwartz kernel $A_{ij,x}(y, z)$ is smooth in x, y , and z , and satisfies the estimate in 2.2. A_{11} maps sections of $\mathcal{S}^+ \otimes E$ to itself, A_{12} maps sections of $\mathcal{S}^- \otimes E$ to sections of $\mathcal{S}^+ \otimes E$, etc. The set of all such operators is denoted $\Psi_{\mathbb{G}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ or $C_{\mathbb{G}}^\infty(\mathcal{G}; \text{Hom}(\mathcal{S} \otimes E))$. If we unitalize $\Psi_{\mathbb{G}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ by adding two copies of \mathbb{C} corresponding to the projections $\pi_{\pm} : C_c^\infty(\mathcal{S} \otimes E) \rightarrow$

$C_c^\infty(\mathcal{S}^{\pm} \otimes E)$, then we get a unital algebra that we denote by $\tilde{\Psi}_{\mathbb{G}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$. Note that $\pi_+ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and $\pi_- = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$. Since the grading operator α for $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ satisfies $\alpha = \pi_+ - \pi_-$, α belongs to $\tilde{\Psi}_{\mathbb{G}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$.

The odd operator D is elliptic, so its analytic index is defined using a parametrix Q for D which is also odd, i.e.

$$Q = Q^{\pm} : C_c^\infty(\mathcal{S}^{\pm} \otimes E) \longrightarrow C_c^\infty(\mathcal{S}^{\mp} \otimes E).$$

Set

$$S_+ = I - Q^- \circ D^+ \text{ and } S_- = I - D^+ \circ Q^-$$

so

$$S_{\pm} : C_c^\infty(\mathcal{S}^{\pm} \otimes E) \longrightarrow C_c^\infty(\mathcal{S}^{\pm} \otimes E).$$

Using embeddings of our bundles in trivial bundles and computing the boundary map in K -theory, it is easy to see that the analytic index of D is the K -theory class [CM91] in $K_0(\Psi^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) = K_0(C_c^\infty(\mathcal{G}; \text{Hom}(\mathcal{S} \otimes E)))$,

$$\text{Ind}_a(D^+) = [e] - [\pi_-],$$

where the idempotent e is given by

$$2.4. \quad e = \begin{pmatrix} S_+^2 & -Q^- \circ (S_- + S_-^2) \\ -S_- \circ D^+ & I - S_-^2 \end{pmatrix}.$$

The class $[e] - [\pi_-]$ lives in the K_0 -group of the unital algebra $\tilde{\Psi}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ but its image in the K_0 -group of $\mathbb{C} \oplus \mathbb{C}$ under the map induced by

$$p : \begin{pmatrix} A_{11} + \lambda I_{\mathcal{S}^+ \otimes E} & A_{12} \\ A_{21} & A_{22} + \mu I_{\mathcal{S}^+ \otimes E} \end{pmatrix} \longmapsto (\lambda, \mu),$$

is trivial. Since this epimorphism admits a splitting homomorphism, it is clear that the kernel of the induced map p_* is isomorphic to the K_0 -group of the non-unital algebra $\Psi^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$. Hence, the index $\text{Ind}_a(D^+) = [e] - [\pi_-]$ is well defined.

Proposition 2.5. *Assume that \mathcal{G} is Hausdorff. Set*

$$P(tD) = \begin{bmatrix} e^{-tD^- D^+} & (-e^{-tD^- D^+ / 2}) \frac{I - e^{-tD^- D^+}}{tD^- D^+} \sqrt{t} D^- \\ -e^{-tD^+ D^- / 2} \sqrt{t} D^+ & I - e^{-tD^+ D^-} \end{bmatrix}.$$

Then, for all $t > 0$, $P(tD)$ is an idempotent in $\tilde{\Psi}_{\mathfrak{G}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ and

$$[P(tD)] - [\pi_-] = \text{Ind}_a^{\mathfrak{G}}(D^+) \in K_0(C_{\mathfrak{G}}^{\infty}(\mathcal{G}; \text{Hom}(\mathcal{S} \otimes E))).$$

Proof. It is classical that all the operators in $P(tD)$ (with the possible exception of the term π_-) are smoothing when restricted to any \tilde{L}_x , so their Schwartz kernels are smooth when restricted to any \tilde{L}_x . Thus to check for smoothness, we need only check that they are smooth transversely, i.e. smooth in the variable $x \in M$. The coefficients of the D^{\pm} are smooth, and Corollary 3.11 of [He95] (which requires that \mathcal{G} be Hausdorff), says that the $e^{-tD^{\pm}D^{\mp}}$ are transversely smooth. We will show presently that $e^{-tD^-D^+/2} \frac{I - e^{-tD^-D^+}}{tD^-D^+} \sqrt{t}D^- = \sqrt{t}D^- e^{-tD^+D^-/2} \frac{I - e^{-tD^+D^-}}{tD^+D^-}$ is also transversely smooth.

By [He95], the Schwartz kernels $P_{t,x}^{\pm}(y, z)$ of the $e^{-tD^{\pm}D^{\mp}}$ satisfy the following estimate. Given a non-negative integer i and non-negative integer multi indices α, β , and γ , and a real number $T > 0$, there is a constant $C > 0$ such that for all $x \in M, y, z \in \tilde{L}_x$, and $0 \leq t \leq T$,

$$\mathbf{2.6.} \quad \left\| \frac{\partial^{i+|\alpha|+|\beta|+|\gamma|} P_{t,x}^{\pm}(y, z)}{\partial t^i \partial x^{\alpha} \partial y^{\beta} \partial z^{\gamma}} \right\| \leq C t^{-(p/2+i+|\alpha|+|\beta|+|\gamma|)} \exp\left[\frac{-d_x(y, z)^2}{4t}\right].$$

It follows immediately that the $e^{-tD^{\pm}D^{\mp}}$ and the $e^{-tD^{\pm}D^{\mp}/2}$ satisfy the estimate in Equation 2.2, and so also $e^{-tD^+D^-/2} \sqrt{t}D^+ = \sqrt{t}D^+ e^{-tD^-D^+/2}$, since the derivatives of the coefficients of D^{\pm} are uniformly bounded on \mathcal{G} .

To handle $e^{-tD^-D^+/2} \frac{I - e^{-tD^-D^+}}{tD^-D^+} \sqrt{t}D^- = \sqrt{t}D^- e^{-tD^+D^-/2} \frac{I - e^{-tD^+D^-}}{tD^+D^-}$, note that

$$\frac{d}{ds} \left[\frac{I - e^{-sD^+D^-}}{D^+D^-} \right] = e^{-sD^+D^-}, \quad \text{so} \quad \frac{I - e^{-tD^+D^-}}{tD^+D^-} = \frac{1}{t} \int_0^t e^{-sD^+D^-} ds.$$

Thus

$$\sqrt{t}D^- e^{-tD^+D^-/2} \frac{I - e^{-tD^+D^-}}{tD^+D^-} = \frac{\sqrt{t}D^-}{t} \int_0^t e^{-(t/2+s)D^+D^-} ds = \frac{\sqrt{t}D^-}{t} \int_{t/2}^{3t/2} e^{-sD^+D^-} ds.$$

A simple calculation using Equation 2.6 above then shows that for fixed t , $\sqrt{t}D^- e^{-tD^+D^-/2} \frac{I - e^{-tD^+D^-}}{tD^+D^-}$ is transversely smooth and that it satisfies the estimate in Equation 2.2.

It is easy to check that the operator $Q(tD) = Q^{\pm}(tD)$ where $Q^-(tD) = \frac{I - e^{-tD^-D^+/2}}{tD^-D^+} \sqrt{t}D^-$ and $Q^+(tD) = \frac{I - e^{-tD^+D^-/2}}{tD^+D^-} \sqrt{t}D^+$, is a parametrix for $\sqrt{t}D$. The corresponding idempotent e given by Equation 2.4 is then $P(tD)$, so the Schwartz analytic index of tD is just $[P(tD)] - [\pi_-]$. Since the index class only depends on the K -theory class of the principal symbol, it is clear that the K -theory class $[P(tD)] - [\pi_-]$ is independent of $t > 0$. \square

3. THE CHERN CHARACTER IN HAEFLIGER COHOMOLOGY

In this section we review the construction of the Chern-Connes character in Haefliger cohomology given in [BH-I]. In view of our definition of the analytic index through the K -group of the unitalization $\tilde{\Psi}_{\mathfrak{G}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$, the Chern character is easy to express in terms of heat kernels. We may regard the connection ∇ on $\mathcal{S} \otimes E$ as an operator of degree one on $C^{\infty}(\mathcal{S} \otimes E \otimes \wedge T^* \mathcal{G})$ where on decomposable sections $\phi \otimes \omega$, $\nabla(\phi \otimes \omega) = (\nabla\phi) \otimes \omega + \phi \otimes d\omega$. The foliation F_s has normal bundle $\nu_s^* \simeq s^*(T^*M)$, and ∇ defines a *quasi-connection* ∇^{ν} acting on $C^{\infty}(\mathcal{S} \otimes E \otimes \wedge \nu_s^*)$ by the composition

$$C^{\infty}(\mathcal{S} \otimes E \otimes \wedge \nu_s^*) \xrightarrow{i} C^{\infty}(\mathcal{S} \otimes E \otimes \wedge T^* \mathcal{G}) \xrightarrow{\nabla} C^{\infty}(\mathcal{S} \otimes E \otimes \wedge T^* \mathcal{G}) \xrightarrow{p_{\nu}} C^{\infty}(\mathcal{S} \otimes E \otimes \wedge \nu_s^*),$$

where i is the inclusion and p_{ν} is induced by the restriction $p_{\nu} : T^* \mathcal{G} \rightarrow \nu_s^*$.

$C^\infty(\mathcal{S} \otimes E \otimes \wedge \nu_s^*)$ is an $\mathcal{A}(M)$ -module where for $\phi \in C^\infty(\mathcal{S} \otimes E \otimes \wedge \nu_s^*)$, and $\omega \in \mathcal{A}(M)$, we set

$$\omega \cdot \phi = p_\nu(s^*(\omega))\phi,$$

where again the map $p_\nu : \mathcal{A}(\mathcal{G}) \rightarrow C^\infty(\wedge \nu_s^*)$ is induced by the projection $p_\nu : T^*\mathcal{G} \rightarrow \nu_s^*$.

Recall $\Psi^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E) \simeq C_c^\infty(\mathcal{G}, \text{Hom}(\mathcal{S} \otimes E))$ the space of uniformly supported regularizing \mathcal{G} -operators. We may consider the algebra

$$\Psi^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E) \widehat{\otimes}_{C^\infty(M)} \mathcal{A}(M)$$

as a subspace of the space of $\mathcal{A}(M)$ -equivariant endomorphisms of $C^\infty(\mathcal{S} \otimes E \otimes \wedge \nu_s^*)$ by using the $\mathcal{A}(M)$ module structure of $C^\infty(\mathcal{S} \otimes E \otimes \wedge \nu_s^*)$.

Denote by $\partial_\nu : \text{End}(C^\infty(\mathcal{S} \otimes E \otimes \wedge \nu_s^*)) \rightarrow \text{End}(C^\infty(\mathcal{S} \otimes E \otimes \wedge \nu_s^*))$ the linear operator given by the graded commutator

$$\partial_\nu(T) = [\nabla^\nu, T].$$

The operator ∂_ν maps the space

$$\mathcal{A}_c(\mathcal{G}, \mathcal{S} \otimes E) := \Psi^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E) \widehat{\otimes}_{C^\infty(M)} \mathcal{A}(M)$$

to itself, and $(\partial_\nu)^2$ is given by the commutator with the curvature $\theta = (\nabla^\nu)^2$ of ∇^ν .

In the same way, we consider the algebra

$$\mathcal{A}_\mathfrak{E}(\mathcal{G}, \mathcal{S} \otimes E) := \Psi_\mathfrak{E}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E) \widehat{\otimes}_{C^\infty(M)} \mathcal{A}(M)$$

where $\Psi_\mathfrak{E}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ is the algebra of superexponentially decaying operators defined in the previous section. Then ∂_ν also acts on $\mathcal{A}_\mathfrak{E}(\mathcal{G}, \mathcal{S} \otimes E)$ with $(\partial_\nu)^2$ given again by the commutator with the zero-th order differential operator θ .

By the Schwartz kernel theorem, the algebra $\mathcal{A}_\mathfrak{E}(\mathcal{G}, \mathcal{S} \otimes E)$ is isomorphic to the algebra

$$C_\mathfrak{E}^\infty(\mathcal{G}; \text{Hom}(\mathcal{S} \otimes E)) \widehat{\otimes}_{C^\infty(M)} \mathcal{A}(M).$$

For any $T \in \mathcal{A}_\mathfrak{E}(\mathcal{G}, \mathcal{S} \otimes E)$, define the trace of T to be the (compactly supported) Haefliger k -form $\text{Tr}(T)$ given by

$$\text{Tr}(T) = \int_F \text{tr}(K(\bar{x})) dx = \int_F \text{tr}(K(x, x)) dx,$$

where K is the smooth Schwartz kernel of T , \bar{x} is the class of the constant path at x , $\text{tr}(K(\bar{x}))$ is the usual trace of $K(\bar{x}) \in \text{End}((\mathcal{S} \otimes E)_{\bar{x}}) \otimes \wedge T^*M_x$ and so belongs to $\wedge T^*M_x$, and dx is the leafwise volume form associated with the fixed orientation of the foliation F . The map

$$\text{Tr} : \mathcal{A}_\mathfrak{E}(\mathcal{G}, \mathcal{S} \otimes E) \longrightarrow \mathcal{A}_c(M/F)$$

is then a graded trace which satisfies $\text{Tr} \circ \partial_\nu = d_H \circ \text{Tr}$, see [BH-I].

Since ∂_ν^2 is not necessarily zero, we used Connes' X -trick to construct a new graded differential algebra $(\tilde{\mathcal{A}}_\mathfrak{E}, \delta)$ out of the graded quasi-differential algebra $(\mathcal{A}_\mathfrak{E}(\mathcal{G}, \mathcal{S} \otimes E), \partial_\nu)$, see [Con94], p. 229. First, note that the curvature operator θ is a multiplier of $\mathcal{A}_\mathfrak{E}(\mathcal{G}, \mathcal{S} \otimes E)$. As a vector space $\tilde{\mathcal{A}}_\mathfrak{E} = M_2(\mathcal{A}_\mathfrak{E}(\mathcal{G}, \mathcal{S} \otimes E))$. An element $\tilde{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \tilde{\mathcal{A}}_\mathfrak{E}$ is homogeneous of degree $\partial \tilde{T} = k$ if

$$k = \partial T_{11} = \partial T_{12} + 1 = \partial T_{21} + 1 = \partial T_{22} + 2.$$

On homogeneous elements of $\tilde{\mathcal{A}}_\mathfrak{E}$, δ is given by

$$\delta \tilde{T} = \begin{pmatrix} \partial_\nu T_{11} & \partial_\nu T_{12} \\ -\partial_\nu T_{21} & -\partial_\nu T_{22} \end{pmatrix} + \begin{pmatrix} 0 & -\theta \\ 1 & 0 \end{pmatrix} \tilde{T} + (-1)^{\partial \tilde{T}} \tilde{T} \begin{pmatrix} 0 & 1 \\ -\theta & 0 \end{pmatrix},$$

and is extended to non-homogenous elements by linearity. A straightforward computation gives $\delta^2 = 0$. For homogeneous $T \in \mathcal{A}_\mathfrak{E}(\mathcal{G}, \mathcal{S} \otimes E)$, the differential δ on $\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} \in \tilde{\mathcal{A}}_\mathfrak{E}$ is given by

$$\delta \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \partial_\nu T & (-1)^{\partial T} T \\ T & 0 \end{pmatrix}.$$

Set

$$\Theta = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix}$$

and define a new product on $\tilde{\mathcal{A}}_{\mathfrak{S}}$ by

$$\tilde{T} * \tilde{T}' = \tilde{T} \Theta \tilde{T}'.$$

This makes $(\tilde{\mathcal{A}}_{\mathfrak{S}}, \delta)$ a graded differential algebra. For simplicity, we shall remove the multiplication $*$ from the notation and write $\tilde{T}\tilde{T}'$ for $\tilde{T} * \tilde{T}'$, when no confusion will occur.

The graded algebra $A_{\mathfrak{S}}(\mathcal{G}, \mathcal{S} \otimes E)$ embeds as a subalgebra of $\tilde{\mathcal{A}}_{\mathfrak{S}}$ by using the map

$$T \mapsto \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}.$$

We shall therefore also denote by T the image in $\tilde{\mathcal{A}}_{\mathfrak{S}}$ of any $T \in A_{\mathfrak{S}}(\mathcal{G}, \mathcal{S} \otimes E)$.

For homogeneous $\tilde{T} \in \tilde{\mathcal{A}}_{\mathfrak{S}}$ define

$$\Phi(\tilde{T}) = \text{Tr}(T_{11}) - (-1)^{\partial \tilde{T}} \text{Tr}(T_{22}\theta),$$

and extend to arbitrary elements by linearity. The map $\Phi : \tilde{\mathcal{A}}_{\mathfrak{S}} \rightarrow \mathcal{A}_c^*(M/F)$ is then a graded trace, and again we have $\Phi \circ \delta = d_H \circ \Phi$, see [BH-I].

The (algebraic) Chern-Connes character in the even case is the morphism

$$\text{ch}_a : K_0(C_{\mathfrak{S}}^{\infty}(\mathcal{G}, \mathcal{S} \otimes E)) = K_0(\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) \longrightarrow H_c^*(M/F)$$

defined as follows. Denote by $\hat{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ the minimal unitalization of $\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$. This amounts to adding a copy of the complex numbers \mathbb{C} , so

$$\hat{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E) = \Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E) \oplus \mathbb{C}.$$

Let $M_N(\hat{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E))$ be the space of $N \times N$ matrices with coefficients in $\hat{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$. Denote by $\text{tr} : M_N(\hat{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) \rightarrow \Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ the usual trace.

Recall the following:

Theorem 3.1. [BH-I] *Let $B = [\tilde{e}_1] - [\tilde{e}_2]$ be an element of $K_0(\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E))$, where $\tilde{e}_1 = (e_1, \lambda_1)$ and $\tilde{e}_2 = (e_2, \lambda_2)$ are idempotents in $M_N(\hat{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E))$. Then the Haefliger forms*

$$(\Phi \circ \text{tr})\left(e_1 \exp\left(\frac{-(\delta e_1)^2}{2i\pi}\right)\right) \text{ and } (\Phi \circ \text{tr})\left(e_2 \exp\left(\frac{-(\delta e_2)^2}{2i\pi}\right)\right)$$

are closed and the Haefliger cohomology class of their difference depends only on B .

Definition 3.2. *The algebraic Chern character $\text{ch}_a(B)$ of B is the Haefliger cohomology class*

$$\text{3.3.} \quad \text{ch}_a(B) = \left[(\Phi \circ \text{tr})\left(e_1 \exp\left(\frac{-(\delta e_1)^2}{2i\pi}\right)\right) \right] - \left[(\Phi \circ \text{tr})\left(e_2 \exp\left(\frac{-(\delta e_2)^2}{2i\pi}\right)\right) \right].$$

In order to effectively compute the Chern character of the index of a generalized Dirac operator for F , we need some further results. The exact sequence of algebras

$$0 \rightarrow \Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E) \xrightarrow{i} \tilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E) \xrightarrow{p} \mathbb{C}^2 \rightarrow 0$$

has a splitting homomorphism $\varrho : \mathbb{C}^2 \rightarrow \tilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ given by $\varrho(\lambda, \mu) = \lambda\pi_+ + \mu\pi_-$. Therefore the kernel of the induced map

$$p_* : K_0(\tilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) \longrightarrow K_0(\mathbb{C}^2) \simeq \mathbb{Z}^2,$$

is isomorphic to the group $K_0(\Psi_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E))$. Denote by p_0 the obvious projection of $\hat{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ onto \mathbb{C} . Then the inclusion map

$$\beta : \hat{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E) \longrightarrow \tilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E),$$

given by $\beta(T, \lambda) = T + \lambda\pi_+ + \lambda\pi_-$ induces the isomorphism

$$\beta_* : K_0(\hat{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) = \text{Ker}(p_{0,*}) \longrightarrow \text{Ker}(p_*) \subset K_0(\tilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)).$$

We shall use the universal graded algebra in the proof of Proposition 3.4 below, so we recall its definition. To any algebra \mathcal{C} , there corresponds a (universal) differential graded algebra $\Omega(\mathcal{C}) = \bigoplus_{n \geq 0} \Omega^n(\mathcal{C})$ which is defined by

$$\Omega^0(\mathcal{C}) := \mathcal{C} \oplus \mathbb{C}, \text{ and for } n \geq 1, \Omega^n(\mathcal{C}) := (\mathcal{C} \oplus \mathbb{C}) \otimes \mathcal{C}^{\otimes n}.$$

The differential $d : \Omega^n(\mathcal{C}) \rightarrow \Omega^{n+1}(\mathcal{C})$ is defined for $a^j \in \mathcal{C}$ and $c \in \mathbb{C}$ by

$$d[(a^0 + c) \otimes a^1 \otimes \cdots \otimes a^n] := 1 \otimes a^0 \otimes a^1 \otimes \cdots \otimes a^n.$$

It is clear that by definition $d^2 = 0$. The space $\Omega^n(\mathcal{C})$ is endowed with a natural right \mathcal{C} -module structure (and hence right $\mathcal{C} \oplus \mathbb{C}$ -module structure) defined by

$$((a^0 + c) \otimes a^1 \otimes \cdots \otimes a^n) a^{n+1} := (-1)^n \sum_{j=0}^n (-1)^j (a^0 + c) \otimes \cdots \otimes a^j a^{j+1} \otimes \cdots \otimes a^{n+1}.$$

The algebra structure of $\Omega(\mathcal{C})$ is defined by setting

$$((a^0 + c) \otimes a^1 \otimes \cdots \otimes a^n)(b^0 \otimes b^1 \otimes \cdots \otimes b^k) := ((a^0 + c) \otimes a^1 \otimes \cdots \otimes a^n) b^0 \otimes b^1 \otimes \cdots \otimes b^k$$

and

$$((a^0 + c) \otimes a^1 \otimes \cdots \otimes a^n) c' \otimes b^1 \otimes \cdots \otimes b^k := c' [(a^0 + c) \otimes a^1 \otimes \cdots \otimes a^n \otimes b^1 \otimes \cdots \otimes b^k].$$

A straightforward verification shows that $(\Omega(\mathcal{C}), d)$ is a differential graded algebra, see [Con85]. We point out that by definition

$$(a^0 + c) da^1 \cdots da^n = (a^0 + c) \otimes a^1 \otimes \cdots \otimes a^n.$$

The following is known to experts. We give the proof for completeness, since it will be used in the sequel.

Proposition 3.4. *Let \tilde{e} and \tilde{e}' be two idempotents in $M_N(\tilde{\Psi}_{\mathfrak{G}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E))$ such that $[\tilde{e}] - [\tilde{e}']$ belongs to the kernel of p_* . Then the Haefliger forms*

$$\begin{aligned} & (\Phi \circ \text{tr}) \left((\tilde{e} - (\varrho \circ p)(\tilde{e})) \exp \left(\frac{-(\delta(\tilde{e} - (\varrho \circ p)(\tilde{e})))^2}{2i\pi} \right) \right) \text{ and} \\ & (\Phi \circ \text{tr}) \left((\tilde{e}' - (\varrho \circ p)(\tilde{e}')) \exp \left(\frac{-(\delta(\tilde{e}' - (\varrho \circ p)(\tilde{e}')))^2}{2i\pi} \right) \right) \end{aligned}$$

are closed and we have the following equality in Haefliger cohomology:

$$\begin{aligned} (\text{ch}_a \circ \beta_*^{-1})([\tilde{e}] - [\tilde{e}']) &= \left[(\Phi \circ \text{tr}) \left((\tilde{e} - (\varrho \circ p)(\tilde{e})) \exp \left(\frac{-(\delta(\tilde{e} - (\varrho \circ p)(\tilde{e})))^2}{2i\pi} \right) \right) \right] \\ &\quad - \left[(\Phi \circ \text{tr}) \left((\tilde{e}' - (\varrho \circ p)(\tilde{e}')) \exp \left(\frac{-(\delta(\tilde{e}' - (\varrho \circ p)(\tilde{e}')))^2}{2i\pi} \right) \right) \right] \end{aligned}$$

Proof. We define for every $k \geq 0$ a multilinear functional $\tilde{\Phi}$ on the unital algebra $\tilde{\Psi}_{\mathfrak{G}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ by the equality

$$\tilde{\Phi}(\tilde{T}^0, \dots, \tilde{T}^k) := \Phi(T^0 \delta T^1 \cdots \delta T^k) + \Phi(\delta(\Lambda^0 T^1) \delta T^2 \cdots \delta T^k),$$

where $\tilde{T}^j = T^j + \Lambda^j \in \tilde{\Psi}_{\mathfrak{G}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ with

$$T^j = \tilde{T}^j - (\varrho \circ p)(\tilde{T}^j) \in \Psi_{\mathfrak{G}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E) \text{ and } \Lambda^j = \varrho \circ p(\tilde{T}^j) = \begin{pmatrix} \lambda^j & 0 \\ 0 & \mu^j \end{pmatrix} = \lambda^j \pi_+ + \mu^j \pi_-.$$

Then $\tilde{\Phi}$ is a functional on the universal differential graded algebra associated with $\tilde{\Psi}_{\mathfrak{G}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$, see [Con85] and also the bivariant constructions in [CQ97, Nis93]. More precisely, we set:

$$\tilde{\Phi}((\tilde{T}^0 + c) d\tilde{T}^1 \cdots d\tilde{T}^k) := \tilde{\Phi}(\tilde{T}^0, \dots, \tilde{T}^k).$$

We then have by definition

$$(\tilde{\Phi} \circ d) = 0$$

on the universal differential graded algebra associated with $\tilde{\Psi}_{\mathfrak{G}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$.

For $\tilde{T}^j = T^j + \Lambda^j \in \tilde{\Psi}_{\mathfrak{S}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$, we have

$$\begin{aligned}
(-1)^k \tilde{\Phi}([\tilde{T}^0 d\tilde{T}^1 \dots d\tilde{T}^k, \tilde{T}^{k+1}]) &= (-1)^k \tilde{\Phi}(\tilde{T}^0 d\tilde{T}^1 \dots d\tilde{T}^k \tilde{T}^{k+1}) - (-1)^k \tilde{\Phi}(\tilde{T}^{k+1} \tilde{T}^0 d\tilde{T}^1 \dots d\tilde{T}^k) \\
&= \tilde{\Phi}(\tilde{T}^0 \tilde{T}^1 d\tilde{T}^2 \dots d\tilde{T}^{k+1}) + \sum_{j=1}^k (-1)^j \tilde{\Phi}(\tilde{T}^0 d\tilde{T}^1 \dots d\tilde{T}^{j-1} d(\tilde{T}^j \tilde{T}^{j+1}) d\tilde{T}^{j+2} \dots d\tilde{T}^{k+1}) \\
&\quad - (-1)^k \tilde{\Phi}(\tilde{T}^{k+1} \tilde{T}^0 d\tilde{T}^1 \dots d\tilde{T}^k) \\
&= \Phi((T^0 T^1 + \Lambda^0 T^1 + T^0 \Lambda^1) \delta T^2 \dots \delta T^{k+1}) + \Phi(\delta(\Lambda^0 \Lambda^1 T^2) \delta T^3 \dots \delta T^{k+1}) \\
&\quad + \sum_{j=1}^k (-1)^j \Phi(T^0 \delta T^1 \dots \delta T^{j-1} \delta(T^j T^{j+1} + \Lambda^j T^{j+1} + T^j \Lambda^{j+1}) \delta T^{j+2} \dots \delta T^{k+1}) \\
&\quad - \Phi(\delta(\Lambda^0 (T^1 T^2 + \Lambda^1 T^2 + T^1 \Lambda^2)) \delta T^3 \dots \delta T^{k+1}) \\
&\quad + \sum_{j=2}^k (-1)^j \Phi(\delta(\Lambda^0 T^1) \delta T^2 \dots \delta T^{j-1} \delta(T^j T^{j+1} + \Lambda^j T^{j+1} + T^j \Lambda^{j+1}) \delta T^{j+2} \dots \delta T^{k+1}) \\
&\quad - (-1)^k \Phi((T^{k+1} T^0 + T^{k+1} \Lambda^0 + \Lambda^{k+1} T^0) \delta T^1 \dots \delta T^k) \\
&\quad - (-1)^k \Phi(\delta(\Lambda^{k+1} \Lambda^0 T^1) \delta T^2 \dots \delta T^k).
\end{aligned}$$

By using a connection which commutes with the grading we insure that $\partial^\nu(\Lambda) = 0$ for any $\Lambda \in \mathbb{C}\pi_+ \oplus \mathbb{C}\pi_-$. Thus, using the definitions of the product and the differential δ , we can easily deduce the following relations for all Λ, T, Λ' , and T' :

$$\begin{aligned}
\mathbf{3.5.} \quad \partial^\nu(\Lambda T) &= \Lambda(\partial^\nu T), \quad \partial^\nu(T\Lambda) = (\partial^\nu T)\Lambda, \quad \theta\Lambda T = \Lambda\theta T, \quad T\Lambda\delta(T') = T\delta(\Lambda T'), \quad \delta(T\Lambda)T' = (\delta T)(\Lambda T'), \\
\delta(T\Lambda)\delta(T') &= \delta(T)\delta(\Lambda T'), \quad T\Lambda\delta(\Lambda' T') = T\delta(\Lambda\Lambda' T') \quad \text{and} \quad \delta(TT') = \delta T T' + T\delta T'.
\end{aligned}$$

It is then a straightforward calculation that

$$\Phi((T^0 T^1 + \Lambda^0 T^1 + T^0 \Lambda^1) \delta T^2 \dots \delta T^{k+1}) +$$

$$\sum_{j=1}^k (-1)^j \Phi(T^0 \delta T^1 \dots \delta T^{j-1} \delta(T^j T^{j+1} + \Lambda^j T^{j+1} + T^j \Lambda^{j+1}) \delta T^{j+2} \dots \delta T^{k+1})$$

collapses to

$$\Phi(\Lambda^0 T^1 \delta T^2 \dots \delta T^{k+1}) + (-1)^k \Phi(T^0 \delta T^1 \dots \delta T^{k-1} (\delta T^k T^{k+1} + \delta(T^k \Lambda^{k+1}))),$$

and

$$\Phi(\delta(\Lambda^0 \Lambda^1 T^2) \delta T^3 \dots \delta T^{k+1}) - \Phi(\delta(\Lambda^0 (T^1 T^2 + \Lambda^1 T^2 + T^1 \Lambda^2)) \delta T^3 \dots \delta T^{k+1}) +$$

$$\sum_{j=2}^k (-1)^j \Phi(\delta(\Lambda^0 T^1) \delta T^2 \dots \delta T^{j-1} \delta(T^j T^{j+1} + \Lambda^j T^{j+1} + T^j \Lambda^{j+1}) \delta T^{j+2} \dots \delta T^{k+1})$$

collapses to

$$-\Phi(\Lambda^0 T^1 \delta T^2 \dots \delta T^{k+1}) + (-1)^k \Phi(\delta(\Lambda^0 T^1) \delta T^2 \dots \delta T^{k-1} (\delta T^k T^{k+1} + \delta(T^k \Lambda^{k+1}))).$$

Substituting and multiplying by $(-1)^k$, we get

$$\begin{aligned}
\tilde{\Phi}([\tilde{T}^0 d\tilde{T}^1 \dots d\tilde{T}^k, \tilde{T}^{k+1}]) &= (-1)^k \Phi(\Lambda^0 T^1 \delta T^2 \dots \delta T^{k+1}) \\
&+ \Phi(T^0 \delta T^1 \dots \delta T^k T^{k+1}) \\
&+ \Phi(T^0 \delta T^1 \dots \delta T^{k-1} \delta(T^k \Lambda^{k+1})) \\
&- (-1)^k \Phi(\Lambda^0 T^1 \delta T^2 \dots \delta T^{k+1}) \\
&+ \Phi(\delta(\Lambda^0 T^1) \delta T^2 \dots \delta T^k T^{k+1}) \\
&+ \Phi(\delta(\Lambda^0 T^1) \delta T^2 \dots \delta T^{k-1} \delta(T^k \Lambda^{k+1})) \\
&- \Phi(T^{k+1} T^0 \delta T^1 \dots \delta T^k) \\
&- \Phi(T^{k+1} \Lambda^0 \delta T^1 \dots \delta T^k) \\
&- \Phi(\Lambda^{k+1} T^0 \delta T^1 \dots \delta T^k) \\
&- \Phi(\delta(\Lambda^{k+1} \Lambda^0 T^1) \delta T^2 \dots \delta T^k).
\end{aligned}$$

The first and the fourth terms on the right cancel. Using 3.5 and the trace property of Φ we have the following equations:

$$\begin{aligned}
0 &= \Phi(T^0 \delta T^1 \dots \delta T^k T^{k+1}) - \Phi(T^{k+1} T^0 \delta T^1 \dots \delta T^k). \\
0 &= \Phi(T^0 \delta T^1 \dots \delta T^{k-1} \delta(T^k \Lambda^{k+1})) - \Phi(\Lambda^{k+1} T^0 \delta T^1 \dots \delta T^k). \\
0 &= \Phi(\delta(\Lambda^0 T^1) \delta T^2 \dots \delta T^k T^{k+1}) - \Phi(T^{k+1} \Lambda^0 \delta T^1 \dots \delta T^k). \\
0 &= \Phi(\delta(\Lambda^0 T^1) \delta T^2 \dots \delta T^{k-1} \delta(T^k \Lambda^{k+1})) - \Phi(\delta(\Lambda^{k+1} \Lambda^0 T^1) \delta T^2 \dots \delta T^k).
\end{aligned}$$

Thus

$$\tilde{\Phi}([\tilde{T}^0 d\tilde{T}^1 \dots d\tilde{T}^k, \tilde{T}^{k+1}]) = 0.$$

Now a classical argument shows that $\tilde{\Phi}$ is then a graded trace on the whole universal algebra associated with $\tilde{\Psi}_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$.

Given the above, we know that for any idempotent \tilde{e} in the matrix algebra $M_N(\tilde{\Psi}_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E))$, the expression

$$(\tilde{\Phi} \circ \text{tr})\left(\tilde{e} \exp\left(\frac{-(d(\tilde{e}))^2}{2i\pi}\right)\right)$$

is a closed Haefliger form and that its cohomology class only depends on the K -theory class $[\tilde{e}]$ of the idempotent \tilde{e} , see for instance [BH-I]. But note that this Haefliger differential form coincides up to exact Haefliger forms with the differential form

$$(\Phi \circ \text{tr})\left((\tilde{e} - (\varrho \circ p)(\tilde{e})) \exp\left(\frac{-(\delta(\tilde{e} - (\varrho \circ p)(\tilde{e})))^2}{2i\pi}\right)\right)$$

which is then also closed and represents the same Haefliger cohomology class. Thus we deduce that the Haefliger class

$$\begin{aligned}
&\left[(\Phi \circ \text{tr})\left((\tilde{e} - (\varrho \circ p)(\tilde{e})) \exp\left(\frac{-(\delta(\tilde{e} - (\varrho \circ p)(\tilde{e})))^2}{2i\pi}\right)\right) \right] - \\
&\left[(\Phi \circ \text{tr})\left((\tilde{e}' - (\varrho \circ p)(\tilde{e}')) \exp\left(\frac{-(\delta(\tilde{e}' - (\varrho \circ p)(\tilde{e}'))^2}{2i\pi}\right)\right) \right],
\end{aligned}$$

is well defined and only depends on the K -theory class $[\tilde{e}] - [\tilde{e}']$. We denote it by $\tilde{\text{ch}}_a([\tilde{e}] - [\tilde{e}'])$. So we have the following morphism

$$\tilde{\text{ch}}_a : K_0(\tilde{\Psi}_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) \longrightarrow \mathbf{H}_c^*(M/F).$$

The above construction applies also to the minimal unitalization $\widehat{\Psi}_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ of the algebra $\Psi_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ and yields a morphism

$$\widehat{\text{ch}}_a : K_0(\widehat{\Psi}_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) \longrightarrow \mathbf{H}_c^*(M/F),$$

whose restriction to $K_0(\Psi_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E))$ is by definition the Chern character ch_a . Note that $\widehat{\text{ch}}_a$ is given by the same formula (3.3), except that the K -theory element is no longer supposed to live in the kernel of

$$p_{0,*} : K_0(\widehat{\Psi}_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) \longrightarrow K_0(\Psi_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)).$$

Now the map $\beta : \widehat{\Psi}_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E) \rightarrow \widetilde{\Psi}_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ induces a well defined morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_0(\Psi_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) & \xrightarrow{i_{0,*}} & K_0(\widehat{\Psi}_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) & \xrightarrow{p_{0,*}} & K_0(\mathbb{C}) \simeq \mathbb{Z} \longrightarrow 0 \\ & & \downarrow id & & \downarrow \beta_* & & \downarrow [\beta]_* \\ 0 & \longrightarrow & K_0(\Psi_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) & \xrightarrow{i_*} & K_0(\widetilde{\Psi}_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) & \xrightarrow{p_*} & K_0(\mathbb{C}^2) \simeq \mathbb{Z}^2 \longrightarrow 0. \end{array}$$

Hence composing with $\widetilde{\text{ch}}_a$ gives the following diagram which is commutative by the very definition of the maps:

$$\begin{array}{ccccc} & & & K_0(\widehat{\Psi}_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) & & \\ & & i_{0,*} \nearrow & \downarrow \beta_* & \searrow \widehat{\text{ch}}_a & \\ 0 & \longrightarrow & K_0(\Psi_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) & & & H_c(M/F). \\ & & i_* \searrow & \downarrow \beta_* & \nearrow \widetilde{\text{ch}}_a & \\ & & & K_0(\widetilde{\Psi}_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) & & \end{array}$$

In particular, $\widetilde{\text{ch}}_a \circ \beta_* = \widehat{\text{ch}}_a$, so

$$\widetilde{\text{ch}}_a \circ \beta_* \circ i_{0,*} = \widehat{\text{ch}}_a \circ i_{0,*} = \text{ch}_a.$$

But,

$$\beta_* \circ i_{0,*} : K_0(\Psi_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) \longrightarrow \text{Ker } p_*,$$

is an isomorphism, so we may define the Chern character directly on the group $K_0(\Psi_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) = \text{Ker } p_*$. The proof is thus complete. \square

Corollary 3.6. *Let D be a generalized Dirac operator for the foliation F acting on the sections of the \mathbb{Z}_2 -graded bundle $\mathcal{S} \otimes E$. Let $P(tD)$ be the associated idempotent in the algebra $\widetilde{\Psi}_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$, as in Proposition 2.5. Set $P_t = P(tD) - \pi_-$. Then for all $t > 0$, the Haefliger form*

$$(\Phi \circ \text{tr}) \left(P_t \exp \left[\frac{-((\delta P_t)^2)}{2i\pi} \right] \right),$$

is closed and as Haefliger classes, we have the equality

$$\text{ch}_a(\text{Ind}_a(D^+)) = \left[(\Phi \circ \text{tr}) \left(P_t \exp \left[\frac{-((\delta P_t)^2)}{2i\pi} \right] \right) \right].$$

Proof. The analytic K -theory index of D in the K -theory group $K_0(\Psi_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E))$ of superexponentially decaying operators is given by

$$\text{Ind}_a(D^+) = [P(tD)] - [\pi_-] \in \text{Ker} \left(K_0(\widetilde{\Psi}_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)) \rightarrow \mathbb{Z}^2 \right).$$

Since the splitting map $\varrho : \mathbb{C}^2 \rightarrow \widetilde{\Psi}_{\mathfrak{E}}^{-\infty}(\mathcal{G}; \mathcal{S} \otimes E)$ is $\varrho(\lambda, \mu) = \lambda\pi_+ + \mu\pi_-$, we have that

$$P(tD) - (\varrho \circ p)(P(tD)) = P_t \text{ and } \pi_- - (\varrho \circ p)(\pi_-) = 0.$$

Now apply Proposition 3.4. \square

In [BH-I] we proved that the Chern character ch_a composed with the topological and analytic index maps of Connes-Skandalis [CS84] yield the same map. As a particular case, for any generalized Dirac operator D with coefficients in a Hermitian bundle E_1 over M , the Chern character of the topological index of D , denoted $\text{ch}_a(\text{Ind}_t(D^+))$, coincides with the Chern character of the analytic index of D , i.e.

$$\text{ch}_a(\text{Ind}_t(D^+)) = \text{ch}_a(\text{Ind}_a(D^+)),$$

and the common value of this Haefliger cohomology class is

$$\text{ch}_a(\text{Ind}_t(D^+)) = \text{ch}_a(\text{Ind}_a(D^+)) = \int_F \widehat{A}(TF) \text{ch}(E_1).$$

Here $\widehat{A}(TF)$ is the usual \widehat{A} genus of the tangent bundle of F , and ch is the usual Chern character of E_1 .

In order to define the Chern character of the index bundle of D , we need to assume that P_0 , the projection onto the kernel of D , is smooth. Classical results imply that P_0 is smooth when restricted to any leaf \widetilde{L}_x , so what we are really assuming is that it is transversely smooth.

Recall that $\alpha = \pi_+ - \pi_-$ is the grading involution for $\mathcal{S} \otimes E = (\mathcal{S}^+ \otimes E) \oplus (\mathcal{S}^- \otimes E)$. Then

$$P_0 = \begin{bmatrix} P_0^+ & 0 \\ 0 & P_0^- \end{bmatrix}, \quad \text{so} \quad \alpha P_0 = \begin{bmatrix} P_0^+ & 0 \\ 0 & -P_0^- \end{bmatrix}$$

is the super-projection onto the leafwise kernel of D , where P_0^\pm is projection onto the kernel of D^\pm . Note that $\partial_\nu \pi_\pm = 0$, provided we use a connection which preserves the splitting $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, which we assume that we do, so $\partial_\nu \alpha = 0$, and $\alpha \theta = \theta \alpha$. Note also that $\alpha P_0 = P_0 \alpha$, so

$$(\partial_\nu(\alpha P_0))^2 = \alpha^2(\partial_\nu P_0)^2 = (\partial_\nu P_0)^2 \quad \text{and} \quad \alpha P_0 \theta \alpha P_0 = \alpha^2 P_0 \theta P_0 = P_0 \theta P_0, \quad \text{which implies} \quad (\delta(\alpha P_0))^2 = (\delta P_0)^2.$$

Proposition 3.7. *The Haefliger form $(\Phi \circ \text{tr})\left(\alpha P_0 \exp\left(\frac{-((\delta(\alpha P_0))^2)}{2i\pi}\right)\right) = (\Phi \circ \text{tr})\left(\alpha P_0 \exp\left(\frac{-((\delta P_0)^2)}{2i\pi}\right)\right)$ is closed, and the Haefliger class it defines depends only on P_0 .*

Proof. Set $U = 2P_0 - 1$ then

$$\alpha U = U \alpha, U^2 = I, U P_0 = P_0 = P_0 U \quad \text{and} \quad U(\delta P_0) = \frac{1}{2}U(\delta U) = -\frac{1}{2}(\delta U)U = -(\delta P_0)U.$$

Thus, for any $k \geq 0$,

$$(d_H \circ \Phi \circ \text{tr})\left(\alpha P_0 (\delta P_0)^{2k}\right) = (\Phi \circ \text{tr})\left(\alpha (\delta P_0)^{2k+1}\right) = (\Phi \circ \text{tr})\left(U^2 \alpha (\delta P_0)^{2k+1}\right).$$

But,

$$(\Phi \circ \text{tr})\left(U^2 \alpha (\delta P_0)^{2k+1}\right) = (-1)^{2k+1} (\Phi \circ \text{tr})\left(U \alpha (\delta P_0)^{2k+1} U\right) = -(\Phi \circ \text{tr})\left(U^2 \alpha (\delta P_0)^{2k+1}\right),$$

so

$$(d_H \circ \Phi \circ \text{tr})\left(\alpha P_0 (\delta P_0)^{2k}\right) = 0.$$

In order to show the independence of the choice of connection, we use the relevant parts of the proof of Theorem 4.1 of [BH-I]. Indeed, it is obvious that the Poincaré argument developed there still applies to the regularizing operator P_0 even though it may be non-compactly supported. \square

Definition 3.8. *The analytic Chern character $\text{ch}_a(P_0)$ of the index bundle of D is the class of the Haefliger form $(\Phi \circ \text{tr})\left(\alpha P_0 \exp\left(\frac{-((\delta(\alpha P_0))^2)}{2i\pi}\right)\right) = (\Phi \circ \text{tr})\left(\alpha P_0 \exp\left(\frac{-((\delta P_0)^2)}{2i\pi}\right)\right)$.*

Finally, an easy induction argument using the fact that for any idempotent e , $e(\partial_\nu e)^{2\ell-1}e = 0$ for all $\ell > 0$, shows that

$$e(\delta e)^{2j} = \begin{pmatrix} e((\partial_\nu e)^2 + e\theta e)^j & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus

$$\mathbf{3.9.} \quad \text{ch}_a(P_0) = \left[(\text{Tr} \circ \text{tr})\left(\alpha P_0 \exp\left(\frac{-((\partial_\nu P_0)^2 + P_0 \theta P_0)}{2i\pi}\right)\right) \right].$$

4. PROOF OF MAIN THEOREM

Denote by P_ϵ the spectral projection for D^2 for the interval $(0, \epsilon)$. Recall that the Novikov-Shubin invariants of D are greater than $k \geq 0$ provided that there is $\beta > k$ so that

$$(\text{Tr} \circ \text{tr})(P_\epsilon) = (\Phi \circ \text{tr})(P_\epsilon) \text{ is } \mathcal{O}(\epsilon^\beta) \text{ as } \epsilon \rightarrow 0.$$

When we say a Haefliger form Ψ depending on ϵ is $\mathcal{O}(\epsilon^\beta)$ as $\epsilon \rightarrow 0$ we mean that there is a constant $C > 0$ so that the function on T , $\|\Psi\|_T \leq C\epsilon^\beta$ as $\epsilon \rightarrow 0$. Here $\|\cdot\|_T$ is the pointwise norm on forms on the transversal T induced from the metric on M . To say that $\|\Psi\|_T \leq C\epsilon^\beta$, means that given any representative $\psi \in \Psi$, there is a constant C_ψ so that $\|\psi\|_T \leq C_\psi\epsilon^\beta$. For a given cover, two representatives differ by a finite number of translations of local forms on transversals to other transversals so if this equation is satisfied for one representative with respect to a given cover, it is satisfied for all representatives with respect to that cover. The fact that for any two good covers of M by foliation charts there is a integer N so that any placque of the first cover intersects at most N placques of the second cover implies easily that this condition does not depend on the choice of good cover.

We now prove our main theorem.

Theorem 4.1. *Assume that \mathcal{G} is Hausdorff, and that the Novikov-Shubin invariants of D are greater than $q/2$. Assume further that the spectral projections P_0 and P_ϵ are transversely smooth (for ϵ sufficiently small), and that $\partial_\nu P_0$ and $\partial_\nu P_\epsilon$ are bounded operators. Then the analytic Chern character of the K -theory index of D equals the analytic Chern character of the index bundle of D , that is*

$$\text{ch}_a(\text{Ind}_a(D^+)) = \text{ch}_a([P_0]).$$

Theorem 4.1 uses estimates on Novikov-Shubin invariants of D to deduce the equality of the whole Chern character of the index bundle with that of the analytic index. We will actually prove the following stronger theorem.

Theorem 4.2. *Assume again that \mathcal{G} is Hausdorff, that the spectral projections P_0 and P_ϵ are transversely smooth (for ϵ sufficiently small), and that $\partial_\nu P_0$ and $\partial_\nu P_\epsilon$ are bounded operators. For a fixed integer k with $0 \leq 2k \leq q$, assume that the Novikov-Shubin invariants of D are greater than k . Then the k^{th} component of the Chern character of the K -theory index of D equals the k^{th} component of the Chern character of the index bundle of D , that is*

$$\text{ch}_a^k(\text{Ind}_a(D^+)) = \text{ch}_a^k([P_0]) \in \mathbf{H}_c^{2k}(M/F).$$

The proof of this theorem is rather long and involves a number of complicated estimates. For easier reading, we will split it into a series of propositions and lemmas. Note that Theorem 4.2 implies Theorem 4.1.

For the rest of this section, let k be a fixed integer in the interval $[0, q/2]$. By Corollary 3.6, we need only show that,

$$\lim_{t \rightarrow \infty} (\Phi \circ \text{tr})\left(P_t(\delta P_t)^{2k}\right) = (\Phi \circ \text{tr})\left(\alpha P_0(\delta(\alpha P_0))^{2k}\right).$$

If we ignore the minus signs in P_t , we see that the diagonal terms give e^{-tD^2} , and the off diagonal terms are given by $(P_t)_{21} = (e^{-tD^2/2}\sqrt{t}D)_{21}$ and $(P_t)_{12} = (e^{-tD^2/2}\frac{I - e^{-tD^2}}{tD^2}\sqrt{t}D)_{12}$. Thus

$$P_t = \pi_+ e^{-tD^2} \pi_+ - \pi_- e^{-tD^2} \pi_- - \pi_- e^{-tD^2/2} \sqrt{t}D \pi_+ - \pi_+ e^{-tD^2/2} \frac{I - e^{-tD^2}}{tD^2} \sqrt{t}D \pi_-.$$

As the connection ∇ used in the definition of ∂_ν preserves the splitting $\mathcal{S} \otimes E = (\mathcal{S}^+ \otimes E) \oplus (\mathcal{S}^- \otimes E)$, $\partial_\nu \pi_\pm = 0$, and we may work with the operators e^{-tD^2} , $e^{-tD^2/2}\sqrt{t}D$, and $e^{-tD^2/2}\frac{I - e^{-tD^2}}{tD^2}\sqrt{t}D$ in what follows instead of the (more notationally complicated) entries of P_t .

We will assume that the reader is familiar with the Spectral Mapping Theorem, see for instance [RS80], and how to use it to compute bounds on norms, strong convergence, etc. This theorem gives that for $\ell \geq 0$,

the norms of the operators $D^\ell e^{-tD^2}$, $D^\ell e^{-tD^2/2} \sqrt{t}D$ and $D^\ell e^{-tD^2/2} \frac{I - e^{-tD^2}}{tD^2} \sqrt{t}D$ are uniformly bounded as $t \rightarrow \infty$. In addition, as $t \rightarrow \infty$, $D^\ell e^{-tD^2/2} \sqrt{t}D$ and $D^\ell e^{-tD^2/2} \frac{I - e^{-tD^2}}{tD^2} \sqrt{t}D$ converge in norm to zero for $\ell \geq 0$, and for $\ell > 0$, $D^\ell e^{-tD^2}$ also converges in norm to zero.

Choose δ so that

$$-1 < \delta < \frac{-k}{\beta} < 0$$

and couple ϵ to t by setting

$$\epsilon = t^\delta.$$

Because of the uniformly bounded geometry of the leaves of F_s , which follows from the fact that all the structures we use on \mathcal{G} are pulled back from the compact manifold M , the leafwise estimates we give below are uniform over all leaves of F_s .

Denote by Q_ϵ the spectral projection for D^2 for the interval $[\epsilon, \infty)$. Since $I = P_0 + P_\epsilon + Q_\epsilon$, the operator $\partial_\nu Q_\epsilon$ is bounded. Now consider

$$P_t = P_0 P_t P_0 + P_\epsilon P_t P_\epsilon + Q_\epsilon P_t Q_\epsilon = \alpha P_0 + P_\epsilon P_t P_\epsilon + Q_\epsilon P_t Q_\epsilon.$$

Proposition 4.3. *As $t \rightarrow \infty$,*

- (i) $\|Q_\epsilon P_t Q_\epsilon\|$ is bounded by a multiple of $e^{-(t^{(1+\delta)/32})}$,
- (ii) $\|\partial_\nu(Q_\epsilon P_t Q_\epsilon)\|$ is bounded by a multiple of $e^{-(t^{(1+\delta)/32})}$,
- (iii) $\|P_\epsilon P_t P_\epsilon\|$ is bounded,
- (iv) $\|\partial_\nu(P_\epsilon P_t P_\epsilon)\|$ is bounded by a multiple of $t^{(\frac{1}{2}+a)}$, for any $a > 0$.

Remark 4.4. *The coefficient $\frac{1}{32}$ in (i) and (ii) can be improved very easily but this does not allow us to improve the assumption on the Novikov-Shubin invariants.*

Proof. Note that the element

$$\partial_\nu(Q_\epsilon P_t Q_\epsilon) = \partial_\nu(Q_\epsilon) P_t Q_\epsilon + Q_\epsilon P_t \partial_\nu(Q_\epsilon) + Q_\epsilon \partial_\nu(P_t) Q_\epsilon$$

and $\|\partial_\nu(Q_\epsilon)\|$ is bounded. We may write $P_t = e^{-tD^2/4} \hat{P}_t = \hat{P}_t e^{-tD^2/4}$ where

$$\hat{P}_t = \begin{bmatrix} e^{-3tD^- D^+/4} & (-e^{-tD^- D^+/4}) \frac{I - e^{-tD^- D^+}}{tD^- D^+} \sqrt{t}D^- \\ -e^{-tD^+ D^-/4} \sqrt{t}D^+ & -e^{-3tD^+ D^-/4} \end{bmatrix}.$$

\hat{P}_t has essentially the same properties as P_t , in particular its norm is bounded independently of t . Since $\|e^{-tD^2/4} Q_\epsilon\| = \|Q_\epsilon e^{-tD^2/4}\| \leq e^{-t\epsilon/4} = e^{-(t^{(1+\delta)/4})}$, we have that $\|P_t Q_\epsilon\|$ and $\|Q_\epsilon P_t\|$ (so also $\|Q_\epsilon P_t Q_\epsilon\|$, $\|\partial_\nu(Q_\epsilon) P_t Q_\epsilon\|$ and $\|Q_\epsilon P_t \partial_\nu(Q_\epsilon)\|$) are bounded by a multiple of $e^{-(t^{(1+\delta)/4})}$. Thus we have (i) of the Proposition, and to establish (ii) we need only consider the term $Q_\epsilon \partial_\nu(P_t) Q_\epsilon$.

Lemma 4.5. $\|Q_\epsilon \partial_\nu(e^{-tD^2/k}) Q_\epsilon\|$ is bounded by a multiple of $e^{-(t^{1+\delta}/8k)}$.

Proof. Recall the foliation Duhamel formula of [He95] (which requires that \mathcal{G} be Hausdorff) which states that

$$\partial_\nu(e^{-tD^2}) = - \int_0^t e^{-sD^2} \partial_\nu(D^2) e^{(s-t)D^2} ds.$$

Thus

$$\begin{aligned} Q_\epsilon \partial_\nu(e^{-tD^2}) Q_\epsilon &= - \int_0^t Q_\epsilon e^{-sD^2} \partial_\nu(D^2) e^{(s-t)D^2} Q_\epsilon ds = \\ &= - \int_{t/2}^t Q_\epsilon e^{-sD^2} \partial_\nu(D^2) e^{(s-t)D^2} Q_\epsilon ds - \int_0^{t/2} Q_\epsilon e^{-sD^2} \partial_\nu(D^2) e^{(s-t)D^2} Q_\epsilon ds. \end{aligned}$$

The norm of the first integral satisfies

$$\begin{aligned} \left\| \int_{t/2}^t Q_\epsilon e^{-sD^2} \partial_\nu(D^2) e^{(s-t)D^2} Q_\epsilon ds \right\| &\leq \int_{t/2}^t \|Q_\epsilon e^{-sD^2} \partial_\nu(D^2) e^{(s-t)D^2} Q_\epsilon\| ds \\ &\leq \int_{t/2}^t \|Q_\epsilon e^{-\frac{s}{2}D^2}\| \|e^{-\frac{s}{2}D^2} \partial_\nu(D^2)\| \|e^{(s-t)D^2} Q_\epsilon\| ds. \end{aligned}$$

Now $\|e^{(s-t)D^2} Q_\epsilon\| \leq 1$, and since $\partial_\nu(D^2)$ is a differential operator of order two with bounded coefficients, $e^{-\frac{1}{2}D^2} \partial_\nu(D^2)$ is a smoothing operator so has bounded norm. Thus

$$\|e^{-\frac{s}{2}D^2} \partial_\nu(D^2)\| \leq \|e^{-\frac{s-1}{2}D^2}\| \|e^{-\frac{1}{2}D^2} \partial_\nu(D^2)\| \leq \|e^{-\frac{1}{2}D^2} \partial_\nu(D^2)\|$$

for $t > 2$, as then $\|e^{-\frac{s-1}{2}D^2}\| \leq 1$ for all $s \geq t/2$. Finally, $\|Q_\epsilon e^{-\frac{s}{2}D^2}\| \leq e^{-s\epsilon/2}$, so the last integral is bounded by a multiple of

$$\epsilon^{-1}(e^{-(t\epsilon/4)} - e^{-(t\epsilon/2)}) = t^{-\delta}(e^{-(t^{1+\delta}/4)} - e^{-(t^{1+\delta}/2)}) < t^{-\delta}e^{-(t^{1+\delta}/4)}.$$

This in turn is bounded by a multiple of $e^{-(t^{1+\delta}/8)}$, for t sufficiently large.

The change of variables $s \rightarrow t-s$ transforms the integral $\int_0^{t/2} Q_\epsilon e^{-sD^2} \partial_\nu(D^2) e^{(s-t)D^2} Q_\epsilon ds$ to the integral $\int_{t/2}^t Q_\epsilon e^{(s-t)D^2} \partial_\nu(D^2) e^{-sD^2} Q_\epsilon ds$, so this satisfies the same estimate. Replacing D^2 by D^2/k then gives the estimate of the lemma. \square

Lemma 4.6. *As $t \rightarrow \infty$, $\|Q_\epsilon \partial_\nu(e^{-tD^2} \sqrt{t}D)Q_\epsilon\|$ is bounded by a multiple of $e^{-(t^{1+\delta}/32)}$.*

Proof. Observe that

$$\begin{aligned} Q_\epsilon \partial_\nu(e^{-tD^2} \sqrt{t}D)Q_\epsilon &= Q_\epsilon \partial_\nu(e^{-tD^2/2} \sqrt{t}D e^{-tD^2/2})Q_\epsilon = \\ &= Q_\epsilon \partial_\nu(e^{-tD^2/2}) \sqrt{t}D e^{-tD^2/2} Q_\epsilon + Q_\epsilon e^{-tD^2/2} \partial_\nu(\sqrt{t}D) e^{-tD^2/2} Q_\epsilon + Q_\epsilon e^{-tD^2/2} \sqrt{t}D \partial_\nu(e^{-tD^2/2}) Q_\epsilon = \\ &= Q_\epsilon \partial_\nu(e^{-tD^2/2}) Q_\epsilon \sqrt{t}D e^{-tD^2/2} Q_\epsilon + Q_\epsilon e^{-tD^2/2} \partial_\nu(\sqrt{t}D) e^{-tD^2/2} Q_\epsilon + Q_\epsilon e^{-tD^2/2} \sqrt{t}D Q_\epsilon \partial_\nu(e^{-tD^2/2}) Q_\epsilon. \end{aligned}$$

The operators $D e^{-tD^2/2}$, $\partial_\nu(D) e^{-tD^2/2}$ and $e^{-tD^2/2} D$ are all smoothing operators with norms bounded independently of t , for t large. The fact that $\|Q_\epsilon e^{-tD^2/2}\| \leq e^{-t\epsilon/2} = e^{-(t^{1+\delta}/2)}$ and the estimate in Lemma 4.5 give that $\|Q_\epsilon \partial_\nu(e^{-tD^2} \sqrt{t}D)Q_\epsilon\|$ is bounded by a multiple of $\sqrt{t}(e^{-(t^{1+\delta}/16)} + e^{-(t^{1+\delta}/2)})$ which is bounded by a multiple of $e^{-(t^{1+\delta}/32)}$, for t large. \square

Lemma 4.7. *As $t \rightarrow \infty$, $\|Q_\epsilon \partial_\nu(e^{-tD^2} \frac{I - e^{-tD^2}}{tD^2} \sqrt{t}D)Q_\epsilon\|$ is bounded by a multiple of $e^{-(t^{1+\delta}/32)}$.*

Proof.

$$\begin{aligned} \|Q_\epsilon \partial_\nu(e^{-tD^2} \frac{I - e^{-tD^2}}{tD^2} \sqrt{t}D)Q_\epsilon\| &= \|Q_\epsilon \partial_\nu(e^{-tD^2} \sqrt{t}D \frac{I - e^{-tD^2}}{tD^2})Q_\epsilon\| \leq \\ &\|Q_\epsilon \partial_\nu(e^{-tD^2} \sqrt{t}D)Q_\epsilon \frac{I - e^{-tD^2}}{tD^2} Q_\epsilon\| + \|Q_\epsilon e^{-tD^2} \sqrt{t}D \partial_\nu(\frac{I - e^{-tD^2}}{tD^2})Q_\epsilon\| \end{aligned}$$

and $\|\frac{I - e^{-tD^2}}{tD^2}\| \leq 1$, so by Lemma 4.6 the first term immediately above satisfies the lemma. If G is

the Green's operator for D , the second term may be written as $Q_\epsilon(G/\sqrt{t})Q_\epsilon e^{-tD^2} tD^2 \partial_\nu(\frac{I - e^{-tD^2}}{tD^2})Q_\epsilon$, and $\|Q_\epsilon G/\sqrt{t}\| \leq (t\epsilon)^{-1/2} = t^{-(1+\delta)/2}$, which is bounded for t large since $1 + \delta > 0$. The operator $tD^2 \frac{I - e^{-tD^2}}{tD^2} = I - e^{-tD^2}$, so

$$tD^2 \partial_\nu(\frac{I - e^{-tD^2}}{tD^2}) = -\partial_\nu(tD^2) \frac{I - e^{-tD^2}}{tD^2} - \partial_\nu(e^{-tD^2}),$$

and

$$Q_\epsilon e^{-tD^2} tD^2 \partial_\nu \left(\frac{I - e^{-tD^2}}{tD^2} \right) Q_\epsilon = -Q_\epsilon e^{-tD^2} \partial_\nu (tD^2) \frac{I - e^{-tD^2}}{tD^2} Q_\epsilon - Q_\epsilon e^{-tD^2} \partial_\nu (e^{-tD^2}) Q_\epsilon.$$

Now,

$$Q_\epsilon e^{-tD^2} \partial_\nu (tD^2) = Q_\epsilon t e^{-tD^2/2} e^{-tD^2/2} \partial_\nu (D^2),$$

and $e^{-tD^2/2} \partial_\nu (D^2)$ is a smoothing operator with norm bounded independently of t , for t large. As

$$\|Q_\epsilon t e^{-tD^2/2}\| \leq t e^{-t\epsilon/2} = t e^{-(t^{1+\delta}/2)} < e^{-(t^{1+\delta}/4)}$$

for t large, the term $Q_\epsilon e^{-tD^2} \partial_\nu (tD^2) \frac{I - e^{-tD^2}}{tD^2} Q_\epsilon$ has norm bounded by a multiple of $e^{-(t^{1+\delta}/4)}$. By Lemma 4.5, the term $Q_\epsilon e^{-tD^2} \partial_\nu (e^{-tD^2}) Q_\epsilon = Q_\epsilon e^{-tD^2} Q_\epsilon \partial_\nu (e^{-tD^2}) Q_\epsilon$ is bounded by a multiple of $e^{-(t^{1+\delta}/8)}$ (actually $e^{-t^{1+\delta}}$ if we use the estimate $\|Q_\epsilon e^{-tD^2}\| \leq e^{-t\epsilon} = e^{-t^{1+\delta}}$). \square

Thus we have the second inequality of Proposition 4.3. The third estimate follows immediately from the fact that both P_t and P_ϵ are bounded.

Lemma 4.8. $\|P_\epsilon \partial_\nu (e^{-tD^2}) P_\epsilon\|$ is bounded by a multiple of $t^{1+(\delta/2)}$.

Note that $1 + (\delta/2) > 1/2$, but by choosing δ close to -1 , we can make $1 + (\delta/2)$ as close to $1/2$ as we please.

Proof.

$$P_\epsilon \partial_\nu (e^{-tD^2}) P_\epsilon = - \int_0^t P_\epsilon e^{-sD^2} \partial_\nu (D^2) e^{(s-t)D^2} P_\epsilon ds = - \int_0^t P_\epsilon e^{-sD^2} P_\epsilon [\partial_\nu (D) D + D \partial_\nu (D)] P_\epsilon e^{(s-t)D^2} P_\epsilon ds.$$

As P_ϵ is a smoothing operator, so are $P_\epsilon \partial_\nu (D)$ and $\partial_\nu (D) P_\epsilon$, since $\partial_\nu (D)$ is a differential operator of order one with bounded coefficients. Since $\epsilon \rightarrow 0$ as $t \rightarrow \infty$, their norms are bounded independently of t for t large. Both $\|P_\epsilon e^{-sD^2}\|$ and $\|e^{(s-t)D^2} P_\epsilon\|$ are bounded by 1, and both $\|P_\epsilon D\|$ and $\|D P_\epsilon\|$ are bounded by $\sqrt{\epsilon}$. Thus $\|P_\epsilon \partial_\nu (e^{-tD^2}) P_\epsilon\|$ is bounded by a multiple of $\int_0^t \sqrt{\epsilon} ds = \sqrt{\epsilon} t = t^{1+(\delta/2)}$. \square

Lemma 4.9. $\|P_\epsilon \partial_\nu (e^{-tD^2} \sqrt{t} D) P_\epsilon\|$ is bounded by a multiple of $t^{(3/2)+\delta}$.

Again note that we can make $3/2 + \delta$ as close to $1/2$ as we please.

Proof.

$$\begin{aligned} \|P_\epsilon \partial_\nu (e^{-tD^2} \sqrt{t} D) P_\epsilon\| &\leq \|P_\epsilon \partial_\nu (e^{-tD^2}) P_\epsilon \sqrt{t} D P_\epsilon\| + \|P_\epsilon e^{-tD^2} P_\epsilon \partial_\nu (\sqrt{t} D) P_\epsilon\| \\ &\leq \|P_\epsilon \partial_\nu (e^{-tD^2}) P_\epsilon\| \sqrt{t} \|D P_\epsilon\| + \sqrt{t} \|P_\epsilon e^{-tD^2} P_\epsilon\| \|\partial_\nu (D) P_\epsilon\| \\ &\leq C_1 (t^{1+(\delta/2)}) \sqrt{t\epsilon} + C_2 t^{1/2} = C_1 t^{(3/2)+\delta} + C_2 t^{1/2} \leq C t^{(3/2)+\delta}. \end{aligned}$$

\square

Lemma 4.10. $\|P_\epsilon \partial_\nu (e^{-tD^2} \frac{I - e^{-tD^2}}{tD^2} \sqrt{t} D) P_\epsilon\|$ is bounded by a multiple of $t^{(3/2)+\delta}$.

Proof. As $P_\epsilon \partial_\nu (e^{-tD^2} \frac{I - e^{-tD^2}}{tD^2} \sqrt{t} D) P_\epsilon = P_\epsilon \partial_\nu (e^{-tD^2} \sqrt{t} D \frac{I - e^{-tD^2}}{tD^2}) P_\epsilon$, we have that

$$P_\epsilon \partial_\nu (e^{-tD^2} \frac{I - e^{-tD^2}}{tD^2} \sqrt{t} D) P_\epsilon = P_\epsilon \partial_\nu (e^{-tD^2} \sqrt{t} D) P_\epsilon \frac{I - e^{-tD^2}}{tD^2} P_\epsilon + P_\epsilon e^{-tD^2} \sqrt{t} D P_\epsilon \partial_\nu \left(\frac{I - e^{-tD^2}}{tD^2} \right) P_\epsilon.$$

By the Lemma 4.9 and the fact that $\|\frac{I - e^{-tD^2}}{tD^2}\| \leq 1$, we need only consider the term

$$\begin{aligned} \|P_\epsilon e^{-tD^2} \sqrt{t} D P_\epsilon \partial_\nu \left(\frac{I - e^{-tD^2}}{tD^2} \right) P_\epsilon\| &\leq \|e^{-tD^2} \sqrt{t} D P_\epsilon\| \|P_\epsilon \partial_\nu \left(\frac{I - e^{-tD^2}}{tD^2} \right) P_\epsilon\| \leq \\ &C \sqrt{t\epsilon} \|P_\epsilon \partial_\nu \left(\frac{I - e^{-tD^2}}{tD^2} \right) P_\epsilon\| = C t^{(1+\delta)/2} \|P_\epsilon \partial_\nu \left(\frac{I - e^{-tD^2}}{tD^2} \right) P_\epsilon\|. \end{aligned}$$

Thus we need only show that $\|P_\epsilon \partial_\nu(\frac{I - e^{-tD^2}}{tD^2})P_\epsilon\|$ is bounded by a multiple of $t^{1+(\delta/2)}$. Note that

$$\frac{d}{dr}\left(\frac{I - e^{-rD^2}}{D^2}\right) = e^{-rD^2}$$

so

$$\frac{d}{dr}(\partial_\nu(\frac{I - e^{-rD^2}}{D^2})) = \partial_\nu(\frac{d}{dr}(\frac{I - e^{-rD^2}}{D^2})) = \partial_\nu(e^{-rD^2}) = -\int_0^r e^{-sD^2} \partial_\nu(D^2) e^{(s-r)D^2} ds.$$

Thus

$$\partial_\nu(\frac{I - e^{-tD^2}}{D^2}) = \int_0^t \frac{d}{dr}(\partial_\nu(\frac{I - e^{-rD^2}}{D^2})) dr = -\int_0^t \int_0^r e^{-sD^2} \partial_\nu(D^2) e^{(s-r)D^2} ds dr,$$

and

$$\begin{aligned} \|P_\epsilon \partial_\nu(\frac{I - e^{-tD^2}}{tD^2})P_\epsilon\| &= \|\frac{1}{t} P_\epsilon \partial_\nu(\frac{I - e^{-tD^2}}{D^2})P_\epsilon\| = \|\frac{1}{t} \int_0^t \int_0^r e^{-sD^2} P_\epsilon \partial_\nu(D^2) P_\epsilon e^{(s-r)D^2} ds dr\| \leq \\ &\frac{1}{t} \int_0^t \int_0^r \|e^{-sD^2}\| \|P_\epsilon[\partial_\nu(D)D + D\partial_\nu(D)]P_\epsilon\| \|e^{(s-r)D^2}\| ds dr \leq \frac{1}{t} \int_0^t \int_0^r C\sqrt{\epsilon} ds dr = Ct^{1+(\delta/2)}. \end{aligned}$$

□

This finishes the proof of Proposition 4.3

□

To finish the proof of Theorem 4.1, first note that the estimates of Proposition 4.3 remain true with ∂_ν replaced by δ . This follows from the fact that for $T \in A_\mathfrak{S}(\mathcal{G}, \mathcal{S} \otimes E) \subset \tilde{\mathcal{A}}_\mathfrak{S}$, δT involves only T and $\partial_\nu T$. Similarly, δP_0 , $\delta(\alpha P_0)$, δP_ϵ , and δQ_ϵ are bounded operators. Finally, for $\tilde{T}_1, \tilde{T}_2 \in \tilde{\mathcal{A}}_\mathfrak{S}$, $\tilde{T}_1 * \tilde{T}_2 = \tilde{T}_1 \Theta \tilde{T}_2$. But multiplication by $\Theta = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix}$ is a bounded operation, so we may ignore it with impunity in the norm estimates of products below.

Since $P_t = \alpha P_0 + P_\epsilon P_t P_\epsilon + Q_\epsilon P_t Q_\epsilon$,

$$\text{tr}(P_t(\delta P_t)^{2k}) = \text{tr}(\alpha P_0(\delta(\alpha P_0))^{2k}) + \text{tr}(\alpha P_0(\delta P_t)^{2k} - \alpha P_0(\delta(\alpha P_0))^{2k}) + \text{tr}(P_\epsilon P_t P_\epsilon(\delta P_t)^{2k}) + \text{tr}(Q_\epsilon P_t Q_\epsilon(\delta P_t)^{2k}).$$

For any integer $\ell \geq 0$,

$$\|D^{2\ell} Q_\epsilon P_t Q_\epsilon(\delta P_t)^{2k}\| = \|D^{2\ell} Q_\epsilon e^{-tD^2/4} Q_\epsilon \hat{P}_t(\delta P_t)^{2k}\| \leq \|D^{2\ell} Q_\epsilon e^{-tD^2/4} Q_\epsilon\| \|\hat{P}_t(\delta P_t)^{2k}\|.$$

Now

$$\delta(P_t) = \delta(\alpha P_0) + \delta(P_\epsilon P_t P_\epsilon) + \delta(Q_\epsilon P_t Q_\epsilon),$$

and $\|\hat{P}_t\|$ is bounded independently of t . So $\|\hat{P}_t(\delta P_t)^{2k}\|$ is bounded by a multiple of

$$\|(\delta P_t)^{2k}\| = \|\delta(\alpha P_0) + \delta(P_\epsilon P_t P_\epsilon) + \delta(Q_\epsilon P_t Q_\epsilon)\|^{2k} \leq Ct^{2k(\frac{1}{2}+a)}$$

where $a > 0$ is a number to be chosen later (as close to zero as we please). On the other hand, for t sufficiently large (so that $t^{1+\delta} > 4\ell$), the maximum of $z^\ell e^{-tz/4}$ on the interval $[\epsilon, \infty)$ occurs at ϵ , so

$$\|D^{2\ell} Q_\epsilon e^{-tD^2/4} Q_\epsilon\| \leq \epsilon^\ell e^{-t\epsilon/4} = t^{\delta\ell} e^{-(t^{1+\delta})/4}$$

so

$$\|D^{2\ell} Q_\epsilon P_t Q_\epsilon(\delta P_t)^{2k}\| \leq Ct^{2k(\frac{1}{2}+a)} t^{\delta\ell} e^{-(t^{1+\delta})/4}$$

which goes to zero as $t \rightarrow \infty$. The proof of Theorem 2.3.13 of [HL90] shows that this implies that $\text{tr}(Q_\epsilon P_t Q_\epsilon(\delta P_t)^{2k})$ is pointwise bounded on M and converges pointwise to zero as $t \rightarrow \infty$. As Φ is integration over a compact set, the bounded convergence theorem gives

$$\mathbf{4.11.} \quad \lim_{t \rightarrow \infty} \Phi \circ \text{tr}(Q_\epsilon P_t Q_\epsilon(\delta P_t)^{2k}) = 0.$$

Now consider $\Phi \circ \text{tr}(P_\epsilon P_t P_\epsilon (\delta P_t)^{2k}) = \Phi \circ \text{tr}(P_\epsilon (\delta P_t)^{2k} P_\epsilon P_t P_\epsilon)$. The proof of Proposition 12 of [HL99] shows that

$$\|\Phi \circ \text{tr}(P_\epsilon (\delta P_t)^{2k} P_\epsilon P_t P_\epsilon)\|_T \leq C \|P_\epsilon (\delta P_t)^{2k} P_\epsilon P_t\| \|\Phi \circ \text{tr}(P_\epsilon)\|_T.$$

Since $\Phi \circ \text{tr}(P_\epsilon)$ is $\mathcal{O}(\epsilon^\beta)$, this is bounded by a multiple of

$$t^{2k(\frac{1}{2}+a)} \epsilon^\beta = t^{2k(\frac{1}{2}+a)} t^{\delta\beta} = t^{2k(\frac{1}{2}+a)+\delta\beta}.$$

Recall that $-1 < \delta < -k/\beta$ and therefore we can choose $a > 0$ so small that

$$2k(\frac{1}{2} + a) + \delta\beta < 0.$$

Then

$$\mathbf{4.12.} \quad \lim_{t \rightarrow \infty} \Phi \circ \text{tr}(P_\epsilon P_t P_\epsilon (\partial_\nu P_t)^{2k}) = 0.$$

Finally, consider the individual terms of $\text{tr}(\alpha P_0 (\delta P_t)^{2k} - \alpha P_0 (\delta P_0)^{2k}) = \text{tr}(P_0 \alpha (\delta P_t)^{2k} - P_0 \alpha (\delta P_0)^{2k})$. Suppose the term $P_0 A$ contains a $\delta(Q_\epsilon P_t Q_\epsilon)$. Then

$$\|P_0 A\| \leq \|P_0\| \|\alpha\| \|\delta(Q_\epsilon P_t Q_\epsilon)\|^\mu \|\delta(\alpha P_0)\|^\beta \|\delta(P_\epsilon P_t P_\epsilon)\|^\gamma$$

where $\mu + \beta + \gamma = 2k$ and $\mu > 0$. Since $\|\alpha\|$ is bounded, Proposition 4.3, gives that as $t \rightarrow \infty$,

$$\|P_0 A\| \leq C e^{-\mu(t^{1+\delta}/32)} t^{\gamma(\frac{1}{2}+a)}.$$

For every positive integer ℓ , $D^{2\ell} P_0 = 0$, so for every integer $\ell \geq 0$, $\|D^{2\ell} P_0 A\| \rightarrow 0$ as $t \rightarrow \infty$. Proceeding as in the proof of Equation 4.11, we have

$$\lim_{t \rightarrow \infty} \Phi \circ \text{tr}(P_0 A) = 0.$$

Now suppose that we have one of the remaining terms. It must contain a term of the form $\delta(P_\epsilon P_t P_\epsilon)$. As $P_\epsilon^2 = P_\epsilon$ and δ is a derivation, we may replace $\delta(P_\epsilon P_t P_\epsilon)$ by

$$\delta(P_\epsilon^2 P_t P_\epsilon) = \delta(P_\epsilon)(P_\epsilon P_t P_\epsilon) + P_\epsilon \delta(P_\epsilon P_t P_\epsilon) = \delta(P_\epsilon)(P_\epsilon P_t P_\epsilon) P_\epsilon + P_\epsilon \delta(P_\epsilon P_t P_\epsilon).$$

Using the trace property of $\Phi \circ \text{tr}$, we get two terms of the form $\Phi \circ \text{tr}(A P_\epsilon)$. As above, the proof of Proposition 12 of [HL99] shows that

$$\|\Phi \circ \text{tr}(A P_\epsilon)\|_T \leq C \|A\| \|\Phi \circ \text{tr}(P_\epsilon)\|_T.$$

Now A is a product of terms of the form $\alpha, P_0, \delta(\alpha P_0), P_\epsilon, \delta(P_\epsilon), P_\epsilon P_t P_\epsilon$, and $\delta(P_\epsilon P_t P_\epsilon)$. Each of these is bounded in norm, except the last which has norm bounded by a multiple of $t^{(\frac{1}{2}+a)}$. As A can contain no more than $2k$ terms of the form $\delta(P_\epsilon P_t P_\epsilon)$, and $\Phi \circ \text{tr}(P_\epsilon)$ is $\mathcal{O}(\epsilon^\beta)$, we have that $\|\Phi \circ \text{tr}(A P_\epsilon)\|_T$ is bounded by a multiple of

$$t^{2k(\frac{1}{2}+a)} \epsilon^\beta = t^{2k(\frac{1}{2}+a)} t^{\delta\beta} = t^{2k(\frac{1}{2}+a)+\delta\beta}.$$

By our choice of a , we have that the limit as $t \rightarrow \infty$ of these terms is zero, just as in the proof of Equation 4.12.

This completes the proof of Theorem 4.2

5. BISMUT SUPERCONNECTIONS

As noted above, in [BH-I] we proved that the Chern character ch_a composed with the topological and analytic index maps of Connes-Skandalis [CS84] yield the same map. In particular, for any Dirac operator D , the Chern character of the topological index of D , coincides with the Chern character of the analytic index of D , i.e.

$$\text{ch}_a(\text{Ind}_t(D^+)) = \text{ch}_a(\text{Ind}_a(D^+)).$$

In [HL99], we proved that $\text{ch}_a(\text{Ind}_t(D^+))$ is equal to the Chern character of the index bundle of D in another sense. We defined a ‘‘connection’’ ∇ on the index bundle $[P_0]$ of D , and defined the Chern character of $[P_0]$ to be the Haefliger class of $\text{Tr}(\alpha e^{-(\nabla^2/2i\pi)})$. We then used a Bismut superconnection for foliations, [He95], to show that $\text{ch}_a(\text{Ind}_t(D))$ contains the Haefliger form $\text{Tr}(\alpha e^{-(\nabla^2/2i\pi)})$, provided that the assumptions of Theorem 4.1 are satisfied, but with the stronger assumption that the Novikov-Shubin invariants of D are

greater than **three** times the codimension of F . We will now show that whenever P_0 is smooth, $\text{ch}_a([P_0])$ contains the Haefliger form $\text{Tr}(\alpha e^{-(\nabla^2/2i\pi)})$, so the two definitions of the Chern character of $[P_0]$ agree.

We first recall the construction of Bismut superconnections for D . See [B86], [BV87], and also [He95]. Let ∇^B be a Bott connection on ν_s^* . If $\omega_1, \dots, \omega_n$ is a local framing for ν_s^* , then $\nabla^B \omega_i = \sum_{j=1}^n \omega_j \otimes \theta_j^i$ where θ_j^i are local one forms on \mathcal{G} and the θ_j^i satisfy $d\omega_i = \sum_{j=1}^n \omega_j \wedge \theta_j^i$. that is, the composition

$$C^\infty(\nu_s^*) \xrightarrow{\nabla^B} C^\infty(\nu_s^* \otimes T^*\mathcal{G}) \xrightarrow{\wedge} C^\infty(\nu_s^* \wedge T^*\mathcal{G})$$

is just $\omega \rightarrow d\omega$. ∇^B induces a connection on $\wedge \nu_s^*$ also denoted ∇^B so that

$$C^\infty(\wedge \nu_s^*) \xrightarrow{\nabla^B} C^\infty(\wedge \nu_s^* \otimes T^*\mathcal{G}) \xrightarrow{\wedge} C^\infty(\wedge \nu_s^* \wedge T^*\mathcal{G})$$

is also just $\omega \rightarrow d\omega$.

Set $\mathcal{V} = TF_s \oplus \nu_s \oplus \nu_s^* = T\mathcal{G} \oplus \nu_s^*$ over \mathcal{G} , and define a symmetric bilinear form g on \mathcal{V} as follows. TF_s and $\nu_s \oplus \nu_s^*$ are orthogonal and $g|_{TF_s}$ is $g_0|_{TF_s}$. The form $g|_{\nu_s \oplus \nu_s^*}$ is given by the canonical duality, i.e. ν_s and ν_s^* are totally isotropic and $g(X, \omega) = \omega(X)$ for $X \in \nu_s, \omega \in \nu_s^*$. In [BV87], p. 455. it is shown that there is a unique connection ∇ , the Bismut connection, on \mathcal{V} so that ∇ preserves ν_s^* and $g, \nabla|_{\nu_s^*} = \nabla^B$ and for all $X, Y \in C^\infty(T\mathcal{G}), \nabla_X Y - \nabla_Y X = [X, Y]$. Note that in general ∇ does not preserve $T\mathcal{G}$ but that for $X, Y \in C^\infty(T\mathcal{G}), \nabla_X Y - \nabla_Y X \in C^\infty(T\mathcal{G})$.

Consider the vector space $V = \mathbb{R}^p \oplus \mathbb{R}^n \oplus \mathbb{R}^{n^*}$. Define a bilinear form Q on V as g was on \mathcal{V} , i.e. \mathbb{R}^p is orthogonal to $\mathbb{R}^n \oplus \mathbb{R}^{n^*}$, $Q|_{\mathbb{R}^p}$ is the usual inner product, and $Q|_{\mathbb{R}^n \oplus \mathbb{R}^{n^*}}$ is given by the canonical duality. Let $C(V, Q)$ be the associated Clifford algebra and set $S_0 = \wedge \mathbb{R}^{n^*} \otimes S$ where S is the spinor space for \mathbb{R}^p with the usual inner product. Let ρ be the representation of the Clifford algebra of \mathbb{R}^p in S . Then S_0 is the spinor space for $C(V, Q)$ with the Clifford multiplication being defined by

$$\begin{aligned} \rho_0(X)(\omega \otimes s) &= (-1)^{\deg \omega} \omega \otimes \rho(X)s \\ \rho_0(Y)(\omega \otimes s) &= -2i(Y)\omega \otimes s \\ \rho_0(\phi)(\omega \otimes s) &= \phi \wedge \omega \otimes s \end{aligned}$$

for $X \in \mathbb{R}^p, Y \in \mathbb{R}^n, \phi \in \mathbb{R}^{n^*}, \omega \in \wedge \mathbb{R}^{n^*}, s \in S$. See [BV87], p. 456 and for general facts about spinors and Clifford algebras, [LM89].

The above fact allows Berline and Vergne to give a beautiful and concise definition of Bismut superconnections for fiber bundles which was extended to foliations in [He95]. Recall that \mathcal{S} is the spinor bundle along the leaves of F_s , and consider the vector bundle $S_0 = \wedge \nu_s^* \otimes \mathcal{S}$ over \mathcal{G} and the bundle of Clifford algebras $C(\mathcal{V})$ over \mathcal{G} associated to \mathcal{V}, g . Then $S_{0,y}$, the fiber over $y \in \mathcal{G}$ of S_0 , is a module for the algebra $C(\mathcal{V})_y$ and we denote the module action also by ρ_0 . The connection ∇ on \mathcal{V} induces a connection ∇ on S_0 ([BV87], p. 456; or more generally [LM89], Ch. 4). Let E be a vector bundle with connection over \mathcal{G} as in Section 2. We shall also denote by ∇ the tensor product connection on $S_0 \otimes E$.

A Bismut superconnection \mathbb{B} for F_s and E is the Dirac type operator on $C_c^\infty(S_0 \otimes E)$ defined as follows. Let X_1, \dots, X_p be a local oriented orthonormal basis of TF_s , and X_{p+1}, \dots, X_{p+n} a local basis of ν_s . Let X_1^*, \dots, X_{p+n}^* be the dual basis in $TF_s \oplus \nu_s^*$, i.e. $X_i^* = X_i$ for $1 \leq i \leq p, X_i^* = \omega_i$, for $p+1 \leq i \leq p+n$ where $\omega_i \in \nu_s^*$ and $\omega_i(X_j) = \delta_{ij}$. Set

$$\mathbb{B} = \sum_{i=1}^{p+n} (\rho_0(X_i^*) \otimes 1) \nabla_{X_i} = \sum_{i=1}^p \rho(X_i) \nabla_{X_i} + \sum_{i=p+1}^{p+n} \omega_i \nabla_{X_i}.$$

\mathbb{B} does not depend on the choice of X_1, \dots, X_{p+n} .

Since $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ is \mathbb{Z}_2 graded and $\wedge \nu_s^*$ is \mathbb{Z} graded, $S_0 = \wedge \nu_s^* \otimes \mathcal{S}$ has a total \mathbb{Z}_2 grading and we write $S_0 = S_0^+ \oplus S_0^-$. We then have an associated \mathbb{Z}_2 grading $S_0 \otimes E = (S_0^+ \otimes E) \oplus (S_0^- \otimes E)$. It is immediate from the fact that ∇ preserves the grading that \mathbb{B} is an odd operator, i.e. \mathbb{B} maps $C^\infty(S_0^+ \otimes E)$ to $C^\infty(S_0^- \otimes E)$ and vice-versa.

Finally, we may use the \mathbb{Z} grading on $\wedge \nu_s^*$ to grade the operator \mathbb{B} , i.e. $\mathbb{B} = \mathbb{B}^{[0]} + \mathbb{B}^{[1]} + \dots$ where $\mathbb{B}^{[i]} : C^\infty(\wedge^k \nu_s^* \otimes S \otimes E) \rightarrow C^\infty(\wedge^{k+i} \nu_s^* \otimes S \otimes E)$.

It is straightforward to check that

Proposition 5.1. *The term $\mathbb{B}^{[1]}$ is a quasi-connection ∇^ν for $E \otimes \wedge \nu_s^*$ as defined in Section 3.*

Recall, [HL99], that a connection on the index bundle of D is defined by

$$\nabla = P_0 \mathbb{B}^{[1]} P_0.$$

For this to be well defined, we must require that P_0 is smooth.

Theorem 5.2. *Suppose that P_0 is smooth. Then $\text{ch}_a([P_0])$ contains the Haefliger form $\text{Tr}(\alpha e^{-(\nabla^2/2i\pi)})$*

Proof. First we calculate ∇^2 .

$$\begin{aligned} \nabla^2 &= P_0 \mathbb{B}^{[1]} P_0 \mathbb{B}^{[1]} P_0 \\ &= P_0 [\mathbb{B}^{[1]}, P_0] \mathbb{B}^{[1]} P_0 + P_0 (\mathbb{B}^{[1]})^2 P_0 \\ &= P_0 [\mathbb{B}^{[1]}, P_0] [\mathbb{B}^{[1]}, P_0] + P_0 [\mathbb{B}^{[1]}, P_0] P_0 \mathbb{B}^{[1]} + P_0 (\mathbb{B}^{[1]})^2 P_0 \\ &= P_0 [\mathbb{B}^{[1]}, P_0] [\mathbb{B}^{[1]}, P_0] + P_0 (\mathbb{B}^{[1]})^2 P_0. \end{aligned}$$

The last equality is a consequence of the relation $P_0 [\mathbb{B}^{[1]}, P_0] P_0 = 0$ which is true since $P_0^2 = P_0$ and since $[\mathbb{B}^{[1]}, \cdot]$ is a derivation. This derivation is precisely ∂_ν , so $(\mathbb{B}^{[1]})^2 = \theta$ as in Section 3. Thus

$$\nabla^2 = P_0 (\partial_\nu P_0)^2 + P_0 \theta P_0,$$

and

$$\nabla^{2k} = (P_0 (\partial_\nu P_0)^2 + P_0 \theta P_0)^k.$$

Note that

$$\partial_\nu(P_0) = \partial_\nu(P_0 P_0) = \partial_\nu(P_0) P_0 + P_0 \partial_\nu(P_0),$$

so

$$\partial_\nu(P_0) P_0 = \partial_\nu(P_0) - P_0 \partial_\nu(P_0).$$

Using this twice, one can easily show that

$$P_0 \partial_\nu(P_0) \partial_\nu(P_0) = P_0 \partial_\nu(P_0) \partial_\nu(P_0) P_0.$$

Then a simple induction argument shows that

$$(P_0 (\partial_\nu P_0)^2 + P_0 \theta P_0)^k = P_0 ((\partial_\nu P_0)^2 + P_0 \theta P_0)^k$$

Thus,

$$\text{Tr}(\alpha \nabla^{2k}) = \text{Tr}(\alpha P_0 ((\partial_\nu P_0)^2 + P_0 \theta P_0)^k),$$

and comparing with Equation 3.9, we see that $\text{ch}_a([P_0])$ contains the Haefliger form $\text{Tr}(\alpha e^{-(\nabla^2/2i\pi)})$. \square

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