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Topological expansion of the 2-matrix model correlation functions: diagrammatic rules for a residue formula

*B. Eynard*¹, *N. Orantin*²

Service de Physique Théorique de Saclay,
F-91191 Gif-sur-Yvette Cedex, France.

Abstract:

We rewrite the loop equations of the hermitian 2-matrix model, in a way which allows to compute all the correlation functions, to all orders in the topological $1/N^2$ expansion, as residues on an algebraic curve. Those residues, can be represented diagrammatically as Feynman graphs of a cubic interaction field theory on the curve.

1 Introduction

The purpose of this article, is to generalize the method invented in [12], for the 2-matrix model. The method of [12] is a diagrammatic technique for computing correlation functions of the 1-matrix model in terms of residues on some algebraic curve.

Random matrix models play an important role in physics and mathematics [29], and have a wealth of applications which are too long to list here. In this article, we consider “formal” random matrix integrals, which are known to be generating functions for counting some classes of discrete surfaces [9, 31, 6, 20, 21]. The basic idea, is to consider a matrix integral, as a deformation of a gaussian matrix integral, and make a formal expansion of the non-gaussian part. Each term of that formal expansion is an expectation value of a gaussian integral, and using Wick’s theorem, each term can be represented by a Feynman graph. Because the integration variables are matrices, the graphs are “fat graphs”, which have a 2-dimensional structure. The Hermitean matrix

¹E-mail: eynard@spht.saclay.cea.fr

²E-mail: orantin@spht.saclay.cea.fr

models enumerate oriented surfaces (whereas other matrix ensembles can enumerate non-oriented surfaces). This Formal expansion equivalent to an enumerating function of Feynman graphs is a standard tool in physics. Random matrices have thus played a role in all theories where one need to sum over surfaces, i.e. string theory and quantum gravity (which would better be called statistical physics on a random lattice).

The partition function, free energy and correlation functions are all generating functions enumerating some kinds of graphs (respectively closed graphs, connected closed graphs, open graphs). There exist several methods for computing the free energy and correlation function, the one we consider here is the “loop equation” method, which is nothing but Schwinger-Dyson, or Ward identities [9, 30]. They implement the Virasoro (W) constraints on the partition function, i.e. the fact that the matrix integral is left unchanged under a change of variable. The loop equations are valid in the formal model, order by order in the expansion parameters.

In the formal model, the size N of matrices, is just a complex parameter, it needs not be an integer, and all observables (free energy, correlation functions) always have a $1/N$ expansion, because for each power of the expansion parameters, there is only a finite number of graphs with a given power of N . The power of N in a graph is its Euler characteristic, and thus the $1/N$ expansion is known as the “topological expansion” [31]. In the formal model, N is thus an expansion parameter, and working order by order in N enumerates only discrete surfaces of a given topology [6].

To large N limit, (i.e. planar topologies), the solution of loop equations is known to be related to Toda hierarchy [8, 33, 32]. For this reason, the large N expansion of matrix models, play an important role in integrable systems, and in many areas of physics [27]. It was suggested by [10] that the low energy effective of some string theory models is also described by matrix models.

In the beginning, the formal matrix models were considered in their 1-cut phase, because a potential which is a small deformation of a quadratic one, must have only one well, i.e. the variable perturbatively explore only one well. However, a $N \times N$ matrix has N eigenvalues, and even though each of them can explore perturbatively only one well, they do not need explore all the same well. That gives “multicut” solutions of matrix models, where the number of eigenvalues near each extremum of the potential is fixed (fixed filling fractions). Multicut solutions play an important role in string theory, as they describe multi-particle states [10, 11]. Multicut solutions correspond to enumerating surfaces with contact terms, which can be called “foam of surfaces” as described in [5, 21].

The link between formal matrix models (which always have a $1/N$ expansion) and convergent matrix integrals (which have a $1/N$ expansion only in the 1-cut case under

certain assumptions), has been better understood after the work of [5]. We emphasize again, that the results developed in this article concern the formal matrix model with fixed filling fractions, and should not be applied to convergent matrix model directly.

Recently, it has progressively become clear that large N expansion of random matrix models has a strong link with algebraic geometry. The free energy and correlation functions have been computed in terms of properties of an algebraic curve. The large N limit of the 1-point correlation function (called the resolvent) is solution of an algebraic equation, which thus defines an algebraic curve. There has been many works which computed free energy and correlation functions in terms of that algebraic curve. The leading order resolvent and free energy were computed in the 1-cut case (algebraic curve of genus zero) in the pioneering work of [6], then some recursive method for computing other correlation functions to leading order were invented by Akemann [1], then some recursive method for computing correlation functions and free energy to all orders in $1/N$ were invented by [3]. Those methods were first limited to 1-matrix case and 1-cut.

Then, for 1-matrix, several works have dealt with multicut: Akeman found the first subleading term for the multicut resolvent and the 2-cut free energy [1, 2], Chekhov found the first subleading term for the multi-cut free energy [7]. Then a (non-recursive) diagrammatic method was invented in [12] to find all correlation functions to all orders, in the multicut case.

The 1-matrix model, corresponds to hyperelliptical curves only. In order to have more general algebraic curves, one needs at least a 2-matrix model. For the 2-matrix models, the loop equations have been known since [30], and have been written in a concise form in [17, 18, 19]. They have been used to find the subleading term of the free energy, first in the genus zero case in [14], then in the genus 1 case in [15], and with arbitrary genus in [16]. The purpose of this article is to generalize the diagrammatic method of [12] to the 2-matrix case.

Outline of the article:

- In sections 2,3 and 4, we introduce the notations and some basic tools of algebraic geometry needed all along the article.
- In section 5, we consider the leading order of the correlation functions, corresponding to the genus zero case. We introduce some Feynman rules giving rise to graphs whose tree level describes the genus zero correlation functions.
- Section 6 is dedicated to the computation of the full $\frac{1}{N^2}$ -expansion. We show that the $\frac{1}{N^{2h}}$ term corresponds to the sum over all h-loop graphs described earlier.

- In section 7, we build an effective non cubic theory generating the same correlation functions.
- In section 8, we study the gaussian case corresponding to the 1-matrix model limit.

2 Definition of the model

In this article, we are interested in the study of the formal-two-matrix-model and the computation of a whole family of some observables of this model. The partition function Z is the formal matrix integral:

$$Z := \int_{H_n \times H_n} dM_1 dM_2 e^{-N \text{Tr}(V_1(M_1) + V_2(M_2) - M_1 M_2)} \quad (2.1)$$

where M_1 and M_2 are two $N \times N$ hermitian matrices, dM_1 and dM_2 the products of Lebesgue measures of the real components of M_1 and M_2 respectively, and V_1 and V_2 two polynomial potentials of degree $d_1 + 1$ and $d_2 + 1$ respectively :

$$V_1(x) = \sum_{k=1}^{d_1+1} \frac{g_k}{k} x^k \quad , \quad V_2(y) = \sum_{k=1}^{d_2+1} \frac{\tilde{g}_k}{k} y^k \quad (2.2)$$

Formal integral means it is computed order by order in powers of the g_k 's (see section 4 or [21]).

In [12], one of the authors solved an analogous problem for the one matrix model, (i.e. the model where V_2 is quadratic). In the large N limit, he computed the full $\frac{1}{N^2}$ expansion of the formal expectation values related to the partition function Z :

$$N^{k-2} \left\langle \text{Tr} \frac{1}{x_1 - M} \text{Tr} \frac{1}{x_2 - M} \dots \text{Tr} \frac{1}{x_k - M} \right\rangle_c \quad (2.3)$$

where c denotes the connected part. This expansion was represented as Feynman graphs of an effective cubic field theory.

The purpose of this article is to extend this description to the two matrix model. Let us consider the expectation values :

$$\overline{w_{k,l}}(\mathbf{x}_K; \mathbf{y}_L) := N^{k+l-2} \left\langle \prod_{i=1}^k \text{tr} \frac{1}{x_i - M_1} \prod_{i=1}^l \text{tr} \frac{1}{y_i - M_2} \right\rangle_c \quad (2.4)$$

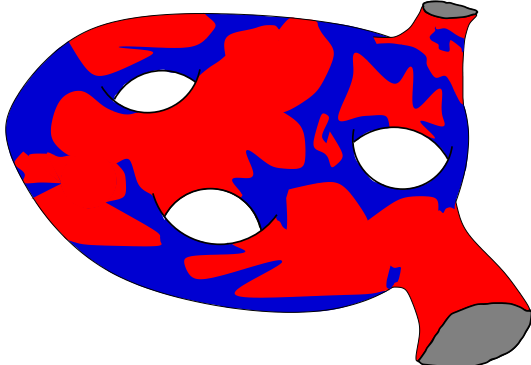
where $\mathbf{x}_K = (x_1, \dots, x_k)$ and $\mathbf{y}_L = (y_1, \dots, y_l)$, and the formal average $\langle \cdot \rangle$ is computed with the measure in Eq. (2.1). Those correlation functions can be expanded as formal series in $\frac{1}{N^2}$ in the large N limit:

$$\overline{w_{k,l}}(\mathbf{x}_K; \mathbf{y}_L) := \sum_{h=0}^{\infty} N^{-2h} \overline{w_{k,l}^{(h)}}(\mathbf{x}_K; \mathbf{y}_L) \quad (2.5)$$

We compute here the full expansion terms $\overline{w_{k,0}^{(h)}}(\mathbf{x}_K)$ and $\overline{w_{0,l}^{(h)}}(\mathbf{y}_L)$ as residues on an algebraic curve. Eventually, we represent them as Feynman graphs of a cubic field theory on this curve.

The $\frac{1}{N^2}$ expansion of such correlation functions is known to enumerate discrete surfaces of a given topology, whose polygons carry a spin + or - (Ising model on a random surface [25, 27]), see [21] for the multicut case i.e. foam of Ising surfaces.

The $\overline{w_{k,l}^{(h)}}$ are generating functions enumerating genus h discrete surfaces with k boundaries of spin + and l boundaries of spin -.



$$\overline{w_{2,0}^{(3)}} = \tag{2.6}$$

Notice that the question of boundaries with non uniform spin, i.e. with changes of boundary conditions has been solved to the leading order only in [13].

3 Some algebraic geometry and notations

In this section, we first introduce some notations used all along this article. We then recall the link existing between matrix models and algebraic geometry through the *master loop equation*. We end this part by recalling some basics in algebraic geometry necessary to lead our computations to their end.

3.1 Notation for sets of variables

We will consider functions of many variables x_1, x_2, x_3, \dots , or of a subset of those variables. In that purpose we introduce the following notations:

Let K be a k -upple of integers:

$$K = (i_1, i_2, \dots, i_k) \tag{3.7}$$

We denote $k = |K|$ the length (or cardinal) of K . for any $j \leq |K|$, we denote K_j the set of all j -upples (i.e. subsets of length j) contained in K :

$$K_j := \{J \subset K, |J| = j\} \tag{3.8}$$

We define the following k -uple of complex numbers:

$$\mathbf{x}_K := (x_{i_1}, x_{i_2}, \dots, x_{i_k}) \quad (3.9)$$

3.2 Correlation functions

For a given k , we define the correlation function:

$$\bar{w}_k(x_1, \dots, x_k) := N^{k-2} \left\langle \prod_{i=1}^k \operatorname{tr} \frac{1}{x_i - M_1} \right\rangle_c, \quad (3.10)$$

i.e., with the previous notations:

$$\bar{w}_{|K|}(\mathbf{x}_K) := N^{|K|-2} \left\langle \prod_{r=1}^{|K|} \operatorname{tr} \frac{1}{x_{i_r} - M_1} \right\rangle_c, \quad (3.11)$$

And we consider its $1/N^2$ expansion:

$$\bar{w}_k(\mathbf{x}_K) = \sum_{h=0}^{\infty} \frac{1}{N^{2h}} \bar{w}_k^{(h)}(\mathbf{x}_K) \quad (3.12)$$

We also define the following auxiliary functions:

$$\bar{u}_k(x, y; \mathbf{x}_K) := N^{|K|-1} \left\langle \operatorname{tr} \frac{1}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \prod_{r=1}^{|K|} \operatorname{tr} \frac{1}{x_{i_r} - M_1} \right\rangle_c \quad (3.13)$$

$$p_k(x, y; \mathbf{x}_K) := N^{|K|-1} \left\langle \operatorname{tr} \frac{V_1'(x) - V_1'(M_1)}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \prod_{r=1}^{|K|} \operatorname{tr} \frac{1}{x_{i_r} - M_1} \right\rangle_c \quad (3.14)$$

$$a_k(x; \mathbf{x}_K) := N^{|K|-1} \left\langle \operatorname{tr} \frac{1}{x - M_1} V_2'(M_2) \prod_{r=1}^{|K|} \operatorname{tr} \frac{1}{x_{i_r} - M_1} \right\rangle_c \quad (3.15)$$

Notice that $\bar{u}_k(x, y; \mathbf{x}_K)$ is a polynomial in y of degree $d_2 - 1$, and $p_k(x, y; \mathbf{x}_K)$ is a polynomial in x of degree $d_1 - 1$ and in y of degree $d_2 - 1$.

It is convenient to renormalize those functions, and define:

$$u_k(x, y; \mathbf{x}_K) := \bar{u}_k(x, y; \mathbf{x}_K) - \delta_{k,0}(V_2'(y) - x) \quad (3.16)$$

and

$$w_k(\mathbf{x}_K) := \bar{w}_k(\mathbf{x}_K) + \frac{\delta_{k,2}}{(x_1 - x_2)^2} \quad (3.17)$$

Let us remark that all those functions have the same kind of topological expansion as $w_k(\mathbf{x}_K)$ and one defines $p_k^{(h)}(x, y; \mathbf{x}_K)$ and $u_k^{(h)}(x, y; \mathbf{x}_K)$ as well like in Eq. (3.12).

For the sake of simplicity, we will often omit the exponent (h) for the leading term $h = 0$.

We define the two functions:

$$Y(x) := V_1'(x) - w_1(x) \quad (3.18)$$

$$X(y) := V_2'(y) - \bar{w}_{0,1}(y) \quad (3.19)$$

which we see below, describe the algebraic curve.

3.3 The master loop equation

Loop equations (also called Schwinger–Dyson equations), proceed from the invariance of the matrix integral Eq. (2.1) under changes of variables, they imply a set of relationships between correlation functions. For the 2-matrix model, loop equations have been known since [30], and written in a more systematic way in [17, 18, 19, 26]. It is well known that in the large N limit, loop equations imply an algebraic equation for the functions w_1 (resp. $\bar{w}_{0,1}$), i.e. for the function $Y(x)$ (resp. $X(y)$), called the master loop equation. Let us briefly recall how to derive it (see [19]):

- the change of variables $M_2 \rightarrow M_2 + \epsilon \frac{1}{x-M_1}$ implies:

$$0 = a_0(x) - x\bar{w}_1(x) + 1 \quad (3.20)$$

- the change of variables $M_1 \rightarrow M_1 + \epsilon \frac{1}{x-M_1} \frac{V_2'(y) - V_2'(M_2)}{y-M_2}$ implies:

$$\bar{w}_1(x)\bar{u}_0(x, y) + \frac{1}{N^2}u_1(x, y; x) = V_1'(x)\bar{u}_0(x, y) - p_0(x, y) - y\bar{u}_0(x, y) + V_2'(y)w_1(x) - a_0(x) \quad (3.21)$$

i.e., putting everything together:

$$(y - Y(x))u_0(x, y) + \frac{1}{N^2}u_1(x, y; x) = (V_2'(y) - x)(V_1'(x) - y) - p_0(x, y) + 1 \quad (3.22)$$

We define:

$$E(x, y) = (V_2'(y) - x)(V_1'(x) - y) - p_0(x, y) + 1 \quad (3.23)$$

The *master loop equation* is thus:

$$(y - Y(x))u_0(x, y) + \frac{1}{N^2}u_1(x, y; x) = E(x, y) \quad (3.24)$$

To large N leading order, we have:

$$(y - Y(x))u_0(x, y) = E(x, y) \quad (3.25)$$

Since $u_0(x, y)$ is a polynomial in y , it has no singularity for y finite and the LHS vanishes for $y = Y(x)$, i.e.:

$$E(x, Y(x)) = 0 \quad (3.26)$$

This defines an algebraic curve $E(x, y) = 0$.

Notice that to leading order we have:

$$u_0(x, y) = \frac{E(x, y)}{y - Y(x)} \quad (3.27)$$

and

$$u_0(x, Y(x)) = E_y(x, Y(x)) \quad (3.28)$$

3.4 Introduction to some algebraic geometry

We use notations similar to [24] or [23].

Let us parameterize the curve $E(x, y) = 0$ with a running point p of a compact Riemann surface \mathcal{E} . It means that we define two analytical meromorphic functions $x(p)$ and $y(p)$ on \mathcal{E} such that:

$$E(x, y) = 0 \Leftrightarrow \exists p \in \mathcal{E} \quad x = x(p), \quad y = y(p) \quad (3.29)$$

The functions x and y are not bijective. Indeed, since $E(x, y)$ has a degree $d_2 + 1$ in y , it has $d_2 + 1$ solutions, i.e. for a given x , there exist $d_2 + 1$ points p on \mathcal{E} such that $x(p) = x$. Thus, the Riemann surface is made of $d_2 + 1$ x -sheets, respectively $d_1 + 1$ y -sheets. Hence, from now on, we use these notations:

$$x(p) = x \Leftrightarrow p = p^{(j)}(x) \quad \text{for } j = 0, \dots, d_2 \quad (3.30)$$

$$y(p) = y \Leftrightarrow p = \tilde{p}^{(j)}(y) \quad \text{for } j = 0, \dots, d_1 \quad (3.31)$$

We will most often omit the exponent 0 corresponding to the physical sheet: $p := p^0$.

For instance, one can write $E(x, y)$ as:

$$\begin{aligned} E(x(p), y(q)) &= -g_{d_1+1} \times \prod_{i=0}^{d_1} (x(p) - x(\tilde{q}^{(i)}(y))) \\ &= -\tilde{g}_{d_2+1} \times \prod_{i=0}^{d_2} (y(q) - y(p^{(i)}(x))) \end{aligned} \quad (3.32)$$

Considering that the $w_k^{(h)}$'s, $u_k^{(h)}$'s and $p_k^{(h)}$'s are multivalued functions in their arguments x , we now work with differentials monovalued on the Riemann surface.

Let us write the differentials :

$$W_{k+1}(p, \mathbf{p}_K) := w_{k+1}(x(p), \mathbf{x}(p_K)) dx(p) \prod_{i=1}^k dx(p_i) \quad (3.33)$$

$$U_k(p, y; \mathbf{p}_K) := u_k(x(p), y; \mathbf{x}(p_K)) dx(p) \prod_{i=1}^k dx(p_i) \quad (3.34)$$

$$P_k(x, y; \mathbf{p}_K) := p_k(x, y; \mathbf{x}(p_K)) \prod_{i=1}^k dx(p_i) \quad (3.35)$$

Note: In the following, the arguments of a function will be called $x(p)$ or $y(r)$ if the function is defined on the basis, and p or r if the function is defined on the Riemann surface - and so multivalued on the basis-.

Let us now review some basic objects we need before going any further.

Behaviours at ∞ . We see from Eq. (3.18), that at large x , we have $Y(x) \sim V_1'(x) - \frac{1}{x} + O(1/x^2)$ in the x -physical sheet. And we see from Eq. (3.19), that at large y , we have $X(y) \sim V_2'(y) - \frac{1}{y} + O(1/y^2)$ in the y -physical sheet. This means that the functions $x(p)$ and $y(p)$ have two poles, ∞_+ and ∞_- on \mathcal{E} . The function $x(p)$ has a simple pole at ∞_+ and a pole of degree d_2 at ∞_- , while the function $y(p)$ has a simple pole at ∞_- and a pole of degree d_1 at ∞_+ . We have:

$$y(p) \underset{p \rightarrow \infty_+}{\sim} V_1'(x(p)) - \frac{1}{x(p)} + O(1/x(p)^2) \quad , \quad x(p) \underset{p \rightarrow \infty_-}{\sim} V_2'(y(p)) - \frac{1}{y(p)} + O(1/y(p)^2) \quad (3.36)$$

In particular:

$$\text{Res}_{\infty_+} y dx = \text{Res}_{\infty_-} x dy = 1 \quad (3.37)$$

Genus and cycles. The curve \mathcal{E} is a compact Riemann-surface with a finite genus $g \leq d_1 d_2 - 1$. If $g = 0$, \mathcal{E} is simply connected, and if $g \neq 0$, there exist $2g$ linearly independent irreducible cycles on \mathcal{E} , such that by removing those $2g$ cycles we get a simply connected domain. It is possible to choose canonically the $2g$ cycles as $\mathcal{A}_i, \mathcal{B}_i$, $i = 1, \dots, g$, such that:

$$\mathcal{A}_i \cap \mathcal{A}_j = 0 \quad , \quad \mathcal{B}_i \cap \mathcal{B}_j = 0 \quad , \quad \mathcal{A}_i \cap \mathcal{B}_j = \delta_{ij} \quad (3.38)$$

Branch points. The x -branch points a_i , $i = 1, \dots, d_2 + 1 + 2g$, are the zeroes of the differential dx , respectively, the y -branch points b_i , $i = 1, \dots, d_1 + 1 + 2g$, are the zeroes of dy . We assume here, that all branch points are simple and distinct. Notice

also, that $E_y(x(p), y(p))$ vanishes (simple zeroes) at the branchpoints (it vanishes in other points too).

Bergmann kernel. On the Riemann surface \mathcal{E} , there exists a unique abelian bilinear differential $B(p, q)$, with one double pole at $p = q$, such that:

$$B(p, q) \underset{p \rightarrow q}{\sim} \frac{dx(p)dx(q)}{(x(p) - x(q))^2} + \text{finite} \quad \text{and} \quad \forall i \quad \oint_{p \in \mathcal{A}_i} B(p, q) = 0 \quad (3.39)$$

It is symmetric:

$$B(p, q) = B(q, p) \quad (3.40)$$

Its expression in terms of theta-functions can be found in [24, 23], it depends only on the complex structure of \mathcal{E} .

Abelian differential of third kind.

On the Riemann surface \mathcal{E} , there exists a unique abelian differential of the third kind $dS_{q,r}(p)$, with two simple poles at $p = q$ and at $p = r$, such that:

$$\text{Res}_{p \rightarrow q} dS_{q,r}(p) = 1 = - \text{Res}_{p \rightarrow r} dS_{q,r}(p) \quad \text{and} \quad \forall i \quad \oint_{\mathcal{A}_i} dS_{q,r}(p) = 0 \quad (3.41)$$

We have:

$$dS_{q,r}(p) = \int_{q'=r}^q B(p, q') \quad (3.42)$$

where the integration path does not intersect any \mathcal{A}_i or \mathcal{B}_i .

$dS_{q,r}(p)$ is a differential on \mathcal{E} in terms of p , but it is a multivalued function of q (and of r). After crossing a cycle \mathcal{B}_i , it has no discontinuity, and after crossing a cycle \mathcal{A}_i , it has a discontinuity:

$$\text{disc}(dS_{q,r}(p)) = dS_{q_+,r}(p) - dS_{q_-,r}(p) = \oint_{q' \in \mathcal{B}_i} B(p, q') \quad (3.43)$$

Note that the discontinuity is independent of q .

Riemann bilinear identity. If ω is a differential form on \mathcal{E} , such that $\oint_{q \in \mathcal{A}_i} \omega(q) = 0$, we have:

$$\begin{aligned} \sum_i \text{Res}_{q \rightarrow z_i} \omega(q) dS_{q,r}(p) &= \sum_{i=1}^g \oint_{q \in \mathcal{A}_i} \text{disc}_{\mathcal{A}_i}(\omega(q) dS_{q,r}(p)) \\ &\quad - \sum_{i=1}^g \oint_{q \in \mathcal{B}_i} \text{disc}_{\mathcal{B}_i}(\omega(q) dS_{q,r}(p)) \\ &= \sum_{i=1}^g \oint_{q \in \mathcal{A}_i} \omega(q) \text{disc}_{\mathcal{A}_i}(dS_{q,r}(p)) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^g \text{disc}_{\mathcal{A}_i}(dS_{q,r}(p)) \oint_{q \in \mathcal{A}_i} \omega(q) \\
&= 0
\end{aligned} \tag{3.44}$$

where the LHS is the sum over all residues on a fundamental domain, the poles z_i are all the poles of ω as well as the pole at $q = p$. This identity is obtained by moving the integration contours on the surface, and taking carefully into account discontinuities along the non-trivial cycles (see [24, 23]).

4 Loop equations and fixed filling fractions

To large N leading order, the loop equation Eq. (3.26) is an algebraic equation:

$$E(x, Y(x)) = 0 \tag{4.45}$$

The coefficients of E are fixed by assumptions on the problem we are considering.

Here, we consider the problem of a formal matrix model, i.e. which is computed as the formal power series expansion of a matrix integral, where the non-quadratic terms in the potentials V_1 and V_2 are treated as perturbations near quadratic potentials. Such a perturbative expansion can be performed only near local extrema of $V_1(x) + V_2(y) - xy$, i.e. near points such that:

$$V_1'(\xi_i) = \eta_i \quad , \quad V_2'(\eta_i) = \xi_i \tag{4.46}$$

which has $d_1 d_2$ solutions. Therefore, if \overline{M}_1 and \overline{M}_2 are diagonal matrices, whose diagonal entries are some ξ_i 's (resp. η_i 's), $(\overline{M}_1, \overline{M}_2)$ is a local extremum of $\text{tr}(V_1(M_1) + V_2(M_2) - M_1 M_2)$ around which we can perform the perturbative expansion.

The choice of such an extremum, around which the perturbative series is computed, is equivalent to the choice of the number of eigenvalues near each pair (ξ_i, η_i) , $i = 1, \dots, d_1 d_2$, i.e. the data of $d_1 d_2$ integers n_i such that:

$$\sum_{i=1}^{d_1 d_2} n_i = N \tag{4.47}$$

This means, that we can choose some contours \mathcal{C}_i , $i = 1, \dots, d_1 d_2$, such that:

$$\frac{1}{2i\pi} \oint_{\mathcal{C}_i} \text{tr} \frac{dx}{x - M_1} = -n_i \tag{4.48}$$

i.e.

$$\frac{1}{2i\pi} \oint_{\mathcal{C}_i} w_1(x) dx = -\frac{n_i}{N} := -\epsilon_i \tag{4.49}$$

If $w_1(x) = V_1'(x) - Y(x)$ is solution of an algebraic equation of genus g , it means that up to a redefinition of contours, the filling fractions are the \mathcal{A} -cycle integrals. We define:

$$\frac{1}{2i\pi} \oint_{\mathcal{A}_i} y dx = -\frac{1}{2i\pi} \oint_{\mathcal{A}_i} x dy = \epsilon_i \quad (4.50)$$

The ϵ_i 's are called filling fractions, and they are given parameters (moduli) of the model. They don't depend on the potential or on any other parameter.

In particular, since all correlation functions $w_k(x_1, \dots, x_k)$ are obtained by derivation of w_1 with respect to the potential V_1 , we have:

$$\frac{1}{2i\pi} \oint_{\mathcal{A}_i} w_k(x_1, \dots, x_k) dx_1 = 0 \quad (4.51)$$

Eq. (4.50) together with the large x and y behaviours Eq. (3.36), are sufficient to determine completely all the coefficients of the polynomial $E(x, y)$, and thus the leading large N resolvents $w_1(x)$ and $\bar{w}_{0;1}(y)$.

In this perturbative model, the free energy as well as all correlation functions always have a formal $1/N^2$ expansion.

In what follows, we assume that the leading resolvent, i.e. the functions $Y(x)$ and $X(y)$ are known, and we refer the reader to the existing literature on that topic, for instance [4, 19, 26, 28].

5 The leading term : genus zero case.

Before considering the full $\frac{1}{N^2}$ expansion, let us focus on the structure of the leading terms corresponding to genus zero surfaces. We establish a general loop-equation whose solution gives a recursive definition of the w_k 's and the u_k 's that can be represented by Feynman graphs.

In this section, we omit the subscript (0), i.e. $w_k^{(0)}(\mathbf{x}_K) \rightarrow w_k(\mathbf{x}_K)$, and we omit the $1/N^2$ terms in the loop equations.

We consider: $K = (1, \dots, k)$.

5.1 The loop-equation

The w_k 's are totally determined by only one generic loop equation considered to its leading order. We build this equation in order to link w_k and w_{k+1} .

- The change of variable $\delta M_2 = \frac{1}{x-M_1} \prod_{i=1}^k \text{tr} \frac{1}{x_i-M_1}$ implies (see [19]):

$$a_k(x; \mathbf{x}_K) = x \bar{w}_{k+1}(x, \mathbf{x}_K) \quad (5.52)$$

• The change of variable $\delta M_1 = \frac{1}{x-M_1} \frac{V_2'(y)-V_2'(M_2)}{y-M_2} \prod_{i=1}^k \text{tr} \frac{1}{x_i-M_1}$ implies to leading order (see [19]):

$$\begin{aligned}
& w_1(x) \bar{u}_k(x, y; \mathbf{x}_K) + \sum_{j=0}^{k-1} \sum_{J \in K_j} \bar{u}_j(x, y; \mathbf{x}_J) \bar{w}_{k-j+1}(x, \mathbf{x}_{K-J}) \\
& + \sum_{j=1}^k \frac{\partial}{\partial x_j} \frac{\bar{u}_{k-1}(x, y; \mathbf{x}_{K-\{j\}}) - \bar{u}_{k-1}(x_j, y; \mathbf{x}_{K-\{j\}})}{x - x_j} \\
= & V_1'(x) \bar{u}_{k,0}(x, y; \mathbf{x}_K) - p_{k,0}(x, y; \mathbf{x}_K) \\
& - y \bar{u}_k(x, y; \mathbf{x}_K) + V_2'(y) \bar{w}_{k+1}(x, \mathbf{x}_K) - a_k(x; \mathbf{x}_K)
\end{aligned} \tag{5.53}$$

i.e. for $k \geq 1$:

$$\begin{aligned}
(y - Y(x)) u_k(x, y; \mathbf{x}_K) & = - \sum_{j=0}^{k-1} \sum_{J \in K_j} u_j(x, y; \mathbf{x}_J) w_{k-j+1}(x, \mathbf{x}_{K-J}) \\
& + \sum_{j=1}^k \frac{\partial}{\partial x_j} \frac{u_{k-1}(x_j, y; \mathbf{x}_{K-\{j\}})}{x - x_j} - p_k(x, y; \mathbf{x}_K)
\end{aligned} \tag{5.54}$$

Under this form, the loop equation is in terms of multivalued functions. It is more appropriate to write it in terms of meromorphic differentials on the Riemann surface:

$$\begin{aligned}
(y(r) - y(p)) U_k(p, y(r); \mathbf{p}_K) & = - \sum_{j=0}^{k-1} \sum_{J \in K_j} \frac{U_j(p, y(r); \mathbf{p}_J) W_{k-j+1}(p, \mathbf{p}_{K-J})}{dx(p)} \\
& + \sum_{j=1}^k d_{p_j} \left(\frac{U_{k-1}(p_j, y(r); \mathbf{p}_{K-\{j\}})}{x(p) - x(p_j)} \frac{dx(p)}{dx(p_j)} \right) \\
& - P_k(x(p), y(r); \mathbf{p}_K) dx(p)
\end{aligned} \tag{5.55}$$

In particular, if one chooses $r = p$, one gets (using Eq. (3.27) and Eq. (3.28)):

$$\begin{aligned}
E_y(x(p), y(p)) W_{k+1}(p, \mathbf{p}_K) & = -P_k(x(p), y(p); \mathbf{p}_K) dx(p) \\
& - \sum_{j=1}^{k-1} \sum_{J \in K_j} \frac{U_j(p, y(p); \mathbf{p}_J) W_{k-j+1}(p, \mathbf{p}_{K-J})}{dx(p)} \\
& + \sum_{j=1}^k d_{p_j} \left(\frac{U_{k-1}(p_j, y(p); \mathbf{p}_{K-\{j\}})}{x(p) - x(p_j)} \frac{dx(p)}{dx(p_j)} \right)
\end{aligned} \tag{5.56}$$

Notice that these two equations Eq. (5.55) and Eq. (5.56) imply by recursion, that W_k and U_k are indeed meromorphic differentials on the curve, in all their variables.

We define:

$$\forall(i, j) \quad R_k^i(p^{(j)}, p_K) := \frac{U_k(p^{(j)}, y(p^{(i)}); p_K)}{E_y(x(p^{(j)}), y(p^{(i)})) dx(p^{(j)})} \tag{5.57}$$

We have already obtained (see Eq. (3.27)) that:

$$R_0^i(p^{(l)}) = \frac{\delta_{i,l}}{dx(p)} \quad (5.58)$$

5.2 k=1

We begin by this special case because it represents some initial condition for the following. In fact, the two correlation functions $W_2(p_1, p_2)$ and $U_1(p_1, y; p_2)$, are the basis of the whole structure of the W_k 's. Moreover, it allows us to show through a simple example the way we proceed further for the general case.

We first derive once again the well known result claiming that the two point function is nothing else but the Bergmann Kernel (see [4] for instance).

Let $o \in \mathcal{E}$ be an arbitrary point on the Riemann surface. Since the abelian differential of the 3rd kind defined in Eq. (3.41) $dS_{q,o}(p)$ behaves as $\frac{dx(p)}{x(p)-x(q)}$ when $q \rightarrow p$, one can write the Cauchy formula under the form:

$$W_{2,0}(p, p_1) = - \operatorname{Res}_{q \rightarrow p} dS_{q,o}(p) W_{2,0}(q; p_1) \quad (5.59)$$

One can see from Eq. (5.56) with $k = 1$, and from Eq. (4.51), that the integrand in the RHS has poles only for $q \rightarrow p$ and $q \rightarrow p_1$, Since $W_{2,0}$ has vanishing \mathcal{A} -cycles due to Eq. (4.51), we can use the Riemann bilinear identity Eq. (3.44), and get:

$$W_{2,0}(p, p_1) = \operatorname{Res}_{q \rightarrow p_1} dS_{q,o}(p) W_{2,0}(q; p_1) \quad (5.60)$$

For $k = 1$, Eq. (5.56) reads:

$$\begin{aligned} E_y(x(p), y(p)) W_2(p, p_1) &= -P_1(x(p), y(p); p_1) dx(p) \\ &\quad + d_{p_1} \left(\frac{U_0(p_1, y(p))}{x(p) - x(p_1)} \frac{dx(p)}{dx(p_1)} \right) \end{aligned} \quad (5.61)$$

and thus we have:

$$\begin{aligned} W_2(p; p_1) &= \operatorname{Res}_{q \rightarrow p_1} dS_{q,o}(p) W_2(q; p_1) \\ &= - \operatorname{Res}_{q \rightarrow p_1} dS_{q,o}(p) \frac{P_1(x(q), y(q); p_1) dx(q)}{E_y(x(q), y(q))} \\ &\quad + \operatorname{Res}_{q \rightarrow p_1} dS_{q,o}(p) \frac{d_{p_1} \left(\frac{U_0(p_1, y(q))}{x(q) - x(p_1)} \frac{dx(q)}{dx(p_1)} \right)}{E_y(x(q), y(q))} \end{aligned} \quad (5.62)$$

Since $P_1(x(q), y(q); p_1)$ is a polynomial in $x(q)$ and $y(q)$, it has no pole at $q = p_1$. For the second term we use Eq. (3.27):

$$\begin{aligned}
W_2(p; p_1) &= \operatorname{Res}_{q \rightarrow p_1} dS_{q,o}(p) W_2(q; p_1) \\
&= d_{p_1} \operatorname{Res}_{q \rightarrow p_1} dS_{q,o}(p) \frac{E(x(p_1), y(q)) dx(q)}{(x(q) - x(p_1))(y(q) - y(p_1))E_y(x(q), y(q))} \\
&= d_{p_1} dS_{p_1,o}(p) \\
&= B(p_1, p)
\end{aligned} \tag{5.63}$$

We thus recover the well known result: the two-points function is equal to the Bergmann kernel on the Riemann surface corresponding to the algebraic equation $E(x, y) = 0$ (cf [4, 26, 19, 28]).

$$\boxed{W_2(p; p_1) = B(p, p_1)} \tag{5.64}$$

Let us now compute $U_1(p, y; p_1)$. For $k = 1$, Eq. (5.55) reads:

$$\begin{aligned}
\frac{(y(r) - y(q))U_1(q, y(r); p_1)}{dx(q)} &= -\frac{W_2(q; p_1)U_0(q, y(r))}{dx(q)^2} - P_1(x(q), y(r); p_1) \\
&\quad + d_{p_1} \left(\frac{U_0(p_1, y(r))}{(x(q) - x(p_1)) dx(p_1)} \right)
\end{aligned} \tag{5.65}$$

take it for $q = r = p^{(i)}$:

$$\begin{aligned}
0 &= -\frac{W_2(p^{(i)}; p_1)U_0(p^{(i)}, y(p^{(i)}))}{dx(p^{(i)})^2} - P_1(x(p^{(i)}), y(p^{(i)}); p_1) \\
&\quad + d_{p_1} \left(\frac{U_0(p_1, y(p^{(i)}))}{(x(p^{(i)}) - x(p_1)) dx(p_1)} \right)
\end{aligned} \tag{5.66}$$

using that $x(p) = x(p^{(i)})$, we have:

$$\begin{aligned}
0 &= -\frac{W_2(p^{(i)}; p_1)U_0(p^{(i)}, y(p^{(i)}))}{dx(p)^2} - P_1(x(p), y(p^{(i)}); p_1) \\
&\quad + d_{p_1} \left(\frac{U_0(p_1, y(p^{(i)}))}{(x(p) - x(p_1)) dx(p_1)} \right)
\end{aligned} \tag{5.67}$$

Now, write Eq. (5.65) with $q = p$ and $r = p^{(i)}$:

$$\begin{aligned}
\frac{(y(p^{(i)}) - y(p))U_1(p, y(p^{(i)}); p_1)}{dx(p)} &= -\frac{W_2(p; p_1)U_0(p, y(p^{(i)}))}{dx(p)^2} - P_1(x(p), y(p^{(i)}); p_1) \\
&\quad + d_{p_1} \left(\frac{U_0(p_1, y(p^{(i)}))}{(x(p) - x(p_1)) dx(p_1)} \right)
\end{aligned} \tag{5.68}$$

and insert Eq. (5.67), you get:

$$(y(p^{(i)}) - y(p))U_1(p, y(p^{(i)}); p_1) = \frac{W_2(p^{(i)}; p_1)U_0(p^{(i)}, y(p^{(i)}))}{dx(p)} - \frac{W_2(p; p_1)U_0(p, y(p^{(i)}))}{dx(p)} \quad (5.69)$$

Using Eq. (3.27), i.e. $U_0(p, y) = \frac{E(x(p), y)}{y - y(p)} dx(p)$ this implies:

$$(y(p^{(i)}) - y(p))U_1(p, y(p^{(i)}); p_1) = W_2(p^{(i)}; p_1)E_y(x(p^{(i)}), y(p^{(i)})) \quad (5.70)$$

Since $U_1(p, y; p_1)$ is a polynomial of degree $d_2 - 1$ in y , we can reconstruct it through the interpolation formula:

$$U_1(p, y; p_1) = \frac{E(x(p), y)}{(y - y(p))} \sum_{i=1}^{d_2} \frac{1}{y - y(p^{(i)})} \frac{(y(p^{(i)}) - y(p))U_1(p, y(p^{(i)}); p_1)}{E_y(x(p^{(i)}), y(p^{(i)}))} \quad (5.71)$$

i.e.

$$U_1(p, y; p_1) = \frac{E(x(p), y)}{(y - y(p))} \sum_{i=1}^{d_2} \frac{W_2(p^{(i)}, p_1)}{y - y(p^{(i)})} \quad (5.72)$$

and in particular, at $y = y(p)$, we have:

$$R_1^0(p, p_1) dx(p) = \frac{U_1(p, y(p); p_1)}{E_y(x(p), y(p))} = \sum_{i=1}^{d_2} \frac{W_2(p^{(i)}, p_1)}{y(p) - y(p^{(i)})} \quad (5.73)$$

and for $i \neq 0$, we have:

$$R_1^i(p, p_1) dx(p) = \frac{U_1(p, y(p^{(i)}); p_1)}{E_y(x(p), y(p^{(i)}))} = \frac{W_2(p^{(i)}, p_1)}{(y(p^{(i)}) - y(p))} \quad (5.74)$$

5.3 $k \geq 2$.

In this section, one proceeds in a very similar fashion as in the previous one. Using the same tricks, we are able to determine two relations defining recursively $W_{k+1}(p, \mathbf{p}_K)$.

Let us suppose that one knows $W_j(\mathbf{p}_J)$ for $j \leq k$ and $R_j^i(p, \mathbf{p}_J)$ for $j \leq k - 1$. The first step consists in the determination of $W_{k+1}(p, \mathbf{p}_K)$ as a function of the lower order correlation functions. The second step leads us to the computation of $R_k^i(p, \mathbf{p}_K)$. Once this is done, one knows the correlation functions one order upper: one can then know them for any k , considering that one knows $W_2(p, p_1)$.

5.3.1 Determination of W_{k+1} .

In this section, we derive a recursive formula for $W_{k+1}(p, \mathbf{p}_K)$.

The Cauchy formula gives:

$$W_{k+1}(p, \mathbf{p}_K) = - \operatorname{Res}_{p' \rightarrow p} W_{k+1}(p', \mathbf{p}_K) dS_{p', o}(p) \quad (5.75)$$

The integrand has poles in p' only at p and the branch points a_s . Using again Riemann bilinear identity Eq. (3.44), we can then move the integration contour and get:

$$W_{k+1}(p, \mathbf{p}_K) = \sum_s \operatorname{Res}_{p' \rightarrow a_s} W_{k+1}(p', \mathbf{p}_K) dS_{p', o}(p) \quad (5.76)$$

We now introduce the loop equation Eq. (5.56) inside this expression and remark that only one term has poles when $p' \rightarrow a_s$. Thus $W_{k+1}(p, \mathbf{p}_K)$ can be written:

$$\begin{aligned} W_{k+1}(p, \mathbf{p}_K) &= - \sum_s \operatorname{Res}_{p' \rightarrow a_s} \sum_{j=1}^{k-1} \sum_{J \in K_j} \frac{U_j(p', y(p'); \mathbf{p}_J)}{E_y(x(p'), y(p'))} \frac{W_{k-j+1}(p', \mathbf{p}_{K-J})}{dx(p')} dS_{p', o}(p) \\ &= - \sum_s \operatorname{Res}_{p' \rightarrow a_s} \sum_{j=1}^{k-1} \sum_{J \in K_j} R_j^0(p', \mathbf{p}_J) W_{k-j+1}(p', \mathbf{p}_{K-J}) dS_{p', o}(p) \end{aligned} \quad (5.77)$$

However, $U_k(p, y; \mathbf{p}_K)$ is a polynomial in y whose degree is equal to $d_2 - 1$. Considering its d_2 values for $y = y(p^{(i)})$ with $i \in [1, d_2]$, the interpolation formula reads:

$$\forall y \quad \frac{(y - y(p))U_k(p, y; \mathbf{p}_K)}{E(x(p), y)} = - \sum_{i=1}^{d_2} \frac{U_k(p, y(p^{(i)}); \mathbf{p}_K)(y(p) - y(p^{(i)}))}{(y - y(p^{(i)}))E_y(x(p), y(p^{(i)}))} \quad (5.78)$$

for $y = y(p)$, this gives:

$$R_k^0(p, \mathbf{p}_K) = - \sum_{i=1}^{d_2} R_k^i(p, \mathbf{p}_K) \quad (5.79)$$

So, in Eq. (5.77), one obtains the recursive formula for $W_k(\mathbf{p}_K)$:

$$W_{k+1}(p, \mathbf{p}_K) = \sum_{i=1}^{d_2} \sum_{j=1}^{k-1} \sum_{J \in K_j} \sum_s \operatorname{Res}_{p' \rightarrow a_s} R_j^i(p'; \mathbf{p}_J) W_{k-j+1}(p', \mathbf{p}_{K-J}) dS_{p', o}(p)$$

(5.80)

The sum over j represents the summation over all partitions of K into two subsets J and $K - J$.

5.3.2 Determination of R_k^i .

In this section, one proceeds recursively, in order to find a recursion formula for R_k^i .

For this purpose, one needs to know an intermediate expression defining the different U_k 's as well as a relation linking the value of $\sum_{j=0}^{k-1} U_j(p^{(i)}, y(p); \mathbf{p}_J) W_{k-j+1}(p^{(i)}, \mathbf{p}_{K-J})$ for different i 's.

Let us rewrite here Eq. (5.55):

$$\begin{aligned}
(y(r) - y(q))U_k(q, y(r); \mathbf{p}_K) &= - \sum_{j=0}^{k-1} \sum_{J \in K_j} \frac{1}{dx(q)} U_j(q, y(r); \mathbf{p}_J) W_{k-j+1}(q, \mathbf{p}_{K-J}) \\
&\quad + \sum_{j=1}^k d_{p_j} \left(\frac{U_{k-1}(p_j, y(r); \mathbf{p}_{K-\{j\}})}{x(q) - x(p_j)} \frac{dx(q)}{dx(p_j)} \right) \\
&\quad - P_k(x(q), y(r); \mathbf{p}_K) dx(q)
\end{aligned} \tag{5.81}$$

- We proceed here exactly as in the case $k = 1$. We use the properties of rational functions defined on the basis and not the Riemann surface.

For $r = q = p^{(i)}$, Eq. (5.81) reads:

$$\begin{aligned}
0 &= - \sum_{j=0}^{k-1} \sum_{J \in K_j} \frac{1}{dx(p^{(i)})} U_j(p^{(i)}, y(p^{(i)}); \mathbf{p}_J) W_{k-j+1}(p^{(i)}, \mathbf{p}_{K-J}) \\
&\quad + \sum_{j=1}^k d_{p_j} \left(\frac{U_{k-1}(p_j, y(p^{(i)}); \mathbf{p}_{K-\{j\}})}{x(p^{(i)}) - x(p_j)} \frac{dx(p^{(i)})}{dx(p_j)} \right) \\
&\quad - P_k(x(p^{(i)}), y(p^{(i)}); \mathbf{p}_K) dx(p^{(i)}) \\
&= - \sum_{j=0}^{k-1} \sum_{J \in K_j} \frac{1}{dx(p)} U_j(p^{(i)}, y(p^{(i)}); \mathbf{p}_J) W_{k-j+1}(p^{(i)}, \mathbf{p}_{K-J}) \\
&\quad + \sum_{j=1}^k d_{p_j} \left(\frac{U_{k-1}(p_j, y(p^{(i)}); \mathbf{p}_{K-\{j\}})}{x(p) - x(p_j)} \frac{dx(p)}{dx(p_j)} \right) \\
&\quad - P_k(x(p), y(p^{(i)}); \mathbf{p}_K) dx(p)
\end{aligned} \tag{5.82}$$

where we have used that $x(p) = x(p^{(i)})$.

Now, write Eq. (5.81) with $r = p^{(i)}$ and $q = p$:

$$\begin{aligned}
&(y(p^{(i)}) - y(p))U_k(p, y(p^{(i)}); \mathbf{p}_K) \\
&= - \sum_{j=0}^{k-1} \sum_{J \in K_j} \frac{1}{dx(p)} U_j(p, y(p^{(i)}); \mathbf{p}_J) W_{k-j+1}(p, \mathbf{p}_{K-J}) \\
&\quad + \sum_{j=1}^k d_{p_j} \left(\frac{U_{k-1}(p_j, y(p^{(i)}); \mathbf{p}_{K-\{j\}})}{x(p) - x(p_j)} \frac{dx(p)}{dx(p_j)} \right)
\end{aligned}$$

$$-P_k(x(p), y(p^{(i)}); \mathbf{p}_K) dx(p) \quad (5.83)$$

and inserting Eq. (5.82) we get:

$$\begin{aligned} & (y(p^{(i)}) - y(p)) U_k(p, y(p^{(i)}); \mathbf{p}_K) \\ = & - \sum_{j=0}^{k-1} \sum_{J \in K_j} \frac{1}{dx(p)} U_j(p, y(p^{(i)}); \mathbf{p}_J) W_{k-j+1}(p, \mathbf{p}_{K-J}) \\ & + \sum_{j=0}^{k-1} \sum_{J \in K_j} \frac{1}{dx(p)} U_j(p^{(i)}, y(p^{(i)}); \mathbf{p}_J) W_{k-j+1}(p^{(i)}, \mathbf{p}_{K-J}) \end{aligned} \quad (5.84)$$

This formula, is in principle sufficient to compute the U_k 's recursively, and then one can compute the R_k^i 's. However, what we need in order to get Feynman rules, is a closed recursion relation for the R_k^i 's themselves. In order to achieve this aim, we show that:

Lemma: for any $k \geq 1$, one has:

$$\begin{aligned} U_k(p, y; \mathbf{p}_K) &= \frac{E(x(p), y) dx(p)}{y - y(p)} \sum_{r=1}^{d_2} \sum_{K_1 \cup \dots \cup K_r = K} \sum_{j_1 \neq j_2 \neq \dots \neq j_r = 1}^{d_2} \\ & \prod_{t=1}^r \frac{W_{|K_t|+1}(p^{(j_t)}, \mathbf{p}_{K_t})}{(y - y(p^{(j_t)})) dx(p)} \end{aligned} \quad (5.85)$$

where the sum over $K_1 \cup \dots \cup K_r = K$ is a sum over all partitions of K into r subsets.

Proof: It can be proven easily by recursive action of $\partial/\partial V_1$, as in [3], however, in order to have a self-contained method, we want to derive it here only from the loop equations.

The proof works by recursion on k . It was proven in the previous section for $k = 1$. Let us assume that, it holds for any $l \leq k - 1$.

Notice, that since both sides of Eq. (5.85) are polynomials of y , from degree $d_2 - 1$, it is sufficient to prove that the equality holds for d_2 values of y , namely, it is sufficient to prove it for $y = y(p^{(i)})$, $i = 1, \dots, d_2$. Therefore, one has to prove that:

$$\begin{aligned} \frac{U_k(p, y(p^{(i)}); \mathbf{p}_K)}{dx(p)} &= \frac{E_y(x(p^{(i)}), y(p^{(i)}))}{y(p^{(i)}) - y(p)} \sum_{r=1}^{d_2} \sum_{K_1 \cup \dots \cup K_r = K} \sum_{j_1 \neq j_2 \neq \dots \neq j_{r-1} \neq 0, i}^{d_2} \\ & \frac{W_{|K_r|+1}(p^{(i)}, \mathbf{p}_{K_r})}{dx(p)} \prod_{t=1}^{r-1} \frac{W_{|K_t|+1}(p^{(j_t)}, \mathbf{p}_{K_t})}{(y - y(p^{(j_t)})) dx(p)} \end{aligned} \quad (5.86)$$

where only the sums in which one of the j_t 's is equal to i contribute.

The recursion hypothesis for $j \leq k-1$, and any $J \in K_j$ gives:

$$\begin{aligned} \frac{U_j(p^{(i)}, y(p^{(i)}); \mathbf{P}_J)}{dx(p)} &= E_y(x(p^{(i)}), y(p^{(i)})) \sum_{r=1}^{d_2} \sum_{J_1 \cup \dots \cup J_r = J} \sum_{j_1 \neq j_2 \neq \dots \neq j_r \neq i} \\ &\quad \prod_{t=1}^r \frac{W_{|J_t|+1}(p^{(j_t)}, \mathbf{P}_{J_t})}{(y(p^{(i)}) - y(p^{(j_t)})) dx(p)} \end{aligned} \quad (5.87)$$

In order to compute $U_j(p, y(p^{(i)}); \mathbf{P}_J)$, one has to keep only terms in the sum such that there exists a t such that $j_t = i$, i.e.

$$\begin{aligned} \frac{U_j(p, y(p^{(i)}); \mathbf{P}_J)}{dx(p)} &= E_y(x(p^{(i)}), y(p^{(i)})) \sum_{r=1}^{d_2} \sum_{J_1 \cup \dots \cup J_r = J} \sum_{j_1 \neq j_2 \neq \dots \neq j_{r-1} \neq 0, i} \\ &\quad \frac{W_{|J_r|+1}(p^{(i)}, \mathbf{P}_{J_r})}{(y(p^{(i)}) - y(p)) dx(p)} \prod_{t=1}^{r-1} \frac{W_{|J_t|+1}(p^{(j_t)}, \mathbf{P}_{J_t})}{(y(p^{(i)}) - y(p^{(j_t)})) dx(p)} \end{aligned} \quad (5.88)$$

Insert that into Eq. (5.84):

$$\begin{aligned} &(y(p^{(i)}) - y(p)) U_k(p, y(p^{(i)}); \mathbf{P}_K) \\ = &-E_y(x(p^{(i)}), y(p^{(i)})) \sum_{j=0}^{k-1} \sum_{J \in K_j} \sum_{r=1}^{d_2} \sum_{J_1 \cup \dots \cup J_r = J} \sum_{j_1 \neq j_2 \neq \dots \neq j_{r-1} \neq 0, i} \\ &W_{k-j+1}(p, \mathbf{P}_{K-J}) \frac{W_{|J_r|+1}(p^{(i)}, \mathbf{P}_{J_r})}{(y(p^{(i)}) - y(p)) dx(p)} \prod_{t=1}^{r-1} \frac{W_{|J_t|+1}(p^{(j_t)}, \mathbf{P}_{J_t})}{(y(p^{(i)}) - y(p^{(j_t)})) dx(p)} \\ &+ E_y(x(p^{(i)}), y(p^{(i)})) \sum_{j=0}^{k-1} \sum_{J \in K_j} \sum_{r=1}^{d_2} \sum_{J_1 \cup \dots \cup J_r = J} \sum_{j_1 \neq j_2 \neq \dots \neq j_r \neq i} \\ &W_{k-j+1}(p^{(i)}, \mathbf{P}_{K-J}) \prod_{t=1}^r \frac{W_{|J_t|+1}(p^{(j_t)}, \mathbf{P}_{J_t})}{(y(p^{(i)}) - y(p^{(j_t)})) dx(p)} \end{aligned} \quad (5.89)$$

The difference between these two summation, keeps only $j_t \neq 0, i$, thus:

$$\begin{aligned} &U_k(p, y(p^{(i)}); \mathbf{P}_K) \\ = &E_y(x(p^{(i)}), y(p^{(i)})) dx(p) \sum_{j=0}^{k-1} \sum_{J \in K_j} \sum_{r=1}^{d_2} \sum_{J_1 \cup \dots \cup J_r = J} \sum_{j_1 \neq j_2 \neq \dots \neq j_r \neq i, 0} \\ &\frac{W_{k-j+1}(p^{(i)}, \mathbf{P}_{K-J})}{(y(p^{(i)}) - y(p)) dx(p)} \prod_{t=1}^r \frac{W_{|J_t|+1}(p^{(j_t)}, \mathbf{P}_{J_t})}{(y(p^{(i)}) - y(p^{(j_t)})) dx(p)} \end{aligned} \quad (5.90)$$

i.e. we have proven the lemma for k , for $y = y(p^{(i)})$, and since both sides are polynomials in y of degree $d_2 - 1$, the equality holds for all y .

•

Remark that this lemma can easily be obtained by recursively applying the $\partial/\partial V_1(x)$ loop insertion operator. We did not use that method here, in order to have a self consistent method based only on loop equations.

Theorem: For all $k \geq 1$, one has:

$$\boxed{\begin{aligned} & \sum_{i=1}^{d_2} \sum_{j=0}^{k-1} \sum_{J \in K_j} U_j(p^{(i)}, y(p); \mathbf{p}_J) W_{k-j+1}(p^{(i)}, \mathbf{p}_{K-J}) \\ = & \sum_{j=1}^{k-1} \sum_{J \in K_j} U_j(p, y(p); \mathbf{p}_J) W_{k-j+1}(p, \mathbf{p}_{K-J}) \end{aligned}} \quad (5.91)$$

Proof of the theorem: Let us simply perform some basic rearrangements:

$$\begin{aligned} & \sum_{i=1}^{d_2} \sum_{j=0}^{k-1} \sum_{J \in K_j} U_j(p^{(i)}, y(p); \mathbf{p}_J) W_{k-j+1}(p^{(i)}, \mathbf{p}_{K-J}) \\ = & \sum_{K_1 \cup L=K} \sum_{j_1=1}^{d_2} W_{|K_1|+1}(p^{(j_1)}, \mathbf{p}_{K_1}) U_{|L|+1}(p^{(j_1)}, y(p); \mathbf{p}_L) \\ = & E_y(x(p), y(p)) dx(p) \sum_{K_1 \cup L=K} \sum_{j_1=1}^{d_2} \sum_{r=1}^{d_2} \sum_{K_2 \cup \dots \cup K_{r+1}=L} \sum_{j_2 \neq j_3 \neq \dots \neq j_r \in [1, d_2] - \{j_1\}} \\ & W_{|K_1|+1}(p^{(j_1)}, \mathbf{p}_{K_1}) \frac{W_{|K_{r+1}|+1}(p, \mathbf{p}_{K_{r+1}})}{(y(p) - y(p^{(j_1)}))} \prod_{a=2}^r \frac{W_{|K_a|+1}(p^{(j_a)}, \mathbf{p}_{K_a})}{(y(p) - y(p^{(j_a)}))} dx(p) \\ = & E_y(x(p), y(p)) dx(p) \sum_{r=1}^{d_2} \sum_{K_1 \cup \dots \cup K_{r+1}=K} \sum_{j_1 \neq j_2 \neq \dots \neq j_r=1}^{d_2} \\ & \prod_{a=1}^r \frac{W_{|K_a|+1}(p^{(j_a)}, \mathbf{p}_{K_a}) W_{|K_{r+1}|+1}(p, \mathbf{p}_{K_{r+1}})}{(y(p) - y(p^{(j_a)}))} dx(p) \\ = & \sum_{K_{r+1} \cup J=K} W_{|K_{r+1}|+1}(p, \mathbf{p}_{K_{r+1}}) U_{|J|}(p, y(p); \mathbf{p}_J) \end{aligned} \quad (5.92)$$

•

This identity simplifies Eq. (5.84) which becomes now:

$$(y(p^{(i)}) - y(p)) R_k^i(p, \mathbf{p}_K) dx(p) =$$

$$W_{k+1}(p^{(i)}, \mathbf{p}_K) + \sum_{j=1}^{k-1} \sum_{J \in K_j} \sum_{l \neq 0, i} \frac{U_j(p^{(l)}, y(p^{(i)}); \mathbf{p}_J) W_{k-j+1}(p^{(l)}, \mathbf{p}_{K-J})}{E_y(x(p), y(p^{(i)})) dx(p)} \quad (5.93)$$

One can now write down the final recursion formula for $R_k^i(p, \mathbf{p}_K)$ in these terms:

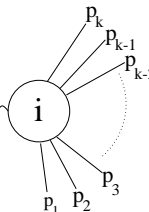
$$R_k^i(p, \mathbf{p}_K) = \frac{W_{k+1}(p^{(i)}, \mathbf{p}_K)}{(y(p^{(i)}) - y(p)) dx(p)} + \sum_{j=1}^{k-1} \sum_{J \in K_j} \sum_{l \neq 0, i} \frac{R_j^i(p^{(l)}, \mathbf{p}_J) W_{k-j+1}(p^{(l)}, \mathbf{p}_{K-J})}{(y(p^{(i)}) - y(p)) dx(p)}$$

(5.94)

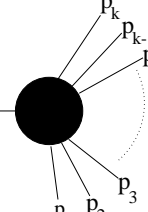
5.4 Diagrammatic solution

This section is the principal part of the article. We define a correspondence between the correlation functions and a system of Feynman-like graphs. To every k -point function, we associate a graph with k external legs and Eq. (5.80) and Eq. (5.94) become two relations describing these graphs as functions of graphs with $j \leq k - 1$ legs thanks to some rules we introduce in this part.

Let R_k^i and W_k be represented by:

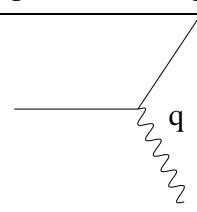
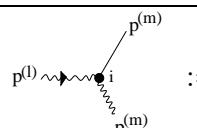
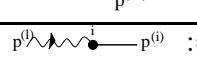
$R_k^i(p, p_K) :=$


and

$W_{k+1}(p, p_K) :=$


(5.95)

Let us introduce also the following propagators and vertices:

non-arrowed propagator:	$p \text{ --- } q := W_2(p, q)$
arrowed propagator:	$p \text{ --- } \blacktriangleright \text{ --- } q := dS_{q,o}(p)$
Residue cubic-vertex:	 $:= \sum_s \text{Res}_{q \rightarrow a_s}$
colored cubic-vertices:	 $:= \frac{(1 - \delta_{l,m})(1 - \delta_{m,i})(1 - \delta_{i,l})}{(y(p^{(i)}) - y(p^{(l)})) dx(p)}$
simple vertex:	 $:= \frac{1}{(y(p^{(i)}) - y(p^{(l)})) dx(p)} (1 - \delta_{i,l})$

One can now simply interpret the recursion relations in terms of graphs:

- Eq. (5.80) can be represented as:

$$= \sum_{i=1}^{d_2} \sum_{j=1}^{k-1} \sum_{J \in K_j} \text{Diagram} \quad (5.96)$$

- Eq. (5.94) can be represented as:

$$+ \sum_{j=1}^{k-1} \sum_{J \in K_j} \sum_{l=1}^{d_2} \text{Diagram} \quad (5.97)$$

From these relations, it is easy to see that $W_{k+1}(p, \mathbf{p}_K)$ is represented by all *binary trees with 1 root and k leaves* following the rules:

- *The vertices have valence 1, 2 or 3;*
- *the edges are arrowed or not; arrowed edges are oscillating or not;*
- *the arrows form a binary skeleton-tree;*
- *from each vertex comes one oscillating and one non-oscillating edge;*
- *two linked indices are different;*
- *the k leaves are non-arrowed propagators finishing at p_j 's.*

5.5 Examples.

Let us briefly review the diagrams induced for the first cases.

One has the following diagrammatic representations:

- $k = 2$:

$$W_2(p, q) = \mathbf{p} \text{ ————— } \mathbf{q} \quad (5.98)$$

and

$$R_1^i(p, p_1) = \mathbf{p} \rightsquigarrow \text{---} \overset{i}{\bullet}_{p^{(i)}} \text{---} \mathbf{p}_1 \quad (5.99)$$

These two diagrams represent the basis of the whole representation: they allow to draw the $k = 3$ correlation functions:

- $k = 3$:

$$\begin{aligned}
 W_3(p, p_1, p_2) &= \sum_{i=1}^{d_2} \mathbf{p} \text{---} \text{---} \text{---} \begin{array}{l} \nearrow \mathbf{p}_1 \\ \searrow \mathbf{p}_2 \end{array} \quad + \quad \mathbf{p} \text{---} \text{---} \text{---} \begin{array}{l} \nearrow \mathbf{p}_2 \\ \searrow \mathbf{p}_1 \end{array} \\
 &= \sum_{i=1}^{d_2} \sum_s \text{Res}_{p' \rightarrow a_s} \left[\frac{B(p'^{(i)}, p_1)B(p', p_2)}{(y(p'^{(i)}) - y(p'))dx(p')} + \frac{B(p'^{(i)}, p_2)B(p', p_1)}{(y(p'^{(i)}) - y(p'))dx(p')} \right] dS_{p', o}(p)
 \end{aligned} \quad (5.100)$$

and

$$\begin{aligned}
 R_2^i(p, p_1, p_2) &= \sum_{j=1}^{d_2} \mathbf{p} \rightsquigarrow \text{---} \overset{i}{\bullet} \text{---} \begin{array}{l} \nearrow \mathbf{p}_1 \\ \searrow \mathbf{p}_2 \end{array} \quad + \quad \mathbf{p} \rightsquigarrow \text{---} \overset{i}{\bullet} \text{---} \begin{array}{l} \nearrow \mathbf{p}_2 \\ \searrow \mathbf{p}_1 \end{array} \\
 &+ \sum_{j \neq i} \mathbf{p} \rightsquigarrow \text{---} \overset{i}{\bullet} \text{---} \begin{array}{l} \nearrow \mathbf{p}_1 \\ \searrow \mathbf{p}_2 \end{array} \quad + \quad \mathbf{p} \rightsquigarrow \text{---} \overset{i}{\bullet} \text{---} \begin{array}{l} \nearrow \mathbf{p}_2 \\ \searrow \mathbf{p}_1 \end{array}
 \end{aligned} \quad (5.101)$$

• $k = 4$:

$$\begin{aligned}
 W_4(p, p_1, p_2, p_3) = & \sum_{i=1}^{d_2} \sum_{j=1}^{d_2} p \rightarrow \text{Diagram 1} \\
 & + \sum_{i=1}^{d_2} \sum_{j=1}^{d_2} p \rightarrow \text{Diagram 2} \\
 & + \sum_{i=1}^{d_2} \sum_{j \neq i=1}^{d_2} p \rightarrow \text{Diagram 3} \\
 & + (\text{permutations of } \{p_1, p_2, p_3\})
 \end{aligned} \tag{5.102}$$

One has to consider all the permutations on the external legs. Thus, W_4 is the sum over 18 different diagrams.

6 Genus expansion

In this section, one determines all the terms of the genus (i.e. $\frac{1}{N^2}$) expansion of the correlation functions.

6.1 Loop equations.

In section (5.1), one has kept only leading terms when performing the changes of variable. Let us now write the $\frac{1}{N^2}$ corrective term for the same changes of variable. One obtains the loop equation :

$$\begin{aligned} & (y(r) - y(p))U_k(p, y(r); \mathbf{P}_K) \\ &= -P_k(x(p), y(r); \mathbf{P}_K)dx(p) - \sum_{j=0}^{k-1} \frac{1}{dx(p)} U_j(p, y(r); \mathbf{P}_J) W_{k-j+1}(p, \mathbf{P}_{K-J}) \\ & \quad - \frac{1}{N^2} \frac{U_{k+1}(p, y(r); \mathbf{P}, \mathbf{P}_k)}{dx(p)} + \sum_j d_{p_j} \left(\frac{U_{k-1}(p_j, y(r); \mathbf{P}_{K-\{j\}})}{x(p)-x(p_j)} \frac{dx(p)}{dx(p_j)} \right) \end{aligned} \tag{6.103}$$

For the following, one should remind the expression of the function $Y(x(p))$:

$$Y(x) := V_1'(x) - \frac{w_1(x)}{dx} \tag{6.104}$$

Then, for $h \geq 1$:

$$Y^{(h)}(x(p)) = -\frac{W_1^{(h)}(p)}{dx(p)} \tag{6.105}$$

Consider now the $\frac{1}{N^2}$ expansion of this equation order by order. The genus h term (corresponding to the $\frac{1}{N^{2h}}$ term) gives:

$$\begin{aligned} & (y(r) - y(p))U_k^{(h)}(p, y(r); \mathbf{P}_K) - \sum_{m=1}^h Y^{(m)}(x(p))U_k^{(h-m)}(p, y(r); \mathbf{P}_K) \\ &= -P_k^{(h)}(x(p), y(r); \mathbf{P}_K)dx(p) \\ & \quad - \sum_{m=0}^h \sum_{j=0}^{k-1} \frac{1}{dx(p)} U_j^{(m)}(p, y(r); \mathbf{P}_J) W_{k-j+1}^{(h-m)}(p, \mathbf{P}_{K-J}) \\ & \quad - \frac{U_{k+1}^{(h-1)}(p, y(r); \mathbf{P}, \mathbf{P}_k)}{dx(p)} + \sum_j d_{p_j} \left(\frac{U_{k-1}^{(h)}(p_j, y(r); \mathbf{P}_{K-\{j\}})}{x(p)-x(p_j)} \frac{dx(p)}{dx(p_j)} \right) \end{aligned} \tag{6.106}$$

When $y(r) = y(p)$:

$$\begin{aligned} & \sum_{m=1}^h Y^{(m)}(x(p))U_k^{(h-m)}(p, y(p); \mathbf{P}_K) \\ &= P_k^{(h)}(x(p), y(p); \mathbf{P}_K)dx(p) + \sum_{m=0}^h \sum_{j=0}^{k-1} \frac{1}{dx(p)} U_j^{(m)}(p, y(p); \mathbf{P}_J) W_{k-j+1}^{(h-m)}(p, \mathbf{P}_{K-J}) \\ & \quad + \frac{U_{k+1}^{(h-1)}(p, y(p); \mathbf{P}, \mathbf{P}_k)}{dx(p)} - \sum_j d_{p_j} \left(\frac{U_{k-1}^{(h)}(p_j, y(p); \mathbf{P}_{K-\{j\}})}{x(p)-x(p_j)} \frac{dx(p)}{dx(p_j)} \right) \end{aligned} \tag{6.107}$$

For the following, let h and k be two given positive integers. Let us consider $W_j^{(m)}$ known for any j if $m < h$ and any $j \leq k$ if $m = h$. One also assume that $R_j^{i,(m)}$ is known for any i and any j if $m < h$ and any $j < k$ if $m = h$. Starting from these assumptions, one computes $W_{k+1}^{(h)}$ and $R_k^{i,(h)}$, what will allow us to know any term recursively.

6.3.1 A recursive formula for $W_{k+1}^{(h)}$

Let us remind Eq. (6.107) in a more suitable way to emphasize that it allows us to compute $W_{k+1}^{(h)}(p, p_K)$ with our assumption:

$$\begin{aligned}
W_{k+1}^{(h)}(p, \mathbf{p}_K) U_0(p, y(p)) = & \\
& - \sum_{m=0}^{h-1} W_1^{(h-m)}(p) U_k^{(m)}(p, y(p); \mathbf{p}_K) \\
& - P_k^{(h)}(p, y(p); \mathbf{p}_K) dx(p)^2 \\
& - \sum_{m=0}^h \sum_{j=0, m+j \neq 0}^{k-1} U_j^{(m)}(p, y(p); \mathbf{p}_J) W_{k-j+1}^{(h-m)}(p, \mathbf{p}_{K-J}) \\
& - U_{k+1}^{(h-1)}(p, y(p); p, \mathbf{p}_k) + \sum_j \sum_j d_{p_j} \left(\frac{U_{k-1}^{(h)}(p_j, y(p); \mathbf{p}_{K-\{j\}}) dx(p)}{x(p)-x(p_j)} \frac{dx(p)}{dx(p_j)} \right) \frac{1}{dx(p)}
\end{aligned} \tag{6.112}$$

Remark that the RHS contains only known terms except $P_k^{(h)}(p, y(p); \mathbf{p}_K)$. Fortunately, it plays no role in Cauchy formula.

Indeed, we write the Cauchy formula, move the integration contour and vanish integrals around the cycles thanks to the same arguments as in the preceding sections. This gives:

$$\begin{aligned}
W_{k+1}^{(h)}(p, \mathbf{p}_K) &= - \operatorname{Res}_{p' \rightarrow p} W_{k+1}^{(h)}(p', \mathbf{p}_K) dS_{p', o}(p) \\
&= \sum_s \operatorname{Res}_{p' \rightarrow a_s} W_{k+1}^{(h)}(p', \mathbf{p}_K) dS_{p', o}(p)
\end{aligned} \tag{6.113}$$

We now introduce Eq. (6.112) inside this formula and keep only terms which have poles at the branch points:

$$\begin{aligned}
W_{k+1}^{(h)}(p, \mathbf{p}_K) = & \\
& - \sum_{m=0}^{h-1} \sum_s \operatorname{Res}_{p' \rightarrow a_s} W_1^{(h-m)}(p') R_k^{(m)}(p'; \mathbf{p}_K) dS_{p', o}(p) \\
& - \sum_{m=0}^h \sum_{j=0, m+j \neq 0}^{k-1} \sum_s \operatorname{Res}_{p' \rightarrow a_s} R_j^{(m)}(p'; \mathbf{p}_J) W_{k-j+1}^{(h-m)}(p', \mathbf{p}_{K-J}) dS_{p', o}(p) \\
& - \sum_s \operatorname{Res}_{p' \rightarrow a_s} R_{k+1}^{(h-1)}(p'; p', \mathbf{p}_k) dS_{p', o}(p)
\end{aligned} \tag{6.114}$$

For convenience, let us note:

$$W_1^{(0)}(p) \equiv W_0^{(1)}(p) \equiv 0 \tag{6.115}$$

Then, the recursive definition of $W_{k+1}^{(h)}(p, p_K)$ reads:

$$\boxed{
\begin{aligned}
W_{k+1}^{(h)}(p, \mathbf{p}_K) = & \\
& \sum_{i=1}^{d_2} \sum_{m=0}^h \sum_{j=0, m+j \neq 0}^k \sum_s \operatorname{Res}_{p' \rightarrow a_s} R_j^{i, (m)}(p'; \mathbf{p}_J) W_{k-j+1}^{(h-m)}(p', \mathbf{p}_{K-J}) dS_{p', o}(p) \\
& + \sum_{i=1}^{d_2} \sum_s \operatorname{Res}_{p' \rightarrow a_s} R_{k+1}^{i, (h-1)}(p'; p', \mathbf{p}_K) dS_{p', o}(p)
\end{aligned}
}$$

(6.116)

6.3.2 A recursive formula for $R_k^{i, (h)}$

The second step consists in the derivation of an equivalent formula for $R_k^{i, (h)}$. We proceed in the same way as for the genus 0 case: we use the rational properties of some of the correlation functions to write the recursive formula, with the aid of a relation similar to Eq. (5.91).

So let us recall to mind Eq. (6.106), and present it in a slightly different way:

$$\begin{aligned}
& (y(r) - y(q)) U_k^{(h)}(q, y(r); \mathbf{p}_K) \\
& + \sum_{m=1}^h \frac{W_1^{(m)}(q) U_k^{(h-m)}(q, y(r); \mathbf{p}_K)}{dx(q)} + \frac{U_{k+1}^{(h-1)}(q, y(r); q, p_k)}{dx(q)} \\
= & -P_k^{(h)}(x(q), y(r); \mathbf{p}_K) dx(q) \\
& - \sum_{m=0}^h \sum_{j=0}^{k-1} \frac{1}{dx(q)} U_j^{(m)}(q, y(r); \mathbf{p}_J) W_{k-j+1}^{(h-m)}(q, \mathbf{p}_{K-J}) \\
& + \sum_j d p_j \left(\frac{U_{k-1}^{(h)}(p_j, y(r); \mathbf{p}_{K-\{j\}})}{x(q) - x(p_j)} \frac{dx(q)}{dx(p_j)} \right)
\end{aligned} \tag{6.117}$$

This equation taken for $r = q = p^{(i)}$ can be written:

$$\begin{aligned}
& \sum_{m=1}^h \frac{W_1^{(m)}(p^{(i)}) U_k^{(h-m)}(p^{(i)}, y(p^{(i)}); \mathbf{p}_K)}{dx(p^{(i)})} + \frac{U_{k+1}^{(h-1)}(p^{(i)}, y(p^{(i)}); q, p_k)}{dx(p^{(i)})} \\
= & -P_k^{(h)}(x(p^{(i)}), y(p^{(i)}); \mathbf{p}_K) dx(p^{(i)}) \\
& - \sum_{m=0}^h \sum_{j=0}^{k-1} \frac{1}{dx(p^{(i)})} U_j^{(m)}(p^{(i)}, y(p^{(i)}); \mathbf{p}_J) W_{k-j+1}^{(h-m)}(p^{(i)}, \mathbf{p}_{K-J}) \\
& + \sum_j d p_j \left(\frac{U_{k-1}^{(h)}(p_j, y(p^{(i)}); \mathbf{p}_{K-\{j\}})}{x(p^{(i)}) - x(p_j)} \frac{dx(p^{(i)})}{dx(p_j)} \right) \\
= & -P_k^{(h)}(x(p), y(p^{(i)}); \mathbf{p}_K) dx(p) \\
& - \sum_{m=0}^h \sum_{j=0}^{k-1} \frac{1}{dx(p)} U_j^{(m)}(p^{(i)}, y(p^{(i)}); \mathbf{p}_J) W_{k-j+1}^{(h-m)}(p^{(i)}, \mathbf{p}_{K-J}) \\
& + \sum_j d p_j \left(\frac{U_{k-1}^{(h)}(p_j, y(p^{(i)}); \mathbf{p}_{K-\{j\}})}{x(p) - x(p_j)} \frac{dx(p)}{dx(p_j)} \right)
\end{aligned} \tag{6.118}$$

where we have emphasized the equality $x(p^{(i)}) = x(p)$.

Let us now write 6.117 for $r = p^{(i)}$ and $q = p$:

$$\begin{aligned}
& (y(p^{(i)}) - y(p))U_k^{(h)}(p, y(p^{(i)}); \mathbf{P}_K) \\
& + \sum_{m=1}^h \frac{W_1^{(m)}(p)U_k^{(h-m)}(p, y(p^{(i)}); \mathbf{P}_K)}{dx(p)} + \frac{U_{k+1}^{(h-1)}(p, y(p^{(i)}); p, p_k)}{dx(p)} \\
= & -P_k^{(h)}(x(p), y(p^{(i)}); \mathbf{P}_K)dx(p) \\
& - \sum_{m=0}^h \sum_{j=0}^{k-1} \frac{1}{dx(p)} U_j^{(m)}(p, y(p^{(i)}); \mathbf{P}_J) W_{k-j+1}^{(h-m)}(p, \mathbf{P}_{K-J}) \\
& + \sum_j d_{p_j} \left(\frac{U_{k-1}^{(h)}(p_j, y(p^{(i)}); \mathbf{P}_{K-\{j\}})}{x(p) - x(p_j)} \frac{dx(p)}{dx(p_j)} \right)
\end{aligned} \tag{6.119}$$

That is to say, using 6.118:

$$\begin{aligned}
(y(p^{(i)}) - y(p))U_k^{(h)}(p, y(p^{(i)}); \mathbf{P}_K) = & \sum_{m=0}^h \sum_{j=0}^k \frac{W_{j+1}^{(m)}(p^{(i)}, \mathbf{P}_J) U_{k-j}^{(h-m)}(p^{(i)}, y(p^{(i)}); \mathbf{P}_{K-J})}{dx(p)} \\
& + \frac{U_{k+1}^{(h-1)}(p^{(i)}, y(p^{(i)}); p^{(i)}, p_k)}{dx} \\
& - \sum_{m=0}^h \sum_{j=0}^k \frac{W_{j+1}^{(m)}(p, \mathbf{P}_J) U_{k-j}^{(h-m)}(p, y(p^{(i)}); \mathbf{P}_{K-J})}{dx(p)} \\
& - \frac{U_{k+1}^{(h-1)}(p, y(p^{(i)}); p, p_k)}{dx}
\end{aligned} \tag{6.120}$$

We now establish a relation similar to Eq. (5.91) in order to present our recursive formula in such a way that it can be graphically interpreted.

In order to achieve this aim, one has to determine an explicit intermediate formula for $U_k^{(h)}(p, y; p_K)$. Let us assume that (for the proof, see appendix A):

$$\begin{aligned}
U_k^{(h)}(p, y(p^{(i)}); \mathbf{P}_K) = & \frac{E_y(x, y(p^{(i)}))}{y(p^{(i)}) - y(p)} \sum_{r=1}^{\min(d_2, k+h)} \sum_{K_1 \cup \dots \cup K_r = K} \sum_{h_\alpha=0}^h \sum_{k_\alpha=|K_\alpha|}^{k+h} \sum_{j_{\alpha, \beta} \neq j_{\alpha', \beta'} \in [1, d_2] - \{i\}} \frac{1}{\Omega} \\
& \frac{W_{k_1+1}^{(h_1)}(p^{(i)}, \mathbf{P}_{K_1}, p^{(j_{1,1})}, \dots, p^{(j_{1, k_1 - |K_1|})}) \left(\prod_{\alpha=2}^r W_{k_\alpha+1}^{(h_\alpha)}(p^{(j_{\alpha, 0})}, \mathbf{P}_{K_\alpha}, p^{(j_{\alpha, 1})}, \dots, p^{(j_{\alpha, k_\alpha - |K_\alpha|})}) \right)}{dx(p)^{r-k-1+\sum k_\alpha} \prod_{\alpha, \beta} y(p^{(i)}) - y(p^{(j_{\alpha, \beta})})}
\end{aligned} \tag{6.121}$$

where Ω is some symmetry factor which does only depend on the $k_\alpha - |K_\alpha|$'s and one has the following constraints:

- $\sum_o (h_\alpha + k_\alpha) = h + k$;
- $0 \leq |K_\alpha| \leq k_\alpha$

One should note that the only external parameter entering these constraints is $k + h$.

It is now possible to derive an equality equivalent to Eq. (5.91). One shows – in appendix B – that:

$$\begin{aligned}
& \sum_{m=0}^h \sum_{j=0; m_j \neq kh}^k W_{j+1}^{(m)}(p, \mathbf{P}_J) U_{k-j}^{(h-m)}(p, y(p); \mathbf{P}_{K-J}) + U_{k+1}^{(h-1)}(p, y(p); p, \mathbf{P}_k) \\
= & \sum_{i=1}^{d_2} \sum_{m=0}^h \sum_{j=0; m_j \neq kh}^k W_{j+1}^{(m)}(p^{(i)}, \mathbf{P}_J) U_{k-j}^{(h-m)}(p^{(i)}, y(p); \mathbf{P}_{K-J})
\end{aligned}$$

$$(6.122) \quad + \sum_{i=1}^{d_2} U_{k+1}^{(h-1)}(p^{(i)}, y(p); p^{(i)}, \mathbf{p}_k)$$

This equality allows us to write:

$$(6.123) \quad \begin{aligned} & (y(p^{(i)}) - y(p))U_k^{(h)}(p, y(p^{(i)}); \mathbf{p}_K) = \\ & \sum_{m=0}^h \sum_{j=0; m_j \neq kh}^k \sum_{l \neq 0, i} \frac{W_{j+1}^{(m)}(p^{(l)}, \mathbf{p}_J)U_{k-j}^{(h-m)}(p^{(l)}, y(p^{(i)}); \mathbf{p}_{K-j})}{dx(p)} \\ & + \sum_{l \neq 0, i} \frac{U_{k+1}^{(h-1)}(p^{(l)}, y(p^{(i)}); p^{(l)}, \mathbf{p}_k)}{dx(p)} + W_{k+1}^{(h)}(p^{(i)}, \mathbf{p}_K)E_y(x, y(p^{(i)})) \end{aligned}$$

That is to say:

$$(6.124) \quad \begin{aligned} R_k^{i, (h)}(p, \mathbf{p}_K) &= \sum_{m=0}^h \sum_{j=0; m_j \neq kh}^k \sum_{l \neq 0, i} \frac{W_{j+1}^{(m)}(p^{(l)}, \mathbf{p}_J)R_{k-j}^{i, (h-m)}(p^{(l)}; \mathbf{p}_{K-j})}{(y(p^{(i)}) - y(p))dx(p)} \\ &+ \sum_{l \neq 0, i} \frac{R_{k+1}^{i, (h-1)}(p^{(l)}; p^{(l)}, \mathbf{p}_k)}{(y(p^{(i)}) - y(p))dx(p)} + \frac{W_{k+1}^{(h)}(p^{(i)}, \mathbf{p}_K)}{(y(p^{(i)}) - y(p))dx(p)} \end{aligned}$$

6.3.3 Diagrammatic solution.

In this section, we express the functions $W_k^{(h)}$ as diagrams, using the Feynman rules defined for the genus 0 case. This way, we generalize the trees obtained in the preceding case to any loop-order: we show that the $\frac{1}{N^{2h}}$ correction can be seen as a h-loop diagrammatic correction.

Let us represent $R_k^{i, (h)}$, and $W_k^{(h)}$ respectively, as white and black disks with h holes:

$$(6.125) \quad W_k^{(h)}(p, p_K) := \text{Diagram of a black disk with } h \text{ holes and } k \text{ external legs } p, p_1, \dots, p_k$$

$$(6.126) \quad R_k^{i, (h)}(p, p_K) := \text{Diagram of a white disk with } h \text{ holes and } k \text{ external legs } p, p_1, \dots, p_k$$

One now can use the preceding Feynman rules in order to represent diagrammatically the two recursive equations.

- The recursive relation giving the $W_k^{(h)}$'s is diagrammatically written as follows:

$$(6.127)$$

- Given lower order $R_l^{i,(m)}$'s and $W_l^{(m)}$'s, one can obtain $R_k^{i,(h)}$ diagrammatically by writing Eq. (6.124):

$$(6.128)$$

This means that $W_{k+1}^{(h)}$ is obtained by *the summation over all Feynman graphs with k external legs and h loops* and following the same prescriptions as the trees corresponding to W_k , i.e.:

- *The vertices have valence 1,2 or 3, there are $2h + k - 1$ tri-valent vertices;*

- The edges, are arrowed or not, the arrowed edges are colored or not;
- The subgraph made of arrowed edges forms a skeleton tree;
- from each tri-valent vertex comes one colored and one non-colored propagator;
- two linked colors are different;
- the k leaves are non-arrowed propagators finishing at p_j 's.

A practical way to draw these graphs is to draw every skeleton tree of arrows, put k non arrowed propagators as leaves and close it with h non arrowed propagators in order to obtain h loops.

6.3.4 Examples.

Let us now carry out two simple examples of graphs obtained this way. We write $W_2^{(1)}$ and $W_1^{(2)}$.

$$\begin{aligned}
W_2^{(1)} = & \sum_{i=1}^{d_2} \sum_{j \in [1, d_2] - \{i\}} \left[\begin{array}{c} \text{Diagram 1} + \text{Diagram 2} \\ \text{Diagram 3} \end{array} \right] \\
& + \sum_{i=1}^{d_2} \sum_{j=1}^{d_2} \left[\begin{array}{c} \text{Diagram 4} + \text{Diagram 5} \\ \text{Diagram 6} + \text{Diagram 7} \end{array} \right]
\end{aligned} \tag{6.129}$$

The diagrams in the equation represent various Feynman-like graphs with vertices i and j , and edges that are either straight lines with arrows or wavy lines. The graphs are arranged in two main groups, each enclosed in large square brackets. The first group contains three diagrams, and the second group contains four diagrams. The diagrams show different ways to connect vertices i and j with propagators, including loops and external lines.

Analytically, this reads:

$$\begin{aligned}
W_2^{(1)}(p, p_1) = & \sum_{i=1}^{d_2} \sum_{j \in [1, d_2] - \{i\}} \sum_s \text{Res}_{p' \rightarrow a_s} \frac{dS_{p', o}(p)}{(y(p'^{(i)}) - y(p'))(y(p'^{(i)}) - y(p'^{(j)}))dx^2(p')} \\
& [B(p', p_1)B(p'^{(i)}, p'^{(j)}) + B(p'^{(i)}, p_1)B(p', p'^{(j)}) + B(p', p'^{(i)})B(p_1, p'^{(j)})] \\
& + \sum_{i=1}^{d_2} \sum_{j=1}^{d_2} \sum_{s, t} \text{Res}_{p' \rightarrow a_s} \text{Res}_{p'' \rightarrow a_t} \frac{dS_{p', o}(p)}{(y(p'^{(i)}) - y(p'))(y(p''^{(j)}) - y(p''))dx(p')dx(p'')} \\
& [B(p', p_1)B(p'', p''^{(j)})dS_{p'', o}(p'^{(i)}) + B(p'^{(i)}, p_1)B(p'', p''^{(j)})dS_{p'', o}(p') \\
& + B(p'', p')B(p_1, p''^{(j)})dS_{p'', o}(p'^{(i)}) + B(p_1, p'')B(p', p''^{(j)})dS_{p'', o}(p'^{(i)})]
\end{aligned} \tag{6.130}$$

$$\begin{aligned}
W_1^{(2)} = & \sum_{i=1}^{d_2} \sum_{j=1}^{d_2} \sum_{k=1}^{d_2} \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \end{array} \right] \\
& + \sum_{i=1}^{d_2} \sum_{j \in [1, d_2] - \{i\}} \sum_{k=1}^{d_2} \left[\begin{array}{c} \text{Diagram 6} \\ \text{Diagram 7} \end{array} \right] \\
& + \sum_{i=1}^{d_2} \sum_{j=1}^{d_2} \sum_{k \in [1, d_2] - \{j\}} \left[\begin{array}{c} \text{Diagram 8} \\ \text{Diagram 9} \end{array} \right] \\
& + \sum_{i=1}^{d_2} \sum_{j \in [1, d_2] - \{i\}} \sum_{k \in [1, d_2] - \{j\}} \left[\begin{array}{c} \text{Diagram 10} \\ \text{Diagram 11} \end{array} \right] \\
& + \left[\begin{array}{c} \text{Diagram 12} \end{array} \right]
\end{aligned}
\tag{6.131}$$

Note: In the 1-matrix limit, the only diagrams remaining are the five first ones, i.e. the one where no colors are linked. Indeed, this limit is obtained diagrammatically by taking the length of the oscillating propagator to zero. When two colors are linked, this gives rise to a four legs vertex which can not exist.

7 An effective non cubic theory

The Feynman graphs described up to now correspond to a cubic theory and then, have the advantage to have only trivalent vertices. Nevertheless, some problems do not need this property to be solved and one can build an effective theory with less propagators but vertices with valence up to $d_2 - 1$. Roughly, it consists in resumming the linked waved vertices into one multivalent vertex:

$$\text{---} \sim \text{---} \quad (7.132)$$

Let us present this effective theory in this section.

7.1 Leading order: Genus 0

Actually, we have already written the equations necessary to define this effective theory. Let us consider Eq. (5.77) and Eq. (5.85):

$$W_{k+1}(p, \mathbf{p}_K) = - \sum_s \text{Res}_{p' \rightarrow a_s} \sum_{j=1}^{k-1} \sum_{J \in K_j} \frac{1}{dx'} \frac{U_j(p', y(p')); \mathbf{p}_J}{E_y(x(p'), y(p'))} W_{k-j+1}(p', \mathbf{p}_{K-J}) dS_{p', o}(p) \quad (7.133)$$

$$U_k(p, y; \mathbf{p}_K) = \frac{E(x(p), y) dx(p)}{y - y(p)} \sum_{r=1}^{d_2} \sum_{K_1 \cup \dots \cup K_r = K} \sum_{j_1 \neq j_2 \neq \dots \neq j_r = 1}^{d_2} \prod_{t=1}^r \frac{W_{|K_t|+1}(p^{(j_t)}, \mathbf{p}_{K_t})}{(y - y(p^{(j_t)}))} dx(p) \quad (7.134)$$

This second equation taken for $y = y(p)$ reads:

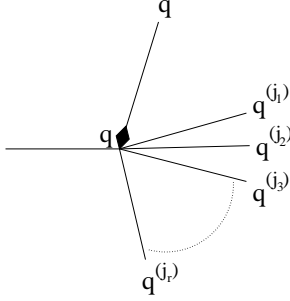
$$\frac{U_k(p, y(p); \mathbf{p}_K)}{E_y(x(p), y(p)) dx(p)} = \sum_{r=1}^{d_2} \sum_{K_1 \cup \dots \cup K_r = K} \sum_{j_1 \neq j_2 \neq \dots \neq j_r = 1}^{d_2} \prod_{t=1}^r \frac{W_{|K_t|+1}(p^{(j_t)}, \mathbf{p}_{K_t})}{(y(p) - y(p^{(j_t)}))} dx(p) \quad (7.135)$$

Introduce it in Eq. (7.133) and get a closed recursive formula for the W_k 's:

$$W_{k+1}(p, \mathbf{p}_K) = - \sum_s \text{Res}_{p' \rightarrow a_s} \sum_{r=1}^{d_2} \sum_{K_0 \cup K_1 \cup \dots \cup K_r = K} \sum_{j_1 \neq j_2 \neq \dots \neq j_r = 1}^{d_2} W_{|K_0|+1}(p', \mathbf{p}_{K_0}) \prod_{t=1}^r \frac{W_{|K_t|+1}(p^{(j_t)}, \mathbf{p}_{K_t})}{(y(p') - y(p^{(j_t)}))} dx(p')$$

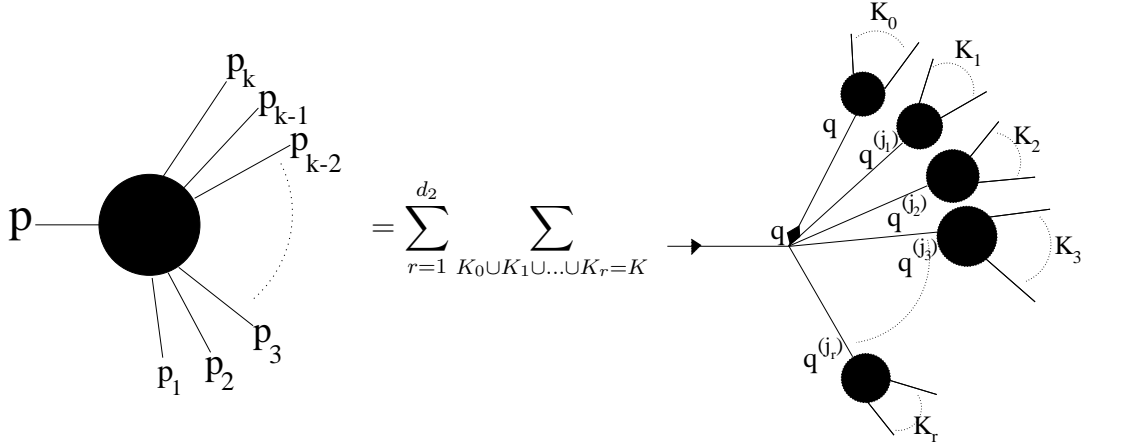
$$(7.136)$$

Let us introduce the following Feynman rules:

non-arrowed propagator:	$p \text{ --- } q := W_2(p, q)$
arrowed propagator:	$p \text{ --> } q := dS_{q,o}(p)$
$r+2$ - vertex ($1 \leq r \leq d_2$) with one marked edge:	 $:= \frac{-\sum_s \sum_{j_1 \neq \dots \neq j_r \neq 0} \text{Res}_{q \rightarrow a_s}}{\prod_{t=1}^r \frac{1}{(y(q) - y(q^{(j_t)})) dx(q)}}$

Remark that one leg of the multiple vertex is marked: on this leg, there is no summation over the different sheets.

Using these rules, one can diagrammatically write the recursive relation as follows:



$$(7.137)$$

From this relation, one can see that $W_{k+1}(p, \mathbf{p}_K)$ is obtained as the *summation over all trees with $k + 1$ external legs* and following the rules:

- The vertices have valence r such as $1 \leq r \leq \min(k + 1, d_2 + 1)$;
- The edges are arrowed;
- One of the legs of each vertex is marked
- The subgraph made of arrowed edges forms a skeleton tree;
- The k leaves are non arrowed propagators ending at p_j 's.

The drawbacks of this effective theory induced by the existence of multivalent vertices is balanced by the simplicity of these vertices and the lack of different propagators.

7.2 Any genus h

Let us now study the extension of this theory to any genus.

Once again, the fundamental equations have already been written. Let us recall to mind the following equations Eq. (6.116) and Eq. (6.121):

$$\begin{aligned}
W_{k+1}^{(h)}(p, \mathbf{p}_K) = & \\
- \sum_{m=0}^h \sum_{j=0, m+j \neq 0}^k \sum_s \text{Res}_{p' \rightarrow a_s} \frac{U_j^{(m)}(p', y(p'); \mathbf{p}_J)}{E_y(x(p'), y(p'))} W_{k-j+1}^{(h-m)}(p', \mathbf{p}_{K-J}) dS_{p', \alpha}(p) & \quad (7.138) \\
- \sum_s \text{Res}_{p' \rightarrow a_s} \frac{U_{k+1}^{(h-1)}(p', y(p'); p', \mathbf{p}_K)}{E_y(x(p'), y(p'))} dS_{p', \alpha}(p) &
\end{aligned}$$

and, for $i \neq 0$:

$$\begin{aligned}
U_k^{(h)}(p, y(p^{(i)}); \mathbf{p}_K) = & \\
\frac{E_y(x, y(p^{(i)}))}{y(p^{(i)}) - y(p)} \sum_{r=1}^{\min(d_2, k+h)} \sum_{K_1 \cup \dots \cup K_r = K} \sum_{h_\alpha=0}^h \sum_{k_\alpha=|K_\alpha|}^{k+h} \sum_{j_{\alpha, \beta} \neq j_{\alpha'}, \beta' \in [1, d_2] - \{i\}} \frac{1}{\Omega} & \\
\frac{W_{k_1+1}^{(h_1)}(p^{(i)}, \mathbf{p}_{K_1}, p^{(j_{1,1})}, \dots, p^{(j_{1, k_1 - |K_1|})}) \left(\prod_{\alpha=2}^r W_{k_\alpha+1}^{(h_\alpha)}(p^{(j_{\alpha,0})}, \mathbf{p}_{K_\alpha}, p^{(j_{\alpha,1})}, \dots, p^{(j_{\alpha, k_\alpha - |K_\alpha|})}) \right)}{dx(p)^{r-k-1+\sum k_\alpha} \prod_{\alpha, \beta} y(p^{(i)}) - y(p^{(j_{\alpha, \beta})})} & \quad (7.139)
\end{aligned}$$

In order to introduce this second formula inside the first one, one has to use the interpolation formula to consider the case where $i = 0$:

$$\begin{aligned}
U_l^{(m)}(p, y(p); \mathbf{p}_L) = & \\
\frac{E_y(x(p), y(p))}{E_y(x(p), y(p))} = & \\
- \sum_{r=1}^{\min(d_2, l+m)} \sum_{L_1 \cup \dots \cup L_r = L} \sum_{m_\alpha=0}^m \sum_{l_\alpha=|L_\alpha|}^{l+m} \sum_{j_1 \neq \dots \neq j_r \in [1, d_2]} \frac{1}{\Omega} & \quad (7.140) \\
\frac{W_{l_1+1}^{(m_1)}(p^{(j_{1,0})}, \mathbf{p}_{L_1}, p^{(j_{1,1})}, \dots, p^{(j_{1, l_1 - |L_1|})}) \prod_{\alpha=2}^r W_{l_\alpha+1}^{(m_\alpha)}(p^{(j_{\alpha,0})}, \mathbf{p}_{L_\alpha}, p^{(j_{\alpha,1})}, \dots, p^{(j_{\alpha, l_\alpha - |L_\alpha|})})}{dx(p)^{r-l-1+\sum l_\alpha} (y(p^{(j_{1,0})}) - y(p)) \prod_{\alpha, \beta} (y(p^{(j_{1,0})}) - y(p^{(j_{\alpha, \beta})}))} &
\end{aligned}$$

Recursively, it is easy to check that it can be written:

$$\begin{aligned}
\frac{U_l^{(m)}(p, y(p); \mathbf{p}_L)}{E_y(x(p), y(p)) dx(p)} = & \\
\sum_{r=1}^{\min(d_2, l+m)} \sum_{L_1 \cup \dots \cup L_r = L} \sum_{m_\alpha=0}^m \sum_{l_\alpha=|L_\alpha|}^{l+m} \sum_{j_{\alpha, \beta} \neq j_{\alpha'}, \beta' \in [1, d_2]} \frac{1}{\Omega'} & \quad (7.141) \\
\prod_{\alpha=1}^r \frac{W_{l_\alpha+1}^{(m_\alpha)}(p^{(j_{\alpha,0})}, \mathbf{p}_{L_\alpha}, p^{(j_{\alpha,1})}, \dots, p^{(j_{\alpha, l_\alpha - |L_\alpha|})})}{dx(p)^{l_\alpha - |L_\alpha| + 1} \prod_{\beta=0}^{l_\alpha - |L_\alpha|} (y(p) - y(p^{(j_{\alpha, \beta})}))} &
\end{aligned}$$

where Ω' is some other symmetry factor depending only on the same parameters as Ω .

One is now able to write an explicit recursion formula for the $W_k^{(h)}$'s that can be graphically represented the Feynman rules introduced in this section. The introduction of Eq. (7.141) in Eq. (7.138) gives:

$$\begin{aligned}
W_{k+1}^{(h)}(p, \mathbf{p}_K) = & \\
- \sum_s \text{Res}_{p' \rightarrow a_s} \sum_{r=1}^{d_2} \sum_{K_0 \cup K_1 \cup \dots \cup K_r = K} \sum_{h_\alpha=0}^h \sum_{k_\alpha=|K_\alpha|}^{k+h} \sum_{j_{\alpha,\beta} \neq j_{\alpha',\beta'} \in [1, d_2]} \frac{1}{\Omega'} & \\
dS_{p',o}(p) W_{|K_0|+1}^{(h_0)}(p', \mathbf{p}_{K_0}) \prod_{\alpha=1}^r \frac{W_{k_\alpha+1}^{(h_\alpha)}(p'^{(j_{\alpha,0})}, \mathbf{p}_{K_\alpha}, p'^{(j_{\alpha,1})}, \dots, p'^{(j_{\alpha,k_\alpha-|K_\alpha|})})}{dx(p')^{k_\alpha-|K_\alpha|+1} \prod_{\beta=0}^{k_\alpha-|K_\alpha|} (y(p')-y(p'^{(j_{\alpha,\beta})}))} & \\
- \sum_s \text{Res}_{p' \rightarrow a_s} \frac{U_{k+1}^{(h-1)}(p', y(p'); p', \mathbf{p}_K)}{E_y(x(p'), y(p'))} dS_{p',\alpha}(p) & \tag{7.142}
\end{aligned}$$

That is to say:

The diagrammatic equation (7.143) illustrates the decomposition of a vertex with genus h and k external legs. On the left, a single vertex is shown with an external leg p and internal legs $p_1, p_2, \dots, p_{k-1}, p_k$. This is equated to a sum of two terms. The first term is a sum over $r=1$ to d_2 and h_α of a sum over partitions $K_0 \cup K_1 \cup \dots \cup K_r = K$. This term shows a central vertex with r legs connecting to r smaller vertices, each with genus h_α and $|K_\alpha|$ external legs. The second term is a sum over $r=1$ to d_2 and h_α of a sum over partitions $K_1 \cup \dots \cup K_r = K$. This term shows a central vertex with r legs connecting to r smaller vertices, each with genus h_α and $|K_\alpha|$ external legs, but with one leg of the central vertex left unattached, representing a self-energy-like contribution.

Remark that we have splitted the diagrams in the RHS in order to reproduce the recursion relation. Nevertheless, the first term in the RHS is nothing else but a particular case of the second term where the marked leg of the vertex is left alone inside one of the W 's.

Hence, the h 'th order expansion term of the correlation function $W_k^{(h)}$ is obtained as the *summation over all Feynman diagrams with k external legs and h loops* following the same rules as exposed in the genus 0 case, i.e.:

- The vertices have valence r such as $1 \leq r \leq d_2 + 1$;
- The edges are arrowed or not;

- One of the legs of each vertex is marked;
- The subgraph made of arrowed edges forms a skeleton tree;
- the k leaves are non arrowed propagators ending at p_j 's.

7.3 Examples

Let us review some simple examples of this description.

$$W_3(p, p_1, p_2) = \text{Diagram 1} + \text{Diagram 2} \quad (7.144)$$

Analytically, this reads:

$$W_3(p, p_1, p_2) = \sum_{i=1}^{d_2} \sum_s \text{Res}_{q \rightarrow a_s} [B(q^{(i)}, p_1)B(q, p_2) + B(q^{(i)}, p_2)B(q, p_1)] \frac{dS_{q,o}(p)}{(y(q^{(i)}) - y(q))dx(q)} \quad (7.145)$$

$$W_1^{(1)}(p) = \text{Diagram 1} = \sum_s \sum_{i=1}^{d_2} \text{Res}_{q \rightarrow a_s} dS_{q,o}(p) \frac{B(q, q^{(i)})}{(y(q^{(i)}) - y(q))dx(q)} \quad (7.146)$$

$$W_2^{(1)}(p, p_1) = \text{Diagram 1} + \text{Diagram 2}$$

$$\begin{aligned}
& + \frac{1}{2} \left[\text{Diagram 1} + \text{Diagram 2} \right] \\
& + \left[\text{Diagram 3} + \text{Diagram 4} \right] \quad (7.147)
\end{aligned}$$

8 The gaussian case: the 1-matrix model limit.

In this section, we are interested in the special case where $d_2 = 1$, i.e. one has a gaussian potential in M_2 . This situation is very important because it links our results to the 1-matrix model studied in [12]. Indeed, when one of the potentials is gaussian – V_2 for example –, the integration over one of the variables – M_2 in this case – is gaussian and can be straightforwardly performed without giving any contribution to the formal expansion. Then, the 2-matrix model with one gaussian potential is equivalent to the 1-matrix model with a potential $V = V_1 - \frac{x^2}{2g_2}$. We check in this part that our results coincide with the ones obtained directly from the 1-matrix model in [12]. Actually, it is a good way to better understand the structure obtained in this case.

Let us note:

$$V_2(y) = \frac{g_2}{2}y^2 \quad (8.148)$$

In this case, the Riemann surface is an hyperelliptic surface with only two x -sheets. The equation $x(p) = x$ has only two solutions. Let us call them p and \bar{p} , i.e. $p^{(0)} = p$ and $p^{(1)} = \bar{p}$. They obey the following relations:

$$x(p) = x(\bar{p}) \quad \text{and} \quad y(p) = -y(\bar{p}) \quad (8.149)$$

The algebraic equation generating the Riemann surface reads:

$$E(x(p), y(r)) = -g_2(y(r) - y(p))(y(r) - y(\bar{p})) = -g_2(y(r)^2 - y(p)^2) \quad (8.150)$$

One can also remark that:

$$U_k(p, y; \mathbf{p}_K) = g_2 W_{k+1}(p, \mathbf{p}_K) \quad (8.151)$$

That is to say:

$$R_k^0(p, \mathbf{p}_K) = \frac{U_k(p, y(p); \mathbf{p}_K)}{E_y(x(p), y(p))} = -\frac{W_{k+1}(p, \mathbf{p}_K)}{2y(p)} \quad (8.152)$$

So that:

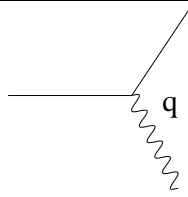
$$R_k^0(\bar{p}, \mathbf{p}_K) = R_k^1(p, \mathbf{p}_K) = \frac{W_{k+1}(p, \mathbf{p}_K)}{2y(p)} \quad (8.153)$$

Diagrammatic rules.

One can now study how the Feynman rules of the two theories introduced earlier behave in this limit.

- **The cubic field theory**

Because V_2 is gaussian, the Feynman rules become:

non-arrowed propagator:	$p \text{ --- } q := W_2(p, q)$
arrowed propagator:	$p \text{ --- } \blacktriangleright \text{ --- } q := dS_{q,o}(p)$
Residue cubic-vertex:	 $:= \sum_s Res_{q \rightarrow a_s}$
simple vertex:	$p \text{ --- } \bullet \text{ --- } \bar{p} := -\frac{1}{2y(p)dx(p)}$

The last component of the Feynman diagrams, the colored cubic-vertex, implies three different x -sheets. Because there exists only two such sheets in the gaussian case, this vertex vanish:

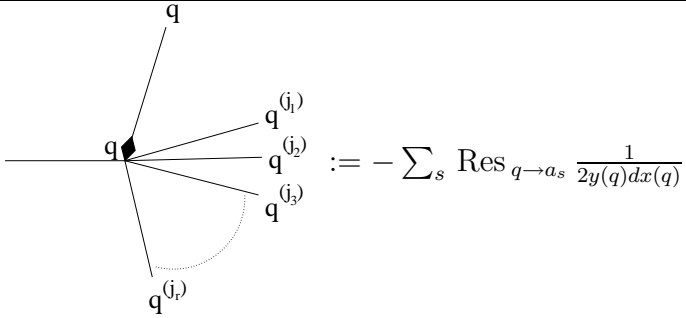
$$\begin{array}{c}
 p^{(m)} \\
 \diagup \\
 p^{(l)} \text{ --- } \blacktriangleright \text{ --- } \bullet \text{ --- } i \\
 \diagdown \\
 p^{(m)}
 \end{array}
 := 0 \quad (8.154)$$

Considered that the bivalent vertices and trivalent vertices only appear together, one can merge them into one whose value is equal to $-\sum_s Res_{q \rightarrow a_s} \frac{1}{2y(q)dx(q)}$, and one recovers [12]:

$$\begin{array}{c}
 \diagup \\
 \text{---} \\
 \diagdown \\
 \bullet \\
 \diagup \\
 q \\
 \diagdown
 \end{array}
 \rightarrow
 \begin{array}{c}
 \diagup \\
 \text{---} \\
 \diagdown \\
 q
 \end{array}
 \quad (8.155)$$

- **The effective theory**

The effect of the gaussian limit on the effective theory is to make it cubic. One obtain the following rules:

non-arrowed propagator:	$p \text{ --- } q := W_2(p, q)$
arrowed propagator:	$p \text{ --- } \blacktriangleright \text{ --- } q := dS_{q,o}(p)$
cubic vertex (only for $r=1$):	

Hence, the two theories turn into only one cubic theory in this limit which is the one derived in [12]. Indeed, the corresponding recursive relation appears to be:

$$\begin{aligned}
W_{k+1}^{(h)}(p, p_K) = & - \sum_l \text{Res}_{q \rightarrow a_l} \frac{W_{k+2}^{(h-1)}(q, q, p_K) dS_{q,o}(p)}{2y(q)} \\
& - \sum_{m=0}^h \sum_{j=0, j+m \neq 0}^k \sum_l \text{Res}_{q \rightarrow a_l} \frac{W_{j+1}^{(m)}(q, p_J) W_{k-j+1}^{(h-m)}(q, p_{K-J}) dS_{q,o}(p)}{2y(q)} \quad (8.156)
\end{aligned}$$

Remark:

Diagrammatically, this limit can be easily interpreted. Starting from the general cubic theory, in order, to obtain the 1-matrix model graphs from the 2-matrix model ones, one only has to take the length of the oscillating propagators to 0. In this case, the graphs containing at least one colored vertex vanish.

Everything works as if the oscillating propagators of the 2-matrix model were unstable particles which decay into stable ones represented by non-oscillating propagators. Then the 1-matrix limit is obtained by taking the life time of these particles to 0.

One shall also note that there is no symmetry factor in the 2-matrix model graphs of the cubic theory whereas there are not well understood ones in the 1-matrix case. The derivation of the 1-matrix model as a limit exhibits how these factors arise. They come from the same contribution given by different diagrams in this limit. This observation exhibits how the 2-matrix model seem more fundamental.

9 Conclusion

In this article, we have generalized the diagrammatic technique of [12] to compute all non-mixed correlation functions of the 2-matrix model, to all orders in the topological expansion.

The result can be represented diagrammatically, with some cubic Feynman rules, which are just convenient notations for writing residues on an algebraic curve.

This shows that the method discovered in [12] is very universal, i.e. it works for all algebraic curves, not only hyperelliptical curves.

The future prospects of that work are to find the diagrammatic rules for computing the free energy to all order in the topological expansion, and also all mixed correlation functions (using the result of [13]). Another possible extension is to work out the multimatrix model, i.e. the chain of matrices as in [19], and in particular the limit of matrix quantum mechanics. We believe that this technique could apply to many other integrable models.

Another question, is to understand the limit of critical points, i.e. when some branchpoints and double points start to coalesce. It seems that the diagrammatic technique should just reduce to consider only residues at branchpoints which become critical. One may expect to recover some relation with the Kontsevich integral, in relationship with KP integrable hierarchies.

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Appendix.

Appendix A: Computation of Eq. (6.121).

In this appendix one proves recursively Eq. (6.121) for any k and h .

Let us suppose that this formula is known for any $U_l^{(m)}$ with $m \leq h-1$ and for any $U_l^{(h)}$ with $l \leq k-1$. One writes it:

$$U_l^{(m)}(p, y(p^{(i)}); \mathbf{p}_L) = \frac{E_y(x, y(p^{(i)}))}{y(p^{(i)})-y(p)} \sum_{r=1}^{\min(d_2, k+h)} \sum_{L_1 \cup \dots \cup L_r = L} \sum_{m_\alpha=0}^m \sum_{l_\alpha=|L_\alpha|}^{l+m} \sum_{j_{\alpha, \beta} \neq j_{\alpha', \beta'} \in [1, d_2] - \{i\}} \frac{1}{\Omega} \frac{W_{l_1+1}^{(m_1)}(p^{(i)}, \mathbf{p}_{L_1}, p^{(j_{1,1})}, \dots, p^{(j_{1, l_1-|L_1|})}) \left(\prod_{\alpha=2}^r W_{l_\alpha+1}^{(m_\alpha)}(p^{(j_{\alpha,0})}, \mathbf{p}_{L_\alpha}, p^{(j_{\alpha,1})}, \dots, p^{(j_{\alpha, l_\alpha-|L_\alpha|})}) \right)}{dx(p)^{r-l-1+\sum L_\alpha} \prod_{\alpha, \beta} y(p^{(i)})-y(p^{(j_{\alpha, \beta})})} \quad (9.157)$$

Let us introduce some shortened notations so that one can write this proof in a few pages.

Considering the sum on the RHS of Eq. (9.157), one can see that there are two different kinds of terms:

- If $l_1 = |L_1|$, one can factorise the term $W_{|L_1|+1}^{(m_1)}(p^{(i)}, \mathbf{p}_{L_1})$. Let us note the sum of these terms $W(p^{(i)}, p_L)W(p_L, p^{(j)})$ where we have noted W instead of $W_{|L_1|+1}^{(m_1)}$ to indicate that these are formal notations;
- the other terms correspond to the sum over all $l_1 \neq |L_1|$. We note it $W(p^{(i)}, p_L, p^{(j)})W(p_L, p^{(j)})$.

Using these notations, one can shortly write Eq. (9.157):

$$U_l^{(m)}(p, y(p^{(i)}); \mathbf{p}_L) = W(p^{(i)}, p_L)W(p_L, p^{(j)}) + W(p^{(i)}, p_L, p^{(j)})W(p_L, p^{(j)}) \quad (9.158)$$

Thus the interpolation formula gives:

$$U_l^{(m)}(p^{(i)}, y(p^{(i)}); p_L) = W(p_L, p^{(j)}) + W(p, p_L)W(p_L, p^{(j)}) + W(p, p_L, p^{(j)})W(p_L, p^{(j)}) \quad (9.159)$$

where the first term corresponds to the sum where all j_β 's are different from i and 0 and there is no W_{l_i} whose argument is p or $p^{(i)}$.

On the other hand, one knows the relation 6.120:

$$U_k^{(h)}(p, y(p^{(i)}); \mathbf{p}_K) = \sum_{m=0}^h \sum_{j=0}^k \frac{W_{j+1}^{(m)}(p^{(i)}, \mathbf{p}_J) U_{k-j}^{(h-m)}(p^{(i)}, y(p^{(i)}); \mathbf{p}_{K-J})}{(y(p^{(i)})-y(p))dx(p)} - \sum_{m=0}^h \sum_{j=0}^k \frac{W_{j+1}^{(m)}(p, \mathbf{p}_J) U_{k-j}^{(h-m)}(p, y(p^{(i)}); \mathbf{p}_{K-J})}{(y(p^{(i)})-y(p))dx(p)} - \frac{U_{k+1}^{(h-1)}(p, y(p^{(i)}); \mathbf{p}_K)}{(y(p^{(i)})-y(p))dx} + \frac{U_{k+1}^{(h-1)}(p^{(i)}, y(p^{(i)}); p^{(i)}, \mathbf{p}_K)}{(y(p^{(i)})-y(p))dx} \quad (9.160)$$

Remark that the terms in the RHS of this equation correspond to the criterion of the hypothesis and one can then express them as a product of W 's following the notations introduced earlier. This reads:

$$\begin{aligned}
& U_k^{(h)}(p, y(p^{(i)}); p_K) = \\
& W(p^{(i)}, p_K)W(p_K, p^{(j)}) + W(p^{(i)}, p_K)W(p, p_K)W(p_K, p^{(j)}) \\
& + W(p^{(i)}, p_K)W(p, p_K, p^{(j)})W(p_K, p^{(j)}) - W(p, p_K)W(p^{(i)}, p_K)W(p_K, p^{(j)}) \\
& - W(p, p_K)W(p^{(i)}, p_K, p^{(j)})W(p_K, p^{(j)}) + W(p^{(i)}, p_K, p^{(j)})W(p_K, p^{(j)}) \\
& + W(p^{(i)}, p, p_K)W(p_K, p^{(j)}) + W(p, p_K)W(p^{(i)}, p_K, p^{(j)})W(p_K, p^{(j)}) \\
& + W(p^{(i)}, p, p_K, p^{(j)})W(p_K, p^{(j)}) + W(p, p_K, p^{(j)})W(p^{(i)}, p_K, p^{(j)})W(p_K, p^{(j)}) \\
& - W(p, p^{(i)}, p_K)W(p_K, p^{(j)}) - W(p^{(i)}, p_K)W(p, p_K, p^{(j)})W(p_K, p^{(j)}) \\
& - W(p, p^{(i)}, p_K, p^{(j)})W(p_K, p^{(j)}) - W(p^{(i)}, p_K, p^{(j)})W(p, p_K, p^{(j)})W(p_K, p^{(j)}) \\
& = W(p^{(i)}, p_K)W(p_K, p^{(j)}) + W(p^{(i)}, p_K, p^{(j)})W(p_K, p^{(j)})
\end{aligned} \tag{9.161}$$

So one has proven the formula for $U_k^{(h)}$.

Because this formula is true for $h=0$, it is true for any k and h .

9.1 Appendix B : Derivation of Eq. (6.122).

One wants to show that:

$$\begin{aligned} & \sum_{m=0}^h \sum_{j=0; m_j \neq kh}^k W_{j+1}^{(m)}(p, p_J) U_{k-j}^{(h-m)}(p, y(p); p_{K-J}) + \frac{U_{k+1}^{(h-1)}(p, y(p); p, p_k)}{dx} = \\ & \sum_{i=1}^{d_2} \sum_{m=0}^h \sum_{j=0; m_j \neq kh}^k W_{j+1}^{(m)}(p^{(i)}, p_J) U_{k-j}^{(h-m)}(p^{(i)}, y(p); p_{K-J}) \\ & + \sum_{i=1}^{d_2} \frac{U_{k+1}^{(h-1)}(p^{(i)}, y(p); p^{(i)}, p_k)}{dx} \end{aligned} \quad (9.162)$$

Let us compute the difference D between the two sides of the equation by the introduction of Eq. (6.121) written with some few different notations which are defined as follows:

- $l = r + h - \sum_{\alpha} h_{\alpha}$;
- $u_{\beta} = \sum_{\epsilon=1}^{\beta} (k_{\epsilon} - |K_{\epsilon}|) - \beta$.

One can then write:

$$\begin{aligned} D &= \sum_{m=0}^h \sum_{j=0; m_j \neq kh}^k W_{j+1}^{(m)}(p, p_J) E_y(x, y(p)) \sum_{i=1}^{d_2} \frac{1}{y(p) - y(p^{(i)})} \\ & \times \sum_{r=1}^{d_2} \sum_{o=1}^r \sum_{h_o=0}^{h-m} \sum_{k_o=0}^{k+h-j-m} \sum_{j_2 \neq \dots \neq j_l \in [1, d_2] - \{i\}} \sum_{K_1 \cup \dots \cup K_r = K} \frac{1}{\Omega} \\ & \frac{W_{k_1+1}^{(h_1)}(p^{(i)}, p_{K_1}, p^{(j_{r+1})}, \dots, p^{(j_{r+u_1})}) \left(\prod_{\beta=2}^r W_{k_{\beta}+1}^{(h_{\beta})}(p^{(j_{\beta})}, p_{K_{\beta}}, p^{(j_{r+u_{\beta-1}+1})}, \dots, p^{(j_{r+u_{\beta}})}) \right)}{\prod_{\gamma=2}^l y(p^{(i)}) - y(p^{(j_{\gamma})})} \\ & - \sum_{i=1}^{d_2} \sum_{m=0}^h \sum_{j=0; m_j \neq kh}^k W_{j+1}^{(m)}(p^{(i)}, p_J) \frac{E_y(x, y(p))}{y(p) - y(p^{(i)})} \\ & \times \sum_{r=1}^{d_2} \sum_{o=1}^r \sum_{h_o=0}^{h-m} \sum_{k_o=0}^{k+h-j-m} \sum_{j_2 \neq \dots \neq j_l \in [1, d_2] - \{i\}} \sum_{K_1 \cup \dots \cup K_r = K} \frac{1}{\Omega} \\ & \frac{W_{k_1+1}^{(h_1)}(p, p_{K_1}, p^{(j_{r+1})}, \dots, p^{(j_{r+u_1})}) \left(\prod_{\beta=2}^r W_{k_{\beta}+1}^{(h_{\beta})}(p^{(j_{\beta})}, p_{K_{\beta}}, p^{(j_{r+u_{\beta-1}+1})}, \dots, p^{(j_{r+u_{\beta}})}) \right)}{\prod_{\gamma=2}^l y(p) - y(p^{(j_{\gamma})})} \\ & + E_y(x, y(p)) \sum_{i=1}^{d_2} \frac{1}{y(p) - y(p^{(i)})} \\ & \times \sum_{r=1}^{d_2} \sum_{o=1}^r \sum_{h_o=0}^{h-1} \sum_{k_o=0}^{k+h} \sum_{j_2 \neq \dots \neq j_l \in [1, d_2] - \{i\}} \sum_{K_1 \cup \dots \cup K_r = K} \frac{1}{\Omega} \\ & \times \left[\frac{W_{k_1+1}^{(h_1)}(p^{(i)}, p, p_{K_1}, p^{(j_{r+1})}, \dots, p^{(j_{r+u_1})}) \left(\prod_{\beta=2}^r W_{k_{\beta}+1}^{(h_{\beta})}(p^{(j_{\beta})}, p_{K_{\beta}}, p^{(j_{r+u_{\beta-1}+1})}, \dots, p^{(j_{r+u_{\beta}})}) \right)}{\prod_{\gamma=2}^l y(p^{(i)}) - y(p^{(j_{\gamma})})} \right. \\ & + \frac{W_{k_1+1}^{(h_1)}(p^{(i)}, p_{K_1}, p^{(j_{r+1})}, \dots, p^{(j_{r+u_1})}) W_{k_2+1}^{(h_2)}(p, p^{(j_2)}, p_{K_2}, p^{(j_{r+u_1+1})}, \dots, p^{(j_{r+u_2})})}{\prod_{\gamma=2}^l y(p^{(i)}) - y(p^{(j_{\gamma})})} \\ & \left. \times \prod_{\beta=3}^r W_{k_{\beta}+1}^{(h_{\beta})}(p^{(j_{\beta})}, p_{K_{\beta}}, p^{(j_{r+u_{\beta-1}+1})}, \dots, p^{(j_{r+u_{\beta}})}) \right] \\ & - \sum_{i=1}^{d_2} \frac{E_y(x, y(p))}{y(p) - y(p^{(i)})} \\ & \times \sum_{r=1}^{d_2} \sum_{o=1}^r \sum_{h_o=0}^{h-1} \sum_{k_o=0}^{k+h} \sum_{j_2 \neq \dots \neq j_l \in [1, d_2] - \{i\}} \sum_{K_1 \cup \dots \cup K_r = K} \frac{1}{\Omega} \\ & \times \left[\frac{W_{k_1+1}^{(h_1)}(p^{(i)}, p, p_{K_1}, p^{(j_{r+1})}, \dots, p^{(j_{r+u_1})}) \left(\prod_{\beta=2}^r W_{k_{\beta}+1}^{(h_{\beta})}(p^{(j_{\beta})}, p_{K_{\beta}}, p^{(j_{r+u_{\beta-1}+1})}, \dots, p^{(j_{r+u_{\beta}})}) \right)}{\prod_{\gamma=2}^l y(p) - y(p^{(j_{\gamma})})} \right. \\ & + \frac{W_{k_1+1}^{(h_1)}(p, p_{K_1}, p^{(j_{r+1})}, \dots, p^{(j_{r+u_1})}) W_{k_2+1}^{(h_2)}(p^{(i)}, p^{(j_2)}, p_{K_2}, p^{(j_{r+u_1+1})}, \dots, p^{(j_{r+u_2})})}{\prod_{\gamma=2}^l y(p^{(i)}) - y(p^{(j_{\gamma})})} \\ & \left. \times \prod_{\beta=3}^r W_{k_{\beta}+1}^{(h_{\beta})}(p^{(j_{\beta})}, p_{K_{\beta}}, p^{(j_{r+u_{\beta-1}+1})}, \dots, p^{(j_{r+u_{\beta}})}) \right] \end{aligned} \quad (9.163)$$

The difference between the two first terms leaves only the terms corresponding to $u_1 \neq 0$ in the first one minus $u_1 \neq 0$ in the second one.

The difference between two last terms will allow us to compensate the preceding ones. Indeed, the terms with $p^{(i)}$ and p together in the same correlation function straightforwardly vanish and one gets the exact opposite to the two first terms remaining.

Thus $D=0$ and the equality 9.162 is proven.

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