



HAL
open science

The Hahn-Banach Theorem implies the existence of a non-Lebesgue measurable set

Matthew Foreman, Friedrich Wehrung

► **To cite this version:**

Matthew Foreman, Friedrich Wehrung. The Hahn-Banach Theorem implies the existence of a non-Lebesgue measurable set. *Fundamenta Mathematicae*, 1991, 138, no. 1, pp.13-19. hal-00004713

HAL Id: hal-00004713

<https://hal.science/hal-00004713v1>

Submitted on 14 Apr 2005

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THE HAHN-BANACH THEOREM IMPLIES THE EXISTENCE OF A NON LEBESGUE-MEASURABLE SET

Matthew Foreman and Friedrich Wehrung
Ohio State University (U.S.A.)
CNRS, Université de Caen. France.

§0. Introduction.

Few methods are known to construct non Lebesgue-measurable sets of reals: most standard ones start from a well-ordering of \mathbb{R} , or from the existence of a non-trivial ultrafilter over ω , and thus need the axiom of choice AC or at least the Boolean Prime Ideal theorem BPI (see [5]). In this paper we present a new way for proving the existence of non-measurable sets using a convenient operation of a discrete group on the Euclidian sphere. The only choice assumption used in this construction is the Hahn-Banach theorem, a weaker hypothesis than BPI (see [9]). Our construction proves that the Hahn-Banach theorem implies the existence of a non-measurable set of reals. This answers questions in [9], [10]. (Since we do not even use the countable axiom of choice, we cannot assume the countable additivity of Lebesgue measure; e.g. the real numbers could be a countable union of countable sets.)

In fact we prove (under Hahn-Banach theorem) that there is no finitely additive, rotation invariant extension of Lebesgue measure to $\mathcal{P}(\mathbb{R}^3)$. Notice that Hahn-Banach implies the existence of a finitely additive, isometry invariant extension of Lebesgue measure to $\mathcal{P}(\mathbb{R}^2)$ (see [14]).

We use standard set-theoretical notation and terminology. For example, if X is any set, $\mathcal{P}(X)$ is the power set of X . If $A \subseteq X$ and $f : X \rightarrow Y$ is a map, then $f[A]$ is the image of A under f . Furthermore, ω is the set of all natural numbers.

We assume ZF throughout this paper; no choice assumption (even countable) is made.

§1. Definitions.

First, let us give one of the many equivalent statements of the Hahn-Banach theorem. We use the version [11]:

The Hahn-Banach Theorem. *Let E be a vector space over the reals, let S be a subspace of E , and f be a linear functional on S . Let p be a map $E \rightarrow \mathbb{R}$ such that whenever $x, y \in E$ and $\lambda \geq 0$, we have $p(\lambda x) = \lambda p(x)$ and $p(x + y) \leq p(x) + p(y)$. Then there is a linear functional \bar{f} on E , extending f , such that $(\forall x \in E)(\bar{f}(x) \leq p(x))$.*

Definition. *If B is a Boolean algebra, a finitely additive probability measure on B (from now on a measure) is a map $\mu : B \rightarrow [0, 1]$ such that $\mu(1_B) = 1$ and $\mu(x \vee y) = \mu(x) + \mu(y)$ whenever $x \wedge y = 0$.*

It is known that $ZF + \text{Hahn-Banach}$ implies that every Boolean algebra has a measure (actually in ZF without choice, this last statement is equivalent to the Hahn-Banach theorem, see [7,15]). It also yields the following statement for collections of Boolean algebras:

Proposition 1. (*ZF+Hahn-Banach theorem*) Let $\langle B_i : i \in I \rangle$ be a sequence of Boolean algebras (with I not necessarily well-orderable). Then there exists $\langle \mu_i : i \in I \rangle$ such that for each $i \in I$, μ_i is a measure on B_i .

Proof. Let $(B, e_i)_{i \in I}$ be the direct sum of $(B_i)_{i \in I}$ in the category of Boolean algebras: so, for every $i \in I$, e_i is an homomorphism $B_i \rightarrow B$ (elements of B are formal Boolean combinations of elements of the B_i with no other relations than those from the B_i ; one can prove that e_i is one-to-one). By the Hahn-Banach theorem there is a measure μ on B . Put $\mu_i = \mu \circ e_i$. ■

Definition. A universally measured space is an ordered pair (Ω, μ) where Ω is a set and μ is a measure on the Boolean algebra $\mathcal{P}(\Omega)$. A group G is said to act by measure preserving transformations on (Ω, μ) when G acts on Ω and $\mu(gA) = \mu(A)$ for all $g \in G$ and $A \in \mathcal{P}(\Omega)$.

We are going to be mainly concerned about the following measure existence statement:

Definition. Let a group G act on a set Ω . $IM(\Omega, G)$ is the statement “there is a G -invariant measure on $\mathcal{P}(\Omega)$ ”.

In the case of a group acting on itself, we get the following classical definition.

Definition. A group G is amenable when there is a measure μ on $\mathcal{P}(G)$ such that $\mu(Ag) = \mu(A)$ for all $g \in G$, $A \in \mathcal{P}(G)$.

Assuming the Hahn-Banach theorem many groups are amenable, including finite groups, solvable groups and their extensions. The best known non-amenable group is the free group on two generators.

Proposition 2. (Classical) [14] - The free group on two generators, F_2 , is not amenable.

For all integers $n \geq 1$, denote by O_n the isometry group of S^{n-1} (with Euclidian norm), $SO_n = \{u \in O_n : \det(u) = +1\}$, where $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is the n -dimensional Euclidian sphere. One can prove in ZFC that $IM(S^n, SO_{n+1})$ does not hold for $n \geq 2$, and thus SO_{n+1} is not amenable (see [14]). On the other hand, in [10] and [13], the authors construct models of $ZF + DC$ in which $IM(S^n, O_{n+1})$ holds for every $n \geq 1$ (in [13], the measure is just normalized Lebesgue-measure).

A group G acts on a set Ω *freely* when for all $g \in G$, $x \in \Omega$, $gx = x$ implies $g = 1$.

§2. The main results.

We start with a classical result.

Proposition 3. Assume $IM(S^2, SO_3)$. Then there is a free measure-preserving action of F_2 on some universally measured space (Ω, μ) .

Proof. Consider a subgroup of SO_3 isomorphic to F_2 , [14] and D the subset of S^2 consisting of the union of all the possible orbits of fixed points of elements of $F_2 \setminus \{1\}$. D is countable since each orbit is effectively countable and it is easy to distinguish fixed points of elements of F_2 acting on S^2 . Hence D is the image of a function with domain $\{0, 1\} \times F_2 \times F_2$. (Recall, we do not know that a countable union of countable sets is

countable.) Let μ be the witness to $IM(S^2, SO_3)$. Since F_2 acts freely on $S^2 \setminus D$, we will be done if we can show $\mu(D) = 0$.

In [14] it is shown that every SO_3 -invariant finitely additive measure on S^2 gives each countable set measure zero. We paraphrase the proof given there and check that it works without AC.

It clearly suffices to find a rotation g such that for all $k \in \omega \setminus \{0\}$, $g^k D \cap D = \emptyset$. Since then $\{g^k D : k \in \omega\}$ is an infinite collection of pairwise disjoint subsets of S^2 of the same μ -measure. Let $\langle a_n : n \in \omega \rangle$ be an enumeration of D . Let ℓ be a line through the origin missing D . Let $A_n = \{g \in SO(3) : g \text{ is a rotation about } \ell \text{ and for some } i \neq j \in \omega, g^n a_i = a_j\}$. Then A_n is countable in a canonical way, since each $g \in A_n$ is determined by a_i and a_j . Hence $\cup A_n$ is countable. Choose a rotation g about ℓ such that $g \notin \cup A_n$ and g has infinite order. Then for all $n \geq 1$, $g^n D \cap D = \emptyset$. ■

Another example is with $IM({}^\omega 2, G)$ where ${}^\omega 2$ is the Cantor space with its canonical metric and G its group isometries (see [12]).

Our main theorem is:

Theorem 4. (*ZF+Hahn-Banach*) - *Let a group G act freely and measure-preserving on a universally measured space (Ω, μ) . Then G is amenable.*

Proof. (Note the similarity to [6].)

Denote by Ω/G the set of orbits of Ω modulo G .

By Proposition 1, there is a sequence $\langle \mu_{[x]} : [x] \in \Omega/G \rangle$ such that for each $[x] \in \Omega/G$, $\mu_{[x]}$ is a measure on $\mathcal{P}([x])$. For each $A \subseteq G$, let $a : \Omega \rightarrow [0, 1]$ be the following function: $a(x) = \mu_{[x]}(Ax)$; define $\lambda : \mathcal{P}(G) \rightarrow [0, 1]$ by $\lambda(A) = \int a(x) d\mu(x)$. Note that $x \mapsto a(x)$ is a measurable function since (Ω, μ) is a universally measured space; the integration here is essentially Lebesgue integration, and it does not appeal to any choice (no limit theorems are needed).

We claim that λ is a measure on $\mathcal{P}(G)$, invariant under right translation.

Note that $\lambda(G) = 1$. If A, B are two disjoint subsets of G and a, b, c are the functions corresponding to $A, B, A \cup B$ respectively, then $(\forall x \in \Omega)(c(x) = a(x) + b(x))$. Hence $\lambda(A \cup B) = \lambda(A) + \lambda(B)$.

Finally, if $B = Ag$ for some $g \in G$ and a, b are the functions corresponding to A and B then, for all $x \in \Omega$,

$$\begin{aligned} b(x) &= \mu_{[x]}(Bx) = \mu_{[x]}(Agx) \\ &= \mu_{[x]}(A(gx)) = \mu_{[gx]}(A(gx)) = a(gx). \end{aligned}$$

Hence $\lambda(B) = \int b(x) d\mu(x) = \int a(gx) d\mu(x) = \int a(x) d\mu(x) = \lambda(A)$ since g is μ -measure preserving. ■

Corollary 1. - *ZF+Hahn-Banach implies not $IM(S^2, SO_3)$. Thus, there is a non-Lebesgue measurable subset of S^2 .*

Proof. Propositions 2, 3 and Theorem 4. ■

Note that in the last part of the statement above, S^2 could be replaced by many other spaces, like \mathbb{R}^n , $n \geq 1$. (See §3 for details).

Corollary 2. *If H is generic for the partial ordering adding ω_1 random reals to a model V of ZFC and $V(\mathbb{R})$ is the smallest model of set theory containing V and reals of $V[H]$, then $V(\mathbb{R})$ does not satisfy the Hahn-Banach theorem.*

Proof. $V(\mathbb{R})$ is the model considered by D. Pincus and R. Solovay in [10]. It satisfies $IM(S^n, SO_{n+1})$ for all $n \geq 1$, and thus $IM(S^2, SO_3)$; we conclude by Corollary 1. ■

Another way to see Corollary 1 is the following:

Corollary 3. *If F_2 acts freely on $\Omega = S^2 \setminus D$ (D as in the proof of Proposition 3) by rotations, and if $\langle \mu_{[x]} : [x] \in \Omega/F_2 \rangle$ is any assignment of finitely additive probability measures $\mu_{[x]}$ on $\mathcal{P}([x])$, then there are $A \subseteq F_2$ and $\alpha \in [0, 1]$ such that $\{x : \mu_{[x]}(Ax) < \alpha\}$ is not Lebesgue measurable. Further the set A can be isolated explicitly (see [14]).* ■

§3. Appendix. Lebesgue measure without countable choice.

Ordinarily, the theory of Lebesgue measure is developed with use of AC_ω . The use of AC_ω allows one to use arbitrary Borel sets. In this section we explore how to use “coded” Borel sets to eliminate the necessity of AC_ω in many applications. For example, we would still like the existence of non-measurable set to be independent from the reference space (here, S^2). The aim of this section is to show how to adapt the proofs of the “classical” theory (with AC_ω) to the study of Lebesgue-measure in a totally choiceless context. The ideas here date from [13].

In order to get as many measurable sets as possible, the classical outer measure construction (see [4]) seems convenient enough. This construction, which we will sketch in \mathbb{R} , works as well in \mathbb{R}^n or in much more abstract spaces.

Define the outer measure of $A \subseteq \mathbb{R}$ by the greatest lower bound of all sums $\sum_{n \in \omega} \text{length}(I_n)$ where I_n are intervals, and $A \subseteq \bigcup_{n \in \omega} I_n$; call it $\mu^*(A)$. Say that A is Lebesgue-measurable when for all $X \subseteq \mathbb{R}$, $\mu^*(X) = \mu^*(X \cap A) + \mu^*(X \setminus A)$. Note $\mathcal{M} = \{A \subseteq \mathbb{R}; A \text{ is Lebesgue-measurable}\}$, $\mu = \mu^* \upharpoonright \mathcal{M}$. It is still possible to prove that \mathcal{M} is a Boolean subalgebra of $\mathcal{P}(\mathbb{R})$ and that μ is a finitely additive function $\mathcal{M} \rightarrow [0, \infty]$, and that \mathcal{M} contains all open sets. But one cannot prove any more that \mathcal{M} is a σ -algebra (since \mathbb{R} can be a countable union of countable sets, see [5]). So, instead of considering Borel subsets of \mathbb{R} , consider those which have a *code*, as e.g. in [12]; a Borel code is essentially a real, encoding the “construction” of some Borel set. Similarly, say that $(A_n)_{n \in \omega}$ is coded sequence of Borel sets when there is a sequence $(c_n)_{n \in \omega}$ such that for every n , c_n is a code for A_n . And then, we can prove the following properties of (μ, \mathcal{M}) :

(a) \mathcal{M} is Boolean subalgebra of $\mathcal{P}(\mathbb{R})$, containing all coded Borel subsets of \mathbb{R} .

(b) μ is a finitely additive map $\mathcal{M} \rightarrow [0, \infty]$, and whenever $(A_n)_{n \in \omega}$ is a disjoint coded sequence of Borel sets, we have:

$$\mu \left(\bigcup_{n \in \omega} A_n \right) = \sum_{n \in \omega} \mu(A_n).$$

(c) A subset $A \subseteq \mathbb{R}$ is in \mathcal{M} iff for all $\varepsilon > 0$ and all coded Borel B with $\mu(B) < \infty$, there are coded Borel F and U such that $F \subseteq A \cap B \subseteq U$ and $\mu(U \setminus F) < \varepsilon$.

(Actually, it is enough to check when B is a bounded interval, and U can be chosen as an open set, F as a closed set.)

(d) μ is σ -finite: there is a coded sequence $(A_n)_{n \in \omega}$ of Borel sets such that $\mathbb{R} = \bigcup_{n \in \omega} A_n$ and $(\forall n \in \omega)(\mu(A_n) < \infty)$. (Take $A_n = [-n, n]$.)

The precautions needed by elimination of AC_ω in the classical proof of (a) and (d) above (see [4]) make the proof somewhat more lengthy, but without real difficulties. Note that in (c), the assumption $\mu(B) < \infty$ does not seem to be removable without countable choice.

Let us call the μ above the Lebesgue measure on \mathbb{R} ; a similar construction yields Lebesgue measure on \mathbb{R}^n , for all $n \geq 1$.

More generally, let us set the following definition:

Definition. A coded Borel space is an ordered pair (Ω, \mathcal{B}) where Ω is a coded Borel subset of the Hilbert cube ${}^\omega[0, 1]$ and \mathcal{B} is the algebra of coded Borel subsets of Ω .

We can naturally extend this definition by taking all isomorphic images; this way, all usual spaces of analysis - like \mathbb{R}^n , S^n , or ${}^\omega 2$, together with their coded Borel subsets, become coded Borel spaces. Anyway, even without using countable choice, it turns out that the following is true:

Proposition 5. Let (Ω, \mathcal{B}) be an uncountable coded Borel space. Then there is a coded Borel isomorphism from (Ω, \mathcal{B}) onto (I, \mathcal{B}_I) , where $I = [0, 1]$ and \mathcal{B}_I is the algebra of coded Borel subsets of I . ■

Here, a coded Borel isomorphism $(\Omega, \mathcal{B}) \rightarrow (I, \mathcal{B}_I)$ is naturally a bijection $f : \Omega \rightarrow I$ such that the neighborhood diagrams of f and f^{-1} are coded Borel.

Now, let us give the new definition of measure we are going to use:

Definition. Let (Ω, \mathcal{B}) be a coded Borel space. A regular measure on (Ω, \mathcal{B}) is a map $\mu : \mathcal{M} \rightarrow [0, \infty]$ such that (μ, \mathcal{M}) satisfies conditions (a) to (d) above, with Ω instead of \mathbb{R} . Say that μ is nonatomic when $(\forall x \in \Omega)(\mu(\{x\}) = 0)$.

The essential isomorphism theorem between these measure spaces is still valid (after a suitable reformation). It can be stated the following way:

Proposition 6. Let μ be a regular, nonatomic measure on a coded Borel space (Ω, \mathcal{B}) , with $\mu(\Omega) = 1$. Then there are $N \subseteq \Omega$, $D \subseteq [0, 1]$ and $f : \Omega \rightarrow [0, 1]$ such that, if ℓ is Lebesgue measure on $[0, 1]$,

- (i) $N \in \mathcal{B}$, D is countable, $\mu(N) = \ell(D) = 0$.
- (ii) f is a coded Borel isomorphism $\Omega \setminus N \rightarrow [0, 1] \setminus D$.
- (iii) For all B in \mathcal{B} , $f[B]$ is coded Borel in $[0, 1]$ and $\mu(B) = \ell(f[B])$.

Outline of Proof (See [11]). First, notice that by (b) and $\mu(\Omega) = 1$, Ω is uncountable. So, by proposition 5, without loss of generality, $\Omega = [0, 1]$ and \mathcal{B} is the algebra of coded Borel subsets of $[0, 1]$. Then, define $f : [0, 1] \rightarrow [0, 1]$ by $f(x) = \mu([0, x])$. Then, D is just

$\{y \in [0, 1] : f^{-1}\{y\} \text{ has nonempty interior}\}$ and N is $f^{-1}[D]$. (iii) is proven by induction on a code of B , and it uses nonatomicity of μ . ■

Now, Proposition 6 has an immediate corollary:

Corollary 1. *Let μ be a regular, nonatomic measure on a coded Borel space (Ω, \mathcal{B}) , with $\mu(\Omega) \neq 0$. Then the following are equivalent:*

- (i) *Every subset of Ω is μ -measurable.*
- (ii) *Every subset of $[0, 1]$ is Lebesgue-measurable.* ■

(To prove (i) \Rightarrow (ii), one has to use σ -finiteness, nonatomicity of μ and $\mu(\Omega) \neq 0$; for (ii) \Rightarrow (i), use characterisation (c) above of μ -measurability).

In particular, every subset of \mathbb{R}^n ($n \geq 1$) is Lebesgue-measurable iff every subset of $[0, 1]$ is Lebesgue-measurable (which is well-known in the classical theory using countable choice). Let LM be the latter statement.

Now, define Lebesgue measure v_n on S^n as being the image under $x \mapsto \frac{x}{\|x\|}$ of Lebesgue measure on $B^{n+1} \setminus \{0\}$, where B^{n+1} is the Euclidian closed ball of \mathbb{R}^{n+1} of volume 1.

Corollary 2. *LM implies $IM(S^n, SO_{n+1})$ for all $n \geq 1$.*

Proof. If LM holds, then v_n is defined on $\mathcal{P}(S^n)$ by the previous corollary; so v_n witnesses $IM(S^n, SO_{n+1})$. ■

More precisely, the result would be the same with a rotation-invariant extension of Lebesgue-measure on $\mathcal{P}(S^2)$; thus, the results of the previous paragraph imply for example that *Hahn-Banach theorem implies nonexistence of a rotation-invariant extension of Lebesgue-measure to a (finitely additive) measure on $\mathcal{P}(\mathbb{R}^3)$.*

Further notes. Theorem 4 could be formulated as follows: *“If G is a nonamenable group acting freely on a set Ω and if μ is a G -invariant finitely additive probability measure defined on a G -invariant subalgebra of $\mathcal{P}(\Omega)$, then Ω has non-measurable subsets (w.r.t. μ)”.* Now, while this paper was printed, the second author showed, under the same hypotheses, that in the G -equidecomposability type semigroup of Ω (see [14]), $n[\Omega] = (n+1)[\Omega]$ for some integer n , effectively computable from the number of pieces necessary to a paradoxical decomposition of G . For the action of F_2 described above, we can get $n = 5$, which is somewhat disappointing since it is not known whether the cancellation law (see [14]) follows from HB (it follows from BPI). But independently, J. Pawlikowski proved using ideas from this paper, that one can actually take $n = 1$, that is, $[\Omega] = 2[\Omega]$; thus, HB implies the Banach-Tarski paradox. See [8] for more details.

BIBLIOGRAPHY

- [1] M. Foreman, “Amenable groups acting on the natural numbers, an independence result”. Bulletin AMS, October 1989, vol. 21, N° 2.
- [2] K. Gödel, “The consistency of the axiom of choice and the generalized continuum hypothesis with the axioms of set theory”. Ann. of Math. Studies N° 3. Princeton University Press, 1940.
- [3] F. Greenleaf, “Invariant means on Topological Groups”. Van Nostrand, Reinhold Company, 1969.
- [4] P. R. Halmos, “Measure Theory”. Van Nostrand, Reinhold Company, 1950.
- [5] T. Jech, “The axiom of choice”. North-Holland Publishing Company.
- [6] A. Kechris, “Amenable equivalence relations and Turing degrees”. Handwritten notes, 1988.
- [7] W. A. J. Luxemburg, “Reduced products of the real number system”. Applications of model theory to algebra, analysis and probability. Holt, Reinhart and Winston, 1969.
- [8] J. Pawlikowski, “The Hahn-Banach Theorem implies the Banach-Tarski Paradox”. To appear.
- [9] D. Pincus, “The strength of Hahn-Banach’s Theorem”. Victoria symposium on non-standard analysis. Springer lecture notes n° 369, 1974 (pp. 203-248).
- [10] D. Pincus and R. Solovay, “Definability of measures and ultrafilters”. Journal of Symbolic Logic 42, n° 2, 1977, (pp. 179-190).
- [11] H. L. Royden, “Real Analysis”. The Macmillan Company, New York, 1968.
- [12] T. Slaman and J. Steel, “Definable functions on Degrees”. Cabal seminar 1981-85, appeared in 1988, Springer Verlag lecture notes.
- [13] R. Solovay, “A model of set theory in which every set of reals is Lebesgue-measurable”. Annals of Math. 1970, (pp. 1-56).
- [14] S. Wagon, “The Banach-Tarski Paradox”. Cambridge University Press, 1984.
- [15] J. Conway, “A course in Functional Analysis”. Springer-Verlag, 1985.