



# Injective positively ordered monoids II

Friedrich Wehrung

## ► To cite this version:

Friedrich Wehrung. Injective positively ordered monoids II. Journal of Pure and Applied Algebra, 1992, 83 (1), pp.83–100. 10.1016/0022-4049(92)90105-O . hal-00004712

**HAL Id: hal-00004712**

**<https://hal.science/hal-00004712>**

Submitted on 14 Apr 2005

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Injective positively ordered monoids II

Friedrich Wehrung

Université de Caen

département de Mathématiques

14032 CAEN CEDEX, FRANCE

## ABSTRACT.

We continue in this paper the study of positively ordered monoids (P.O.M.'s) initiated in [39]. We prove that injective P.O.M.'s are the retracts of the powers of  $\overline{\mathbb{P}}=[0, \infty]$ . We also characterize the natural P.O.M.-homomorphism from a given refinement P.O.M. to its bidual, with *e.g.* applications to decomposition spaces. As another application, we prove that a refinement P.O.M. admits a ‘Banach limit’ if and only if it embeds into a power of  $\overline{\mathbb{P}}$ .

## INTRODUCTION

In [39], we have given an ‘arithmetical’ characterization of injective objects in the category of ‘positively ordered monoids’ (from now on P.O.M.'s), equipped with its natural notion of embedding, and sketched the general arithmetical structure of these objects and some weaker structures, as *e.g.* refinement P.O.M.'s, strong refinement P.O.M.'s, refinement  $\ell$ -P.O.M.'s, complete P.O.M.'s, one of the main differences with most similar theories being the lack of additive cancellation property. Injective P.O.M.'s appeared there as ‘ideal objects’, rather amenable on the ‘computational’ point of view. In particular, all of them are divisible weak cardinal algebras (*see also* [31], [32]). Here, we shall start from an arbitrary P.O.M.  $A$ , and give embedding criteria of  $A$  into an injective P.O.M. — if this is possible, then we shall say that  $A$  is *regular*. As it turns out, many P.O.M.'s are not regular, unlike what happens for abelian groups or Boolean algebras (there are ‘not enough injective P.O.M.'s’); one of the most noticeable exceptions is the case of *equidecomposability types* P.O.M.'s — *see* [13], [17], [26], [27], [35], [36], [37] for examples: most of the time, these are not regular. Furthermore, the study of these spaces needs ordinary rather *ad hoc* ‘geometric’ techniques. We will be able to use our algebraic approach to give some non trivial general information about them (corollary 2.15).

On the other hand, many P.O.M.'s are regular, as *e.g.* weak cardinal algebras (this has been proved independantly in [32]), or the P.O.M.'s  $K(A)$  introduced here (just before 2.10). The most fundamental regular P.O.M. is  $\overline{\mathbb{P}} = [0, \infty]$ , as it turns out that injective P.O.M.'s are exactly the retracts of the powers of  $\overline{\mathbb{P}}$  (corollary 1.6) — the usual way to express it is by saying that  $\overline{\mathbb{P}}$  is a *cogenerator* of the class of regular P.O.M.'s.

We show also a strange connection between two rather different notions, the notion of injectivity and the notion of *Banach limit*. A Banach limit over a P.O.M.  $A$  is by definition a shift-invariant P.O.M.-homomorphism from the P.O.M.  $\mathcal{B}(A)$  of  $A$ -bounded sequences of elements of  $A$  to some P.O.M. containing  $A$ , sending for all  $a$  in  $A$  the constant sequence with value  $a$  on  $a$ . It is well-known (*see e.g.* [19]) that there are Banach limits over the positive reals. It is then not difficult to prove that there is a Banach limit over every regular P.O.M.. The converse is not true (even for positive cones of abelian ordered groups), as we show it with example 1.14. But for *refinement P.O.M.'s* ([39], definition 1.6), we could prove that existence of a Banach limit is *equivalent* with regularity (corollary 2.18). This is done by elaborating for refinement P.O.M.'s a new regularity criterion (theorem 2.16) which is much more wieldy than the general one (theorem 1.5). In particular, this criterion implies immediately the main result of [32].

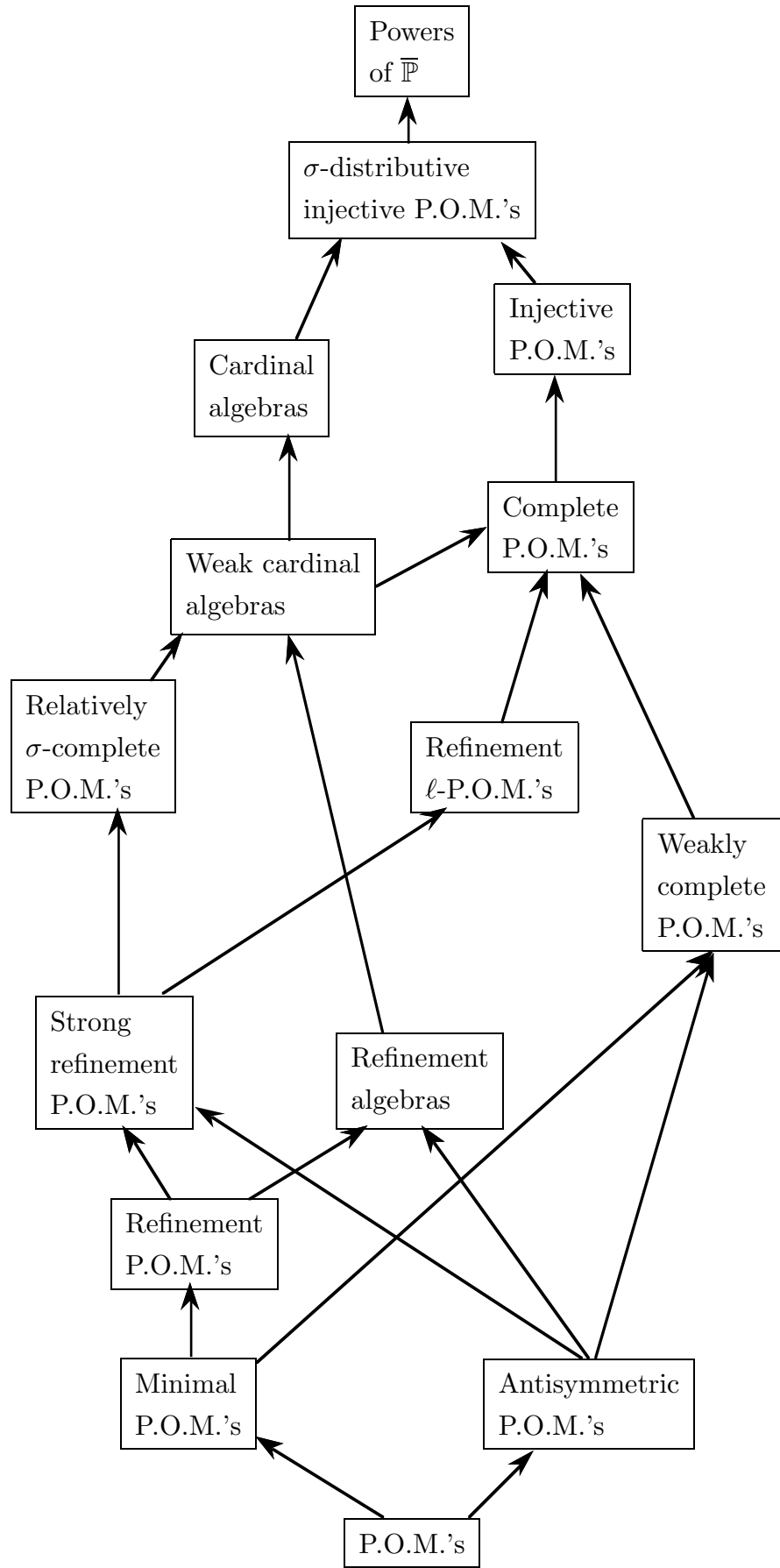
We now summarize the organization of our paper.

— In chapter 1, we characterize the natural evaluation map from a given P.O.M. to its bidual (theorem 1.2), and we deduce from it a first regularity criterion (theorem 1.5). We observe that there are always Banach limits over regular P.O.M.'s, and we give a partial converse (proposition 1.13) and counterexamples.

— In chapter 2, we show that many of the statements proved by Tarski in [35] for cardinal algebras are in fact ‘approximately true’ in *all* refinement P.O.M.'s (2.1 to 2.9). These results are unified by theorem 2.12. This theorem is proved *via* the introduction of a *functor*, denoted by  $K$ , from the category of refinement P.O.M.'s to itself. For every refinement P.O.M.  $A$ ,  $K(A)$  is roughly speaking the P.O.M. of ‘limits’ of all sequences of elements of  $A$ , subjected to no other relations than the ones ensuring that  $K(A)$  embeds into a power of  $\overline{\mathbb{P}}$ . Thus the universal sentences true in  $K(A)$  are those which are ‘approximately true’ in  $A$ .

This allows us to characterize the evaluation map of a given refinement P.O.M in a much more convenient fashion than the one given in theorem 1.2 (theorem 2.14). We deduce applications to equidecomposability types P.O.M.'s (corollary 2.15), a new regularity criterion (theorem 2.16), the existence of Banach limits (corollary 2.18).

We shall use throughout this paper the notations and definitions used in [39]. We recall on the following picture the different classes of P.O.M.'s used in [39] and in this work. As in [39], an arrow from a class  $A$  to a class  $B$  indicates strict inclusion of  $B$  into  $A$ . We also recall that this diagram is *complete*, *i.e.* that its transitive closure shows exactly all the inclusion relations between the considered classes (*see* [39]).



## 1. COGENERATING INJECTIVE P.O.M.'s.

We shall show here that  $\overline{\mathbb{P}}$  is the appropriate pivot around which can be built a global theory of duality for P.O.M.'s (there are more general pivots in the case where we put restrictions on the class of P.O.M.'s in which we are working; this will appear in a forthcoming paper).

**1.1. Definition.** Let  $A$  be a P.O.M.. The *dual* of  $A$  is  $\text{Hom}(A, \overline{\mathbb{P}})$ , denoted by  $A^*$ ; the *bidual* of  $A$  is  $(A^*)^* = A^{**}$ . We denote by  $(x \mapsto x^{**})$  the natural homomorphism  $A \rightarrow A^{**}$  (i.e.  $x^{**} : u \mapsto u(x)$ ).

Call an element  $a$  of a P.O.M.  $A$  *singular* when  $(\exists n \in \mathbb{N})(n+1)a \leq na)$ . If  $a, b$  are in  $A$ , let  $E(a, b)$  be the following statement:

“For all  $n$  in  $\mathbb{N}$ , there is  $k$  in  $\mathbb{N}$  such that  $n2^k a \leq (n+1)2^k b$ .”

**1.2. Theorem.** Let  $A$  be a P.O.M.,  $a, b$  in  $A$ . Then  $a^{**} \leq b^{**}$  if and only if  $E(a, b)$  holds.

**Proof.** Suppose first that  $E(a, b)$  holds. Let  $n$  in  $\mathbb{N} \setminus \{0\}$ ; thus there exists  $k$  in  $\mathbb{N} \setminus \{0\}$  such that  $nka \leq (n+1)kb$ . Then for all  $u$  in  $A^*$ ,  $nku(a) \leq (n+1)ku(b)$ , thus  $u(a) \leq (1 + 1/n)u(b)$ . This holds for all  $n$ , thus the conclusion follows.

Suppose that  $a^{**} \leq b^{**}$ . This means that  $(\forall u \in A^*)(u(a) \leq u(b))$ , or, by injectivity of  $\overline{\mathbb{P}}$ ,  $(\forall u \in (\mathbb{N}a + \mathbb{N}b)^*)(u(a) \leq u(b))$ .

Moreover, by [39], lemma 3.7 (iii), this is equivalent to

$$(\forall u \in (\mathbb{N}b)^*)(u^*(a) \leq u(b)) \quad (*)$$

by using the notation of [39], 3.7.

**Claim 1.**  $a \in A|b$ .

**Proof of claim.** Let  $(u : \mathbb{N}b \rightarrow \overline{\mathbb{P}}, x \mapsto 0)$ . Then  $u \in (\mathbb{N}b)^*$ , and it is easy to see that  $u^*(a)$  is equal to 0 if  $a \in A|b$ ,  $\infty$  otherwise. However, by (\*), only the first possibility is acceptable. ■ Claim 1.

**Claim 2.** For all  $\varepsilon > 0$ , there are  $p, m, n$  in  $\mathbb{N}$  such that  $n > 0$ ,  $m \leq \varepsilon n$  and  $pb + na \leq (p + m + n)b$ .

**Proof of claim.** Suppose first that  $b$  is not singular. Thus we can define  $(u : \mathbb{N}b \rightarrow \overline{\mathbb{P}}, nb \mapsto n)$ , and  $u \in (\mathbb{N}b)^*$ . It is easy to see that

$$u^*(a) = \bigwedge \left\{ \frac{q-p}{n} : p, q, n \in \mathbb{N} \text{ and } n > 0 \text{ and } pb + na \leq qb \right\}.$$

By (\*),  $u^*(a) \leq 1$ , which easily implies the conclusion of the claim.

Now, suppose that  $b$  is singular. Then there are positive integers  $k, l$  such that  $(k+1)b \leq kb$  and  $a \leq lb$  (use claim 1). Now, let  $\varepsilon > 0$ ; let  $m = 0$ ,  $n = 1$  and  $p \in \mathbb{N}$  such that  $p \geq k, nl$ ; thus  $2pb \leq pb$ . So  $n > 0$ ,  $m \leq n\varepsilon$ , and

$$2pb + na \leq 2pb + nlb \leq pb + nlb \leq (p+p)b \leq (2p+m+n)b.$$

Hence, the conclusion of the claim holds again. ■ Claim 2.

**Claim 3.** For all  $n$  in  $\mathbb{N}$ , there is  $k$  in  $\mathbb{N} \setminus \{0\}$  such that  $nka \leq (n+1)kb$ .

**Proof of claim.** Let  $n$  in  $\mathbb{N}$ . If  $n = 0$  then take  $k = 1$ , so suppose  $n \neq 0$ . Using claim 2 for  $\varepsilon = 1/2n$ , we see that there are  $p, m, q$  in  $\mathbb{N}$  such that  $m \leq q/2n$  and  $q > 0$  and  $pb + qa \leq (p+m+q)b$ . Thus for all  $k$  in  $\mathbb{N}$ , we have  $kqa \leq pb + kqa \leq (p+k(m+q))b$ . Multiplying both sides by  $2n$  yields  $2knqa \leq (2np+2knq+kq)b$ . Taking  $k \geq 2np/q$  yields  $(2kq)na \leq (2knq+2kq)b = (2kq)(n+1)b$ . The conclusion follows. ■ Claim 3.

Now let  $n$  in  $\mathbb{N} \setminus \{0\}$ . By claim 3, there exists  $k$  in  $\mathbb{N} \setminus \{0\}$  such that  $2nka \leq (2n+1)kb$ . Let  $(x \mapsto \log x)$  be the logarithm function with basis 2 on  $(0, \infty)$ ; for every real  $x$ , let  $[x]$  be the largest integer  $\leq x$ , and let  $\langle x \rangle = x - [x]$ . Let  $\varepsilon = \log(\frac{2n+2}{2n+1})$ . Since  $\{\langle \log m \rangle : m \in \mathbb{N} \setminus \{0\}\}$  is dense in  $[0, 1]$ , there is  $m$  in  $\mathbb{N} \setminus \{0\}$  such that

$$1 - \varepsilon < \langle \log m + \log(\frac{(2n+1)k}{n+1}) \rangle < 1.$$

Put  $l = [\log m + \log(\frac{(2n+1)k}{n+1})] + 1$ . It follows that

$$\log(\frac{(2n+1)km}{n+1}) < l < \log(\frac{(2n+1)km}{n+1}) + \varepsilon,$$

whence it follows easily that

$$(2n+1)km < (n+1)2^l \text{ and } 2^l n < 2mkn.$$

Thus, we have

$$\begin{aligned} n2^l a &\leq m \cdot (2nk)a \\ &\leq m \cdot ((2n+1)k)b \\ &\leq ((n+1)2^l)b, \end{aligned}$$

which concludes the proof. ■

**1.3. Definition.** A *cone* is the positive cone of an abelian, ordered group. A cone is a *CR-cone* when it satisfies the multiplicative  $\leq$ -cancellation property and the finite refinement property.

As the following example shows, one cannot in general bound the  $k$  of the definition of  $E(a, b)$ , even for cones.

**1.4. EXAMPLE.** *A cone in which there are elements  $a$  and  $b$  such that  $a^{**} \leq b^{**}$ , but for all  $k$  in  $\mathbb{N} \setminus \{0\}$  and all  $n \geq k$ ,  $nka \not\leq (n+1)kb$ .*

Let  $(f : \mathbb{R}_+ \rightarrow \mathbb{R}, x \mapsto x - \sqrt{x})$ . It is easy to verify that  $f(x+y) \geq f(x) + f(y)$  for all  $x, y \geq 0$ . Hence, the subset  $A$  of  $\mathbb{R} \times \mathbb{R}$  defined by

$$(x, y) \in A \Leftrightarrow (y \geq 0 \text{ and } x + f(y) \geq 0)$$

is a cone. Put  $a = (1, 0)$  and  $b = (0, 1)$ . Then for all  $n$  in  $\mathbb{N}$ ,  $(n^2 - n)a \leq n^2b$  (thus  $a^{**} \leq b^{**}$ ) but for all  $k$  in  $\mathbb{N} \setminus \{0\}$  and for all  $n \geq k$ ,  $nka \not\leq (n+1)kb$ .

This example is to be put in contrast with the following theorem:

**1.5. Theorem.** *Let  $A$  be a P.O.M.. Then the following are equivalent:*

- (i) *The natural map  $A \rightarrow A^{**}$  is an embedding;*
- (ii)  *$A$  embeds into a power of  $\overline{\mathbb{P}}$ ;*
- (iii)  *$A$  is antisymmetric and satisfies the following statement:*

$$(\forall a, b \in A) \left( (\forall n \in \mathbb{N}) (na \leq (n+1)b) \Rightarrow a \leq b \right);$$

- (iv)  *$A$  embeds into an injective P.O.M..*

**Proof.** (i) $\Leftrightarrow$ (ii), (ii) $\Rightarrow$ (iii) are trivial. (iii) $\Rightarrow$ (i) Suppose that  $A$  satisfies (iii). For all  $a, b$  in  $A$  such that  $2a \leq 2b$ , it is easy to see that  $na \leq (n+1)b$  for all  $n$  in  $\mathbb{N}$ , whence  $a \leq b$  by assumption. Thus  $A$  satisfies the multiplicative 2- $\leq$ -cancellation property. Using theorem 1.2, (i) follows. (ii) $\Rightarrow$ (iv) is trivial since powers of  $\overline{\mathbb{P}}$  are injective. Finally, to prove that (iv) $\Rightarrow$ (iii), it is sufficient to prove that every injective P.O.M.  $E$  satisfies (iii). First,  $E$  is antisymmetric by [39], 3.11. Next, let  $a, b$  in  $E$  such that  $na \leq (n+1)b$  for all  $n$ . For all  $n$  in  $\mathbb{N} \setminus \{0\}$ , the multiplication by  $n$  is an automorphism of  $E$  ([39], 3.11), and thus we obtain  $a \leq b + (1/n)b$ ; hence (using completeness of  $E$  — [39], definition 2.15), if  $c = \bigwedge \{(1/n)b : n \in \mathbb{N} \setminus \{0\}\}$ , then  $a \leq b + c$ . But  $c \ll b$  by [39], 2.12, whence  $a \leq b$ . The conclusion follows. ■

If  $A$  satisfies one of the conditions (i) to (iv), we will say that  $A$  is *regular*. The embedding of a regular P.O.M. into its bidual yields a natural metric structure on this P.O.M., studied in [38]. By (ii), the class of regular (or injective) P.O.M.'s is *cogenerated* by  $\overline{\mathbb{P}}$ , i.e. every regular (or injective) P.O.M. embeds into a power of  $\overline{\mathbb{P}}$ . This yields easily another characterization of injective P.O.M.'s:

**1.6. Corollary.** *A P.O.M. is injective if and only if it is a retract of a power of  $\overline{\mathbb{P}}$ .* ■

**1.7. EXAMPLE.** The space of all Lebesgue-measurable functions from  $[0, 1]$  to  $\overline{\mathbb{P}}$  modulo null sets is an injective P.O.M.. So is  $\overline{\mathbb{P}}^{\mathbb{P}}$  (see [39], 3.13), but none is a product of copies of  $\overline{\mathbb{P}}$  or **2**.

Theorem 1.5 has also another consequence about partially ordered abelian groups:

**1.8. Corollary.** *Let  $A$  be an abelian, directed partially ordered group. Then  $A$  embeds into a complete  $\ell$ -group if and only if  $A_+$  embeds into an injective P.O.M..*

**Proof.** It is shown in [1] that  $A$  embeds into a complete  $\ell$ -group if and only if it satisfies  $(\forall a, b)((\forall n \in \mathbb{N})(na \leq b) \Rightarrow a \leq 0)$ . The conclusion follows easily from theorem 1.5. ■

So, corollary 1.8 makes the connection between the embedding criterion of theorem 1.5 and the usual criterion for partially ordered groups.

Theorem 1.5 can also be used for the *refinement  $\ell$ -P.O.M.*'s introduced in [39], definition 1.22; we use the Archimedean property presented in [39], definition 2.1 (*i.e.*  $A$  is Archimedean if and only if for all  $a, b$  in  $A$ ,  $(\forall n \in \mathbb{N})(na \leq b)$  implies  $a + b = b$ ).

**1.9. Corollary.** *A refinement  $\ell$ -P.O.M. is regular if and only if it is Archimedean.*

Note that the corresponding property for abelian  $\ell$ -groups is well-known (see [1]).

**Proof.** Similar as for [39], corollary 1.26, by using [39], 1.17 and 1.25. ■

Now, let  $A$  be a P.O.M.. Put

$$\mathcal{B}(A) = \{(a_n)_n \in A^\omega : (\exists a \in A)(\forall n \in \omega)(a_n \leq a)\},$$

equipped with its natural [componentwise] P.O.M. structure, and let  $s : \mathcal{B}(A) \rightarrow \mathcal{B}(A)$ ,  $(a_n)_n \mapsto (a_{n+1})_n$  be the *shift-endomorphism*. Say that a *Banach limit over  $A$*  is a P.O.M.-homomorphism  $\lambda$  from  $\mathcal{B}(A)$  to some P.O.M.  $B$  containing  $A$  such that for all  $a$  in  $A$ , if  $\bar{a}$  denotes the constant sequence with value  $a$ , then  $\lambda(\bar{a}) = a$ , and  $\lambda(sx) = \lambda(x)$  for all  $x$  in  $\mathcal{B}(A)$ .

**1.10. EXAMPLE.** *A CR-cone without a Banach limit.* Let  $A$  be the P.O.M. of equidecomposability types of polyhedra of  $\mathbb{R}^3$ : only polyhedra are allowed as pieces and subsets of affine subspaces of dimension at most 2 are identified to zero. Let  $a$  (resp.  $b$ ) denote the equidecomposability type of the regular tetrahedron (resp. the cube) of volume 1. It is well-known that  $a \neq b$  (this is Dehn's theorem, the solution of Hilbert's third problem). On the other hand,  $A$  is the positive cone of an abelian, directed ordered group



$G$  (this is a consequence of Zylev's theorem, see [2]). Let  $c$  be an element of  $A \setminus \{0\}$  such that  $c \leq a, b$  (for example,  $c$  is the equidecomposability type of the cube of volume  $\frac{1}{8}$ ). Consider both following elements of  $G^\omega$ :

$$u = (c, a - b + c, 2a - 2b + c, 3a - 3b + c, \dots)$$

$$v = (a - c, b - c, 2b - a - c, 3b - 2a - c, \dots)$$

It is proved in [2] that  $a$  and  $b$  are 'infinitely close' in the following sense:

$$(\forall x \in A \setminus \{0\})(-x \leq a - b \leq x).$$

It follows that *both  $u$  and  $v$  are in  $A^\omega$* . Furthermore, it is immediate that  $u + v = \bar{a}$  and  $u + sv = \bar{b}$ . Hence,

*$\bar{a}$  and  $\bar{b}$  are equidecomposable modulo the monogenic semigroup of transformations of  $A^\omega$  generated by the shift mapping (we will say 'shift-equidecomposable'), although  $a \neq b$  (The same idea actually yields that  $\bar{x} = \bar{y}$  if and only if  $x$  and  $y$  have the same volume).* It follows easily that  $A$  does not have a Banach limit.

The following theorem shows a connection in one direction between regular P.O.M.'s and Banach limits.

**1.11. Theorem.** *Let  $A$  be a regular P.O.M.. Then there is a Banach limit over  $A$ .*

**Proof.** By 1.5 (ii), we may assume without loss of generality that  $A$  is a sub-P.O.M. of a power of  $\bar{\mathbb{P}}$ , say  $\bar{\mathbb{P}}^I$  for some set  $I$ . Since the shift  $(n \mapsto n + 1)$  generates a monogenic, thus amenable semigroup of transformations of  $\omega$ , there is a shift-invariant finitely additive probability measure  $\mu : \mathcal{P}(\omega) \rightarrow [0, 1]$  — see [19]. Define  $\lambda : \mathcal{B}(A) \rightarrow \bar{\mathbb{P}}^I$  by  $\lambda(a) = (\int a_{ni} d\mu(n))_{i \in I}$  if we put  $a = (a_n)_{n \in \omega}$ ,  $a_n = (a_{ni})_{i \in I}$ . It is easy to see that  $\lambda$  is a Banach limit over  $A$ . ■

For arbitrary P.O.M.'s, the following example shows that the converse of this proposition is false:

**1.12. EXAMPLE.** *A non-regular P.O.M. over which there exists a Banach limit.*

Just equip  $[0, \infty]$  with its *coarse* preordering ( $\leq_c = \bar{\mathbb{P}} \times \bar{\mathbb{P}}$ ). Any Banach limit on  $\bar{\mathbb{P}}$  (given by the proof of 1.11) is also a Banach limit over this P.O.M..

Still, one can state and prove a partial converse of theorem 1.11:

**1.13. Proposition.** *Let  $A$  be a P.O.M. over which there exists a Banach limit. Then  $A$  satisfies the multiplicative = - and  $\leq$ -cancellation property. Furthermore, if  $A$  is minimal, then it is antisymmetric.*

**Proof.** Let  $\lambda : \mathcal{B}(A) \rightarrow B$  be a Banach limit over  $A$ . For all  $a$  in  $A$  and  $m$  in  $\mathbb{N} \setminus \{0\}$ , we have

$$a = \lambda((a, a, \dots)) = \lambda((a_0, a_1, \dots))$$

where  $a_n = ma$  when  $m$  divides  $n$ , 0 otherwise. The first statement follows immediately. Now assume that  $A$  is minimal. Let  $a, b$  in  $A$  such that  $a \leq b$  and  $b \leq a$ , so that there are  $u$  and  $v$  in  $A$  such that  $b = a + u$  and  $a = b + v$ . Put  $x = \lambda((nu)_n)$  and  $y = \lambda((nv)_n)$ . Since  $a = a + x + y$  and  $u + x = x$ , we have  $a = a + u = b$ . ■

Strangely, even for minimal P.O.M.'s, the converse of theorem 1.11 is *false* (see definition 1.3):

**1.14. EXAMPLE.** *A non-regular CR-cone over which there exists a Banach limit.*

For every abelian ordered groups  $G$  and  $H$ , define an abelian ordered group  $G \bowtie H$  with underlying set  $G \times H$  and with positive cone  $((G_+ \setminus \{0\}) \times H_+) \cup \{(0, 0)\}$ .

Let  $A$  be the positive cone of  $\mathbb{Z} \bowtie \mathbb{Z}$ . We show that  $A$  satisfies the required conditions. Obviously,  $A$  is a CR-cone.

Let  $p : A \rightarrow \mathbb{N}$  and  $q : A \rightarrow \mathbb{N}$  be respectively the first and the second projection. Thus  $p$  and  $q$  are P.O.M.-homomorphisms. Let  $\mu$  be a shift-invariant positive linear functional on the space of all bounded real sequences such that  $\mu(1) = 1$  (see [19]).

**Claim 1.** *Let  $a = (a_n)_n$  in  $\mathcal{B}(\mathbb{N})$ .*

*Then  $\mu(a) = 0$  if and only if  $\mu(\{n \in \omega : a_n \neq 0\}) = 0$ .*

**Proof of claim.** Let  $X = \{n \in \omega : a_n \neq 0\}$ . If  $1_X$  is the indicator function of  $X$ , then, since  $a$  is  $\mathbb{N}$ -valued and bounded, there is a constant  $C \in \mathbb{N}$  such that  $1_X \leq a \leq C \cdot 1_X$ . The conclusion follows. ■ Claim 1.

**Claim 2.** *Let  $a$  in  $\mathcal{B}(A)$ . Then  $(\mu(p \circ a), \mu(q \circ a)) \in (\mathbb{R} \bowtie \mathbb{R})_+$ .*

**Proof of claim.** Immediate by definition of  $\mathbb{R} \bowtie \mathbb{R}$  and claim 1. ■ Claim 2.

By claim 2, one can define a P.O.M.-homomorphism  $\lambda$  from  $\mathcal{B}(A)$  to  $(\mathbb{R} \bowtie \mathbb{R})_+$  by  $\lambda(a) = (\mu(p \circ a), \mu(q \circ a))$ . Then  $\lambda$  is a Banach limit over  $A$ . On the other hand, put  $a = (1, 0)$  and  $b = (1, 1)$ . Then  $a \not\leq b$  but  $na \leq (n+1)b$  for all  $n$  in  $\mathbb{N}$ , whence  $A$  is not regular. ■

On the other hand, we shall see in the next chapter that the converse of theorem 1.11 is *true* for *refinement P.O.M.'s* (corollary 2.18).

## 2. CASE OF REFINEMENT P.O.M.'s.

In this chapter, the finite refinement property will play a decisive role, in finding for refinement P.O.M.'s a more efficient regularity criterion (theorem 2.16) than the one of theorem 1.5. Our first purpose is the following: given a refinement P.O.M.  $A$ , we try to construct a canonical P.O.M.-homomorphism  $\varepsilon_A$  from  $A$  into some relatively  $\sigma$ -complete P.O.M. (see [39], 2.1) in a canonical way. Of course, we cannot always get an embedding (e.g. if  $A$  does not satisfy the multiplicative  $\leq$ -cancellation property), but if such an embedding is possible, our construction will give one of them (without claiming universality of  $\varepsilon_A$ , which is indeed not the case).

From now on until 2.12, we fix a refinement P.O.M.  $A$ . We define, for all  $n$  in  $\mathbb{N}$ , relations on  $A$  by

$$a \ll_n b \Leftrightarrow (\exists m \in \mathbb{N})(2^{m+n}a \leq 2^m b)$$

$$a \leq_n^c b \Leftrightarrow (\exists d)(a \leq b + d \text{ and } d \ll_n c)$$

$$a \leq_n b \Leftrightarrow a \leq_n^b b, \text{ i.e. } (\exists d)(a \leq b + d \text{ and } d \ll_n c)$$

$$a \equiv_n b \Leftrightarrow (\exists u, v)(u \ll_n a \text{ and } v \ll_n b \text{ and } a + u = b + v)$$

Note that  $a \equiv_n b$  implies  $a \leq_n b$  and  $b \leq_n a$ .

For  $n = -\infty$ , say that the relations above are always satisfied.

From 2.1 to 2.9, our purpose will be to try to prove in  $A$  enough properties of relatively  $\sigma$ -complete P.O.M.'s “modulo  $\equiv_n$ ,  $n$  large enough”.

**2.1. Lemma.** *Let  $a, b, c, d, a_0, a_1, b_0, b_1$  be in  $A$ ,  $m, n$  in  $\mathbb{N}$ . Then*

- (i)  $(a \ll_n b \text{ and } m \leq n) \Rightarrow a \ll_m b$ . Similarly for  $a \leq_n^c b$ ,  $a \leq_n b$ ,  $a \equiv_n b$ .
- (ii)  $2a \ll_n 2b \Leftrightarrow a \ll_n b$ .
- (iii)  $(a_0 \ll_n b_0 \text{ and } a_1 \ll_n b_1) \Rightarrow a_0 + a_1 \ll_n b_0 + b_1$ .
- (iv)  $a \ll_n b \Rightarrow a \leq_n b$ .
- (v)  $((a \ll_{n+1} b \text{ and } b \leq_0 c) \text{ or } (a \leq_0 b \text{ and } b \ll_{n+1} c)) \Rightarrow a \ll_n c$ .
- (vi)  $(a \leq_{n+1}^d b \text{ and } b \leq_{n+1}^d c) \Rightarrow a \leq_n^d c$ .
- (vii)  $a_0 \leq_n^b a_1 \Rightarrow a_0 + c \leq_n^b a_1 + c$ .
- (viii)  $(a \leq_{n+2} b \text{ and } b \leq_{n+1} c) \Rightarrow a \leq_n c$ .
- (ix)  $(a \equiv_{n+2} b \text{ and } b \equiv_{n+2} c) \Rightarrow a \equiv_n c$ .

**Proof.** Easy, left to the reader. ■

**2.2. Lemma.** Let  $a, b, c$  in  $A$ , let  $n$  in  $\mathbb{N}$ .

- (i) If  $a + c = b + c$ , then there are  $u \ll_n a$ ,  $v \ll_n b$  and  $d$  such that  $a = d + u$  and  $b = d + v$ .
- (ii) If  $a + c \leq b + c$ , then there is  $d \ll_n c$  such that  $a \leq b + d$ .

**Proof.** Immediate from [39], 1.11. ■

**2.3. Lemma.** Let  $a, b, c$  in  $A$ ,  $n$  in  $\mathbb{N}$  such that  $2nc \leq a + b$ . Then there are  $a', b'$  such that  $c = a' + b'$  and  $na' \leq a$  and  $nb' \leq b$ .

**Proof.** We can suppose  $n > 0$ . Let  $u, v$  such that  $u \leq a$ ,  $v \leq b$  and  $2nc = u + v$ . By [39], 1.9, there are  $x_k$  ( $k \leq 2n$ ) such that the following holds:

$$u = \sum_{k \leq 2n} kx_k, \quad v = \sum_{k \leq 2n} (2n - k)x_k, \quad c = \sum_{k \leq 2n} x_k.$$

Take  $a' = \sum_{n \leq k \leq 2n} x_k$ ,  $b' = \sum_{k < n} x_k$ . It easy to see that  $a', b'$  satisfy the required conditions. ■

**2.4. Lemma.** Let  $a, b, c$  in  $A$ ,  $n$  in  $\mathbb{N}$  such that  $c \ll_{n+1} a + b$ . Then there are  $a' \ll_n a$ ,  $b' \ll_n b$  such that  $c = a' + b'$ .

**Proof.** Immediate by 2.3. ■

**2.5. Lemma.** Let  $n$  in  $\mathbb{N}$ ,  $a, b, x_0, x_1, h_0, h_1$  in  $A$  such that  $b = x_0 + h_0 = x_1 + h_1$  and  $h_0, h_1 \ll_{n+1} a$ . Then there are  $x, h$  such that  $x \leq x_0, x_1$  and  $b = x + h$  and  $h \ll_n a$ .

**Proof.** Use the finite refinement property. There is a refinement matrix:

	$x_1$	$h_1$
$x_0$	$x$	$p$
$h_0$	$q$	$r$

Put  $h = p + q + r$ . Then  $x$  and  $h$  satisfy the required conditions. ■

**2.6. Lemma.** Let  $n$  in  $\mathbb{N}$ ,  $a, b, x_0, x_1$  in  $A$  such that  $a \leq b \leq_{n+3} a + x_0, a + x_1$ . Then there is  $x$  in  $A$  such that the following holds:

- (i)  $x \leq_{n+2} x_0, x_1$ .
- (ii)  $a + x \leq b \leq_n a + x$ .

**Proof.** Let  $c$  such that  $b = a + c$ . For all  $i < 2$ , there is  $e_i \ll_{n+3} a + x_i$  such that  $b \leq a + x_i + e_i$ . By 2.4, there are  $u_i \ll_{n+2} a$  and  $v_i \ll_{n+2} x_i$  such that  $e_i = u_i + v_i$ . Put  $y_i = x_i + v_i$ , so that we have  $y_i \leq_{n+2} x_i$ . Thus, we have  $a + c \leq a + (u_i + y_i)$  for  $i < 2$ ,

thus, by 2.2, (ii), there is  $w_i \ll_{n+2} a$  such that  $c \leq (u_i + y_i) + w_i$ . Thus  $c = z_i + h_i$  for some  $z_i \leq y_i$ ,  $h_i \leq u_i + w_i$ . Thus  $h_i \ll_{n+1} a$ . By 2.5, there are  $x \leq z_0, z_1$  and  $h \ll_n a$  such that  $c = x + h$ . Then  $x$  satisfies the required conditions. ■

In the proof above, assume the stronger hypothesis  $b \leq a + x_i$ ; then we can take  $u_i = v_i = 0$ , thus  $x \leq x_i$  and  $h_i \ll_{n+2} a$ . Actually, we will need only the following weaker statement:

**2.7. Lemma.** *Let  $n$  in  $\mathbb{N}$ ,  $a, b, c$  in  $A$  such that  $a \leq b \leq a + c$ . Then there is  $d \leq c$  such that  $a + d \leq b \leq_n a + d$ .* ■

Now, we are ready to prove the ‘approximation of the finite interpolation lemma’ (as in [35], 2.28).

**2.8. Lemma.** *Let  $n$  in  $\mathbb{N}$ , let  $a_0, a_1, b_0, b_1$  in  $A$  such that  $a_0, a_1 \leq_{n+1} b_0, b_1$ . Then there is  $c$  in  $A$  such that  $a_0 \leq c$  and  $a_1 \leq_{n+1} c \leq_n b_0, b_1$ .*

**Proof.** Since  $a_0, a_1 \leq_{n+1} b_i$  for  $i = 0, 1$ , we see easily that there are  $d_0, d_1, e_0, e_1$  such that

$$a_0 + d_0 = a_1 + d_1 \equiv_n b_0, \quad (1)$$

$$a_0 + e_0 = a_1 + e_1 \equiv_n b_1. \quad (2)$$

By (1) and the finite refinement property, there are  $r_1, r_2, r_3, r_4$  such that the following is a refinement matrix:

	$a_1$	$d_1$
$a_0$	$r_1$	$r_2$
$d_0$	$r_3$	$r_4$

By (2),  $r_1 \leq a_1 \leq a_0 + e_0 = r_1 + (e_0 + r_2)$ . By lemma 2.7, there is  $s \leq e_0 + r_2$  such that  $r_1 + s \leq a_1 \leq_{n+1} r_1 + s$ . Thus  $s = s_0 + s_1$  for some  $s_0 \leq e_0$ ,  $s_1 \leq r_2$ . Put  $c = a_0 + s_0$ ; it is immediate that  $c$  satisfies the required conditions. ■

**2.9. Lemma.** *Let  $n, k$  in  $\omega$ , let  $a, b, x_i$  ( $i \leq k$ ) in  $A$  such that  $(\forall i \leq k)(b \leq_{n+5k} a + x_i)$ . Then there is  $x$  in  $A$  such that the following holds:*

(i)  $(\forall i \leq k)(x \leq_{n+1} x_i)$ ;

(ii)  $b \leq_n a + x$ .

**Proof.** For  $k = 0$  it is trivial. Suppose  $k = 1$ . So we have  $a, b \leq_{n+5} a + x_0, a + x_1$ , thus there is  $c$  such that  $a \leq c$  and  $b \leq_{n+5} c \leq_{n+4} a + x_0, a + x_1$ . Thus we have  $a \leq c \leq_{n+4}$

$a + x_0, a + x_1$ . By lemma 2.6, there is  $x \leq_{n+3} x_0, x_1$  such that  $a + x \leq c \leq_{n+1} a + x$ . But  $b \leq_{n+2} c$ , thus  $b \leq_n a + x$  by 2.1, (viii). We conclude easily by induction on  $k$ . ■

*Notation.* If  $\varphi(x)$  is any formula with  $x$  as a free variable, we write

$$(\forall^\infty n)\varphi(n) \text{ if and only if } (\exists m \in \omega)(\forall n \in \omega)(n \geq m \Rightarrow \varphi(n)).$$

Now, define a binary relation  $\equiv$  on  $A^\omega$  by

$$a \equiv b \Leftrightarrow (\forall m \in \mathbb{N})(\forall^\infty n)(a_n \equiv_m b_n)$$

(where  $a = (a_n)_n, b = (b_n)_n$ ).

An alternative definition is the following one: let  $\mathcal{F}$  be the set of all increasing maps  $\varphi : \omega \rightarrow \mathbb{N} \cup \{-\infty\}$  such that  $\lim_{n \rightarrow \infty} \varphi(n) = +\infty$ . Then it is easy to see that

$$a \equiv b \Leftrightarrow (\exists \varphi \in \mathcal{F})(\forall n \in \omega)(a_n \equiv_{\varphi(n)} b_n).$$

It is then immediate to verify that  $\equiv$  is an equivalence relation on  $A^\omega$  which is compatible with the addition. We will denote by  $K(A)$  the quotient monoid  $A^\omega / \equiv$ , equipped with its minimal preordering. If  $a = (a_n)_n \in A^\omega$ , then we denote by  $[a] = [a_n]_n$  its equivalence class modulo  $\equiv$ . For every  $a$  in  $A$ , we will denote by  $\varepsilon(a)$  the class of the constant sequence with value  $a$ . Then  $\varepsilon$  is the natural homomorphism from  $A$  to  $K(A)$ . As suggested by the notation,  $K$  is a *functor*. But  $K(A)$  seems to be characterized by no special universal property with respect to  $A$ ; the interest of  $K$  is in fact essentially *technical* (from more than one point of view, *see* [38]): without its introduction, we would have had to state and prove ‘approximative versions’ of several theorems proved in [39], chapter 2, about relatively  $\sigma$ -complete P.O.M.’s, including the difficult [39], proposition 2.9, which would have made this paper considerably longer and less legible (*see* the Introduction).

**2.10. Lemma.**  $K(A)$  satisfies the finite refinement property.

**Proof.** Let  $a, b, a', b'$  in  $A^\omega$  such that  $[a] + [a'] = [b] + [b']$ . By definition, there are  $\varphi \in \mathcal{F}$  and  $(c_n)_n, (d_n)_n \in A^\omega$  such that  $\forall n \in \omega, c_n \ll_{\varphi(n)+1} a_n + a'_n$  and  $d_n \ll_{\varphi(n)+1} b_n + b'_n$  and  $a_n + a'_n + c_n = b_n + b'_n + d_n$ . By 2.4, for all  $n$ , there are  $u_n \ll_{\varphi(n)} a_n, u'_n \ll_{\varphi(n)} a'_n, v_n \ll_{\varphi(n)} b_n, v'_n \ll_{\varphi(n)} b'_n$  such that  $c_n = u_n + u'_n$  and  $d_n = v_n + v'_n$ . Thus, there is a refinement matrix

	$b_n + v_n$	$b'_n + v'_n$
$a_n + u_n$	$p_n$	$q_n$
$a'_n + u'_n$	$r_n$	$s_n$

Hence,  $[a] = [a + u] = [p] + [q]$ , and similarly,  $[a'] = [r] + [s]$ ,  $[b] = [p] + [r]$ ,  $[b'] = [q] + [s]$ . ■

**2.11. Lemma.** *Let  $a, b$  in  $A^\omega$  such that  $(\forall m \in \mathbb{N})(m[a] \leq [b])$ . Then  $[a] \ll [b]$ .*

**Proof.** For all  $m$ , we have  $2^{m+1}a \leq b$ , thus  $(\forall^\infty n)(a_n \ll_m b_n)$ , thus  $(\forall^\infty n)(b_n \equiv_m a_n + b_n)$ . The result follows. ■

**2.12. Theorem.**  *$K(A)$  is a relatively  $\sigma$ -complete P.O.M..*

**Proof.** It remains to prove that part (i) of [39], definition 2.1, holds. However, this can be done by an easy diagonal argument, using the result of lemma 2.9. ■

Now, we shall give an improvement for refinement P.O.M's of theorem 1.2. We will need the following lemma, which is also an improvement of the multiplicative  $\leq$ -cancellation property.

**2.13. Proposition.** *Let  $A$  be a strong refinement P.O.M. satisfying the multiplicative  $\leq$ -cancellation property. Let  $m, n$  in  $\mathbb{N}$  such that  $m \leq n$ , let  $a, b$  in  $A$  such that  $ma \leq nb$ . Then there is  $c$  in  $A$  such that  $a \leq b + c$  and  $mc \leq (n - m)b$ .*

**Proof.** For  $m = n$  it is trivial, so suppose that  $m < n$ . So  $ma, mb \leq nb$ , thus, by [35], theorem 2.39 (valid in the finite case for strong refinement P.O.M.'s with multiplicative cancellation, with the same proof), there is  $d \in A$  such that  $a, b \leq d$  and  $md \leq nb$ . Let  $c$  such that  $d = b + c$ . By pseudo-cancellation,  $mc \leq (n - m)b$ , so that  $c$  satisfies the required conditions. ■

Before stating theorem 2.14, recall the definitions of the statements  $a \ll_n b$ ,  $a \leq_n b$  seen at the beginning of this chapter and  $E(a, b)$  seen just before theorem 1.2 ( $a, b$  live in some P.O.M.):

$$a \ll_n b \Leftrightarrow (\exists m \in \mathbb{N})(2^{m+n}a \leq 2^m b),$$

$$a \leq_n b \Leftrightarrow (\exists d \ll_n b)(a \leq b + d),$$

$$E(a, b) \Leftrightarrow (\forall n \in \mathbb{N})(\exists k \in \mathbb{N})(n2^k a \leq (n + 1)2^k b).$$

**2.14. Theorem.** *(Embedding theorem). Let  $A$  be a refinement P.O.M. Then for all  $a, b$  in  $A$ , the following are equivalent:*

- (i)  $a^{**} \leq b^{**}$ ;
- (ii)  $E(a, b)$ ;
- (iii)  $(\forall n \in \mathbb{N})(a \leq_n b)$ .

**Proof.** The equivalence (i) $\Leftrightarrow$ (ii) has been proved in 1.2 (it does not depend on the fact that  $A$  is a refinement P.O.M). (iii) $\Rightarrow$ (ii) is easy. We prove that (ii) $\Rightarrow$ (iii). So, let

$a, b$  in  $A$  and assume that  $E(a, b)$  holds. Let  $\varepsilon$  be the natural map from  $A$  to  $K(A)$ , put  $\alpha = \varepsilon(a)$ ,  $\beta = \varepsilon(b)$ ,  $\gamma = \varepsilon(c)$ . By 1.2, for every  $n$  in  $\mathbb{N}$ , there is  $k$  in  $\mathbb{N}$  such that  $2^{n+3}2^k a \leq (2^{n+3} + 1)2^k b$ . Since  $K(A)$  satisfies the multiplicative  $\leq$ -cancellation property,  $2^{n+3}\alpha \leq (2^{n+3} + 1)\beta$ . Thus, by 2.12 and 2.13, there is  $\gamma$  in  $K(A)$  such that  $\alpha \leq \beta + \gamma$  and  $2^{n+3}\gamma \leq \beta$ . Write  $\gamma = [c_l]_l$ . There is  $l$  in  $\mathbb{N}$  such that  $a \leq_{n+3} b + c_l$  and  $2^{n+3}c_l \leq_0 b$ . Since  $c_l \ll_{n+3} 2^{n+3}c_l$ , 2.1 (v) implies that  $c_l \ll_{n+2} b$ . By 2.4, there are  $u \ll_{n+2} b$  and  $v \ll_{n+2} c_l$  such that  $a \leq b + c_l + u + v$ . But  $v \ll_{2n+4} b$ , thus  $c_l + u + v \ll_n b$ , so that  $a \leq_n b$ . This holds for all  $n$  in  $\mathbb{N}$ , which concludes the proof.  $\blacksquare$

We can give an immediate application of this theorem to the study of *equidecomposability types*. We recall here the corresponding definition (*see e.g.* [37]). Let  $G$  be a group acting on a set  $\Omega$ , let  $\mathcal{B}$  be a Boolean subalgebra of subsets of  $\Omega$  which is closed under  $G$ , let  $S(\mathcal{B})$  be the space of all  $\mathcal{B}$ -measurable functions from  $\Omega$  to  $\mathbb{N}$  with finite range. Define on  $S(\mathcal{B})$  binary relations  $\leq_G$  and  $\equiv_G$  by

$$\varphi \leq_G \psi \Leftrightarrow (\exists n \in \omega)(\exists_{i < n} g_i \in G)(\exists_{i < n} \varphi_i)(\varphi \leq \sum_i \varphi_i \text{ and } \sum_i g_i \varphi_i \leq \psi),$$

and

$$\varphi \equiv_G \psi \Leftrightarrow (\exists n \in \omega)(\exists_{i < n} g_i \in G)(\exists_{i < n} \varphi_i)(\varphi = \sum_i \varphi_i \text{ and } \sum_i g_i \varphi_i = \psi).$$

Furthermore, let  $S(\mathcal{B})/G$  be the quotient structure of  $(S(\mathcal{B}), +, 0, \leq_G)$  by  $\equiv_G$ ; we call it the equidecomposability types P.O.M. of  $\mathcal{B}$  modulo  $G$  (it is defined another [equivalent] way in [36]). For each  $X$  in  $\mathcal{B}$ , denote by  $[X]$  the equivalence class of  $1_X$  modulo  $\equiv_G$ . Then theorem 2.14 yields immediately the following consequence:

**2.15. Corollary.** *Let  $X$  and  $Y$  in  $\mathcal{B}$ . Then the following are equivalent:*

- (i) *For every  $G$ -invariant finitely additive measure  $\mu$  from  $\mathcal{B}$  to  $\overline{\mathbb{P}}$ ,  $\mu(X) \leq \mu(Y)$ .*
- (ii) *For every  $n$  in  $\mathbb{N}$ , there are disjoint  $Y' \leq_G Y$  and  $Z$  in  $\mathcal{B}$  such that  $X \subseteq Y' \cup Z$  and  $[Z] \ll_n [Y]$ .*  $\blacksquare$

On the other hand, it is not difficult to show that equidecomposability types P.O.M.'s are almost never regular. Concerning regular P.O.M.'s, theorem 2.14 yields us the following regularity criterion, which is roughly speaking the P.O.M.-analogue of the real analysis statement “the general term of a convergent series goes to 0 as the index goes to infinity”:

**2.16. Theorem.** *Let  $A$  be a refinement P.O.M. Then the following are equivalent:*

- (i)  *$A$  is regular;*



(ii)  $A$  is antisymmetric, and satisfies the following statement:

$$(\forall a, b)(\forall_{n \in \omega} c_n) \left( (\forall n \in \omega) (a \leq b + c_n \text{ and } \sum_{i < n} c_i \leq b) \Rightarrow a \leq b \right).$$

**Proof.** (i) $\Rightarrow$ (ii) Let  $E$  be an injective P.O.M. containing  $A$ , and let  $a, b, c_n$  ( $n \in \omega$ ) be as in the statement above. Let  $c = \bigwedge_n c_n$ ; then  $a \leq b + c$  and  $nc \leq b$  for all  $n$ , whence, by the Archimedean property of  $A$ ,  $c \ll b$ , whence  $a \leq b$ . (ii) $\Rightarrow$ (i) Let  $A$  satisfy (ii). So  $A$  is by assumption antisymmetric. Next, we have the

**Claim.**  $A$  satisfies the multiplicative  $2\text{-}\leq$ -cancellation property.

**Proof of claim.** Let  $a, b$  in  $A$  such that  $2a \leq 2b$ . As in the proof of [39], 2.8, we find a sequence  $(c_n)_{n \in \omega}$  such that for all  $n$ ,  $a \leq b + c_n$  and  $\sum_{i < n} c_i \leq b$ . By assumption,  $a \leq b$ . ■ Claim .

Now, let  $a, b$  in  $A$  such that for all  $n$ ,  $a \leq_n b$ . By the claim, there exists a sequence  $(c_n)_n$  of elements of  $A$  such that for all  $n$ ,  $2^{n+1}c_n \leq b$  and  $a \leq b + c_n$ . Using the claim, we easily obtain that  $\sum_{i < n} c_i \leq b$  for all  $n$ ; by assumption,  $a \leq b$ . We conclude by 2.14. ■

Using this criterion, we get immediately the following

**2.17. Corollary.** Every relatively  $\sigma$ -complete P.O.M. is regular. ■

In particular, we reprove the main result of [32] that [the P.O.M. associated with] every weak cardinal algebra embeds into a power of  $\overline{\mathbb{P}}$ .

We obtain also that the converse of theorem 1.11 for refinement P.O.M.'s is true:

**2.18. Corollary.** Let  $A$  be a refinement P.O.M. Then  $A$  is regular if and only if there is a Banach limit over  $A$ .

**Proof.** The ‘only if’ direction follows from 1.11. Conversely, suppose that there exists a Banach limit  $\lambda : \mathcal{B}(A) \rightarrow B$  where  $B$  is a P.O.M. containing  $A$ ; since  $\lambda$  is also a P.O.M.-homomorphism from  $\mathcal{B}(A)$  to  $B$  equipped with its minimal preordering, we can assume without loss of generality that  $B$  is *minimal*. By 1.13,  $A$  is antisymmetric. Now, let  $a, b, c_n$  ( $n \in \omega$ ) be elements of  $A$  such that for all  $n$ ,  $a \leq b + c_n$  and  $\sum_{i < n} c_i \leq b$ . Let  $\gamma = \lambda((c_n)_n)$  and  $\delta = \lambda((\sum_{i < n} c_i)_n)$ . Then we obtain  $a \leq b + \gamma$  and  $\gamma \ll \delta$  and  $\delta \leq b$ ; but  $B$  is minimal, thus  $\gamma \ll b$ , whence  $a \leq b$ . We conclude using 2.16. ■

## BIBLIOGRAPHY

- [1] G. BIRKHOFF, “Lattice theory”, American Mathematical Society, Providence, Rhode Island, 1961.
- [2] W.G. BOLTJANSKII, “Hilbert’s third problem”, translated by R. Silverman, Washington, D.C.: Winston, 1978.
- [3] R. BRADFORD, “Cardinal addition and the axiom of choice”, *Annals of Mathematical Logic* 3, pp. 111-196, 1971.
- [4] C.C. CHANG and H.J. KEISLER, “Model Theory”, North Holland Publishing Company, 1973.
- [5] J.P.R. CHRISTENSEN, “Topology and Borel structure”, North Holland Publishing Company, 1974.
- [6] R. CHUAQUI, “Cardinal algebras and measures invariant under equivalence relations”, *Transactions of the American Mathematical Society* 142, pp. 61-79, 1969.
- [7] R. CHUAQUI, “Cardinal algebras of functions and integration”, *Fundamenta Mathematicae* 71, pp. 77-84, 1971.
- [8] A.B. CLARKE, “On the representation of cardinal algebras by directed sums”, *Transactions of the American Mathematical Society* 91, pp. 161-192, 1959.
- [9] P.F. CONRAD, “Introduction à la théorie des groupes réticulés”, conferences at the Institut Henri Poincaré (Paris), May-June 1967, written by B. Gostiaux and S. Wolfenstein.
- [10] P.C. DELIYANNIS, “Group representations and cardinal algebras”, *Canadian Journal of Mathematics*, vol. XXII, no. 4, pp. 759-772, 1970.
- [11] C. DELLACHERIE and P.A. MEYER, “Théorie discrète du potentiel”, Hermann, 1983.
- [12] H. DOBBERTIN, “Refinement monoids, Vaught Monoids, and Boolean Algebras”, *Mathematische Annalen* 265, pp. 475-487, 1983.
- [13] R. DOUGHERTY and M. FOREMAN, “Banach-Tarski decompositions using sets with the property of Baire”, *preprint*.
- [14] P.A. FILLMORE, “An Archimedean property of cardinal algebras”, *Michigan Mathematical Journal* 11, pp. 365-367, 1964.
- [15] P.A. FILLMORE, “The dimension theory of certain cardinal algebras”, *Transactions of the American Mathematical Society* 117, pp. 21-36, 1965.
- [16] E. FÖLNER, “On groups with full Banach mean value”, *Mathematica Scandinavica* 3, pp. 243-254, 1955.
- [ ] M. FOREMAN, *see* [13].

- [17] M. FOREMAN and F. WEHRUNG, “The Hahn-Banach theorem implies the existence of a non-measurable set”, to appear in *Fundamenta Mathematicae*.
- [18] K.R. GOODEARL, “Partially ordered abelian groups with the interpolation property”, Mathematical surveys and monographs, number 20, American Mathematical Society, 1986.
- [19] F. GREENLEAF, “Invariant means on topological groups”, Van Nostrand, Reinhold Company, 1969.
- [20] P.R. HALMOS, “Measure Theory”, D. Van Nostrand, 1958.
- [21] W. HANF, “On some fundamental problems concerning isomorphism of Boolean algebras”, *Mathematica Scandinavica* 5, pp. 205-217, 1957.
- [22] T. JECH, “The axiom of choice”, North-Holland Publishing Company, Amsterdam, 1973.
- [23] T. JECH, “Set theory”, Academic press, 1978.
- [24] S. KINOSHITA, “A solution of a problem of R. Sikorski”, *Fundamenta Mathematicae* 40, pp. 39-41, 1953.
- [25] S. KOPPELBERG, “General theory of Boolean Algebras”, in “Handbook of Boolean Algebras”, vol. 1, pp. 1-307, edited by J.D. Monk with R. Bonnet, Elsevier, Amsterdam, 1989.
- [26] M. LACZCOVITCH, “Equidecomposability and discrepancy; a solution of Tarski’s circle-squaring problem”, *Journal für die reine und angewandte Mathematik* 404, pp. 77-117, 1990.
- [27] J. MYCIELSKI, “The Banach-Tarski paradox for the hyperbolic plane”, *Fundamenta Mathematicae* 132, 1989.
- [28] I. NAMIOKA, “Følner’s condition for amenable semi-groups”, *Mathematica Scandinavica* 15, pp. 18-28, 1964.
- [29] D. PINCUS, “The strength of the Hahn-Banach theorem”, *Victoria symposium of Non-Standard Analysis*, Springer Lecture Notes Number 369, pp. 203-248, 1974.
- [30] D. PINCUS and R. SOLOVAY, “Definability of measures and ultrafilters”, *Journal of Symbolic Logic* 42, Number 2, pp. 179-190, 1977.
- [31] K.P.S. BHASKARA RAO and R.M. SHORTT, “The dual of a refinement algebra”, *Lecture Notes in pure and applied mathematics, General Topology and Applications, 5th North East Conference* (Marcel Dekker), vol. 134, pp. 335-367, 1991.
- [32] K.P.S. BHASKARA RAO and R.M. SHORTT, “Weak cardinal algebras”, to appear in *Proceedings of the New York Academy of Sciences, General Topology and its applications, 6th North East Conference*.
- [33] R.M. SHORTT, “Duality for cardinal algebras”, *Forum Mathematicum* 2, pp. 433-450, 1990.
- [ ] R.M. SHORTT, see [31], [32].
- [ ] R. SOLOVAY, see [30].

- [34] A. TARSKI, “Algebraische Fassung des Maßproblems”, *Fundamenta Mathematicae* 31, pp. 47-66, 1938.
- [35] A. TARSKI, “Cardinal Algebras”, New York: Oxford, 1949.
- [36] S. WAGON, “The Banach Tarski-paradox”, Cambridge University Press, 1984.
- [ ] F. WEHRUNG, *see* [17].
- [37] F. WEHRUNG, “Théorème de Hahn-Banach et paradoxes continus et discrets”, *Comptes Rendus de l'Académie des Sciences*, t. 310, série I, pp. 303-306, 1990.
- [38] F. WEHRUNG, “Metric properties of positively ordered monoids”, *to appear in Forum Mathematicum*.
- [39] F. WEHRUNG, “Injective positively ordered monoids I”, *preprint*.