# THE HAHN-BANACH THEOREM IMPLIES THE EXISTENCE OF A NON LEBESGUE-MEASURABLE SET 

Matthew Foreman and Friedrich Wehrung<br>Ohio State University (U.S.A.)<br>CNRS, Université de Caen. France.

## §0. Introduction.

Few methods are known to construct non Lebesgue-measurable sets of reals: most standard ones start from a well-ordering of $\mathbb{R}$, or from the existence of a non-trivial ultrafilter over $\omega$, and thus need the axiom of choice AC or at least the Boolean Prime Ideal theorem BPI (see [5]). In this paper we present a new way for proving the existence of non-measurable sets using a convenient operation of a discrete group on the Euclidian sphere. The only choice assumption used in this construction is the Hahn-Banach theorem, a weaker hypothesis than BPI (see [9]). Our construction proves that the Hahn-Banach theorem implies the existence of a non-measurable set of reals. This answers questions in [9], [10]. (Since we do not even use the countable axiom of choice, we cannot assume the countable additivity of Lebesgue measure; e.g. the real numbers could be a countable union of countable sets.)

In fact we prove (under Hahn-Banach theorem) that there is no finitely additive, rotation invariant extension of Lebesgue measure to $\mathcal{P}\left(\mathbb{R}^{3}\right)$. Notice that Hahn-Banach implies the existence of a finitely additive, isometry invariant extension of Lebesgue measure to $\mathcal{P}\left(\mathbb{R}^{2}\right)$ (see [14]).

We use standard set-theoretical notation and terminology. For example, if $X$ is any set, $\mathcal{P}(X)$ is the power set of $X$. If $A \subseteq X$ and $f: X \rightarrow Y$ is a map, then $f[A]$ is the image of $A$ under $f$. Furthermore, $\omega$ is the set of all natural numbers.

We assume $Z F$ throughout this paper; no choice assumption (even countable) is made.

## §1. Definitions.

First, let us give one of the many equivalent statements of the Hahn-Banach theorem. We use the version [11]:
The Hahn-Banach Theorem. Let $E$ be a vector space over the reals, let $S$ be a subspace of $E$, and $f$ be a linear functional on $S$. Let $p$ be a map $E \rightarrow \mathbb{R}$ such that whenever $x, y \in E$ and $\lambda \geq 0$, we have $p(\lambda x)=\lambda p(x)$ and $p(x+y) \leq p(x)+p(y)$. Then there is a linear functional $\bar{f}$ on $E$, extending $f$, such that $(\forall x \in E)(\bar{f}(x) \leq p(x))$.

Definition. If $B$ is a Boolean algebra, a finitely additive probability measure on $B$ (from now on a measure) is a map $\mu: B \rightarrow[0,1]$ such that $\mu\left(1_{B}\right)=1$ and $\mu(x \vee y)=\mu(x)+\mu(y)$ whenever $x \wedge y=0$.

It is known that $Z F+$ Hahn-Banach implies that every Boolean algebra has a measure (actually in $Z F$ without choice, this last statement is equivalent to the Hahn-Banach theorem, see $[7,15]$ ). It also yields the following statement for collections of Boolean algebras:

Proposition 1. ( $Z F+H a h n-B a n a c h ~ t h e o r e m) ~ L e t ~\left\langle B_{i}: i \in I\right\rangle$ be a sequence of Boolean algebras (with I not necessarily well-orderable). Then there exists $\left\langle\mu_{i}: i \in I\right\rangle$ such that for each $i \in I, \mu_{i}$ is a measure on $B_{i}$.

Proof. Let $\left(B, e_{i}\right)_{i \in I}$ be the direct sum of $\left(B_{i}\right)_{i \in I}$ in the category of Boolean algebras: so, for every $i \in I, e_{i}$ is an homomorphism $B_{i} \rightarrow B$ (elements of $B$ are formal Boolean combinations of elements of the $B_{i}$ with no other relations than those from the $B_{i}$; one can prove that $e_{i}$ is one-to-one). By the Hahn-Banach theorem there is a measure $\mu$ on $B$. Put $\mu_{i}=\mu \circ e_{i}$.

Definition. A universally measured space is an ordered pair $(\Omega, \mu)$ where $\Omega$ is a set and $\mu$ is a measure on the Boolean algebra $\mathcal{P}(\Omega)$. A group $G$ is said to act by measure preserving transformations on $(\Omega, \mu)$ when $G$ acts on $\Omega$ and $\mu(g A)=\mu(A)$ for all $g \in G$ and $A \in \mathcal{P}(\Omega)$.

We are going to be mainly concerned about the following measure existence statement:
Definition. Let a group $G$ act on a set $\Omega . \operatorname{IM}(\Omega, G)$ is the statement "there is a $G$ invariant measure on $\mathcal{P}(\Omega)$ ".

In the case of a group acting on itself, we get the following classical definition.
Definition. A group $G$ is amenable when there is a measure $\mu$ on $\mathcal{P}(G)$ such that $\mu(A g)=$ $\mu(A)$ for all $g \in G, A \in \mathcal{P}(G)$.

Assuming the Hahn-Banach theorem many groups are amenable, including finite groups, solvable groups and their extensions. The best known non-amenable group is the free group on two generators.

Proposition 2. (Classical) [14] - The free group on two generators, $F_{2}$, is not amenable.
For all integers $n \geq 1$, denote by $O_{n}$ the isometry group of $S^{n-1}$ (with Euclidian norm), $S O_{n}=\left\{u \in O_{n}: \operatorname{det}(u)=+1\right\}$, where $S^{n}=\left\{x \in \mathbb{R}^{n+1}:\|x\|=1\right\}$ is the $n$ dimensional Euclidian sphere. One can prove in ZFC that $I M\left(S^{n}, S O_{n+1}\right)$ does not hold for $n \geq 2$, and thus $S O_{n+1}$ is not amenable (see [14]). On the other hand, in [10] and [13], the authors construct models of $Z F+D C$ in which $\operatorname{IM}\left(S^{n}, O_{n+1}\right)$ holds for every $n \geq 1$ (in [13], the measure is just normalized Lebesgue-measure).

A group $G$ acts on a set $\Omega$ freely when for all $g \in G, x \in \Omega, g x=x$ implies $g=1$.

## §2. The main results.

We start with a classical result.
Proposition 3. Assume $\operatorname{IM}\left(S^{2}, S O_{3}\right)$. Then there is a free measure-preserving action of $F_{2}$ on some universally measured space $(\Omega, \mu)$.

Proof. Consider a subgroup of $S O_{3}$ isomorphic to $F_{2},[14]$ and $D$ the subset of $S^{2}$ consisting of the union of all the possible orbits of fixed points of elements of $F_{2} \backslash\{1\}$. $D$ is countable since each orbit is effectively countable and it is easy to distinguish fixed points of elements of $F_{2}$ acting on $S^{2}$. Hence $D$ is the image of a function with domain $\{0,1\} \times F_{2} \times F_{2}$. (Recall, we do not know that a countable union of countable sets is
countable.) Let $\mu$ be the witness to $I M\left(S^{2}, S O_{3}\right)$. Since $F_{2}$ acts freely on $S^{2} \backslash D$, we will be done if we can show $\mu(D)=0$.

In [14] it is shown that every $\mathrm{SO}_{3}$-invariant finitely additive measure on $S^{2}$ gives each countable set measure zero. We paraphrase the proof given there and check that it works without AC.

It clearly suffices to find a rotation $g$ such that for all $k \in \omega \backslash\{0\}, g^{k} D \cap D=\emptyset$. Since then $\left\{g^{k} D: k \in \omega\right\}$ is an infinite collection of pairwise disjoint subsets of $S^{2}$ of the same $\mu$-measure. Let $\left\langle a_{n}: n \in \omega\right\rangle$ be an enumeration of $D$. Let $\ell$ be a line through the origin missing $D$. Let $A_{n}=\{g \in S O(3): g$ is a rotation about $\ell$ and for some $i \neq j \in \omega$, $\left.g^{n} a_{i}=a_{j}\right\}$. Then $A_{n}$ is countable in a canonical way, since each $g \in A_{n}$ is determined by $a_{i}$ and $a_{j}$. Hence $\cup A_{n}$ is countable. Choose a rotation $g$ about $\ell$ such that $g \notin \cup A_{n}$ and $g$ has infinite order. Then for all $n \geq 1, g^{n} D \cap D=\emptyset$.

Another example is with $I M\left({ }^{\omega} 2, G\right)$ where ${ }^{\omega} 2$ is the Cantor space with its canonical metric and $G$ its group isometries (see [12]).

Our main theorem is:
Theorem 4. (ZF+Hahn-Banach) - Let a group $G$ act freely and measure-preserving on a universally measured space $(\Omega, \mu)$. Then $G$ is amenable.

Proof. (Note the similarity to [6].)
Denote by $\Omega / G$ the set of orbits of $\Omega$ modulo $G$.
By Proposition 1, there is a sequence $\left\langle\mu_{[x]}:[x] \in \Omega / G\right\rangle$ such that for each $[x] \in \Omega / G$, $\mu_{[x]}$ is a measure on $\mathcal{P}([x])$. For each $A \subseteq G$, let $a: \Omega \rightarrow[0,1]$ be the following function: $a(x)=\mu_{[x]}(A x)$; define $\lambda: \mathcal{P}(G) \rightarrow[0,1]$ by $\lambda(A)=\int a(x) d \mu(x)$. Note that $x \mapsto a(x)$ is a measurable function since $(\Omega, \mu)$ is a universally measured space; the integration here is essentially Lebesgue integration, and it does not appeal to any choice (no limit theorems are needed).

We claim that $\lambda$ is a measure on $\mathcal{P}(G)$, invariant under right translaton.
Note that $\lambda(G)=1$. If $A, B$ are two disjoint subsets of $G$ and $a, b, c$ are the functions corresponding to $A, B, A \cup B$ respectively, then $(\forall x \in \Omega)(c(x)=a(x)+b(x))$. Hence $\lambda(A \cup B)=\lambda(A)+\lambda(B)$.

Finally, if $B=A g$ for some $g \in G$ and $a, b$ are the functions corresponding to $A$ and $B$ then, for all $x \in \Omega$,

$$
\begin{aligned}
b(x) & =\mu_{[x]}(B x)=\mu_{[x]}(A g x) \\
& =\mu_{[x]}(A(g x))=\mu_{[g x]}(A(g x))=a(g x) .
\end{aligned}
$$

Hence $\lambda(B)=\int b(x) d \mu(x)=\int a(g x) d \mu(x)=\int a(x) d \mu(x)=\lambda(A)$ since $g$ is $\mu$ measure preserving.

Corollary 1. - ZF+Hahn-Banach implies not $\operatorname{IM}\left(S^{2}, S O_{3}\right)$. Thus, there is a nonLebesgue measurable subset of $S^{2}$.

Proof. Propositions 2, 3 and Theorem 4.

Note that in the last part of the statement above, $S^{2}$ could be replaced by many other spaces, like $\mathbb{R}^{n}, n \geq 1$. (See $\S 3$ for details).
Corollary 2. If $H$ is generic for the partial ordering adding $\omega_{1}$ random reals to a model $V$ of $Z F C$ and $V(\mathbb{R})$ is the smallest model of set theory containing $V$ and reals of $V[H]$, then $V(\mathbb{R})$ does not satisfy the Hahn-Banach theorem.

Proof. $V(\mathbb{R})$ is the model considered considered by D. Pincus and R. Solovay in [10]. It satisfies $I M\left(S^{n}, S O_{n+1}\right)$ for all $n \geq 1$, and thus $I M\left(S^{2}, S O_{3}\right)$; we conclude by Corollary 1.

Another way to see Corollary 1 is the following:
Corollary 3. If $F_{2}$ acts freely on $\Omega=S^{2} \backslash D$ ( $D$ as in the proof of Proposition 3) by rotations, and if $\left\langle\mu_{[x]}:[x] \in \Omega / F_{2}\right\rangle$ is any assignment of finitely additive probability measures $\mu_{[x]}$ on $\mathcal{P}([x])$, then there are $A \subseteq F_{2}$ and $\alpha \in[0,1]$ such that $\left\{x: \mu_{[x]}(A x)<\alpha\right\}$ is not Lebesgue measurable. Further the set $A$ can be isolated explicitly (see [14]).

## §3. Appendix. Lebesgue measure without countable choice.

Ordinarily, the theory of Lebesgue measure is developed with use of $A C_{\omega}$. The use of $A C_{\omega}$ allows one to use arbitrary Borel sets. In this section we explore how to use "coded" Borel sets to eliminate the necessity of $A C_{\omega}$ in many applications. For example, we would still like the existence of non-measurable set to be independent from the reference space ( here, $S^{2}$ ). The aim of this section is to show how to adapt the proofs of the "classical" theory (with $A C_{\omega}$ ) to the study of Lebesgue-measure in a totally choiceless context. The ideas here date from [13].

In order to get as many measurable sets as possible, the classical outer measure construction (see [4]) seems convenient enough. This construction, which we will sketch in $\mathbb{R}$, works as well in $\mathbb{R}^{n}$ or in much more abstract spaces.

Define the outer measure of $A \subseteq \mathbb{R}$ by the greatest lower bound of all sums $\sum_{n \in \omega} \operatorname{length}\left(I_{n}\right)$ where $I_{n}$ are intervals, and $A \subseteq \bigcup_{n \in \omega} I_{n}$; call it $\mu^{*}(A)$. Say that $A$ is Lebesgue-measurable when for all $X \subseteq \mathbb{R}, \mu^{*}(X)=\mu^{*}(X \cap A)+\mu^{*}(X \backslash A)$. Note $\mathcal{M}=\{A \subseteq \mathbb{R} ; A$ is Lebesgue-measurable $\}, \mu=\mu^{*} \mid \mathcal{M}$. It is still possible to prove that $\mathcal{M}$ is a Boolean subalgebra of $\mathcal{P}(\mathbb{R})$ and that $\mu$ is a finitely additive function $\mathcal{M} \rightarrow[0, \infty]$, and that $\mathcal{M}$ contains all open sets. But one cannot prove any more that $\mathcal{M}$ is a $\sigma$-algebra (since $\mathbb{R}$ can be a countable union of countable sets, see [5]). So, instead of considering Borel subsets of $\mathbb{R}$, consider those which have a code, as e.g. in [12]; a Borel code is essentially a real, encoding the "construction" of some Borel set. Similarly, say that $\left(A_{n}\right)_{n \in \omega}$ is coded sequence of Borel sets when there is a sequence $\left(c_{n}\right)_{n \in \omega}$ such that for every $n, c_{n}$ is a code for $A_{n}$. And then, we can prove the following properties of $(\mu, \mathcal{M})$ :
(a) $\mathcal{M}$ is Boolean subalgebra of $\mathcal{P}(\mathbb{R})$, containing all coded Borel subsets of $\mathbb{R}$.
(b) $\mu$ is a finitely additive map $\mathcal{M} \rightarrow[0, \infty]$, and whenever $\left(A_{n}\right)_{n \in \omega}$ is a disjoint coded sequence of Borel sets, we have:

$$
\mu\left(\bigcup_{n \in \omega} A_{n}\right)=\sum_{n \in \omega} \mu\left(A_{n}\right) .
$$

(c) A subset $A \subseteq \mathbb{R}$ is in $\mathcal{M}$ iff for all $\varepsilon>0$ and all coded Borel $B$ with $\mu(B)<\infty$, there are coded Borel $F$ and $U$ such that $F \subseteq A \cap B \subseteq U$ and $\mu(U \backslash F)<\varepsilon$.
(Actually, it is enough to check when $B$ is a bounded interval, and $U$ can be chosen as an open set, $F$ as a closed set.)
(d) $\mu$ is $\sigma$-finite: there is a coded sequence $\left(A_{n}\right)_{n \in \omega}$ of Borel sets such that $\mathbb{R}=\bigcup_{n \in \omega} A_{n}$ and $(\forall n \in \omega)\left(\mu\left(A_{n}\right)<\infty\right)$. (Take $A_{n}=[-n, n]$.)

The precautions needed by elimination of $A C_{\omega}$ in the classical proof of (a) and (d) above (see [4]) make the proof somewhat more lengthy, but without real difficulties. Note that in (c), the assumption $\mu(B)<\infty$ does not seem to be removable without countable choice.

Let us call the $\mu$ above the Lebesgue measure on $\mathbb{R}$; a similar construction yields Lebesgue measure on $\mathbb{R}^{n}$, for all $n \geq 1$.

More generally, let us set the following definition:
Definition. A coded Borel space is an ordered pair $(\Omega, \mathcal{B})$ where $\Omega$ is a coded Borel subset of the Hilbert cube ${ }^{\omega}[0,1]$ and $\mathcal{B}$ is the algebra of coded Borel subsets of $\Omega$.

We can naturally extend this definition by taking all isomorphic images; this way, all usual spaces of analysis - like $\mathbb{R}^{n}, S^{n}$, or ${ }^{\omega} 2$, together with their coded Borel subsets, become coded Borel spaces. Anyway, even without using countable choice, it turns out that the following is true:

Proposition 5. Let $(\Omega, \mathcal{B})$ be an uncountable coded Borel space. Then there is a coded Borel isomorphism from $(\Omega, \mathcal{B})$ onto $\left(I, \mathcal{B}_{I}\right)$, where $I=[0,1]$ and $\mathcal{B}_{I}$ is the algebra of coded Borel subsets of I.

Here, a coded Borel isomorphism $(\Omega, \mathcal{B}) \rightarrow\left(I, \mathcal{B}_{I}\right)$ is naturally a bijection $f: \Omega \rightarrow I$ such that the neighborhood diagrams of $f$ and $f^{-1}$ are coded Borel.

Now, let us give the new definition of measure we are going to use:
Definition. Let $(\Omega, \mathcal{B})$ be a coded Borel space. A regular measure on $(\Omega, \mathcal{B})$ is a map $\mu: \mathcal{M} \rightarrow[0, \infty]$ such that $(\mu, \mathcal{M})$ satisfies conditions (a) to (d) above, with $\Omega$ instead of $\mathbb{R}$. Say that $\mu$ is nonatomic when $(\forall x \in \Omega)(\mu(\{x\})=0)$.

The essential isomorphism theorem between these measure spaces is still valid (after a suitable reformation). It can be stated the following way:

Proposition 6. Let $\mu$ be a regular, nonatomic measure on a coded Borel space $(\Omega, \mathcal{B})$, with $\mu(\Omega)=1$. Then there are $N \subseteq \Omega, D \subseteq[0,1]$ and $f: \Omega \rightarrow[0,1]$ such that, if $\ell$ is Lebesgue measure on $[0,1]$,
(i) $N \in \mathcal{B}, D$ is countable, $\mu(N)=\ell(D)=0$.
(ii) $f$ is a coded Borel isomorphism $\Omega \backslash N \rightarrow[0,1] \backslash D$.
(iii) For all $B$ in $\mathcal{B}, f[B]$ is coded Borel in $[0,1]$ and $\mu(B)=\ell(f[B])$.

Outline of Proof (See [11]). First, notice that by (b) and $\mu(\Omega)=1, \Omega$ is uncountable. So, by proposition 5 , without loss of generality, $\Omega=[0,1]$ and $\mathcal{B}$ is the algebra of coded Borel subsets of $[0,1]$. Then, define $f:[0,1] \rightarrow[0,1]$ by $f(x)=\mu([0, x])$. Then, $D$ is just
$\left\{y \in[0,1]: f^{-1}\{y\}\right.$ has nonempty interior $\}$ and $N$ is $f^{-1}[D]$. (iii) is proven by induction on a code of $B$, and it uses nonatomicity of $\mu$.

Now, Proposition 6 has an immediate corollary:
Corollary 1. Let $\mu$ be a regular, nonatomic measure on a coded Borel space ( $\Omega, \mathcal{B}$ ), with $\mu(\Omega) \neq 0$. Then the following are equivalent:
(i) Every subset of $\Omega$ is $\mu$-measurable.
(ii) Every subset of $[0,1]$ is Lebesgue-measurable.
(To prove (i) $\Rightarrow$ (ii), one has to use $\sigma$-finiteness, nonatomicity of $\mu$ and $\mu(\Omega) \neq 0$; for (ii) $\Rightarrow$ (i), use characterisation (c) above of $\mu$-measurability).

In particular, every subset of $\mathbb{R}^{n}(n \geq 1)$ is Lebesgue-measurable iff every subset of $[0,1]$ is Lebesgue-measurable (which is well-known in the classical theory using countable choice). Let $L M$ be the latter statement.

Now, define Lebesgue measure $v_{n}$ on $S^{n}$ as being the image under $x \mapsto \frac{x}{\|x\|}$ of Lebesgue measure on $B^{n+1} \backslash\{0\}$, where $B^{n+1}$ is the Euclidian closed ball of $\mathbb{R}^{n+1}$ of volume 1.
Corollary 2. LM implies $I M\left(S^{n}, S O_{n+1}\right)$ for all $n \geq 1$.
Proof. If $L M$ holds, then $v_{n}$ is defined on $\mathcal{P}\left(S^{n}\right)$ by the previous corollary; so $v_{n}$ witnesses $\operatorname{IM}\left(S^{n}, S O_{n+1}\right)$.

More precisely, the result would be the same with a rotation-invariant extension of Lebesgue-measure on $\mathcal{P}\left(S^{2}\right)$; thus, the results of the previous paragraph imply for example that Hahn-Banach theorem implies nonexistence of a rotation-invariant extension of Lebesgue-measure to a (finitely additive) measure on $\mathcal{P}\left(\mathbb{R}^{3}\right)$.

Further notes. Theorem 4 could be formulated as follows: "If $G$ is a nonamenable group acting freely on a set $\Omega$ and if $\mu$ is a $G$-invariant finitely additive probability measure defined on a $G$-invariant subalgebra of $P(\Omega)$, then $\Omega$ has non-measurable subsets (w.r.t. $\mu$ )". Now, while this paper was printed, the second author showed, under the same hypotheses, that in the $G$-equidecomposability type semigroup of $\Omega$ (see [14]), $n[\Omega]=(n+1)[\Omega]$ for some integer $n$, effectively computable from the number of pieces necessary to a paradoxical decomposition of $G$. For the action of $F_{2}$ described above, we can get $n=5$, which is somewhat disappointing since it is not known whether the cancellation law (see [14]) follows from HB (it follows from BPI). But independently, J. Pawlikowski proved using ideas from this paper, that one can actually take $n=1$, that is, $[\Omega]=2[\Omega]$; thus, HB implies the Banach-Tarski paradox. See [8] for more details.

## BIBLIOGRAPHY

[1] M. Foreman, "Amenable groups acting on the natural numbers, an independence result". Bulletin AMS, October 1989, vol. 21, N ${ }^{\circ} 2$.
[2] K. Gödel, "The consistency of the axiom of choice and the generalized continuum hypothesis with the axioms of set theory". Ann. of Math. Studies N ${ }^{\circ}$ 3. Princeton University Press, 1940.
[3] F. Greenleaf, "Invariant means on Topological Groups". Van Nostrand, Reinhold Company, 1969.
[4] P. R. Halmos, "Measure Theory". Van Nostrand, Rienhold Company, 1950.
[5] T. Jech, "The axiom of choice". North-Holland Publishing Company.
[6] A. Kechris, "Amenable equivalence relations and Turing degrees". Handwritten notes, 1988.
[7] W. A. J. Luxemburg, "Reduced products of the real number system". Applications of model theory to algebra, analysis and probability. Holt, Reinhart and Winston, 1969.
[8] J. Pawlikowski, "The Hahn-Banach Theorem implies the Banach-Tarski Paradox". To appear.
[9] D. Pincus, "The strength of Hahn-Banach's Theorem". Victoria symposium on non-standard analysis. Springer lecture notes n ${ }^{\circ}$ 369, 1974 (pp. 203-248).
[10] D. Pincus and R. Solovay, "Definability of measures and ultrafilters". Journal of Symbolic Logic 42, n ${ }^{\circ}$ 2, 1977, (pp. 179-190).
[11] H. L. Royden, "Real Analysis". The Macmillan Compnay, New York, 1968.
[12] T. Slaman and J. Steel, "Definable functions on Degrees". Cabal seminar 1981-85, appeared in 1988, Springer Verlag lecture notes.
[13] R. Solovay, "A model of set theory in which every set of reals in Lebesguemeasurable". Annals of Math. 1970, (pp. 1-56).
[14] S. Wagon, "The Banach-Tarski Paradox". Cambridge University Press, 1984.
[15] J. Conway, "A course in Functional Analysis". Springer-Verlag, 1985.

