

Injective positively ordered monoids I

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ABSTRACT.

We define in this paper a certain notion of completeness for a wide class of commutative (pre)ordered monoids (from now on P.O.M.'s). This class seems to be the natural context for studying structures like measurable function spaces, equidecomposability types of spaces, partially ordered abelian groups and cardinal algebras. Then, we can prove that roughly speaking, spaces of measures with values in complete P.O.M.'s are complete P.O.M.'s. Furthermore, this notion of completeness yields us an 'arithmetical' characterization of injective P.O.M.'s.

INTRODUCTION

A well-known result of A. Tarski states, for a given commutative monoid A and a given element a of A , a criterion for the existence of a monoid homomorphism from A to the extended positive real line $\overline{\mathbb{P}}$ sending a to 1: the condition is $(\forall n \in \mathbb{N})(\neg \exists x)((n+1)a + x = na)$ (see [34], [36]). In fact, his proof shows slightly more: it is a Hahn-Banach like property, stating the injectivity of $\overline{\mathbb{P}}$ - not in the category of commutative monoids, where, as it is well-known, there are no nontrivial injective objects, but in a certain category of preordered monoids, which we will call 'positively ordered monoids' - from now on P.O.M.'s; by definition, a P.O.M. is a commutative monoid equipped with a preordering which is compatible with the addition and which makes every element positive (see 1.1). This theorem of Tarski is fundamental in decomposition theory, where it allows one to connect existence of invariant measures and non existence of paradoxical decompositions.

The proof of the injectivity of $\overline{\mathbb{P}}$ appears as rather 'arithmetical', which brings the expectation that Tarski's argument could be reproduced in more general (and interesting on their own) structures. We choose here this property of injectivity to initiate a study of spaces which, basically, are P.O.M.'s, but in fact are equipped with some additional structure. Particular cases of these spaces have been considered in the study of different theories, as *e.g.* the theory of abelian groups, the theory of Boolean algebras (see [25]), or the theory of ordered groups (see [1], [9]). In these examples, all the elements are 'cancellable' (the terminology 'finite' is used in [35], definition 4.10); sometimes, one infinite

element is adjoined. Needless to say, in the latter case, proofs about these structures often have to separate the finite and the infinite case.

An important exception to the latter rule is the theory of *cardinal algebras*, initiated by A. Tarski in [35]. A cardinal algebra is by definition a commutative monoid equipped with an infinite operation, defined on all countable sequences, satisfying some simple attributes of what an infinite addition ‘should be’. Isomorphism types of structures which are, in some sense, countably complete form a cardinal algebra; also spaces of positive real-valued (possibly infinite) functions are also often cardinal algebras — *see e.g.* [6], [7], [10], [15], and of course [35]. In cardinal algebras, there are many infinite elements, which, except in representation theorems (*see* [8], [15]), do not really play any special arithmetical role. Furthermore, these structures seem to offer a convenient arithmetical environment, much explored in the first chapters of [35] (but not all about this had been said at that time, *see* [3], [14], [32]).

Still, the definition of cardinal algebras is purely arithmetical, and moreover, it depends on the artificial introduction of an infinite addition and several axioms about it, even for the mere study of first-order properties — as for example the famous *multiplicative cancellation property*

$$(\forall x, y)(mx = my \Rightarrow x = y) \text{ (all } m \text{ in } \mathbb{N} \setminus \{0\}),$$

valid in any cardinal algebra ([35], 2.31). This suggests an enrichment of the environment of cardinal algebras. Actually, Tarski’s book itself calls for such an enrichment — several weakenings of the definition of cardinal algebras are proposed, as for example *refinement algebras* or *generalized cardinal algebras*. We propose here an *algebraic* enrichment (as opposed to arithmetical), based on the possibility to extend P.O.M.-homomorphisms. The strongest of all these possibilities is of course injectivity; this one we characterize arithmetically (theorem 3.11), where it turns out that injectivity is a form of completeness: surprisingly, this characterization bears very close similarities with the definition of cardinal algebras. There comes another surprise: although in most aspects, injective P.O.M.’s enjoy a much stronger completeness character than cardinal algebras, they often do *not* satisfy the refinement postulate ([35], axiom VI of definition of cardinal algebras). Is this loss important? It turns out no, at least as far as most first-order properties are concerned. The corresponding weakening of the definition of cardinal algebra (the notion of *weak cardinal algebra*), obtained by replacing the refinement postulate by the *finite* refinement postulate, has been introduced independently in [32] and here (definition 2.2). Note finally that many mathematical structures yield natural definitions of P.O.M.’s which do not necessarily satisfy the finite refinement property (see examples in 1.5).

We now summarize the organization of our paper.

— In chapter 1, we recall the definition of the very important *finite refinement property*, which has already been studied in many places (*e.g.* [12], [31], [37]), and then the *pseudo-cancellation property* (definition 1.12), which will be the substitute of the classical cancellation property throughout this work. The combination of these two axioms will yield *strong refinement P.O.M.’s*, which will be substitutes for [positive cones of] ordered groups with the finite refinement property, and the *refinement ℓ -P.O.M.’s*, which will be substitutes for [positive cones of] ℓ -groups; furthermore, the latter will provide us with a simple way to derive the finite refinement property, *e.g.* in ‘dual spaces’ of P.O.M.’s (see example 1.24).

— In chapter 2, we introduce *complete P.O.M.’s*, which are designed to concentrate the ‘ordered structures-part’ of the characterization of injective P.O.M.’s. These are roughly speaking substitutes for [positive cones of] complete ℓ -groups. This allows us to generalize in a non-trivial way (divisibility is not used) the main results of [31], [33] — see theorems 2.33, 2.38, 2.42. These results can also be viewed as results of ‘algebraization of real analysis (or measure theory)’.

— Chapter 3 is mainly devoted to give a complete arithmetical characterization of injective P.O.M.’s (theorem 3.11). This characterization would be useless without any proper arithmetical study of the corresponding structures; this is also one of the goals of chapters 1 and 2. Other more algebraic characterizations will appear in [39].

Chapter 1 has been to some extent designed to provide computational facility in stating and proving the results of chapters 2 and 3. Its results will also be essential in the forthcoming [39].

To avoid repeating proofs, Tarski’s monograph [35] will be often referred to throughout our work.

The desire to keep this paper down to a reasonable size has forced us to omit entirely some closely related subjects, as for example the study of *injective closures*, or an algebraic theory of P.O.M.’s, although there is a lot of non trivial information which can already be said about these (resulting *e.g.* in the fact, mentioned in remark 3.15, that divisible weak cardinal algebras are countably injective).

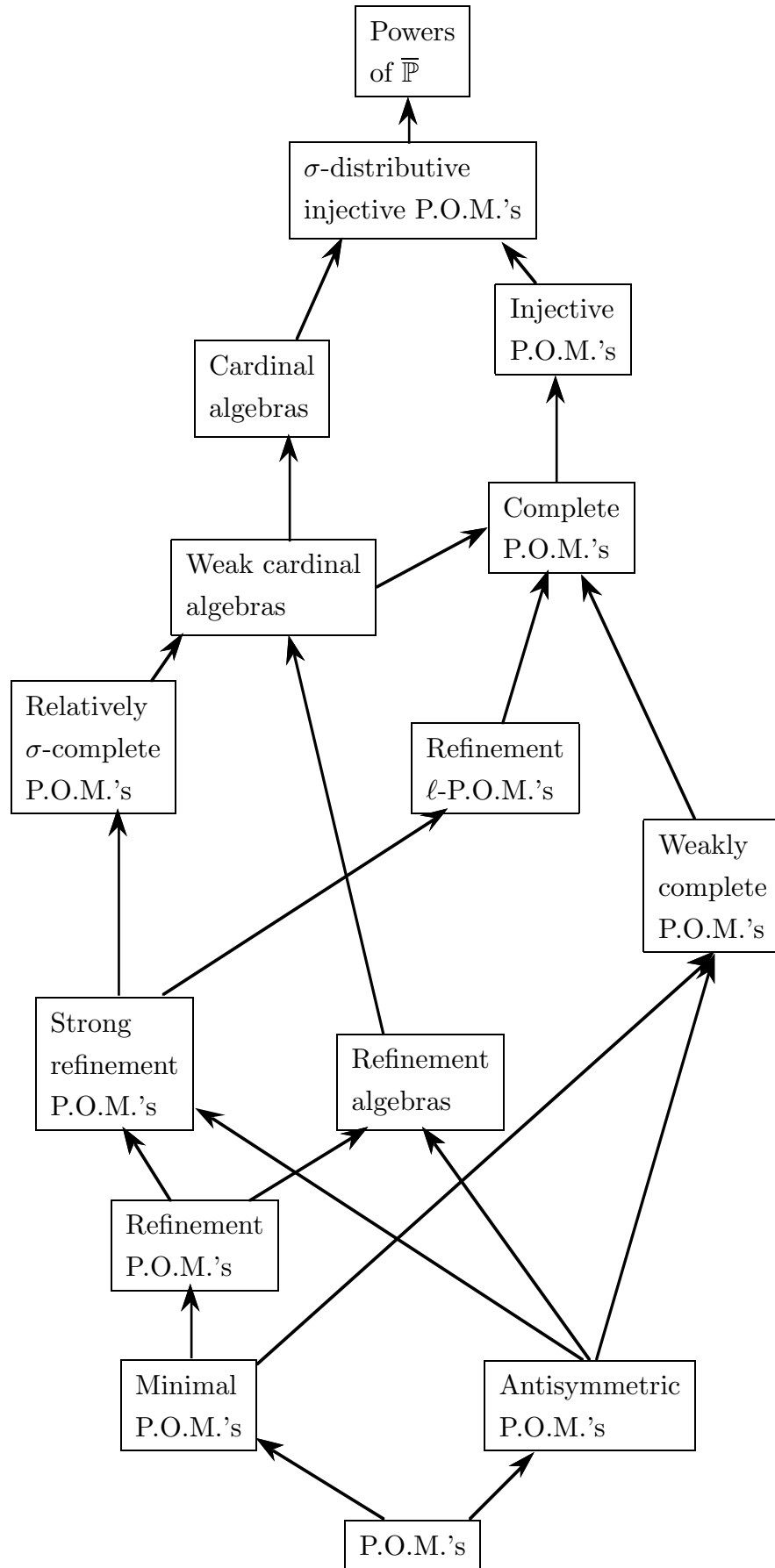
We will use basic set-theoretical notation and terminology. If $(x_i)_{i \in I}$ is a family and there is no ambiguity on I , then we will denote it by $(x_i)_i$; similar conventions apply to $\sum_i x_i$, $\bigcup_i x_i$, *etc.* . If X , Y are two preordered sets and f is a map from X to Y , then we say that f is *increasing* (resp. *decreasing*) when for all x , y in X such that $x \leq y$, we have $f(x) \leq f(y)$ (resp. $f(y) \leq f(x)$). Homomorphisms, unless specified otherwise, will

be P.O.M.-homomorphisms, *i.e.* increasing monoid homomorphisms. The class of natural numbers will sometimes be denoted by \mathbb{N} when it is considered as a monoid (or a P.O.M.), and by ω when it is considered as an ordinal (of course, as the reader might suspect, there will be cases where a dilemma will arise around this...). Finally, most of the objects we will consider in this work will be *sets*, but proper classes (as *e.g.* the class *ON* of all ordinals, or the class *CARD* of all — non necessarily well-ordered — cardinals considered *e.g.* as a monoid) will sometimes be considered; no paradoxes will arise here from this. Finally, most of our results will be proved in set theory plus axiom of choice (even if many of them have ‘choiceless’ versions).

For convenience of the reader, we show on next page a picture of the different principal classes of P.O.M.’s used throughout this paper and its continuation, [39]. If *A* and *B* are two classes of P.O.M.’s, then an arrow from *A* to *B* indicates strict inclusion of *B* into *A*. This diagram is *complete*, in the sense that its transitive closure shows exactly all the inclusion relations between the classes considered (see in particular example 1.20).

Acknowledgements

I wish to thank deeply my dear Estella for having helped me to type the preliminary version of this manuscript. I also wish to thank Patrick Dehornoy for having gone through it and suggested many improvements on its form; and the referee for having read carefully the whole paper, suggesting many improvements on both its form and substance.



1. POSITIVELY ORDERED MONOIDS. STRONG REFINEMENT P.O.M.'s, REFINEMENT ℓ -P.O.M.'s.

1.1. Definition. A *positively ordered monoid* (from now on a P.O.M.) is a structure $\underline{A} = (A, +, 0, \leq)$ where $(A, +, 0)$ is a commutative monoid and \leq is a (partial) preordering of A satisfying the following conditions:

- (i) $(\forall a)(0 \leq a)$;
- (ii) $(\forall a, b, c)(a \leq b \Rightarrow a + c \leq b + c)$ (i.e. \leq is compatible with the addition).

We say that $(A, +, 0)$ is the underlying monoid of \underline{A} .

From now on, we will often make the usual convention of identifying a structure and its underlying set when there is no ambiguity. We shall denote by $+_A$, 0_A , \leq_A respectively the addition, the zero, and the preordering of \underline{A} if A is its underlying set. We put

$$a \equiv_A b \iff a \leq_A b \text{ and } b \leq_A a$$

If A is a commutative monoid, we can define two preorderings on A : the *coarse* preordering $\leq_c = A \times A$, and the *minimal* preordering \leq defined by

$$a \leq b \iff (\exists c \in A)(a + c = b)$$

Both preorderings defined above define P.O.M.'s.

A P.O.M. A will be called *antisymmetric* when its preordering is antisymmetric, and similarly for coarse, minimal, etc. . If ρ is either $=$ or \leq , we will say that A satisfies the *additive ρ -cancellation property* when it satisfies the statement

$$(\forall a, b, c)(a + c \rho b + c \Rightarrow a \rho b).$$

Say that A is *cancellative* when it satisfies the additive $=$ -cancellation property and the additive \leq -cancellation property.

For all m in $\mathbb{N} \setminus \{0\}$, A satisfies the *m - ρ -cancellation property* when it satisfies the statement

$$(\forall a, b)(ma \rho mb \Rightarrow a \rho b).$$

A satisfies the *multiplicative ρ -cancellation property* when it satisfies the *m - ρ -cancellation property* for all m in $\mathbb{N} \setminus \{0\}$. We will often identify a minimal P.O.M. with its underlying monoid.

We now introduce some further notation and terminology which will be useful in the sequel.

If a, b are elements of some P.O.M. A , $a \ll b$ will always be the statement $a + b = b$. Note that \ll is transitive but not necessarily irreflexive; when $a \ll a$, *i.e.* $a + a = a$, we say, as in [35], that a is *idem-multiple*. If X, Y are two subsets of A , we write $X \leq Y$ instead of $(\forall x \in X)(\forall y \in Y)(x \leq y)$. When X (resp. Y) is $\{a\}$, we simply write $a \leq Y$ (resp. $X \leq a$). If $X = \{a_1, \dots, a_m\}$ and $Y = \{b_1, \dots, b_n\}$, we write

$$a_1, \dots, a_m \leq b_1, \dots, b_n$$

Of course, the same conventions apply to \ll or any other binary relation on A .

The following lemma will be used very often in the sequel:

1.2. Lemma. *Let A be a P.O.M., let a, b be in A . Then*

- (i) *If A is minimal, then $a \ll b$ and $b \leq c$ implies $a \ll c$;*
- (ii) *If A is antisymmetric, then $a \leq b$ and $b \ll c$ implies $a \ll c$. Furthermore, if A is minimal, then this last property characterizes antisymmetry.*

Proof. Easy. ■

We now turn to a very important property, the finite refinement property. It has been studied in many places (see *e.g.* [12], [18], [31], [32], [33], [35], [37]).

1.3. Definition. Let A be a commutative monoid, let R, S be binary relations on A , let m, n be in ω , let a_i ($i < m$), b_j ($j < n$) be in A .

- (i) A (R, S) -refinement of $(a_i)_i, (b_j)_j$ is a finite sequence $(c_{ij})_{ij}$ such that

$$(\forall i < m)(a_i R \sum_{j < n} c_{ij}) \text{ and } (\forall j < n)(\sum_{i < m} c_{ij} S b_j)$$

(as usual, the sum of the empty sequence is 0). Equivalently, we say that the following is a *refinement matrix*:

	$S b_0$	$S b_1$	\dots	$S b_{n-1}$
$a_0 R$	c_{00}	c_{01}	\dots	$c_{0,n-1}$
$a_1 R$	c_{10}	c_{11}	\dots	$c_{1,n-1}$
\vdots	\vdots	\vdots	\ddots	\vdots
$a_{m-1} R$	$c_{m-1,0}$	$c_{m-1,1}$	\dots	$c_{m-1,n-1}$

- (ii) A has the (mR, Sn) -refinement property when any two finite sequences $(a_i)_{i < m}$ and $(b_j)_{j < n}$ such that

$$\sum_i a_i S \circ R \sum_j b_j$$

have a (R, S) -refinement;

(iii) A has the *finite (R, S) -refinement property* when it has the (mR, Sn) -refinement property for all m, n in $\omega \setminus \{0\}$.

Usually, when R, S are the equality, we will drop them from the notations above.

1.4. EXAMPLES. There are many examples of P.O.M.'s with the finite refinement property. Let us mention the following ones:

(1) Abelian groups (we see on this example that the mention $m, n \neq 0$ in the (iii) of previous definition is relevant).

(2) Let A be an abelian ℓ -group (see [9]). Then the positive cone A_+ of A satisfies the finite refinement property (for a generalisation see [35], 13.21). For example, for any topological space X , the space $C(X, \mathbb{R}_+)$ of all continuous maps from X to \mathbb{R}_+ satisfies the finite refinement property.

(3) The P.O.M. $\overline{\mathbb{P}} = ([0, \infty], +, 0, \leq)$ of all positive (possibly infinite) reals; $+$ and \leq are respectively its natural sum and (linear) ordering. This is in a sense the most fundamental example, as we will see in [39]. Note also the sub-P.O.M. $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ of $\overline{\mathbb{P}}$.

(4) The P.O.M.'s $\mathbf{1} = (\{0\}, +, 0, \leq)$ equipped with trivial $+$ and \leq , and $\mathbf{2} = (\{0, 1\}, +, 0, \leq)$ equipped with the addition with neutral element 0 and such that $1 + 1 = 1$. Both of them are (isomorphic to) sub-P.O.M.'s of $\overline{\mathbb{P}}$ (example (3)) - respectively $\{0\}$ and $\{0, \infty\}$ - and even *retracts* of $\overline{\mathbb{P}}$ (if $A \subseteq B$ are P.O.M.'s, we say that A is a retract of B when there is a homomorphism r from B onto A such that $r|_A = id$). In fact, $\mathbf{2}$ plays a similar role for the *idem-multiple* P.O.M.'s (*i.e.* satisfying $(\forall x)(2x = x)$) as $\overline{\mathbb{P}}$ does for general P.O.M.'s.

(5) The (proper) class *CARD* of all (non-necessarily well-ordered) cardinals; addition is the ordinary cardinal sum, *i.e.* it is defined by $a + b = |A \cup B|$ whenever $a = |A|$, $b = |B|$ and $A \cap B = \emptyset$, and the preordering is the minimal one; the classical Cantor-Bernstein theorem states that this preordering is an ordering.

(6) The (proper) class of isomorphism types of Boolean algebras, equipped with the sum defined by $\text{type}(A) + \text{type}(B) = \text{type}(A \times B)$ and the minimal preordering defined by $\text{type}(A) \leq \text{type}(B)$ if and only if there is a Boolean algebra C such that $A \times C \cong B$. It turns out that \leq is *not* antisymmetric (even when restricted to countable Boolean algebras, see [24]).

(7) Let E be a preordered vector space of dimension at most 2, let P be a convex cone of E such that $P \subseteq E_+$. Then it can be shown that $(P, +, 0)$ equipped with the restriction

of the preordering of E satisfies the finite (\leq, \leq) , $(\leq, =)$, $(=, \leq)$ and $(=, =)$ -refinement properties. Note that the corresponding property in dimension ≥ 3 fails, as easy examples show it.

(8) The space $D_+^1(\mathbb{R})$ of all real-valued positive differentiable functions on \mathbb{R} satisfies the finite refinement property.

We will see several less elementary examples of P.O.M.'s with the finite refinement property in the sequel.

1.5. EXAMPLES. Let us mention the following natural examples of P.O.M.'s, which do not necessarily enjoy minimality or the finite refinement property:

(1) Let A be a module over a ring R . Then the space of R -submodules of A , equipped with the “Minkowski sum” $U + V = \{u + v : u \in U, v \in V\}$ and the inclusion relation is a minimal, idem-multiple P.O.M..

(2) Let A be a commutative ring. Then the space of all ideals of A , equipped with the “addition” defined by the ideal *product* $I + J = \text{ideal generated by all products } x \cdot y, x \in I \text{ and } y \in J$, and the inverse inclusion, is a P.O.M.. If A is a Dedekind domain, it is minimal and satisfies the finite refinement property (it is even isomorphic to a power of \mathbb{N}).

(3) Let \mathcal{L} be a first-order language, let \mathcal{T} be a theory written in \mathcal{L} whose class of models is *closed under finite direct products* (see [4]). Let \mathcal{M} be the class of isomorphism types of models of \mathcal{T} , equipped with the embeddability preordering (defined by $[A] \leq [B]$ if and only if there exists a \mathcal{L} -embedding from A into B), to which we adjoin an extra element 0. Then \mathcal{M} , equipped with the addition defined by $[A] + [B] = [A \times B]$ and neutral 0, is a P.O.M. (see also the Prologue of [18] for connected matters). Compare with examples (5) and (6) of 1.4.

1.6. Definition A *refinement P.O.M.* is a minimal P.O.M. satisfying the finite refinement property.

Note that it is sufficient to check the $(2, 2)$ -refinement property, and that then, the finite (R, S) -refinement property holds whenever R and S are either $=$ or \leq .

Note also that it can be shown that every minimal P.O.M. can be embedded into a refinement P.O.M (which seems to indicate that the finite refinement property is not a very drastic condition to satisfy for a given minimal P.O.M.). Details about this will be given in a forthcoming paper.

Note finally that a submonoid of a refinement P.O.M is not necessarily a refinement P.O.M. This brings us to a classical definition.

1.7. Definition. Let A be a P.O.M.. A subset I of A is an *ideal* of A when it is a submonoid of A which is *convex*, i.e. satisfies the statement

$$(\forall a \in I)(\forall b \in A)(b \leq a \Rightarrow b \in I).$$

For all a in A , we denote by $A|a$ the ideal generated by a , i.e. $\{x \in A : (\exists n \in \mathbb{N})(x \leq na)\}$.

1.8. Lemma Any ideal of a refinement P.O.M is a refinement P.O.M.

Proof. Obvious. ■

A very useful consequence of the definition of refinement P.O.M is the following lemma 1.9. Note that it is just a simple generalisation of [35], corollary 2.5.

1.9. Lemma. Let n in $\mathbb{N} \setminus \{0\}$, let A be a refinement P.O.M, let a, b, c in A . Then the following are equivalent:

- (i) $a + b = nc$;
- (ii) There are $x_k (k \leq n)$ in A such that the following holds:

$$\begin{cases} a = \sum_{k \leq n} kx_k \\ b = \sum_{k \leq n} (n - k)x_k \\ c = \sum_{k \leq n} x_k \end{cases}$$

Proof. (ii) \Rightarrow (i) is obvious. We prove (i) \Rightarrow (ii) by induction on n . For $n = 1$ it is trivial, so assume it is true for n , and let $a + b = (n + 1)c$, i.e. $a + b = nc + c$. By the finite refinement property, there are u, v, u', v' such that $a = u + u', b = v + v', nc = u + v$ and $c = u' + v'$. By the induction hypothesis, there are $t_k (k \leq n)$ such that the following holds:

$$u = \sum_{k \leq n} kt_k, v = \sum_{k \leq n} (n - k)t_k, \text{ and } c = \sum_{k \leq n} t_k \quad (1)$$

Hence $u' + v' = \sum_{k \leq n} t_k$, thus there are $u_k, v_k (k \leq n)$ such that

$$u' = \sum_{k \leq n} u_k, v' = \sum_{k \leq n} v_k, \text{ and } (\forall k \leq n)(u_k + v_k = t_k) \quad (2)$$

Then, using $a = u + u', b = v + v'$ and (1), (2), it is easy to see that

$$a = \sum_{k \leq n+1} kx_k, b = \sum_{k \leq n+1} (n + 1 - k)x_k, c = \sum_{k \leq n+1} x_k$$

where x_k is equal to v_0 for $k = 0$, u_n for $k = n$, and $u_{k-1} + v_k$ for $1 \leq k \leq n$. ■

Now, if A is a P.O.M. and $a \in A$, we define a preordering $\leq (\text{mod } a)$ and an equivalence relation $\equiv (\text{mod } a)$ by putting

$$x \leq y (\text{mod } a) \Leftrightarrow x + a \leq y + a$$

and

$$x \equiv y (\text{mod } a) \Leftrightarrow x + a = y + a.$$

It is easy to see that $\equiv (\text{mod } a)$ is compatible with both the addition and $\leq (\text{mod } a)$.

1.10. Definition. We denote by $\frac{A}{a}$ the quotient structure of $(A, +, 0, \leq (\text{mod } a))$ by $\equiv (\text{mod } a)$.

Now, let us return back to refinement P.O.M's. Most of the refinement P.O.M's we shall consider are not cancellative — among the examples in 1.4, only (1), (2), (7), (8) do satisfy it, and then their study is rather related to other research areas, as *e.g.* group theory. The best approximation we can give to additive cancellation and which holds in the general case seems to be the following:

1.11. Lemma. *Let A be a refinement P.O.M, let a, b, c in A , let n in \mathbb{N} .*

- (i) *If $a + c = b + c$, then there are d, u, v such that $nu, nv \leq c$ and $a = d + u$, $b = d + v$;*
- (ii) *If $a + c \leq b + c$, then there is d such that $a \leq b + d$ and $nd \leq c$.*

Proof. (This is a finite version of theorem 2.6 of [35]).

- (i) Using the finite refinement property, define inductively $a_k, b_k, c_k, d_k, (k \in \omega)$ by $a_0 = a, b_0 = b, c_0 = c$, and if $a_k + c_k = b_k + c_k$, then the following is a refinement matrix:

	b_k	c_k
a_k	d_k	a_{k+1}
c_k	b_{k+1}	c_{k+1}

thus $a_{k+1} + c_{k+1} = b_{k+1} + c_{k+1}$ and the induction hypothesis is satisfied. An easy induction yields $a = (\sum_{k < n} d_k) + a_n, b = (\sum_{k < n} d_k) + b_n, c = (\sum_{k < n} a_{k+1}) + c_n = (\sum_{k < n} b_{k+1}) + c_n$; take $d = \sum_{k < n} d_k, u = a_n, v = b_n$.

- (ii) Immediate from (i) and minimality. ■

Now, we shall define the refinement P.O.M's where an 'optimal' version of 1.11 holds.

1.12. Definition. Let A be an antisymmetric refinement P.O.M. We say that A is a *strong refinement P.O.M.* when it satisfies the following '*pseudo - cancellation property*':

$$(\forall a, b, c)(a + c \leq b + c \Rightarrow (\exists d \ll c)(a \leq b + d)).$$

Note that since A is minimal, we could replace $a + c \leq b + c$ by $a + c = b + c$ in the formulation of the pseudo-cancellation property. Note also that there are refinement algebras (see [35], definition 11.25) which are *not* strong refinement P.O.M.'s, although we do not know any simple example.

1.13. Lemma. *Any strong refinement P.O.M. satisfies the finite $(=, \ll)$ -refinement property.*

Proof. As in [35], theorem 2.19. ■

1.14. Proposition. *Let A be an antisymmetric refinement P.O.M. Then A is a strong refinement P.O.M. if and only if it satisfies the following statement:*

$$(\forall a, b, c)(a + c = b + c \Rightarrow (\exists d)(\exists u \ll c)(\exists v \ll c)(a = d + u \text{ and } b = d + v));$$

Proof. Suppose first that A is a strong refinement P.O.M., and let a, b, c in A such that $a + c = b + c$. Thus $a + c \leq b + c$, thus, by the pseudo-cancellation property and then the $(1 =, \leq 2)$ -refinement property, there are d and u in A such that $a = d + u$ and $d \leq b$ and $u \ll c$. Let e such that $b = d + e$. Then $a + c = b + c$ yields $e \ll d + c$, whence, by previous lemma, $e = h + v$ for some $h \ll d$ and $v \ll c$. Thus $b = d + v$ and d, u, v satisfy the required condition. The converse is trivial. ■

We introduce a bit of notation before going on. If A is an antisymmetric P.O.M. and $X \subseteq A$, we denote by $\bigwedge X$ (resp. $\bigvee X$) the g.l.b. (resp. l.u.b.) of X if it exists. If $X = \{a, b\}$, we just write $a \wedge b$ (resp. $a \vee b$). In the general case, we write $X + Y = \{x + y : x \in X, y \in Y\}$, $a + X = \{a\} + X$, and similarly for \wedge, \vee , etc. .

We refer to [35] for the proofs of both following useful lemmas, which appeal only to the structures of refinement P.O.M or strong refinement P.O.M.:

1.15. Lemma. *Let A be a refinement P.O.M, let a, b in A such that $a \wedge b$ is defined. Then $a \vee b$ is defined, and for every c in A , $a \wedge b + c = a$ implies $b + c = a \vee b$.*

Proof. See [35], theorem 3.4. ■

1.16. Lemma. *Let A be a strong refinement P.O.M.. Then the following holds:*

- (i) *For all finite subsets X, Y of A such that $X \leq Y$, there is c in A such that $X \leq c \leq Y$ (finite interpolation property).*
- (ii) *Let a, b in A , let $X \subseteq A$ finite nonempty such that $a + X = \{b\}$. Then there are $c \leq X$ and $d \geq X$ in A such that $a + c = a + d = b$.*
- (iii) *Let $a \in A$, $X \subseteq A$ finite nonempty such that $\bigvee X$ is defined. Then $\bigvee(a + X)$ is defined and equal to $a + \bigvee X$.*

(iv) Let a, b in A , $X \subseteq A$ finite nonempty such that $b \leq a + X$. Then there is $c \leq X$ in A such that $b \leq a + c$.

(v) Let $a \in A$, $X \subseteq A$ finite nonempty such that $\bigwedge X$ is defined. Then $\bigwedge(a + X)$ is defined and equal to $a + \bigwedge X$.

(vi) Let $a \in A$, $X \subseteq A$ finite nonempty such that $a \vee X$ and $\bigwedge X$ are defined. Then each of $a \vee \bigwedge X$ and $\bigwedge(a \vee X)$ is defined if and only if the other is defined, and then they are equal.

(vii) Let $a \in A$, $X \subseteq A$ finite nonempty such that $a \wedge X$ and $\bigvee X$ are defined. Then each of $a \wedge \bigvee X$ and $\bigvee(a \wedge X)$ is defined if and only if the other is defined, and then they are equal.

Proof. As in [35], respectively theorems 2.28, 2.25, 3.26, 2.29, 3.25, 3.30, 3.32. ■

One of the interesting features of strong refinement P.O.M.'s is that their class is closed under the quotient operation defined in 1.10:

1.17. Proposition. Let A be a strong refinement P.O.M., let a in A . Then $\frac{A}{a}$ is a strong refinement P.O.M.. Furthermore, $\frac{A|a}{a}$ is the positive cone of an abelian ordered group.

Proof. The only problem is for the proof of the finite refinement property. However, using proposition 1.14, this is a straightforward exercise. ■

In 1.18 and 1.19, A is a fixed antisymmetric P.O.M. satisfying the pseudo-cancellation property.

1.18. Definition. Let a in A . If $\bigvee\{na : n \in \mathbb{N}\}$ is defined, then we denote its value by ∞a .

1.19. Lemma. Let a in A . If ∞a is defined, then $a \ll \infty a$ and for all n in $\mathbb{N} \setminus \{0\}$, we have $\infty(na) = \infty a$.

Proof. The second part is obvious. Suppose $\infty a = b$. Then $2a \leq b$, thus there is $c \geq a$ such that $a + c = b$. For all n in \mathbb{N} , $a + na \leq a + c$, thus, by pseudo-cancellation, there is $e \ll a$ such that $na \leq c + e$, but $a \leq c$, thus $c + e = c$; by definition of b , we get $b \leq c$. Since \leq is antisymmetric, we get $b = c$. ■

1.20. EXAMPLE. Let $\mathbb{N}[c]$ be the minimal P.O.M. generated by two elements 1 and c such that $1 + c = c$: so its elements are of the form n or nc , $n \in \mathbb{N}$ (with $0 \cdot c = 0$). It is easy to see that $\mathbb{N}[c]$ is a strong refinement P.O.M.. If we take $a = 1$, we get $\infty a = c$. Note that here, ∞a is not idem-multiple (because $2c \neq c$). This strong refinement P.O.M.

is also an example of weakly complete strong refinement P.O.M. which is *not* a refinement algebra (see [35], definition 11.25).

The following lemma will be instrumental in the coming study of refinement ℓ -P.O.M.'s, and also in chapter 3.

1.21. Lemma. *Let A be a minimal P.O.M. satisfying the pseudo-cancellation property, let a, b, a', b' in A satisfying $a' + b \leq a + b'$ and $a' \leq b'$. Then there is c in A such that $b \leq a + c$ and $b' = a' + c$.*

Proof. Let d in A such that $a' + d = b'$. Then we have $a' + b \leq a' + a + d$, thus, by pseudo-cancellation, $b \leq a + d + e$ for some $e \ll a'$. Put $c = d + e$. ■

1.22. Definition. A *refinement ℓ -P.O.M.* is a minimal, antisymmetric P.O.M. satisfying the following properties:

- (i) Pseudo -cancellation property;
- (ii) $(\forall a, b) (a \wedge b \text{ is defined})$;
- (iii) $(\forall a, b, c)(a \wedge b + c = (a + c) \wedge (b + c))$ (distributivity of $+$ on \wedge).

Note that we do not put the finite refinement property among the hypotheses. The reason is that it is redundant:

1.23. Proposition. *Every refinement ℓ -P.O.M. is a strong refinement P.O.M..*

Proof. Let A be a refinement ℓ -P.O.M.. We prove that it satisfies the (2,2) - refinement property. So let $a + a' = b + b'$ in A . By distributivity of $+$ on \wedge , we have

$$a + a' \wedge b' = (a + a') \wedge (a + b') = (b + b') \wedge (a + b') = a \wedge b + b'.$$

Thus, by applying lemma 1.21 twice, we get u', u'' such that

$$a = a \wedge b + u' \text{ and } b' \leq a' \wedge b' + u' \tag{1}$$

and

$$b' = a' \wedge b' + u'' \text{ and } a \leq a \wedge b + u'' \tag{2}$$

Now, let $u = u' \wedge u''$. It is immediate, using again distributivity of $+$ on \wedge and (1), (2), that $a \wedge b + u = a$ and $a' \wedge b' + u = b'$. Similarly, there is v such that $a \wedge b + v = b$ and $a' \wedge b' + v = a'$, hence we have the following refinement matrix

	b	b'
a	$a \wedge b$	u
a'	v	$a' \wedge b'$

which concludes the proof. ■

1.24. EXAMPLE. Let A be a refinement P.O.M. Consider the P.O.M. $\text{Hom}(A, \overline{\mathbb{P}})$ of all P.O.M.-homomorphisms from A to $\overline{\mathbb{P}}$ (equipped with componentwise addition and preordering) — recall that $\overline{\mathbb{P}}$ is $[0, \infty]$ equipped with the canonical $+, 0, \leq$. If u, v are in $\text{Hom}(A, \overline{\mathbb{P}})$, then $u \wedge v$ is simply defined by

$$u \wedge v(a) = \bigwedge \{u(x) + v(y) : x + y = a\} \text{ (all } a \text{ in } A)$$

and it is easy to check that we get a refinement ℓ -P.O.M.. The fact that $\text{Hom}(A, \overline{\mathbb{P}})$ satisfies the finite refinement property is the main result of [31], but the proof presented here is in addition also valid for $\text{Hom}(A, \overline{\mathbb{N}})$ and actually in a much more general context, as we will see in chapter 2.

Now, let us mention the next proposition, analogue to 1.17. Note that by 1.15, 1.16 and 1.23, if A is a refinement ℓ -P.O.M., then $a \vee b$ is defined for all a, b in A and (A, \wedge, \vee) is a distributive lattice.

1.25. Proposition. *Let A be a refinement ℓ -P.O.M., let a in A . Then $\frac{A}{a}$ is a refinement ℓ -P.O.M., and the projection $A \rightarrow \frac{A}{a}$ is a P.O.M. - and - lattice homomorphism.*

Proof. Straightforward. ■

1.26. Corollary. *Every refinement ℓ -P.O.M. satisfies the multiplicative \leq -cancellation property.*

Proof. Let A be a refinement ℓ -P.O.M., let m in $\mathbb{N} \setminus \{0\}$, let a, b in A such that $ma \leq mb$. Let $B = A|b$; then a and b are in B . Let $x \mapsto [x]$ be the canonical projection from B to $\frac{B}{b}$. Thus, we get $m[a] \leq m[b]$. But by 1.17, $\frac{B}{b}$ is the positive cone of some abelian ordered group C ; furthermore, by 1.25, C is a ℓ -group. But it is well-known (see [1]) that C satisfies the statement

$$(\forall x)(mx \geq 0 \Rightarrow x \geq 0),$$

thus it follows that $[a] \leq [b]$, i.e. $a + b \leq 2b$. By the pseudo-cancellation property, it follows that $a \leq b$. ■

Finally, we will need in chapter 2 the following lemma:

1.27. Lemma. *Let A be a refinement ℓ -P.O.M., let a, b, c, u in A such that $a \wedge b + u = a$. Then $a \wedge (b + c) = a \wedge c + u \wedge c$.*

Proof. We just reproduce the usual argument for ℓ -groups:

$$a \wedge (b + c) = a \wedge (a + c) \wedge (b + c) = a \wedge (a \wedge b + c) = (a \wedge b + u) \wedge (a \wedge b + c) = a \wedge b + u \wedge c. \blacksquare$$

2. NOTIONS OF COMPLETENESS FOR P.O.M.'s.

The aim of this chapter is to define some ‘natural’ notion of completeness for P.O.M.’s as close as possible in its consequences to the definition of completeness for Boolean algebras — or ordered abelian groups, and satisfied by some ‘canonical’ examples as *e.g.* the P.O.M. $\overline{\mathbb{P}}^I$ for any set I . For many reasons (some of them out of the scope of this paper), we are convinced that our definition of completeness (definition 2.15) listed in this chapter is the relevant one according to this goal. One of the consequences of this definition will be a ‘well-behaved’ arithmetic.

The motivation of the next definition is rather technical, and will appear wholly in [39]; presently, we will just note that it implies the multiplicative \leq -cancellation property (see 2.9).

2.1. Definition. Let E be a refinement P.O.M. We say that it is *relatively σ -complete* when it satisfies both following conditions:

- (i) Let a, b, c_n ($n \in \omega$) in E such that $a \leq b + c_n$ for all n . Then there is c in E such that for all n , $c \leq c_n$ and $a \leq b + c$;
- (ii) E is *Archimedean*, *i.e.* for all a, b in E such that $(\forall n \in \mathbb{N})(na \leq b)$, we have $a \ll b$.

Note that (ii) is not redundant in definition 2.1, as shows *e.g.* the example of positive cones of non-Archimedean ℓ -groups (see [1]).

An important class of relatively σ -complete P.O.M.’s is the class of *cardinal algebras*, studied widely in [35]. For the convenience of the reader, we will recall here that a cardinal algebra is a structure $(E, +, 0, \sum)$, where $+$ is a two-placed operation on E , $0 \in E$, and \sum is a map from E^ω to E (we write as usual $\sum_{n \in \omega} a_n$, or simply $\sum_n a_n$ for $\sum(a_n)_{n \in \omega}$) satisfying the following axioms:

- (CA 0) $(E, +, 0)$ is a commutative monoid.
- (CA 1) For all $(a_n)_n$ in E^ω , $\sum_n a_n = a_0 + \sum_n a_{n+1}$.
- (CA 2) For all $(a_n)_n, (b_n)_n$ in E^ω , $\sum_n (a_n + b_n) = \sum_n a_n + \sum_n b_n$.
- (CA 3) (Refinement postulate) Let a, b in E , $(c_n)_n$ in E^ω such that $a + b = \sum_n c_n$. Then there are $(a_n)_n, (b_n)_n$ in E^ω such that $(\forall n \in \omega)(c_n = a_n + b_n)$ and $a = \sum_n a_n$, $b = \sum_n b_n$.
- (CA 4) (Remainder postulate) Let $(a_n)_n, (b_n)_n$ in E^ω such that $(\forall n \in \omega)(a_n = a_{n+1} + b_n)$. Then there is a in E such that $(\forall n \in \omega)(a_n = a + \sum_i b_{n+i})$.

2.2. Definition. A *weak cardinal algebra* is a structure $(E, +, 0, \sum)$ satisfying (CA 0), (CA 1), (CA 2), (CA 4) above and the finite refinement property.

Among other things, it is proved in [35] that if E is a cardinal algebra, then the monoid $(E, +, 0)$, equipped with its minimal preordering (which turns out to be an ordering) is a strong refinement P.O.M. (see [35], 1.30, 2.3, 2.6), that $\sum_i a_i$ is necessarily the l.u.b. of all finite sums $\sum_{i < n} a_i$ for $n \in \omega$ ([35], 2.21), thus that \sum is uniquely determined by $(E, +)$. Actually, one can verify that *many theorems proved in [35] to be valid for cardinal algebras are also valid for weak cardinal algebras, including those just listed above*. For example, *every weak cardinal algebra is a relatively σ -complete P.O.M.* (by naturally identifying $(E, +, 0, \sum)$ with $(E, +, 0, \leq)$) - see [35], 2.22 and 2.29.

2.3. EXAMPLES.

(1) The already defined $\overline{\mathbb{P}}, \overline{\mathbb{N}}, \mathbf{2}, \mathbf{1}$ (see 1.4, examples (3) and (4)); in these examples, \sum is defined by $\sum_n a_n = \bigvee \{a_0 + a_1 + \dots + a_n : n \in \omega\}$ ($= \infty$ if the sum is divergent).

(2) Let Ω be a set, let \mathcal{B} be a sub- σ -algebra of $\mathcal{P}(\Omega)$, let \mathcal{I} be a σ -ideal of \mathcal{B} . Let $\mathbf{M}_{\mathcal{B}/\mathcal{I}}(\Omega)$ be the space of all equivalence classes of \mathcal{B} -measurable functions from Ω to $\overline{\mathbb{P}}$ modulo the equivalence relation associated with \mathcal{I} (defined by $f \equiv g \Leftrightarrow \{x : f(x) \neq g(x)\} \in \mathcal{I}$), equipped with its natural definition of (componentwise) infinite addition. The origin of the following result goes back to R. Chuaqui (see [7]):

$\mathbf{M}_{\mathcal{B}/\mathcal{I}}(\Omega)$ is a cardinal algebra.

The non-trivial part of the proof is for the refinement postulate; to find a refinement of $(f_m)_m$ and $(g_n)_n$, the idea is to express the entries of the refinement matrix by ‘polynomials’ (with the operations $+, -, \wedge, \vee, \bigwedge, \bigvee$) in the f_m and g_n in the case where $\sum_n f_n$ has finite values, and to write any function as a countably infinite sum of finite-valued functions in the general case.

A more specialized example is the following one:

(3) Let Ω be a topological space; then the quotient $\mathbf{B}(\Omega, \overline{\mathbb{P}})$ of all maps from Ω to $\overline{\mathbb{P}}$ with the Baire property by equivalence on residual sets is a cardinal algebra.

(4) The space of all σ -additive $\overline{\mathbb{P}}$ -valued (or $\overline{\mathbb{N}}$ -valued) measures on any countably complete Boolean algebra is a cardinal algebra. (see [33] and also theorem 2.42).

(5) Also some proper classes can be considered as cardinal algebras, as e.g. *CARD* (we define $\sum_n a_n$ to be the cardinal of $\bigcup_n A_n$ where $|A_n| = a_n$ and the A_n are mutually disjoint), or the class of isomorphism types of σ -complete Boolean algebras (we define $\sum_n a_n$ as the type of $\prod_n A_n$ where the type of A_n is a_n).

2.4. EXAMPLES. Some examples of weak cardinal algebras are the following:

(1) $\mathbb{N}^2 \cup \{\infty\}$ (where $\sum_n a_n$ is defined to be ∞ when the sum is divergent).

(2) $\text{Hom}(A, \overline{\mathbb{P}})$ when A is a refinement P.O.M (see theorem 2.33). If A is *e.g.* the P.O.M. of bounded positive maps from ω to $\overline{\mathbb{P}}$, then $\text{Hom}(A, \overline{\mathbb{P}})$ is *not* a cardinal algebra (it does not satisfy the infinite refinement property).

For our purposes here, all relevant properties of (weak) cardinal algebras which we will need will be consequences of 2.1. From 2.5 to 2.9, we work in a given relatively σ -complete P.O.M. E .

2.5. Lemma. *Let a, b, c in E such that $a \leq b$ and $b \ll c$. Then $a \ll c$.*

Proof. For all n in \mathbb{N} , $na \leq nb$ because $a \leq b$, and $nb \leq c$ because $b \ll c$. So $na \leq c$, and we conclude by the Archimedean property of E . ■

2.6. Lemma. *\leq is antisymmetric.*

Proof. Immediate from 1.2 and 2.5. ■

2.7. Proposition. *E is a strong refinement P.O.M..*

By hypothesis and 2.6, it suffices to prove the pseudo-cancellation property. So let a, b, c in E such that $a + c \leq b + c$. Using the finite refinement property and 1.11, for all n in \mathbb{N} , there is c_n in E such that $nc_n \leq c$ and $a \leq b + c_n$. By the hypothesis (i), there is d in E such that $(\forall n \in \omega)(d \leq c_n)$ and $a \leq b + d$. But for all n , we have $nd \leq nc_n \leq c$, thus $d \ll c$ by the Archimedean property of E . ■

Now, we will generalize the proof of the multiplicative cancellation property for cardinal algebras ([35], 2.31). Despite the small number of (sometimes not completely trivial) changes brought to the original proof, we shall present here for the convenience of the reader a complete proof, simpler than (but very similar to) the one in [35].

2.8. Lemma. *Let $T \subseteq E \times E \times E$ such that for all (a, b, c) in T , there is (a', b', c') in T such that*

$$a + c = 2a' + c' \text{ and } b + c = a' + b' + c'.$$

Then for all a, b, c in T , $a + c \leq b + c$.

Proof. Define inductively (a_n, b_n, c_n) ($n \in \omega$) in T by $a_0 = a$, $b_0 = b$, $c_0 = c$ and for all n ,

$$a_n + c_n = 2a_{n+1} + c_{n+1} \text{ and } b_n + c_n = a_{n+1} + b_{n+1} + c_{n+1} \quad (1)$$

For all n in ω , we have, by (1), $a + c \leq b + c + a_{1+n}$. By the hypothesis (i), there is d such that

$$(\forall n \in \omega)(d \leq a_{1+n}) \quad (2)$$

and

$$a + c \leq b + c + d \quad (3)$$

But by (1), we have, for all n , $b + c \geq \sum_{i < n} a_{1+i}$, thus $b + c \geq nd$ by (2); therefore, by the Archimedean property of E , $d \ll b + c$, thus, by (3), $a + c \leq b + c$. \blacksquare

2.9. Proposition. *E satisfies the multiplicative \leq -cancellation property (thus also the $=$ -cancellation property since \leq is antisymmetric).*

Proof. Let m in $\mathbb{N} \setminus \{0\}$, put $T_m = \{(a, b, c) \in E^3 : ma + c \leq mb + c\}$. To conclude, it is clearly sufficient to prove that T_m satisfies the hypothesis of 2.8.

Suppose first that $m = 2$. If $(a, b, c) \in T_2$, then there is d such that $2a + c + d = 2b + c$, thus $(a + c) + (a + c + d) = 2(b + c)$. By 1.9 (for $n = 2$), there are a', b' such that

$$a + c = 2a' + c', \quad a + c + d = 2b' + c', \quad b + c = a' + b' + c'$$

Thus $(a', b', c') \in T_2$, and so we conclude for $m = 2$.

Now, suppose $m \in \mathbb{N} \setminus \{0\}$ arbitrary, and let (a, b, c) in T_m . Let l in \mathbb{N} such that $2^l \leq m < 2^{l+1}$. Then we get $2^l a + c \leq 2^{l+1} b + c$, thus $a + c \leq 2b + c$ by the case $m = 2$, thus there is d such that $a + d + c = 2b + c$. Let $(x \mapsto [x])$ the canonical projection from E to $\frac{E}{c}$, so that we have $[a] + [d] = 2[b]$. But by 1.17, $\frac{E}{c}$ is a strong refinement P.O.M., thus by 1.9 (for $n = 2$), there are u, v, w such that

$$[a] = 2[u] + [w], \quad [d] = 2[v] + [w], \quad [b] = [u] + [v] + [w] \quad (1)$$

Since $m[a] \leq m[b]$, we get

$$2m[u] + m[w] \leq m[u] + m[v] + m[w],$$

thus, since $\frac{E}{c}$ satisfies the pseudo-cancellation property,

$$m[u] + [w] \leq m[v] + [w] + [e] \text{ for some } [e] \ll [u] \text{ (so } e + u + c = u + c) \quad (2)$$

Now, put $a' = u$, $b' = v + e$, $c' = c + w$. Using (1) and (2), it is easy to check that we have

$$2a' + c' = a + c, \quad a' + b' + c' = b + c$$

and

$$ma' + c' \leq mb' + c' \text{ (thus } (a', b', c') \in T_m),$$

which concludes the proof. ■

Now, we shall explore various strengthenings of the notion of relative σ -completeness, which will ultimately lead to the notions of complete P.O.M.'s (definition 2.15) and of injective P.O.M.'s (definition 3.1).

2.10. Definition. Let E be a minimal, antisymmetric P.O.M.. We say that E is *weakly complete* when it satisfies the following conditions:

- (i) Every subset X of E has a g.l.b., denoted by $\bigwedge X$;
- (ii) $+$ is *distributive on \bigwedge* , i.e.:

$$(\forall x \in E)(\forall X \subseteq E)(\bigwedge(a + X) = a + \bigwedge X).$$

Note that then, E has always a largest element, $\bigwedge \emptyset$, which we will denote by ∞_E , or ∞ if there is no ambiguity. So E satisfies $(\forall x)(\infty + x = \infty)$. Note also that for all x , ∞x exists, and that more generally, every subset of E has a l.u.b. . This entitles us to define $\sum_{i \in I} x_i$ for every family $(x_i)_{i \in I}$ of elements of E , by putting

$$\sum_{i \in I} x_i = \bigvee \left\{ \sum_{i \in J} x_i : J \subseteq I \text{ finite} \right\}$$

Now, we shall extend our definition. From 2.11 to 2.14, we show that many possible extensions are equivalent.

2.11. Lemma. Let E be a weakly complete P.O.M., let $(a_n)_n, (b_n)_n$ be in E^ω such that $(\forall n \in \omega)(a_n \leq a_{n+1} + b_n)$, let $a = \bigwedge_{n \in \omega} a_n$. Then $a_0 \leq a + \sum_n b_n$.

Proof. For all n , we have $a_0 \leq a_n + \sum_{i < n} b_i \leq a_n + \sum_{i \in \omega} b_i$, thus, by taking the g.l.b.'s of both sides, we get the result. ■

2.12. Proposition. Let E be a weakly complete P.O.M.. Then the following are equivalent:

- (i) E is relatively σ -complete;
- (ii) E satisfies the pseudo-cancellation property;
- (iii) E satisfies the finite refinement property;
- (iv) E satisfies the finite (\leq, \leq) -refinement property.

Proof. (i) \Rightarrow (ii) is immediate by 2.7. (ii) \Rightarrow (i) Assume (ii). If a, b, c_n ($n \in \omega$) are in E such that $a \leq b + c_n$ for all n , let $c = \bigwedge_{n \in \omega} c_n$. Then $a \leq b + c$ by distributivity of $+$

on \bigwedge , and $c \leq c_n$ for all n . On the other hand, if a, b are in E such that $na \leq b$ for all n , then, since $\infty a = \bigvee \{na : n \in \mathbb{N}\}$ exists, we have $\infty a \leq b$, but by 1.18, $a \ll \infty a$, hence $a \ll b$. Thus E is relatively σ -complete. (ii) \Rightarrow (iii) Assume (ii). Then E is a refinement ℓ -P.O.M. (see 1.22), thus it satisfies the finite refinement property (see 1.23). (iii) \Rightarrow (iv) is trivial because E is minimal. We conclude by showing that (iv) \Rightarrow (ii). Assume (iv). Let a, b, c in E such that $a + c \leq b + c$. Define inductively a_n, b_n, c_n, d_n ($n \in \omega$) by $a_0 = a, b_0 = b, c_0 = c$ and if $a_n + c_n \leq b_n + c_n$, then we have the following refinement matrix:

	$\leq b_n$	$\leq c_n$
$a_n \leq$	d_n	a_{n+1}
$c_n \leq$	b_{n+1}	c_{n+1}

so that the induction hypothesis is maintained. Now, let $e = \bigwedge_{n \in \omega} a_n$. Since $a_n \leq a_{n+1} + d_n$ for all n , 2.11 implies that $a \leq e + \sum_n d_n$. Since $b_n \geq b_{n+1} + d_n$ for all n , we have $b \geq \sum_n d_n$. Hence, $a \leq b + e$. Finally, if $d = \bigwedge_n c_n$, then, for all n , $e + d \leq a_{n+1} + c_{n+1} \leq c_n$, thus $e + d \leq d$, hence $e \ll d$; but $d \leq c$, thus $e \ll c$, which concludes the proof. \blacksquare

2.13. EXAMPLE. This example shows the relevance of previous proposition: equip $E = \{0, 1, 2\}$ with the addition defined by $x \oplus y = (x + y) \wedge 2$, and its natural (linear) ordering. Then $2 \oplus 1 = 1 \oplus 1 = 2$, but no $x \ll 1$ satisfies $2 \leq 1 \oplus x$. Thus E is a weakly complete P.O.M. without the pseudo-cancellation property.

The last obstacle towards our definition of completeness is the satisfaction of the *absorption property*, formulated in the next definition (see also [35], 1.47).

2.14. Proposition *Let E be a weakly complete P.O.M. satisfying the pseudo-cancellation property. Then the following are equivalent:*

- (i) *For all $a \in E, X \subseteq E$ such that $X \ll a$, we have $\bigvee X \ll a$ (absorption property);*
- (ii) *For all $a \in E, X \subseteq E, X \neq \emptyset$, we have $a + \bigvee X = \bigvee(a + X)$ (property which we will call ‘distributivity of $+$ on $\bigvee \neq \emptyset$ ’).*

Proof. Similar argument as for the proof of 3.26 of [35], adapted to arbitrary l.u.b.’s. We present it here.

(ii) \Rightarrow (i) Assume (ii) satisfied; let $a \in E, X \subseteq E, X \neq \emptyset, X \ll a$. If $X \neq \emptyset$, then we have

$$a + \bigvee X = \bigvee(a + X) = \bigvee\{a\} = a,$$

which is still true if $X = \emptyset$.

(i) \Rightarrow (ii) Assume (i) satisfied; let $a \in E$, $X \subseteq E$, $X \neq \emptyset$, let $b = \bigvee X$, $c = \bigvee(a + X)$. It is obvious that $c \leq a + b$. Conversely, since $X \neq \emptyset$, $a \leq c$, thus there is d such that $a + d = c$. For all $x \in X$, $a + x \leq a + d$, thus there is $y \ll a$ such that $x \leq d + y$. Therefore, if $Y = \{y \in E : y \ll a\}$ and $e = \bigvee Y$, then $e \ll a$ by hypothesis and $(\forall x \in X)(x \leq d + e)$, thus $b \leq d + e$. Hence $a + b \leq a + d + e = c$. ■

2.15. Definition. A *complete P.O.M.* is a weakly complete P.O.M. satisfying the pseudo-cancellation property and the absorption property.

2.16. EXAMPLE. Let Ω be a set, let \mathcal{B} be a countably complete Boolean subalgebra of subsets of Ω , let \mathcal{I} be a countably complete ideal of sets in \mathcal{B} such that $B = \mathcal{B}/\mathcal{I}$ is countably saturated (thus complete). Then the P.O.M. of all equivalence classes modulo \mathcal{I} of \mathcal{B} -measurable functions from Ω to $\overline{\mathbb{P}}$ (or $\overline{\mathbb{N}}$) is a complete P.O.M..

2.17. Proposition. Every complete P.O.M. satisfies the statement

$$(\forall x)(2(\infty x) = \infty x)$$

Proof. Using the distributivity of $+$ on $\bigvee \neq \emptyset$ and 1.19, we have

$$\infty x + \infty x = \infty x + \bigvee \{nx : n \in \mathbb{N}\} = \bigvee \{\infty x + nx : n \in \mathbb{N}\} = \infty x.$$

2.18. EXAMPLE. Let $M = \mathbb{N}[c]$ be the P.O.M. of example 1.20, let $E = M \cup \{\infty\}$. It is easy to check that E is a weakly complete P.O.M. satisfying the pseudo-cancellation property. However, $\mathbb{N} \ll c$ but $\bigvee \mathbb{N} = c$ and $2c \neq c$, so that E is not a complete P.O.M..

A remarkable feature of complete P.O.M.'s is given by next proposition and the following example.

2.19. Proposition. Let E be a complete P.O.M.. Then for all $a \in E$, $X \subseteq E$, we have

$$a \vee \bigwedge X = \bigwedge (a \vee X)$$

Proof. As in [35], theorem 2.30. We present it here:

Let $b = \bigwedge X$, $c = a \vee b$, $d = \bigwedge (a \vee X)$. Clearly, $c \leq d$. Conversely, there is some e such that $e + b = c$, thus $\bigwedge (e + X) = c$ since E is weakly complete, thus, for all x in X , $a \leq e + x$, hence $d \leq a \vee x \leq (e + x) \vee x = e + x$, hence $d \leq \bigwedge (e + X) = c$. ■

2.20. EXAMPLE. $\mathbb{N}^2 \cup \{\infty\}$ is a complete P.O.M. where \wedge is not distributive on \vee : if $a = (1, 0)$, $b = (0, 1)$ and $X = \{nb : n \in \mathbb{N}\}$, then $a \wedge \bigvee X = a$ but $\bigvee(a \wedge X) = 0$.

Now we shall consider two important operations, the *inf difference* and the *sup difference*, and prove several identities involving them and the other operations. From 2.21 to 2.26, we work in a fixed complete P.O.M. E .

2.21. Definition. For all a, b in E , we put

$$a \setminus b = \bigwedge \{x \in E : a \leq b + x\} \text{ (inf difference of } a \text{ and } b),$$

$$a - b = \bigvee \{x \in E : b + x \leq a\} \text{ (sup difference of } a \text{ and } b).$$

Note also that $b \not\leq a$ implies $a - b = 0$. In fact, only the case $b \leq a$ will be considered when the expression $a - b$ will be used. Note that in that case, $a \setminus b \leq a - b$.

Notational convention. When we write any term involving $\bigwedge, \bigvee, \sum, \wedge, \vee, \setminus, -, +$, we put stress first on $\bigwedge, \bigvee, \sum, \wedge, \vee$, then on $\setminus, -,$ then on $+$. For example,

$$a - b \wedge \bigvee X + c = (a - (b \wedge \bigvee X)) + c.$$

2.22. Lemma. For all a, b, c, a', b' in E , we have

- (i) $b + (a \setminus b) = a \vee b$;
- (ii) $a \setminus b = a \vee b \setminus b = a \setminus a \wedge b$;
- (iii) $a \wedge b + a \setminus b = a$;
- (iv) $a + (b - a) = b$ if $a \leq b$;
- (v) $a \wedge (b + c) = a \wedge b + (a \setminus b) \wedge c = a \wedge b + (a - a \wedge b) + c$;
- (vi) $a + (b - c) = (a + b) - c$ if $c \leq b$. Thus, $a \ll b$ implies $a \ll b - c$;
- (vii) $b + c \leq a$ if and only if $b \leq a$ and $c \leq a - b$, and then $(a - b) - c = a - (b + c)$;
- (viii) $(a' + b') - (a + b) = (a' - a) + (b' - b)$ if $a \leq a'$ and $b \leq b'$.

Proof. Easy, using 1.27 for the proofs of (ii) and (v). ■

One of the most noticeable properties of complete P.O.M.'s is the following easily proved

2.23. Proposition. Every complete P.O.M. is a weak cardinal algebra. ■

Note that by example 2.20, a complete P.O.M. is not necessarily a cardinal algebra.

Using lemma 2.22, we can get a more precise picture of the problem of the distributivity of \wedge on \bigvee . First, we extend in the obvious way definition 1.3 to transfinite m and n . Then, we set the

2.24. Definition. Let κ be a cardinal. We say that E is κ -distributive when for all $a \in E$ and $X \subseteq E$ of cardinal $\leq \kappa$, we have $a \wedge \bigvee X = \bigvee(a \wedge X)$.

Note that E is always κ -distributive when κ is finite (see lemma 1.16, (vii)).

2.25. Proposition. Let κ be an infinite cardinal. Then the following are equivalent:

- (i) For all a, b_ξ ($\xi \leq \kappa$) in E with $(b_\xi)_\xi$ increasing, we have $a \wedge \bigvee_\xi b_\xi = \bigvee_\xi(a \wedge b_\xi)$;
- (ii) E is κ -distributive;
- (iii) E satisfies the $(1 =, \leq \kappa)$ -refinement property.

Proof. (i) \Leftrightarrow (ii) easy. (ii) \Rightarrow (iii) Assume (ii). Let $a \leq \sum_{\xi < \kappa} b_\xi$ in E . Put $\bar{b}_\xi = \sum_{i < \xi} b_i$ for all $\xi \leq \kappa$, $a_\xi = (a \setminus \bar{b}_\xi) \wedge b_\xi$ for all $\xi < \kappa$. Using 2.22 (v) and κ -distributivity, it is easy to prove by induction that $\sum_{i < \xi} a_i = a \wedge \bar{b}_\xi$ for all $\xi \leq \kappa$. Thus $a = \sum_{\xi < \kappa} a_\xi$ with $(\forall \xi < \kappa)(a_\xi \leq b_\xi)$. Finally, the proof of (iii) \Rightarrow (ii) proceeds as in [35], lemma 3.31. ■

Note that in [15], it is shown that in weak cardinal algebras, the $(1 =, \leq \omega)$ -refinement property is equivalent to the (ω, ω) -refinement property.

Many other arithmetical properties can be proved about complete P.O.M.'s (including relations involving infinite meet and join and the sup - and - inf differences). Let us just mention the following ones:

2.26. Lemma. Let $X \subseteq E$, $m \in \bar{\mathbb{N}}$. Then we have:

- (i) $\bigvee(m \cdot X) = m \cdot \bigvee X$;
- (ii) If $m < \infty$, then $\bigwedge(m \cdot X) = m \cdot \bigwedge X$.

Proof. Virtually the same as in [35], theorems 3.27 and 3.28 (where it is proved for cardinal algebras, but works also for weak cardinal algebras and in our context); we do not reproduce it here. ■

We shall now see how to construct new complete P.O.M.'s using old ones. Consider first the following question:

(*) If A is a P.O.M. and E is a complete P.O.M., is $\text{Hom}(A, E)$ complete?

There are various counterexamples to (*), e.g. when A is the P.O.M. of linear subspaces of \mathbb{R}^2 equipped with the usual sum and the inclusion, and $E = \mathbf{2}$. Here, A does not satisfy the $(1 =, \leq 2)$ -refinement property. But when A is $\{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ :$

$x \leq y\}$, equipped with componentwise $+$, 0 and \leq , then A is a convex cone of \mathbb{R}^2 included in the cone of positive elements, thus (see 1.4, example (7)) A satisfies the finite (\leq, \leq) , $(\leq, =)$, $(=, \leq)$ refinement properties. However, $\text{Hom}(A, \overline{\mathbb{P}})$ is not even minimal (if $u : (x, y) \mapsto x$, $v : (x, y) \mapsto y$, then $u \leq v$ but there is no w in $\text{Hom}(A, \overline{\mathbb{P}})$ such that $u + w = v$). So we have to make some stronger assumption on A in order to get a positive answer.

From 2.27 to 2.32, let A be a refinement P.O.M. and E a complete P.O.M..

2.27. Lemma. $\text{Hom}(A, E)$ has a largest element, $\overline{\infty}$.

Proof. It is $(\overline{\infty} : x \mapsto (0 \text{ if } x \leq 0, \infty \text{ if } x > 0))$. The verification is trivial. ■

Now, for any u, v in $\text{Hom}(A, E)$ such that $u \leq v$, define

$$v - u : A \rightarrow E, x \mapsto v(x) - u(x).$$

2.28. Lemma. $v - u \in \text{Hom}(A, E)$, and $u + (v - u) = v$. Thus, $\text{Hom}(A, E)$ is minimal.

Proof. By 2.22 (viii) and minimality of A , we have $v - u \in \text{Hom}(A, E)$. The second assertion comes from 2.22 (iv). ■

Now, for any $X \subseteq \text{Hom}(A, E)$, define $u_X : A \rightarrow E$ by

$$u_X(a) = \bigwedge \left\{ \sum_{i < n} u_i(a_i) : n \in \omega \setminus \{0\}, \sum_{i < n} a_i = a, u_i \in X \right\} \text{ if } X \neq \emptyset, \text{ and } u_\emptyset = \overline{\infty}.$$

2.29. Lemma. u_X is the g.l.b. of X in $\text{Hom}(A, E)$.

Proof. Put $u = u_X$. For $X = \emptyset$, it is just 2.27; suppose $X \neq \emptyset$. Using distributivity of $+$ on \bigwedge , we get immediately $(\forall a, b \in A)(u(a + b) \leq u(a) + u(b))$, and using the finite refinement property of A , we get immediately $(\forall a, b \in A)(u(a) + u(b) \leq u(a + b))$. Since $X \neq \emptyset$, the definition of u_X yields $u(0) = 0$. And A is minimal, thus $u \in \text{Hom}(A, E)$. By definition, if $v \in \text{Hom}(A, E)$ and $v \leq X$, then $v \leq u$. Thus u is $\bigwedge X$ in $\text{Hom}(A, E)$. ■

2.30. Lemma. $+$ is distributive on \bigwedge in $\text{Hom}(A, E)$.

Proof. Immediate by definition of u_X and the corresponding property in E . ■

2.31. Lemma. $\text{Hom}(A, E)$ satisfies the pseudo-cancellation property.

Proof. Suppose $u, v, w \in \text{Hom}(A, E)$ and $u + w = v + w$. Then, by 2.22, (viii), $u \leq u + (w - w) = v + (w - w)$ and $w - w \in \text{Hom}(A, E)$ by 2.28, hence the conclusion holds. ■

2.32. Lemma. $\text{Hom}(A, E)$ satisfies the absorption property.

Proof. Let $X \subseteq \text{Hom}(A, E)$ nonempty, let $u \in \text{Hom}(A, E)$ such that $X \ll u$. Let $(w : A \rightarrow E, x \mapsto \sum_{v \in X} v(x))$. Then $w \in \text{Hom}(A, E)$, $\bigvee X \leq w$, and $w \ll u$. Thus $\bigvee X \ll u$. ■

Now, from 2.28 to 2.32 (and definition 2.15), we immediately get the

2.33. Theorem. Let A be a refinement P.O.M., let E be a complete P.O.M.. Then $\text{Hom}(A, E)$ is a complete P.O.M.. ■

2.34. REMARK. 2.33 admits an easy generalization to what we could call ‘partial P.O.M.’s’ (the addition is not defined everywhere). This yields for example the following result:

If B is a Boolean algebra and E is a complete P.O.M., then the P.O.M. of all E -valued measures on B is a complete P.O.M..

($u : B \rightarrow E$ is a *measure* when $u(0) = 0$ and $u(x \vee y) = u(x) + u(y)$ whenever $x \wedge y = 0$).

The following proposition will allow us to extend theorem 2.33, and has also some independent interest:

2.35. Proposition. Any retract of a complete P.O.M. is a complete P.O.M..

Proof. Let E be a retract of a complete P.O.M. F ; this means that E is a sub-P.O.M. of F and there is a P.O.M.-homomorphism π from F to E such that $\pi|_E = \text{id}$. For every subset X of E , denote by $\bigwedge_F X$ its g.l.b. in F ; then it is easy to check that $\pi(\bigwedge_F X)$ is the g.l.b. of X in E . The rest of the verifications are straightforward. ■

Now, we shall extend theorem 2.33 to countably additive homomorphisms instead of just homomorphisms. The natural context is when A is a *cardinal algebra* and E is a complete P.O.M.; we denote by $\text{Hom}^\sigma(A, E)$ the P.O.M. of σ -additive homomorphisms $A \rightarrow E$, i.e. of those homomorphisms u such that $u(\sum_n x_n) = \sum_n u(x_n)$ for all $(x_n)_n$ in A^ω .

In 2.36 and 2.37, let A be a cardinal algebra, let E be a complete P.O.M.. We denote by H the P.O.M. $\text{Hom}(A, E)$, and H^σ the P.O.M. $\text{Hom}^\sigma(A, E)$. For each u in H , define $u' : A \rightarrow E$ by $u'(a) = \bigwedge \{ \sum_{i < \omega} u(a_i) : (a_i)_i \in A^\omega, \sum_i a_i = a \}$.

2.36. Lemma. Let u in H . Then $u' \in H$, $u' \leq u$, and $u' = u$ if and only if u is σ -additive. Furthermore, the map $(u \mapsto u')$ is a P.O.M.-homomorphism from H to H .

Proof. Easy, using the (ω, ω) -refinement property. ■

Now, for each u in H , define a ON - sequence $(u_\xi)_\xi$ by $u_0 = u$, $u_{\xi+1} = (u_\xi)'$, and $u_\lambda = \bigwedge_{\xi < \lambda} u_\xi$ if λ is a limit ordinal (the fact that $u_\lambda \in H$ is immediate by distributivity of $+$ on \bigwedge). There is an ordinal θ such that $u_\theta = u_{\theta+1}$. Define $\pi(u) = u_\theta$.

2.37. Lemma. *Let u in H . Then $\pi(u)$ is the largest v in H^σ such that $v \leq u$. Furthermore, π is a retraction from H onto H^σ .*

Proof. Put $w = \pi(u)$. Then $w = w'$ by definition. The rest follows immediately from 2.36. ■

Now, 2.35 and 2.37 imply immediately our result:

2.38. Theorem. *Let A be a cardinal algebra, let E be a complete P.O.M.. Then $\text{Hom}^\sigma(A, E)$ is a complete P.O.M..* ■

Note that this implies in particular that $\text{Hom}^\sigma(A, E)$ satisfies the finite refinement property.

2.39. REMARK. It is easy to generalize theorem 2.38 to *generalized cardinal algebras*. As a possible application, we get *e.g.* the following:

Let B be a σ -complete Boolean algebra and E a complete P.O.M., then the P.O.M. of σ -additive E -valued measures on B is a complete P.O.M..

Finally, we shall give some more information in the context of 2.38 in the case where E is *linearly ordered*. Say that an element α of E is *isolated* when $\alpha < \bigwedge\{x \in E : \alpha < x\}$.

In 2.40 and 2.41, let A be a cardinal algebra, let E be a linearly ordered complete P.O.M.; put $H = \text{Hom}(A, E)$, $H^\sigma = \text{Hom}^\sigma(A, E)$.

2.40. Lemma. *Let u, v in H^σ . Then $u \wedge v$ calculated in H and in H^σ are the same.*

Proof. Let w be $u \wedge v$ calculated in H . We have seen in 2.29 that w is given by the formula

$$w(a) = \bigwedge \{u(x) + v(y) : x + y = a\}.$$

Let $a = \sum_n a_n$ in A , let $\alpha = \sum_n w(a_n)$, let ε in E such that $\alpha < \alpha + \varepsilon$, with ε minimum with that property if α is isolated. By lemma 2.26, for each n , there are x_n, y_n such that $x_n + y_n = a_n$ and $2^{n+1}(u(x_n) + v(y_n)) < 2^{n+1}w(a_n) + \varepsilon$. Thus, using 2.9, it follows easily that $u(\sum_{i < n} x_i) + v(\sum_{i < n} y_i) \leq \alpha + \varepsilon$ and furthermore, that the inequality is strict if α is isolated. Put $x = \sum_n x_n$, $y = \sum_n y_n$, so that $w(a) \leq u(x) + v(y)$. Since u and v are in H^σ , it follows easily that $w(a) \leq \alpha$; since w is finitely additive, the converse is true. ■

2.41. Lemma. Let $X \subseteq H^\sigma$ be directed (i.e. such that $(\forall x, y \in X)(\exists z \in X)(x, y \leq z)$). Then $\bigvee X$ is the same calculated in H and in H^σ .

Proof. Let $u = \bigvee X$ calculated in H . Since X is directed, we have

$$(\forall a \in A)(u(a) = \bigvee \{v(a) : v \in X\}).$$

We prove that u is σ -additive. So, let $a = \sum_{n \in \omega} a_n$ in A ; for all v in X , we have, since v is σ -additive,

$$v(a) \leq \sum_n v(a_n) \leq \sum_n u(a_n),$$

thus, taking l.u.b. of both sides in v , we get $u(a) \leq \sum_n u(a_n)$; the converse inequality holds because u is finitely additive. ■

2.42. Theorem. Let A be a cardinal algebra, let E be a linearly ordered complete P.O.M.. Then $\text{Hom}^\sigma(A, E)$ is a ω -distributive complete P.O.M.. In particular, it is a cardinal algebra.

Proof. Put $H = \text{Hom}(A, E)$, $H^\sigma = \text{Hom}^\sigma(A, E)$. By 2.25, 2.40 and 2.41, it is sufficient to prove that if $u \in H^\sigma$ and $(v_n)_{n \in \omega}$ is an increasing sequence of elements of H^σ , then $u \wedge \bigvee_n v_n = \bigvee_n (u \wedge v_n)$ in H ; actually, our proof will not use the fact that the v_n are countably additive.

Let $\bar{u} = \bigvee_n (u \wedge v_n)$, $v = \bigvee_n v_n$. Obviously, we have $\bar{u} \leq u \wedge v$. Conversely, let a in A ; put $\alpha = \bar{u}(a)$, let ε in E such that $\alpha < \alpha + \varepsilon$, and ε minimum with this property if α is isolated. We construct sequences $(x_n)_n$ and $(y_n)_n$ of elements of A the following way. Let x_0 and y_0 such that $x_0 + y_0 = a$ and $2(u(x_0) + v_0(y_0)) \leq 2 \cdot u \wedge v_0(a) + \varepsilon$, the inequality being strict if α is isolated. Now suppose that x_n and y_n are constructed such that

$$x_n + y_n = a \text{ and } 2^{n+1}(u(x_n) + v_n(y_n)) \leq 2^{n+1} \cdot u \wedge v_n(a) + \varepsilon, \quad (1)$$

the last inequality being strict if α is isolated. There are h_n and y_{n+1} such that

$$y_n = h_n + y_{n+1} \text{ and } 2^{n+3}(u(h_n) + v_{n+1}(y_{n+1})) < 2^{n+3} \cdot u \wedge v_{n+1}(y_n) + \varepsilon. \quad (2)$$

Put $x_{n+1} = x_n + h_n$. Thus, it follows that

$$\begin{aligned} 2^{n+1}(u(x_n) + u \wedge v_n(y_n)) &\leq 2^{n+1}(u(x_n) + v_n(y_n)) \\ &\leq 2^{n+1} \cdot u \wedge v_n(a) + (2^{n+1} - 1)\varepsilon. \end{aligned}$$

Writing $a = x_n + y_n$ and $u \wedge v_n(y_n) \leq u \wedge v_{n+1}(y_n)$, we obtain that

$$2^{n+1}(u(x_n) + u \wedge v_{n+1}(y_n)) \leq 2^{n+1}(u \wedge v_n(x_n) + u \wedge v_{n+1}(y_n)) + (2^{n+1} - 1)\varepsilon. \quad (3)$$

It follows that

$$\begin{aligned}
2^{n+3}(u(x_{n+1}) + v_{n+1}(y_{n+1})) &\leq 2^{n+3}(u(x_n) + u \wedge v_{n+1}(y_n)) + \varepsilon \text{ (by (2))} \\
&\leq 2^{n+3}(u \wedge v_n(x_n) + u \wedge v_{n+1}(y_n)) + (2^{n+3} - 3)\varepsilon \text{ (by(3))} \\
&\leq 2^{n+3} \cdot u \wedge v_{n+1}(a) + (2^{n+3} - 3)\varepsilon \\
&\leq 2^{n+3} \cdot u \wedge v_{n+1}(a) + (2^{n+3} - 2)\varepsilon,
\end{aligned}$$

the last inequality being strict when α is isolated and $\alpha + \varepsilon < \alpha + 2\varepsilon$. If α is isolated and $\alpha + \varepsilon = \alpha + 2\varepsilon$, then ε is idem-multiple $> \alpha$ and the above calculation shows easily that $u(x_{n+1}) + v_{n+1}(y_{n+1}) < \varepsilon$ (note that since E is linearly ordered and by 2.9, $x < \varepsilon$ and $y < \varepsilon$ implies $x + y < \varepsilon$). Hence in every case, (1) is satisfied with n replaced by $n + 1$, the inequality being strict when α is isolated. So we have constructed our sequences $(x_n)_n$ and $(y_n)_n$. Since A is a cardinal algebra and $(x_n)_n$ is increasing, $x = \bigvee_n x_n$ exists in A , and for all n , $x \leq a \leq x + y_n$. Since A is a cardinal algebra, there is y such that $a = x + y$ and $(\forall n \in \omega)(y \leq y_n)$. Now, for all n, k in ω , we have, using (1),

$$\begin{aligned}
2^{n+k+1}(u(x_{n+k}) + v_n(y)) &\leq 2^{n+k+1}(u(x_{n+k}) + v_{n+k}(y_{n+k})) \\
&\leq 2^{n+k+1} \cdot \alpha + (2^{n+k+1} - 1)\varepsilon,
\end{aligned}$$

the last inequality being strict if α is isolated.

Thus, if α is isolated, then $u(x_{n+k}) + v_n(y) \leq \alpha$ by assumption. Since $u \wedge v(a) \leq u(x) + v(y)$, we obtain, making k , then n go to infinity, $u \wedge v(a) \leq \alpha + \varepsilon$, and $u \wedge v(a) \leq \alpha$ if α is isolated. The conclusion follows. \blacksquare

2.43. QUESTION. Does the conclusion of theorem 2.42 still hold for arbitrary σ -distributive complete P.O.M.'s?

3. INJECTIVE P.O.M.'s.

The ultimate notion of completeness we will present in this work is a seemingly slight strengthening of the notion of complete P.O.M.. The corresponding definition will be much simplified, and it will furthermore admit a simple algebraic equivalent (theorem 3.11).

3.1. Definition. Let E be a P.O.M..

- (i) For all m in $\mathbb{N} \setminus \{0\}$, E is *m-divisible* when $mE = E$; E is \mathbb{N} -divisible when it is *m-divisible* for all m in $\mathbb{N} \setminus \{0\}$.
- (ii) E is *injective* when for every sub-P.O.M. A of a P.O.M. B , every P.O.M.-homomorphism from A to E extends to a P.O.M.-homomorphism from B to E .

It is well-known that the definition corresponding to (ii) for abelian groups yields exactly the \mathbb{N} -divisible abelian groups. However, we will see that the situation is very different in the case of P.O.M.'s; still, many similarities will appear.

From 3.2 to 3.9, we give ourselves a weakly complete P.O.M. E , satisfying 2=-cancellation and 2-divisibility (i.e. $(\forall x)(\exists! y)(y = 2x)$). We denote by $(x \mapsto \frac{x}{2^n})$ the inverse automorphism of $(x \mapsto 2^n x)$ (all n in \mathbb{N}).

3.2. Lemma. *For all a in E , $2(\infty a) = \infty(2a) = \infty a$.*

Proof. The first equality comes from the fact that $(x \mapsto 2x)$ is an automorphism of E , the second one from the definition. ■

3.3. Lemma. *E satisfies the pseudo-cancellation property.*

Proof. Let a, b, c in E such that $a + c \leq b + c$. Thus $2^n a + c \leq 2^n b + c$ for all n in \mathbb{N} , thus $a + \frac{c}{2^n} \leq b + \frac{c}{2^n}$. Let $d = \bigwedge \{ \frac{c}{2^n} : n \in \mathbb{N} \}$. Then, by distributivity of $+$ on \bigwedge , $a + d \leq b + d$, and moreover, $2^n d \leq c$ for all n . Thus $\infty d \leq c$, but ∞d is idem-multiple by 3.2, thus $d \ll c$. ■

3.4. Lemma. *E is a complete P.O.M..*

Proof. It suffices, by definition, to prove the absorption property. So let a in E , $X \subseteq E$ such that $X \ll a$; let $b = \bigvee X$. Then $2^n X \leq a$ for all n in \mathbb{N} , thus $\bigvee 2^n X \leq a$, thus, since $(x \mapsto 2^n x)$ is an automorphism of E , $2^n b \leq a$. Thus $\infty b \leq a$, hence $b \ll a$ by 3.2. ■

By 2.9, we see in particular that E satisfies the multiplicative \leq -cancellation property. The theory of *real multiples* in E follows, with methods whose origin goes back to [35] (end of chapter 2); see also [6], chapter 1.

First, let D be the set of positive dyadic numbers, i.e. $D = \{ \frac{p}{2^n} : p, n \in \mathbb{N} \}$. If $p, n \in \mathbb{N}$ and $x \in E$, we define $\frac{p}{2^n} x = \frac{px}{2^n}$. It is easy to see that this definition is coherent (by 2-divisibility for ‘existence’, and 2=-cancellation for ‘uniqueness’). Finally, for every (α, x) in $\mathbb{R}_+ \times E$, we define $\alpha x = \bigvee \{ rx : r \in D \text{ and } r \leq \alpha \}$. All the relevant information about real products that we will need is concentrated in the following

3.5. Lemma. *The map $\alpha \mapsto h_\alpha$ (where $h_\alpha : x \mapsto \alpha x$) is a homomorphism from $(\mathbb{R}_+, +, \cdot, 0, 1)$ to the space of P.O.M.-endomorphisms of E .*

Proof. The proofs presented in [6], chapter 1 can easily be adapted to the present context, after having replaced \mathbb{Q}_+ by D . ■

Note that lemma 3.5 implies immediately that E is \mathbb{N} -divisible. Now, we shall prove that E is injective. The proof here will be similar to the proof of Tarski's theorem (the basic reference is [34]; see also [35] 14.13, or [36] 9.1), which actually shows injectivity of $\overline{\mathbb{P}}$.

In 3.6 and 3.7, let B be a P.O.M., A a sub-P.O.M. of B , and u in $\text{Hom}(A, E)$.

3.6. Definition. For all b in B , we define (using 3.5)

$$u_*(b) = \bigvee \left\{ \frac{u(y) \setminus u(x)}{n} : x, y \in A \text{ and } n \in \mathbb{N} \setminus \{0\} \text{ and } y \leq x + nb \right\},$$

$$u^*(b) = \bigwedge \left\{ \frac{u(y) - u(x)}{n} : x, y \in A \text{ and } n \in \mathbb{N} \setminus \{0\} \text{ and } x + nb \leq y \right\}.$$

In a measure-theoretic analogy, $u_*(b)$ and $u^*(b)$ correspond respectively to inner and outer measure of b .

3.7. Lemma. For all b in B , the following holds:

(i) For all x, y in A , n in \mathbb{N} , we have

$$y \leq x + nb \Rightarrow u(y) \leq u(x) + nu_*(b)$$

and

$$x + nb \leq y \Rightarrow u(x) + nu^*(b) \leq u(y);$$

(ii) $u_*(b) \leq u^*(b)$;

(iii) Let I_b be defined as follows:

$$I_b = \{\beta \in E : (\forall x, y \in A)(\forall m, n \in \mathbb{N})(x + mb \leq y + nb \Rightarrow u(x) + m\beta \leq u(y) + n\beta)\}$$

Then $u_*(b)$ and $u^*(b)$ are elements of I_b . In fact, $u_*(b) = \min(I_b)$, $u^*(b) = \max(I_b)$.

Proof. (i) is easy and uses distributivity of $+$ on \bigwedge and $\bigvee \neq \emptyset$. Let us prove (ii). It is sufficient to prove that for all x, x', y, y' in E , m, n in $\mathbb{N} \setminus \{0\}$ such that

$$x' \leq x + mb \text{ and } y + nb \leq y', \tag{1}$$

we have

$$\frac{u(x') \setminus u(x)}{m} \leq \frac{u(y') - u(y)}{n}, \tag{2}$$

which can be written

$$u(nx') \setminus u(nx) \leq u(my') - u(my). \quad (3)$$

Using 1.21, with $a = u(nx)$, $b = u(nx')$, $a' = u(my)$, $b' = u(my')$ - note that by (1), $y \leq y'$ thus $u(my) \leq u(my')$ - we see that it is sufficient to prove

$$u(my) + u(nx') \leq u(nx) + u(my'); \quad (4)$$

but by (1), $my + nx' \leq nx + my'$, so (4) holds.

Now, we prove (iii); we do it for example for $u_*(b)$, the proof would be similar for $u^*(b)$. (Note the similarity with the proof of 9.1 of [36]).

So let x, y in A , m, n in \mathbb{N} such that

$$x + mb \leq y + nb \quad (5)$$

Putting $\beta = u_*(b)$, we have to prove

$$u(x) + m\beta \leq u(y) + n\beta \quad (6)$$

We argue by cases.

Case 1. $m = n = 0$. Trivial since u is a homomorphism.

Case 2. $m = 0, n > 0$. Immediate by (i).

Case 3. $m > 0, n = 0$. Using successively (ii) and (i), we get $u(x) + m\beta \leq u(x) + mu^*(b) \leq u(y)$.

Case 4. $m > 0, n > 0$. By distributivity of $+$ on $\bigvee \neq \emptyset$, it suffices to show that for all c, d in A and p in $\mathbb{N} \setminus \{0\}$ such that

$$d \leq c + pb, \quad (7)$$

we have

$$u(x) + m \frac{u(d) \setminus u(c)}{p} \leq u(y) + n\beta \quad (8)$$

By (5) and (7), we get immediately $px + md \leq (py + mc) + pnb$, thus, by case 2, $pu(x) + mu(d) \leq pu(y) + mu(c) + pnb$. Now, using again 1.21 (with $a = mu(c)$, $b = mu(d)$, $a' = pu(x)$, $b' = pu(y) + pn\beta$ - the fact that $a' \leq b'$ comes from $x \leq y + nb$ and case 2), we get $m(u(d) \setminus u(c)) \leq p(u(y) + n\beta) - pu(x)$.

Hence, we get (8) by adding $pu(x)$ on both sides and dividing by p . The fact that $u_*(b) = \bigwedge I_b$ and $u^*(b) = \bigvee I_b$ is immediate by definition. ■

Note that even in the case $E = \overline{\mathbb{P}}$, there are cases where $u_*(b) < u^*(b)$ and $I_b = \{u_*(b), u^*(b)\}$ (so that it is not an interval).

3.8. Corollary. *Every homomorphism from A to E extends to a homomorphism from $A + \mathbb{N}b$ to E .*

Proof. This is because $I_b \neq \emptyset$. ■

Now, an easy application of Zorn's lemma yields the

3.9. Proposition. *E is an injective P.O.M..* ■

3.10. EXAMPLE. **1**, **2** and $\overline{\mathbb{P}}$ are injective P.O.M.'s (see 1.4, examples (3) and (4)). Note that a direct proof is very easy for **1** and **2**; on the other hand, one can prove that injectivity of $\overline{\mathbb{P}}$ is equivalent to the Hahn-Banach extension theorem (which can itself be considered as an injectivity property of \mathbb{R} in the category of normed linear spaces...).

Rather surprisingly, the converse of 3.9 is true, as the next theorem will show.

3.11. Theorem. *Let E be a P.O.M.. Then the following are equivalent:*

- (i) E is injective;
- (ii) E is weakly complete, 2-divisible and satisfies the pseudo-cancellation property;
- (iii) E is weakly complete, 2-divisible and satisfies the 2=-cancellation property;
- (iv) E is complete and \mathbb{N} -divisible.

Proof. (ii) \Rightarrow (iii) By 2.12 and 2.9, E satisfies the multiplicative \leq -cancellation property; (iii) follows. (iii) \Rightarrow (iv) follows from 3.4 and 3.5. (iv) \Rightarrow (iii) comes from 2.12 and 2.9. (iii) \Rightarrow (i) is 3.9. Now we prove (i) \Rightarrow (ii). So let E be an injective P.O.M..

Claim 1. E is 2-divisible.

Proof. For every a in E , $(u : 2\mathbb{N} \rightarrow E, 2n \mapsto na)$ extends to some $v \in \text{Hom}(\mathbb{N}, E)$. If $b = v(1)$, then $2b = a$. ■ Claim 1.

Claim 2. E is minimal.

Proof. Let $a \leq b$ in E . Let M be the sub-P.O.M. of \mathbb{N}^2 generated by $\alpha = (1, 0)$ and $\beta = (1, 1)$. Using the fact that $a \leq b$, it is easy to see that there is a [unique] P.O.M.-homomorphism u from M to E sending α on a and β on b . Since E is injective, there is v in $\text{Hom}(\mathbb{N}^2, E)$ extending u . Put $c = v((0, 1))$; then $a + c = b$. ■ Claim 2.

Claim 3. *Let a in E ; then there is \bar{a} in E such that $2\bar{a} = \bar{a}$ and $a \ll \bar{a}$.*

Proof. Let $u : \mathbb{N} \rightarrow E, n \mapsto na$. Then u is a homomorphism from \mathbb{N} to E , thus it extends to some homomorphism v from $\overline{\mathbb{N}}$ to E ; put $\bar{a} = v(\infty)$. It is immediate that \bar{a} satisfies the required conditions. ■ Claim 3.

Actually, we could even have proved that E has a unique largest element, but the proof, although not more difficult, could not have been generalized to some weaker definitions of injectivity, as *e.g.* countable injectivity which we will use at the end of this chapter; also, existence of a largest element will automatically follow later from completeness of E .

Claim 4. Let a in E such that $2a \leq a$. Then $E|a$ is an injective P.O.M..

Proof. $F = E|a$ is an ideal, thus a sub-P.O.M. of E . Let A be a sub-P.O.M. of a P.O.M. B , let u in $\text{Hom}(A, F)$. Define the *lexicographical product* $\mathbb{N} \times_{lex} A$ to be the product $\mathbb{N} \times A$ equipped with the preordering defined by

$$(m, x) \leq_{lex} (n, y) \Leftrightarrow (m < n \text{ or } (m = n \text{ and } x \leq_A y)).$$

It is easy to verify that $\mathbb{N} \times_{lex} A$ is again a P.O.M.. Now, using the fact that $2a \leq a$, it is easy to see that there is a [unique] P.O.M.-homomorphism \bar{u} from $\mathbb{N} \times_{lex} A$ to E such that $\bar{u}(n, x) = na + u(x)$ for all n in \mathbb{N} , x in A . Since E is injective, there is an extension of \bar{u} to a homomorphism \bar{v} from $\mathbb{N} \times_{lex} B$ to E . Put $v(x) = \bar{v}((0, x))$ (all x in A). Then v extends u , $v \in \text{Hom}(B, E)$, and for all x in A , $v(x) \leq \bar{v}((1, 0)) = a$, so that $v \in \text{Hom}(B, F)$. ■ Claim 4.

Now, we shall prove that \leq is antisymmetric.

Claim 5. Let a, b in E such that $a \ll b$. Then there is \bar{a} in E such that $2\bar{a} = \bar{a}$ and $a \leq \bar{a} \leq b$.

Proof. Let M be the sub-P.O.M. of $\overline{\mathbb{N}} \times \mathbb{N}$ generated by $\alpha = (1, 0)$ and $\beta = (\infty, 1)$. Using the hypothesis, it is easy to verify that there is a [unique] P.O.M.-homomorphism u from M to E sending α on a and β on b . Since E is injective, u has an extension to some v in $\text{Hom}(\overline{\mathbb{N}} \times \mathbb{N}, E)$. Put $\bar{a} = v((\infty, 0))$. It is immediate to check that \bar{a} satisfies the required conditions. ■ Claim 5.

Claim 6. Let a, b, c in E such that $a \leq b$ and $b \ll c$. Then $a \ll c$.

Proof. By claim 5, there is \bar{b} such that $2\bar{b} = \bar{b}$ and $b \leq \bar{b} \leq c$. By claim 4, $E|\bar{b}$ is an injective P.O.M.; since $a \in E|\bar{b}$, claim 3 implies the existence of \bar{a} in $E|\bar{b}$ such that $2\bar{a} = \bar{a}$ and $a \ll \bar{a}$. Since $\bar{a} \leq \bar{b} \leq c$ and E is minimal (claim 2), we get $a \ll c$. ■ Claim 6.

Claim 7. \leq is antisymmetric.

Proof. Immediate from claim 6 and 1.2. ■ Claim 7.

Claim 8. *E satisfies the finite refinement property.*

Proof. It suffices to check the (2,2)-refinement property. Let a, a', b, b' in E such that $a + a' = b + b'$. Let R be the sub-P.O.M. of \mathbb{N}^4 generated by $\alpha = (1, 1, 0, 0)$, $\alpha' = (0, 0, 1, 1)$, $\beta = (1, 0, 1, 0)$, $\beta' = (0, 1, 0, 1)$. Using the hypothesis and the easily proved fact that R is *minimal*, it is easy to verify that there exists a [unique] P.O.M.-homomorphism u from R to E sending α on a , α' on a' , β on b , β' on b' . Now, u admits an extension to a P.O.M.-homomorphism v from \mathbb{N}^4 to E . Then we have the following refinement matrix:

	b	b'
a	$v((1, 0, 0, 0))$	$v((0, 1, 0, 0))$
a'	$v((0, 0, 1, 0))$	$v((0, 0, 0, 1))$

which concludes the proof. ■ Claim 8.

Claim 9. *E is a strong refinement P.O.M..*

Proof. According to claims 2, 7 and 8, it suffices to prove the pseudo-cancellation property. So let a, b, c in E such that $a + c = b + c$. Let S be the sub-P.O.M. of \mathbb{N}^4 generated by $\alpha = (1, 1, 0, 0)$, $\beta = (1, 0, 1, 0)$, $\gamma = (0, 0, 0, 1)$ and $\gamma' = (0, 1, 0, 1)$. It is easy to verify that there is a [unique] P.O.M.-homomorphism u from S to E sending α on a , β on b and γ, γ' on c . Since E is injective, u has an extension to some v in $\text{Hom}(B, E)$. Put $e = v((0, 1, 0, 0))$, then $a \leq b + e$ and $e \ll c$. ■ Claim 9.

Hence, by 1.16 (i), E satisfies the interpolation theorem (for *finite* subsets). The next claim is an infinite version of it:

Claim 10. *Let U, V be two subsets of E such that $U \leq V$. Then there is c in E such that $U \leq c$ and $c \leq V$.*

Proof. Let I be the set of all $i = (p, q, x)$ where $p \subseteq U$, $q \subseteq V$, p and q finite, $x \in E$, and $p \leq x$ and $x \leq q$; write $p = p_i$, $q = q_i$, $x = x_i$. Let \mathcal{F} be the (proper) filter on I generated by the $F_{pq} = \{i \in I : p \subseteq p_i \text{ and } q \subseteq q_i\}$ for all finite $p \subseteq U$, $q \subseteq V$; the fact that $F_{pq} \neq \emptyset$ comes from the finite interpolation theorem. Let *E be the reduced power E^I / \mathcal{F} . Since E is injective, there is a retraction ρ from *E onto E . Put $c = \rho([x_i : i \in I])$. Then for all u in U , we have $u \leq x_i$ for \mathcal{F} -almost all i (those in $F_{\{u\}\emptyset}$), thus $u \leq c$. Similarly, $c \leq v$ for all v in V . ■ Claim 10.

Claim 11. *Every subset of E has a g.l.b. .*

Proof. If $X \subseteq E$, let $Y = \{y \in E : (\forall x \in X)(y \leq x)\}$. For any c in E such that $Y \leq c$ and $c \leq X$ (given by claim 10), we have $c = \bigwedge X$. ■ Claim 11.

Claim 12. $+$ is distributive on \bigwedge .

Proof. Let a in E , $X \subseteq E$. Put $b = \bigwedge X$, $c = \bigwedge(a + X)$. Obviously, $a + b \leq c$. Conversely, let I be the set of finite subsets of X , let \mathcal{F} be the filter on I generated by $\{F_p : p \in I\}$ where $F_p = \{q \in I : p \subseteq q\}$. Let *E be the reduced power E^I/\mathcal{F} . Since E is injective, there is a retraction ρ from *E onto E . Let $d = \rho([\bigwedge p : p \in I])$. For all $p \in I$, we have $c \leq \bigwedge(a + p) = a + \bigwedge p$ (by 1.16, (v)), thus $c \leq a + d$ since $\rho|_E = id$. Moreover, if $x \in X$, then for \mathcal{F} -almost all p (those in $F_{\{x\}}$), $\bigwedge p \leq x$, thus $d \leq x$. Hence $c \leq a + d \leq a + \bigwedge X = a + b$. ■ Claim 12.

By claims 11, 12, 1 and 9, (ii) is satisfied. This concludes the proof. ■

3.12. REMARK. We see in the proofs of claims 2 and 3 how important it was in the definition of injective P.O.M.'s (3.1) to say that the inclusion map from A into B is not only an injective homomorphism (which is here the same as a monomorphism) but an *embedding*: otherwise, the proof of claim 2 would have shown that E is a group, then the proof of claim 3 would have allowed us to conclude $E = \{0\}$ (thus we reprove the well-known fact that the only injective object in the category of commutative monoids is $\{0\}$).

3.13. Corollary. Let E be a complete P.O.M.. Then the P.O.M. E^\flat with the same underlying set as E where the addition is replaced by the meet \vee (and the zero and the ordering are the same) is an injective P.O.M., of which all elements are idem-multiple.

Proof. An immediate consequence of 2.19 and 3.11. ■

We show now briefly how an injectivity concept can be connected with a decomposition problem.

It has been proved (see [26]) that the disc and the square with unit area of \mathbb{R}^2 are equidecomposable (with a very large number of pieces) *using only translations*. A still open problem is whether these pieces can be measurable. An even more general problem could be stated as follows:

Let an amenable group G act by isometries on \mathbb{R}^n ($n \in \mathbb{N} \setminus \{0\}$); is it true that any two measurable equidecomposable subsets of \mathbb{R}^n are equidecomposable using measurable pieces?

Note right now that every abelian group is amenable, in particular the group of translations of \mathbb{R}^2 .

We will not solve this problem here, but we will show how its analogue for ‘continuous’ equidecomposability is true. In fact, we will even show that under a mild set-theoretical hy-

pothesis (weaker than the Continuum hypothesis), it even holds for universally measurable sets.

Define as Mokobodzki does in [11] a *medial measure* to be a universally measurable [finitely additive] probability measure μ on $\mathcal{P}(\omega)$ vanishing on singletons. Are there any medial measures? The answer is “yes, under some mild set-theoretical assumptions” (see [5] or [11]); note that it is not the case when we replace universal measurability by the Baire property (see [5]). The simplest of these assumptions is the Continuum hypothesis, but Martin’s axiom (and even weakened forms of it, see [11]) work as well. Thus, if we consider the following statement of set theory:

ML: “*There are medial measures,*”

then ML is consistent with the axioms of set theory.

It is now time to recall some basic facts about amenable groups (see [19]). If G is a group, then a [finitely additive] probability measure μ on $\mathcal{P}(G)$ is a *right-invariant* (resp. *left-invariant*) *mean* on G when for all g in G and all A in $\mathcal{P}(G)$, we have $\mu(Ag) = \mu(A)$ (resp. $\mu(gA) = \mu(A)$); G is *amenable* when there is a right-invariant (resp. left-invariant, resp. right-and-left invariant - see [19]) mean on G . We shall now see the connection between these preliminaries and injective structures and decompositions. We start with a statement similar to 3.9; there are no essential modifications to bring to the proof.

3.14. Lemma. *Let Ω be a set, let \mathcal{B} a countably complete Boolean subalgebra of $\mathcal{P}(\Omega)$. Let $E = \mathcal{M}(\mathcal{B})$ be the set of all \mathcal{B} -measurable functions from Ω to $\overline{\mathbb{P}}$. Then E is ‘countably injective’, i.e. for all countable P.O.M.’s $A \subseteq B$, every homomorphism from A to E extends to a homomorphism from B to E . ■*

3.15. REMARK. There are other countably injective P.O.M.’s as the $\mathcal{M}(\mathcal{B})$ above. In fact, we know a complete ‘arithmetical’ characterization of countably injective P.O.M.’s; it is in particular noteworthy that divisible weak cardinal algebras are always countably injective. These results need further technical tools, and will appear elsewhere.

Now, for any group G , let us define a G -P.O.M. to be a P.O.M. A equipped with an action of G on A by automorphisms. We define countably injective G -P.O.M.’s just the same way as we defined countably injective P.O.M.’s in 3.14 with G -homomorphisms (i.e. satisfying $u(g.x) = g.u(x)$ for all g in G and all x in X) instead of just homomorphisms. In fact, we shall be interested in a special class of G -P.O.M.’s.

3.16. Definition. Let a group G act by Borel automorphisms on a Borel space Ω ;

equip the space of all universally measurable maps from Ω to $\overline{\mathbb{P}}$ with the canonical action of G by translations. We will denote by $\text{UM}(\Omega)$ the corresponding G -P.O.M..

Our essential tool, which seems in our opinion to have an independent interest, will be the following:

3.17. Theorem. *Assume that there exists a medial measure. Let a countable amenable group G act by Borel automorphisms on a Borel space Ω . Then $\text{UM}(\Omega)$ is a countably injective G -P.O.M..*

Proof. Put $E = \text{UM}(\Omega)$. Let $A \subseteq B$ be two countable G -P.O.M.'s, let u be a G -homomorphism from A to E . Since E satisfies the conditions of 3.14, u extends to a homomorphism v from B to E ; the problem is that v may not be a G -homomorphism. Using Følner's condition for amenability (see [16], [28]), it is not difficult to prove that there is a universally measurable right-invariant mean μ on G ; define a map w from B to E by

$$(\forall b \in B)(\forall t \in \Omega)(w(b)(t) = \int v(gb)(gt) d\mu(g)).$$

(The definition of integral here is essentially the same as Lebesgue integration, except that μ is only finitely additive so that the limit theorems of integration are not available). The fact that w still extends u is immediate. Moreover, w has range in E since μ is universally measurable. Finally, it is straightforward to check that w is a P.O.M.-homomorphism and, using G -invariance of μ , that w is in fact a G -homomorphism. ■

As a special case of G -injectivity, we will use the following

3.18. Corollary. *In the context of 3.17, put $E = \text{UM}(\Omega)$, and let A be a countable sub- G -P.O.M. of E , let B be a countable G -P.O.M. containing A . Then there is a G -homomorphism ρ from B to E such that $\rho|_A$ is the inclusion map from A into E . ■*

Now, we are ready to conclude about continuous decompositions. Define the equidecomposability relation \equiv_G on $\overline{\mathbb{P}}^\Omega$ (which we will call 'continuous equidecomposability' since the pieces we use in our decompositions are rather $\overline{\mathbb{P}}$ -valued functions than the usual characteristic functions, see [37]) by $\varphi \equiv_G \psi$ (or: φ and ψ are G -equidecomposable) if and only if there are n in $\omega \setminus \{0\}$, φ_i ($i < n$) in $\overline{\mathbb{P}}^\Omega$ and g_i ($i < n$) in G such that $\varphi = \sum_{i < n} \varphi_i$ and $\psi = \sum_{i < n} g_i \varphi_i$; moreover, if the φ_i are in $\text{UM}(\Omega)$, we say that $\varphi \equiv_G \psi$ with pieces in $\text{UM}(\Omega)$. Applying 3.18 after having replaced G by the subgroup H generated by the g_i , with $A = H$ -P.O.M. generated by $\{\varphi, \psi\}$ and B generated by the φ_i , we get immediately the following statement:

3.19. Corollary. *Assume that there is a medial measure. Let an amenable group G act by Borel automorphisms on a Borel space Ω . Then two universally measurable functions from Ω to $\overline{\mathbb{P}}$ are [continuously] G -equidecomposable if and only if they are [continuously] G -equidecomposable with universally measurable pieces.* ■

As an example, it follows easily that the square and the disc of unit area in \mathbb{R}^2 are ‘continuously equidecomposable’ using translations and universally measurable pieces. In particular, we get the weaker statement that the square and the disc of unit area in \mathbb{R}^2 are continuously equidecomposable using Lebesgue-measurable pieces, modulo Lebesgue-null sets. Since this is an absolute statement of set theory (*see* [23]), it is also true in set theory without the axiom of choice. We do not know any ‘direct’ proof of this fact.

Concerning a possible extension of these results to *discrete* equidecomposability, we do not know the answer; but a negative counterexample in this direction is the following: let $\Omega = S^1$ be the unit circle of \mathbb{R}^2 , and let g be a rotation (around the origin) with irrational angle (in radians); consider the group G generated by g . Then an easy argument (reasoning on each G -orbit) shows that there is a subset X of Ω such that $\Omega = X \cup gX$ and $X \cap gX = \emptyset$. However, an easy measure-theoretic argument shows that such a set cannot be measurable for the canonical Lebesgue measure on S^1 ; thus, the analogue of 3.18 *fails* for discrete equidecomposability.

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