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Control of stochasticity in magnetic field lines

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Abstract. We present a method of control which is able to create barriers to magnetic field line diffusion by a small modification of the magnetic perturbation. This method of control is based on a *localized* control of chaos in Hamiltonian systems. The aim is to modify the perturbation (of order ε) *locally* by a small control term (of order ε^2) which creates invariant tori acting as barriers to diffusion for Hamiltonian systems with two degrees of freedom. The location of the invariant torus is enforced in the vicinity of the chosen target (at a distance of order ε due to the angle dependence). Given the importance of confinement in magnetic fusion devices, the method is applied to two examples with a loss of magnetic confinement. In the case of locked tearing modes, an invariant torus can be restored that aims at showing the current quench and therefore the generation of runaway electrons. In the second case, the method is applied to the control of stochastic boundaries allowing one to define a transport barrier within the stochastic boundary and therefore to monitor the volume of closed field lines.

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1. Introduction

Controlling chaotic transport is a key challenge in many branches of physics like for instance, in particle accelerators, free electron lasers or in magnetically confined fusion plasmas [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. It covers a large variety of strategies. On the one hand, one can aim at recovering near integrable systems while on the other hand, one can request the existence of a boundary with strongly reduced transport that will act as some transport barrier. Another feature that will prove to be essential when implementing the control scheme is the relative cost of such a control device with respect to its merits. The cost that is considered here will be characterized by the magnitude and complexity of the Fourier spectrum of the control function with respect to that generating the spurious stochastic transport. In the case of systems such that it is essential to control a global transport property without significantly altering the original system under investigation nor its overall chaotic structure, one can restrict the control scheme to that of restoring a local transport barrier. In accordance with the latter idea, the control strategy that we develop here is based on building barriers by adding a small apt perturbation which is localized in phase space, hence confining all the trajectories. This local control strongly reduces the cost of the control scheme since only a small fraction of the phase space is modified. The counterpart is that the impact of such a localized reduction in transport must be efficient enough to be meaningful regarding the control strategy.

In the examples addressed above, the magnetic field is designed to confine the charged particles. Hamiltonian dynamics allows one to describe the motion of the charged particles in such a magnetic field, and the confinement volumes readily take the form of invariant tori. The Hamiltonian formalism appears to be particularly well suited to investigate the control of such systems. In this article, we consider the class of Hamiltonian systems that can be written in the form $H = H_0 + \varepsilon V$ i.e. an integrable Hamiltonian H_0 (with action-angle variables) plus a small perturbation εV . Provided the perturbation is not too small, the explicit dependence of εV on the angle variables breaks the invariance of the action variables and in a generic situation yields chaotic transport. The idea for controlling this transport is to modify the perturbation *slightly* and *locally* and create regular structures (like invariant tori).

In Refs. [14, 15, 16], an explicit method of control was provided for generating f such that the controlled Hamiltonian $H_0 + \varepsilon V + \varepsilon^2 f$ is integrable. We point out that the control term we constructed is of order ε^2 , i.e. much smaller than the perturbation which is of order ε . This method of control has been applied experimentally to increase the kinetic coherence of an electron beam in a Traveling Wave Tube [17]. One possible drawback of this approach is that the control term depends on all the variables and has to be applied on all the phase space. Here we provide a method to construct control terms f with a finite support in phase space, i.e. localized in phase space, such that the controlled Hamiltonian $H_c = H_0 + \varepsilon V + \varepsilon^2 f$ has invariant tori whose explicit expressions

are known. For Hamiltonian systems with two degrees of freedom, these invariant tori act as barriers to diffusion in phase space. Since the method is localized, we point out that the resonant islands are not modified by our control.

In the present paper, this original method is applied to the control of chaotic transport in fusion plasmas. However, these examples are sufficiently general so that one can readily transpose the approach to other fields of physics. Let us consider two examples where the control of chaotic transport appears to be of particular interest.

In a first example, one addresses the issue of large scale stochastic transport between two resonant surfaces. In tokamaks, this situation is met when two tearing modes develop [18] and when their amplitude is large enough to satisfy the so-called overlap criterion proposed by Chirikov [19]. This leads to chaotic transport from one resonant surface to the other. The control scheme can target two results, first to maintain a closed magnetic surface, hence an invariant torus, so that confinement can be sustained until some other action is taken to suppress the tearing modes, second, should the tearing mode interaction have led to a pre-disruptive phase, to generate a transport barrier sufficiently robust to slow the current quench and provide conditions for a controlled disruption with no generation of runaway electrons [20].

In a second example, one considers the situation of an external modification of the magnetic equilibrium, for instance in the case of an ergodic divertor [21], where specific coils are implemented to generate chaotic transport in the outermost magnetic surfaces. A similar situation can be met in stellarators where the magnetic equilibrium can be such that a series of resonances in the plasma boundary creates a region of chaotic transport there. In these cases, it can be of interest to control this chaotic boundary layer by restoring a closed magnetic surface at a given radius. In the latter case, one should consider a control scheme with similar properties to that generating the boundary layer with stochastic transport. This would ensure that a similar coil set to that generating the perturbation of the magnetic equilibrium can be used for the control purpose, hence ensuring some effectiveness in terms of cost and feasibility.

In Sec. 2, we provide an explicit construction of the localized control term. The equation of the created invariant torus is given explicitly. In Sec. 3, the formula of the control term for magnetic field lines is derived as well as the explicit formula for the magnetic surface which has been created by the localized control term. In Sec. 4, the two examples of magnetic perturbations are considered. It is shown on these examples that a small and apt modification of the poloidal flux is able to create a robust magnetic surface which prevents the diffusion of magnetic field lines. In the Appendix, we give a proof of existence of the control term.

2. Control of chaos in Hamiltonian systems

In this section we present a control method of Hamiltonian systems which is directly applicable to magnetic field lines. This version of localized control follows the one developed in Ref. [22, 23]. In order to understand the procedure to build the control

scheme, let us first analyze the properties of a specific class of Hamiltonian systems that will provide the background for the control strategy. In the following \mathbf{A} refers to the action variables and $\boldsymbol{\theta}$ to the angle variables in a phase space of dimension $2L$ (hence both vectors are of dimension L). Let us consider Hamiltonian systems of the form $H(\mathbf{A}, \boldsymbol{\theta}) = H_0(\mathbf{A}) + \varepsilon V(\mathbf{A}, \boldsymbol{\theta})$. Using a suitable expansion, these Hamiltonian systems take the form

$$H(\mathbf{A}, \boldsymbol{\theta}) = \boldsymbol{\omega} \cdot \mathbf{A} + \varepsilon v(\boldsymbol{\theta}) + w(\mathbf{A}, \boldsymbol{\theta}), \quad (1)$$

where $(\mathbf{A}, \boldsymbol{\theta}) \in \mathbb{R}^L \times \mathbb{T}^L$, \mathbb{T} being an angle space (torus) $[-\pi, \pi[$ in a standard approach. The three contributions to $H(\mathbf{A}, \boldsymbol{\theta})$ are respectively $H_0(\mathbf{A}) = \boldsymbol{\omega} \cdot \mathbf{A}$, the main term governing the integrable motion in the vicinity of $\mathbf{A} = \mathbf{0}$, a perturbation $\varepsilon v(\boldsymbol{\theta})$ and a higher order term in \mathbf{A} , $w(\mathbf{A}, \boldsymbol{\theta})$ which can be written in the form

$$w(\mathbf{A}, \boldsymbol{\theta}) = \varepsilon \mathbf{w}_1(\boldsymbol{\theta}) \cdot \mathbf{A} + w_2(\mathbf{A}, \boldsymbol{\theta}),$$

where w_2 is quadratic in the actions, i.e. $w_2(\mathbf{0}, \boldsymbol{\theta}) = 0$ and $\partial_{\mathbf{A}} w_2(\mathbf{0}, \boldsymbol{\theta}) = \mathbf{0}$.

The vector $\boldsymbol{\omega}$ contains the frequencies of the quasi-periodic motion defined by $\dot{\mathbf{A}} = \mathbf{0}$ and $\dot{\boldsymbol{\theta}} = \boldsymbol{\omega}$. Furthermore, $\boldsymbol{\omega}$ is a non-resonant vector of \mathbb{R}^L , i.e. there is no non-zero $\mathbf{k} \in \mathbb{Z}^L$ such that $\boldsymbol{\omega} \cdot \mathbf{k} = 0$. Without restricting the computation of the control term, one can assume that $w(\mathbf{0}, \boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in \mathbb{T}^L$ and that $\int_{\mathbb{T}^L} v(\boldsymbol{\theta}) d^L \boldsymbol{\theta} = 0$ (the average of v is set to zero). We consider a region near $\mathbf{A} = \mathbf{0}$. We notice that for $\varepsilon = 0$ and for any w not necessarily small, the Hamiltonian H has an invariant torus at $\mathbf{A} = \mathbf{0}$. The controlled Hamiltonian we construct is given by

$$H_c(\mathbf{A}, \boldsymbol{\theta}) = \boldsymbol{\omega} \cdot \mathbf{A} + \varepsilon v(\boldsymbol{\theta}) + w(\mathbf{A}, \boldsymbol{\theta}) + \varepsilon^2 f(\boldsymbol{\theta}) \Omega(\mathbf{A}, \boldsymbol{\theta}), \quad (2)$$

where Ω is a smooth characteristic function of a region around a targeted invariant torus. We wish to stress that the actual control term $\varepsilon^2 f$ that we construct only depends on the angle variables $\boldsymbol{\theta}$ and is of order ε^2 (see Appendix). We prove that the control term is :

$$f(\boldsymbol{\theta}) = -\varepsilon^{-2} w(-\varepsilon \Gamma \partial_{\boldsymbol{\theta}} v, \boldsymbol{\theta}), \quad (3)$$

where $\partial_{\boldsymbol{\theta}} v$ denotes the first derivatives of v with respect to $\boldsymbol{\theta}$: $\partial_{\boldsymbol{\theta}} v = \sum_{\mathbf{k} \in \mathbb{Z}^L} i \mathbf{k} v_{\mathbf{k}} e^{i \mathbf{k} \cdot \boldsymbol{\theta}}$ and where the linear operator Γ is the inverse operator of the convective shift $\dot{\boldsymbol{\theta}} \cdot \partial_{\boldsymbol{\theta}}$ relative to the unperturbed dynamics governed by $H_0(\mathbf{A})$. Its explicit form acting on a function $v(\boldsymbol{\theta}) = \sum_{\mathbf{k} \in \mathbb{Z}^L} v_{\mathbf{k}} e^{i \mathbf{k} \cdot \boldsymbol{\theta}}$ is :

$$\Gamma v(\boldsymbol{\theta}) = \sum_{\mathbf{k} \in \mathbb{Z}^L \setminus \{\mathbf{0}\}} \frac{v_{\mathbf{k}}}{i \boldsymbol{\omega} \cdot \mathbf{k}} e^{i \mathbf{k} \cdot \boldsymbol{\theta}},$$

The operators Γ and $\partial_{\boldsymbol{\theta}}$ commute so that $\partial_{\boldsymbol{\theta}}(\Gamma v) = \Gamma(\partial_{\boldsymbol{\theta}} v)$.

There is a significantly large freedom in choosing the function Ω . It is sufficient to have $\Omega(\mathbf{A}, \boldsymbol{\theta}) = 1$ for $\|\mathbf{A}\| \leq \varepsilon$. For instance, $\Omega(\mathbf{A}, \boldsymbol{\theta}) = 1$ would be a possible and simpler choice, however representing a long-range control since the control term $f(\boldsymbol{\theta})$ would be applied on all phase space. On the opposite way, we can design a function

Ω such that the control is localized around the created invariant torus : We denote \mathcal{T}_α and \mathcal{T}_β two neighborhoods (in phase space) of the targeted invariant torus such that $\mathcal{T}_\alpha \subset \mathcal{T}_\beta$. The characteristic function is chosen such that $\Omega(x) = 1$ if $x \in \mathcal{T}_\alpha$ and $\Omega(x) = 0$ if $x \notin \mathcal{T}_\beta$, and Ω is smooth on all phase space. We choose the characteristic function Ω to be

$$\Omega(\mathbf{A}, \boldsymbol{\theta}) = \Omega_{loc}(\|\mathbf{A} + \varepsilon\Gamma\partial_{\boldsymbol{\theta}}v\|), \quad (4)$$

where $\Omega_{loc}(x) = 1$ if $x \leq \alpha$ and $\Omega_{loc}(x) = 0$ if $x \geq \beta$ and, e.g., a polynomial or another function for $x \in]\alpha, \beta[$ such that Ω_{loc} is smooth on \mathbb{R}^+ . The main advantage of this step function Ω is that the control needs less energy (only in the part of phase space where the control is localized) and also it does not change the other part of phase space.

We rigorously prove that H_c has an invariant torus located at $\mathbf{A} = -\varepsilon\Gamma\partial_{\boldsymbol{\theta}}v$. For Hamiltonian systems with two degrees of freedom, such an invariant torus acts as a barrier to diffusion. For the construction of the control term, we notice that we do not require that the quadratic part of w is small in order to have a control term of order ε^2 . Moreover, if the lowest order in powers of \mathbf{A} of w is $n \geq 2$ then the control term is of order ε^n .

In order to derive the expression of the control term, we consider the canonical transformation generated by

$$F(\mathbf{A}', \boldsymbol{\theta}) = \boldsymbol{\theta} \cdot \mathbf{A}' - \varepsilon\Gamma v(\boldsymbol{\theta}),$$

which is a translation in action by $\varepsilon\Gamma\partial_{\boldsymbol{\theta}}v$, i.e. $(\mathbf{A}', \boldsymbol{\theta}') = (\mathbf{A} + \varepsilon\Gamma\partial_{\boldsymbol{\theta}}v, \boldsymbol{\theta})$. The Hamiltonian H_c is mapped onto

$$\tilde{H}_c(\mathbf{A}', \boldsymbol{\theta}') = \boldsymbol{\omega} \cdot \mathbf{A}' + w(\mathbf{A}' - \varepsilon\Gamma\partial_{\boldsymbol{\theta}}v, \boldsymbol{\theta}') + \varepsilon^2 f(\boldsymbol{\theta}')\Omega(\|\mathbf{A}'\|).$$

The translation in action is such that the contribution εv is canceled out since $\boldsymbol{\omega} \cdot \Gamma\partial_{\boldsymbol{\theta}}v = v$. The action $\mathbf{A}' = \mathbf{0}$ is found to be conserved by the flow of \tilde{H}_c since

$$\frac{d\mathbf{A}'}{dt} = -\partial_{\boldsymbol{\theta}}w(\mathbf{A}' - \varepsilon\Gamma\partial_{\boldsymbol{\theta}}v, \boldsymbol{\theta}') + \varepsilon(\Gamma\partial_{\boldsymbol{\theta}}^2v)\partial_{\mathbf{A}}w(\mathbf{A}' - \varepsilon\Gamma\partial_{\boldsymbol{\theta}}v, \boldsymbol{\theta}') - \varepsilon^2(\partial_{\boldsymbol{\theta}}f)\Omega(\|\mathbf{A}'\|),$$

and since the control term $f(\boldsymbol{\theta})$ defined in Eq. (3), is such that :

$$\partial_{\boldsymbol{\theta}}f = -\varepsilon^{-2}\partial_{\boldsymbol{\theta}}w(-\varepsilon\Gamma\partial_{\boldsymbol{\theta}}v, \boldsymbol{\theta}) + \varepsilon^{-1}(\Gamma\partial_{\boldsymbol{\theta}}^2v)\partial_{\mathbf{A}}w(-\varepsilon\Gamma\partial_{\boldsymbol{\theta}}v, \boldsymbol{\theta}).$$

As a consequence, in the domain such that $\Omega(\|\mathbf{A}'\|) = 1$, we find $\mathbf{A}' = \mathbf{0}$ implies $d\mathbf{A}'/dt = \mathbf{0}$, and consequently $\mathbf{A}' = \mathbf{0}$ is an invariant surface for \tilde{H}_c .

3. Magnetic field lines

The magnetic field line dynamics in a toroidal geometry can be written in a Hamiltonian form [24, 25, 26]

$$\begin{aligned} \frac{d\theta}{d\varphi} &= \frac{\partial H}{\partial \psi}, \\ \frac{d\psi}{d\varphi} &= -\frac{\partial H}{\partial \theta}, \end{aligned}$$

where φ which plays the role of effective time is the toroidal angle, ψ is the normalized toroidal flux and H is the poloidal flux. The poloidal angle θ is the conjugate variable to the action ψ . We consider the following class of Hamiltonian systems :

$$H(\psi, \theta, \varphi) = H_0(\psi) + \varepsilon H_1(\psi, \theta, \varphi). \quad (5)$$

We denote $Q(\psi) = H_0'(\psi)$. The quantity $q = 1/Q$ is the safety factor. For $\varepsilon = 0$, we recover the unperturbed magnetic equilibrium such that $d\psi/d\varphi = 0$, $\psi = \psi_0$ are invariant tori also characterized by the rotational transform $q(\psi_0) = d\varphi/d\theta$. We select a magnetic surface by its unperturbed action $\psi = \psi_0$ where one wants to build a barrier to diffusion. We expand H_0 into :

$$H_0(\psi) = Q(\psi_0)(\psi - \psi_0) + \sum_{l=1}^{\infty} \frac{1}{(l+1)!} Q^{(l)}(\psi_0)(\psi - \psi_0)^{l+1}.$$

We denote $\omega = Q(\psi_0) = 1/q(\psi_0)$ the winding ratio of the selected magnetic surface. Following the notation of Sec. 2, we have

$$v(\theta, \varphi) = H_1(\psi_0, \theta, \varphi),$$

and

$$w(\psi, \theta, \varphi) = H_0(\psi) - Q(\psi_0)(\psi - \psi_0) + \varepsilon[H_1(\psi, \theta, \varphi) - H_1(\psi_0, \theta, \varphi)].$$

We expand w :

$$w(\psi, \theta, \varphi) = \varepsilon \partial_\psi H_1(\psi_0, \theta, \varphi)(\psi - \psi_0) + \sum_{l=1}^{\infty} \frac{1}{(l+1)!} [Q^{(l)}(\psi_0) + \varepsilon \partial_\psi^{l+1} H_1(\psi_0, \theta, \varphi)] (\psi - \psi_0)^{l+1}.$$

We notice that $w(\psi_0, \theta, \varphi) = 0$ and that $\partial_\psi w(\psi_0, \theta, \varphi) = \varepsilon \partial_\psi H_1(\psi_0, \theta, \varphi)$ is of order ε . Thus this Hamiltonian satisfies the requirements to construct a localized control term of order ε^2 .

Following Refs. [27, 28, 29], the Fourier expansion of H_1 is given by

$$H_1(\psi_0, \theta, \varphi) = \sum_{m,n} H_{mn}(\psi_0) \cos(m\theta - n\varphi + \chi_{mn}),$$

for some constant phases χ_{mn} . In order to apply the control procedure described in the previous section, we consider that φ is an angle variable and E its conjugate action. In this way, we map the “non-autonomous” Hamiltonian system with 1.5 degrees of freedom into an “autonomous” Hamiltonian of the form (1) with two degrees of freedom where the actions are $\mathbf{A} = (\psi - \psi_0, E)$ and the angles are $\boldsymbol{\theta} = (\theta, \varphi)$. Hamiltonian (5) has the form (1) with $\boldsymbol{\omega} = (\omega, 1)$.

The control term is given by

$$f(\theta, \varphi) = [\partial_\psi H_1(\psi_0, \theta, \varphi)] \Gamma \partial_\theta H_1(\psi_0, \theta, \varphi) - \sum_{l=1}^{\infty} \frac{(-\varepsilon)^{l-1}}{(l+1)!} (Q^{(l)}(\psi_0) + \varepsilon \partial_\psi^{l+1} H_1(\psi_0, \theta, \varphi)) (\Gamma \partial_\theta H_1(\psi_0, \theta, \varphi))^{l+1}, \quad (6)$$

where

$$\Gamma \partial_\theta H_1(\psi_0, \theta, \varphi) = \sum_{m,n} \frac{m H_{mn}(\psi_0)}{m\omega - n} \cos(m\theta - n\varphi + \chi_{mn}). \quad (7)$$

Using this control term, the controlled Hamiltonian has the invariant torus whose equation is

$$\psi = \psi_0 - \varepsilon \Gamma \partial_\theta H_1(\psi_0, \theta, \varphi), \quad (8)$$

The difference between ψ and ψ_0 , of order ε , is a function of ψ_0 both via the dependence of H_1 on ψ_0 and that of the operator Γ with respect to the frequency $\omega = Q(\psi_0)$. The dominant term of the control is given by considering only $l = 1$ in Eq. (6), which leads to

$$f_2(\theta, \varphi) = [\partial_\psi H_1(\psi_0, \theta, \varphi)] \Gamma \partial_\theta H_1(\psi_0, \theta, \varphi) - \frac{Q'(\psi_0)}{2} (\Gamma \partial_\theta H_1(\psi_0, \theta, \varphi))^2, \quad (9)$$

where $\Gamma \partial_\theta H_1$ is given by Eq. (7). The full control term is given as a series in ε :

$$\varepsilon^2 f = \sum_{s=2}^{\infty} \varepsilon^s f_s,$$

where f_s is given by

$$f_s(\theta, \varphi) = \frac{(-1)^s}{(s-1)!} (\partial_\psi^{s-1} H_1) (\Gamma \partial_\theta H_1)^{s-1} + \frac{(-1)^{s+1}}{s!} Q^{(s-1)}(\psi_0) (\Gamma \partial_\theta H_1)^s, \quad (10)$$

for $s \geq 2$.

In summary, the controlled Hamiltonian we consider is

$$H_c = \int \frac{d\psi}{q(\psi)} + \varepsilon H_1(\psi, \theta, \varphi) + \varepsilon^2 f(\theta, \varphi) \Omega(\|\psi - \psi_0 + \varepsilon \Gamma \partial_\theta H_1(\psi_0, \theta, \varphi)\|), \quad (11)$$

where f is given by Eq. (6) for the complete control term, or by Eq. (9) if one wants to use only the dominant term, and $\Gamma \partial_\theta H_1$ is given by Eq. (7). The result in Sec. 2 ensures that the controlled Hamiltonian H_c with the control term f given by Eq. (6) has an invariant surface whose equation is given by Eq. (8). These expressions provide the basis to investigate the control of the stochastic transport governed by magnetic perturbations in fusion devices in the two cases, core tearing modes and boundary stochastic layers as done in Section 4.

4. Control of stochastic transport in fusion devices

4.1. Control of the loss of confinement governed by coupled tearing modes

We consider a Hamiltonian system (5) where ψ is the normalized toroidal flux, $\psi = 1$ at the plasma boundary, with a q -profile as chosen in Refs. [30, 31, 29]

$$q(\psi) = \frac{4}{(2-\psi)(2-2\psi+\psi^2)}. \quad (12)$$

In this expression, $q(\psi)$ is a monotonic function of ψ that varies between 1 on axis, for $\psi = 0$, and 4 at the plasma edge, $\psi = 1$. With such an expression, the slope $d \log(q)/d\psi$

is approximately constant and equal to $\log(q(\psi = 1))$. The results do not depend on the precise form of the safety factor profile and can be readily extended to any profile. We are interested here in the case of two tearing modes with low rational numbers, in practice $(m, n) = (3, 2)$ and $(m, n) = (2, 1)$ where m is the poloidal mode number and n the toroidal mode number, locked to one another. In the case analyzed here both perturbations have equal magnitude characterized by ε . The latter parameter is assumed to be small, but such that the onset to stochastic transport is reached with a magnetic perturbation of the form

$$H_1(\psi, \theta, \varphi) = \cos(2\theta - \varphi) + \cos(3\theta - 2\varphi), \quad (13)$$

so that the Hamiltonian of the system is

$$H = \int \frac{d\psi}{q(\psi)} + \varepsilon H_1(\psi, \theta, \varphi). \quad (14)$$

The resonant surfaces are found to be located on $\psi_{3,2} \approx 0.266$ ($q(\psi_{3,2}) = 3/2$) and $\psi_{2,1} \approx 0.456$ ($q(\psi_{2,1}) = 2$). Expanding to second order the Hamiltonian H in the vicinity of the two resonant surfaces and retaining the resonant term of H_1 , one recovers the characteristic Hamiltonian of the pendulum [32] that allows one to define the so-called unperturbed island width δ in units of ψ :

$$\delta = 2 \left(\frac{\varepsilon}{\frac{-d(1/q(\psi))}{d\psi}} \right)^{1/2}.$$

A Poincaré surface of section of the dynamics given by Eqs. (14)-(13) is represented on Fig. 1 for $\varepsilon = 0.004$. For this value of ε , the island widths are $\delta_{3,2} \approx 0.125$ and $\delta_{2,1} \approx 0.147$ so that a Chirikov parameter $\sigma = (\delta_{2,1} + \delta_{3,2})/(\psi_{2,1} - \psi_{3,2})$ can be computed and is about 1.4, hence larger than the reference value for the onset of large scale transport between the resonant surfaces $\psi_{3,2}$ and $\psi_{2,1}$ as readily observed on Fig. 1. However, ε is small enough that significant remnant islands are still present.

The control term we apply is given by Eq. (9) where

$$\begin{aligned} \partial_\psi H_1(\psi_0, \theta, \varphi) &= 0, \\ \Gamma \partial_\theta H_1(\psi_0, \theta, \varphi) &= \frac{2}{2\omega - 1} \cos(2\theta - \varphi) + \frac{3}{3\omega - 2} \cos(3\theta - 2\varphi). \end{aligned}$$

For ψ_0 , we choose $\psi_0 = 0.35$ hence between the two resonant surfaces $\psi_{3,2} \approx 0.266$ and $\psi_{2,1} \approx 0.456$. Choosing other values of ψ_0 is equivalent to moving the barrier one wants to create. The expression of the partial control term f_2 is given by Eq. (9) :

$$f_2(\theta, \varphi) = \left(- \frac{d(1/q(\psi))}{d\psi} \right) \Big|_{\psi=\psi_0} \left(\frac{2 \cos(2\theta - \varphi)}{2\omega - 1} + \frac{3 \cos(3\theta - 2\varphi)}{3\omega - 2} \right)^2, \quad (15)$$

where $\omega = 1/q(\psi_0) = (2 - \psi_0)(2 - 2\psi_0 + \psi_0^2)/4$ and where we used the fact that H_1 does not depend on ψ . For the present value of ψ_0 one finds $q(\psi_0) \approx 1.7$ and thus $\omega \approx 0.587$. The full control term f creates an invariant torus whose location is given by

$$\psi = \psi_0 - \varepsilon \left(\frac{2}{2\omega - 1} \cos(2\theta - \varphi) + \frac{3}{3\omega - 2} \cos(3\theta - 2\varphi) \right). \quad (16)$$

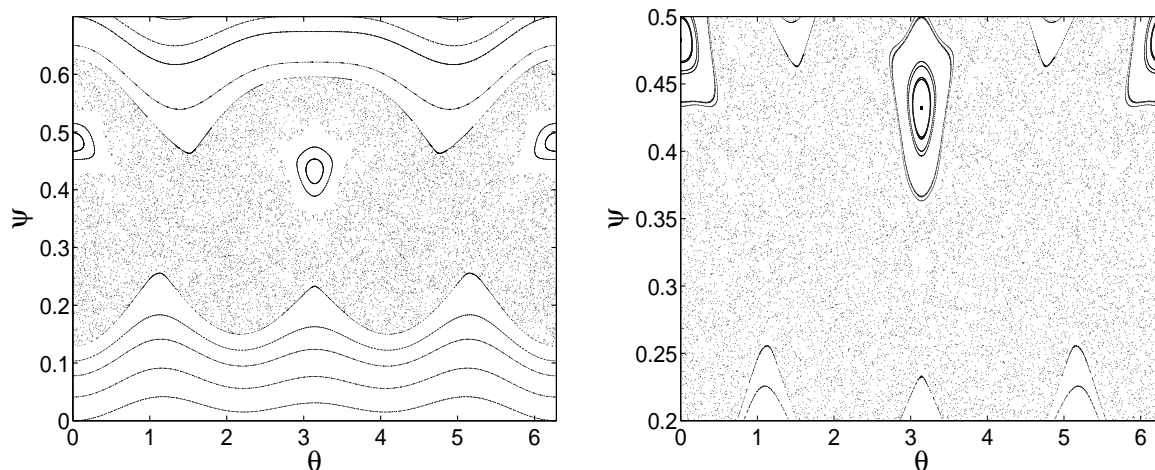


Figure 1. Poincaré surface of section of Hamiltonian (14) with H_1 given by Eq. (13) with $\varepsilon = 0.004$ for $\psi \in [0, 0.7]$ (left panel) and for $\psi \in [0.2, 0.5]$ (right panel).

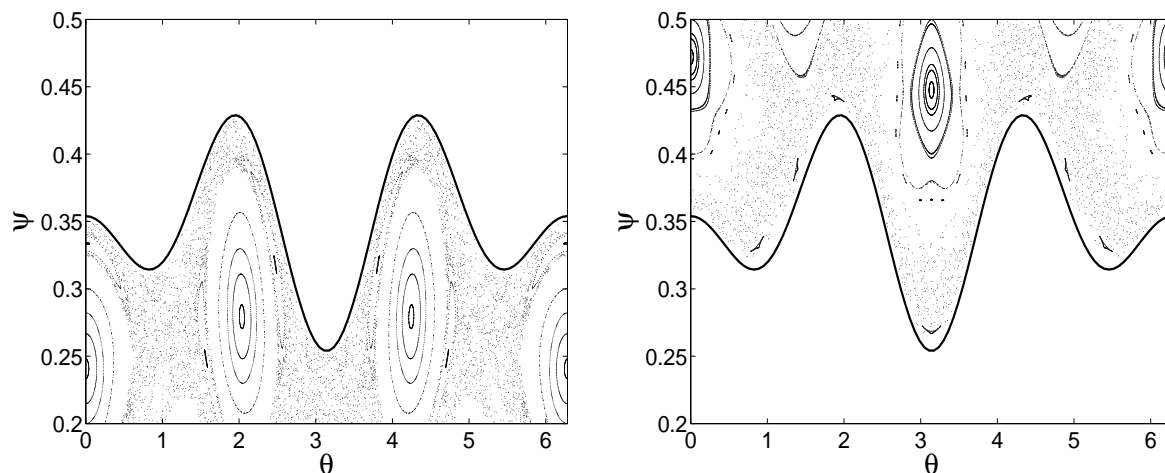


Figure 2. Poincaré surface of section of Hamiltonian (14) with the control term f_2 given by Eq. (15) with $\varepsilon = 0.004$ and $\psi_0 = 0.35$ using $\Omega = 1$, thus with no localization of the control term : with initial conditions below (left panel) or above (right panel) the surface given by Eq. (16) plotted in bold.

The magnitude of the angle modulation of the invariant torus labeled by ψ is relatively large (larger than 40ε).

For the localization function Ω , we use two choices : The first one is $\Omega(|x|) = 1$ for all $|x| \in \mathbb{R}^+$, i.e. without localization. The Poincaré sections of trajectories of the controlled Hamiltonian $H_0 + \varepsilon H_1 + \varepsilon^2 f_2$ are represented in Fig. 2 for trajectories started from below or from above the invariant torus given by Eq. (16) and represented by the bold curve. We notice in particular that the trajectories started from below (resp. above) the invariant torus remain below (resp. above) it. The impact of the control term is noticeable away from the target torus ψ_0 since one can observe much larger remnant islands in the vicinity of the two resonant surfaces compared to the case without control.

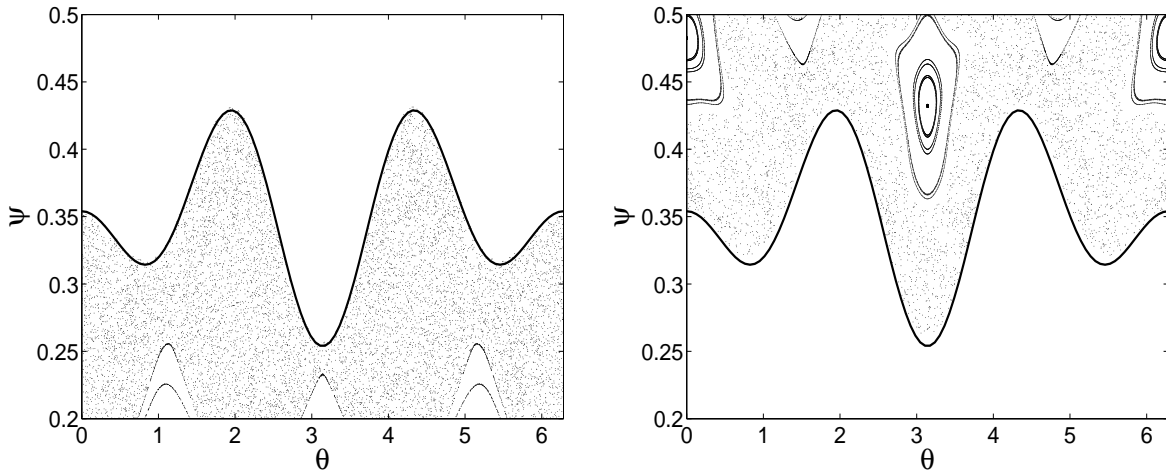


Figure 3. Poincaré surface of section of Hamiltonian (14) with the control term f_2 given by Eq. (15) with $\varepsilon = 0.004$ and $\psi_0 = 0.35$ using $\Omega = \Omega_{loc}$: with initial conditions below (left panel) or above (right panel) the surface given by Eq. (16) plotted in bold.

The second choice of Ω is a localization function. We choose $\Omega_{loc}(|\delta\psi|) = 1$ for $|\delta\psi| \leq \delta\psi_\alpha$, $\Omega_{loc}(|\delta\psi|) = 0$ for $|\delta\psi| \geq \delta\psi_\beta$ and a third order polynomial for $|\delta\psi| \in]\delta\psi_\alpha, \delta\psi_\beta[$ for which Ω_{loc} is a C^1 -function, i.e. $\Omega_{loc}(|\delta\psi|) = 1 - (|\delta\psi| - \delta\psi_\alpha)^2(3\delta\psi_\beta - \delta\psi_\alpha - 2|\delta\psi|)/(\delta\psi_\beta - \delta\psi_\alpha)^3$. In principle, one can choose arbitrarily small values for $\delta\psi_\alpha$ and $\delta\psi_\beta$ if one uses the full control term f given by Eq. (6). However, since f is given by a series, it is more practical to consider the truncated control term f_2 (or a truncation of the series which gives the control term f). Then the value of $\delta\psi_\alpha$ has to be not too small such that the set $\{(\mathbf{A}, \boldsymbol{\theta}) \text{ s.t. } \Omega_{loc}(\mathbf{A}, \boldsymbol{\theta}) = 1\}$ contains the invariant torus which, for f_2 , is ε^3 -close to the one obtained using the complete control term. This leads to a restriction for $\delta\psi_\alpha$, $\delta\psi_\alpha \gtrsim 10^{-3}$ to embed the lowest order in the control term expansion. For the numerics, we choose $\delta\psi_\alpha = 0.01$ and $\delta\psi_\beta = 0.02$. A Poincaré surface of section of the dynamics of H_c given by Eq. (11) with the control term f_2 is represented on Fig. 3 for $\varepsilon = 0.004$ and for $\Omega = \Omega_{loc}$. The bold curve corresponds to the invariant torus given by Eq. (16) that has been created by the addition of the control term which is localized around this surface.

From these figures, we clearly see that the upper and lower parts of phase space are very similar to the ones of Fig. 1 (without control). More precisely, we notice that the structure of the resonant islands is not modified, even the neighboring ones. What has changed is the dynamics in the neighborhood of the bold curve due to the action of the localized control. There is now an invariant torus which prevents the motion to diffuse from the lower part to the upper part of phase space. These two parts are invariant by the dynamics of the controlled Hamiltonian.

The invariant torus created by the localized control persists to higher values of ε . For $\varepsilon \geq 0.1$, the trajectories start to diffuse through the broken invariant surface. The diffusion can be reduced or even suppressed by taking into account higher order terms f_s for $s \geq 3$ in the control term series. We point out that the value of ε for which the

partial control term f_2 is efficient depends on the choice of ψ_0 . There is freedom to choose the initial surface ψ_0 provided that $q(\psi_0)$ is sufficiently irrational.

Let us define the norm of a function $f(\theta, \varphi) = \sum_{m,n} f_{mn} \cos(m\theta - n\varphi)$ as $\|f\| = \max_{m,n} |f_{mn}|$. If we compare the relative size of the control we obtain :

$$\frac{\|\varepsilon^2 f_2\|}{\|\varepsilon H_1\|} \approx 35\varepsilon,$$

typically 14% of εH_1 for $\varepsilon = 0.004$. The magnitude of the control term computed in such a way appears to be a small fraction of the magnitude of the initial perturbation that led to the stochastic transport. Moreover, the control only acts on a finite and small portion of the phase space $[0, 1] \times \mathbb{T}^2$ around the invariant surface (the size of the support of Ω_{loc} is 4%).

The locking of two tearing modes and the onset of large scale chaotic transport govern the thermal and current quench in a disruptive scenario. While the control term discussed in the present work cannot restore the full magnetic confinement, it leads to a transport barrier that can drastically modify the behavior of the current quench. Indeed, the transport barrier will not significantly lower the confinement loss in the outermost part of the discharge while it will decouple the core plasma from this poorly confined region. This should strongly reduce the decay rate of the plasma current in the current quench phase, a feature of strong relevance to minimize the generation of runaway electrons. The control term would then provide the conditions for a soft landing of the plasma and a safer operation of the device.

4.2. Control of stochastic boundary plasmas

For the second example, we consider a magnetic perturbation which models the magnetic field lines in an ergodic divertor [34, 27, 28, 33, 29]. We use the q -profile given by Eq. (12) and the following magnetic perturbation

$$H_1(\psi, \theta, \varphi) = \sum_m (-1)^m \frac{\sin[(m - m_0)\theta_d]}{\pi(m - m_0)} \psi^{\bar{m}/2} \cos(m\theta - n\varphi), \quad (17)$$

where $n = 2$, $\theta_d = \pi/3$, $m_0 = 6$ and the sum ranges from $m_0 - 4$ to $m_0 + 4$. We have chosen $\bar{m} = m$. In experiments, the strong modulation of the toroidal magnetic field between the inner region and the outer region of the torus can lead to a significant departure from this approximation. This is the case for the Tore Supra experiment with $\bar{m} \sim m/2$ and the Textor DED experiment with $\bar{m} \sim 2m$ [35]. However, such aspects of the perturbation, while important for the engineering constraints, do not modify significantly the computation of the control term. Also, the values chosen for the mode numbers are relevant for the Textor DED, a rather low m and n configuration that is required to provide a reasonable stochastic boundary in spite of the DED coil location on the high field side [35]. Few modes are then effective in the spectrum given by Eq. (17) and the largest extent of the perturbed edge region is achieved when the main mode of the spectrum, $m_0 = 6$ and $n = 2$ (hence $q(\psi_{6,2}) = 3$) is located within the

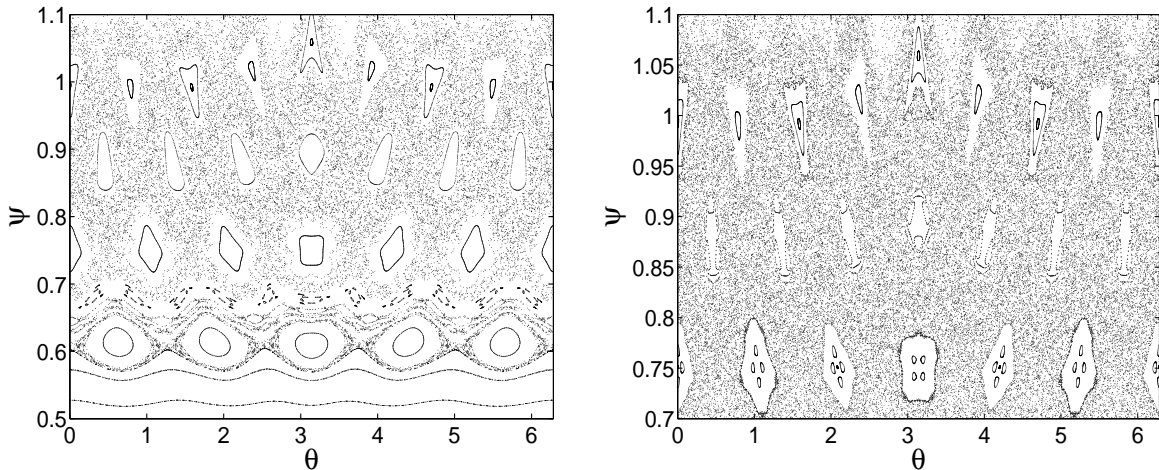


Figure 4. Poincaré surface of section of H given by Eq. (14) with H_1 given by Eq. (17) with $\varepsilon = 0.003$ for $\psi \in [0.5, 1.1]$ (left panel) and for $\psi \in [0.71, 1]$ (right panel).

edge region, at $\psi_{6,2} \approx 0.75$ for $q_{edge} = 4$ for the chosen profile of the safety factor. The region chosen for the control, $\psi_0 \approx 0.9$, $q(\psi_0) \approx 3.6$ thus appears to be located in the vicinity of the mode (7, 2) of the spectrum characterized by a large amplitude (~ 0.8 to be compared to 1 for the main component (6, 2)).

One well documented operational limit of stochastic boundary plasmas is achieved when the stochastic region extends over a too large radial extent [36, 37, 38] and when the stochastic boundary actually reaches the wall, namely when all invariant tori between the stochastic region and the wall are destroyed [36, 37, 38].

A Poincaré surface of section of the dynamics is represented on Fig. 4 for $\varepsilon = 0.003$ where we notice that magnetic field lines diffuse up to outer-edge of the plasma ($\psi = 1$) and to the wall.

The control term we apply is given by Eq. (9) where

$$\begin{aligned} \partial_\psi H_1(\psi_0, \theta, \varphi) &= \sum_m (-1)^m \frac{\sin[(m - m_0)\theta_d]}{\pi(m - m_0)} \frac{m}{2} \psi_0^{m/2-1} \cos(m\theta - n\varphi), \\ \Gamma \partial_\theta H_1(\psi_0, \theta, \varphi) &= \sum_m (-1)^m \frac{\sin[(m - m_0)\theta_d]}{\pi(m - m_0)} \psi_0^{m/2} \frac{m}{m\omega - n} \cos(m\theta - n\varphi). \end{aligned}$$

We use the same characteristic function Ω_{loc} as in the first example with $\delta\psi_\alpha = 0.02$ and $\delta\psi_\beta = 0.03$. We choose $\psi_0 \approx 0.92$. A Poincaré section of Hamiltonian (14) with H_1 given by Eq. (17) and with the control term f_2 given by Eq. (9) for this example with $\varepsilon = 0.003$ is represented on Fig. 5.

As for the first example, we clearly see that the upper and lower parts of phase space are very similar to the ones of Fig. 4. In other words, this localized control does not affect the diffusivity of magnetic lines above the magnetic surface around $\psi = \psi_0$. However the two parts are disconnected by the dynamics since they are invariant by the dynamics of the controlled Hamiltonian. A magnetic surface whose equation is given by Eq. (8) has been created and acts as a barrier to the diffusion toward the border

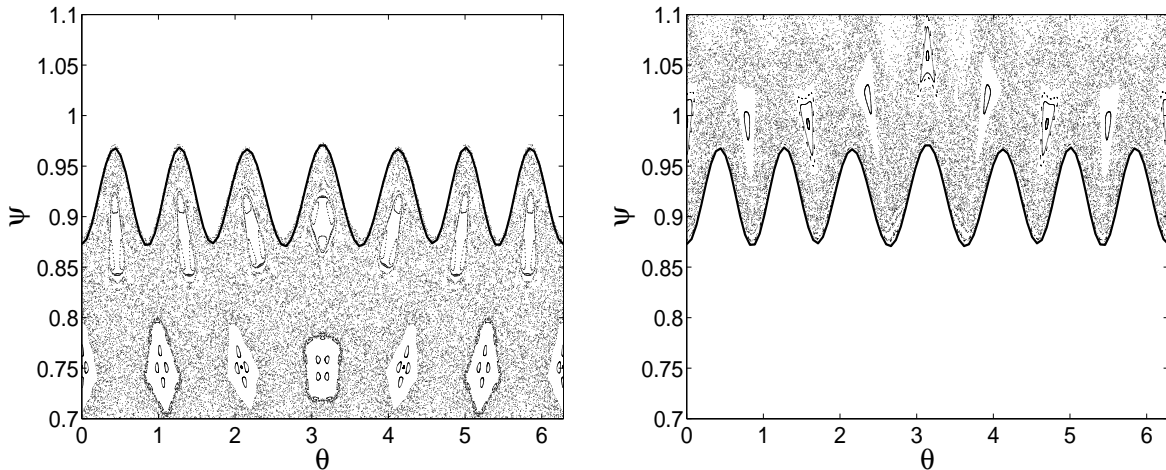


Figure 5. Poincaré surface of section with H_1 given by Eq. (17) using $\Omega = \Omega_{loc}$, with the control term $\varepsilon^2 f_2$ given by Eq. (9) with $\varepsilon = 0.003$ and $\psi_0 \approx 0.92$: with initial conditions below (left panel) or above (right panel) the surface given by Eq. (8).

of the plasma. As stated above, there is experimental evidence that indicates that such a transport barrier is sufficient to decouple the core plasma from the edge plasma [36, 37, 38].

The norm (as defined in the previous section) of the control f_2 is about 15% of εH_1 for $\varepsilon = 0.003$. Moreover, the control only acts on a finite and small portion (around 7%) of the phase space $[0, 1] \times \mathbb{T}^2$ around the invariant surface.

For $\varepsilon \geq 0.003$, the truncated control term f_2 is not sufficient to create the barrier. Therefore, one has to take into account more terms in the series expansion of the control term. A Poincaré section of Hamiltonian (11) with H_1 given by Eq. (17), with the control term $\varepsilon^2 f_2$ and with the control term $\varepsilon^2 f_2 + \varepsilon^3 f_3$ given by Eq. (10) for $\varepsilon = 0.004$ is represented on Fig. 6.

We clearly see that f_2 is not sufficient to create an absolute barrier to diffusion in contrast with the control term obtained with the addition of f_3 . For higher values of ε , more terms of the series can be included if necessary. However we notice that there is still an effective barrier to diffusion which prevents most of the trajectories to diffuse toward the edge. In order to measure the efficiency of the control in the case where the partial control term f_2 is not sufficient to create a barrier, we have plotted in Fig. 7 the probability distribution function of trajectories launched from below the barrier (8) in the chaotic sea for a fixed elapse of time. This function is averaged over the angles θ so that the barrier is not strictly localized on $\psi_0 = 0.92$ due to the angle dependence (set by Eq. (8)). We notice that f_2 is still efficient to reduce the diffusion of trajectories since most of the trajectories are such that $\psi \leq 1$.

Moreover, there is the freedom in simplifying the control term by removing some Fourier components. This can be analyzed when considering the control term computed for the interaction of two tearing modes (see Eq. (15) in Sec. 4.1), and generalized to any set of modes, (m_1, n_1) and (m_2, n_2) with amplitude A_1 and A_2 characterized by a

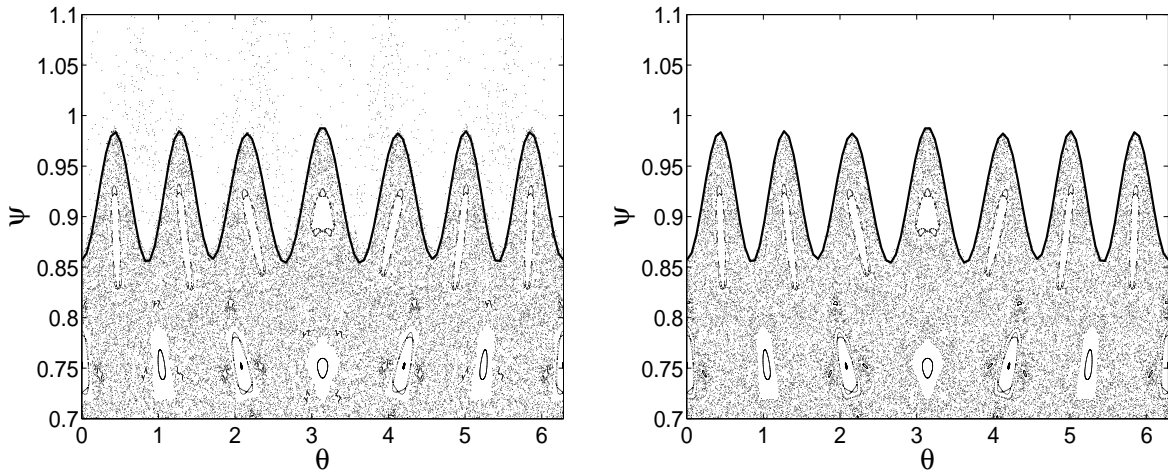


Figure 6. Poincaré surface of section with H_1 given by Eq. (17) with the control term $\varepsilon^2 f_2$ (left panel) and with $\varepsilon^2 f_2 + \varepsilon^3 f_3$ (right panel) given by Eq. (10) with $\varepsilon = 0.004$, $\psi_0 \approx 0.92$ and $\Omega = 1$, and with initial conditions below the surface given by Eq. (8).

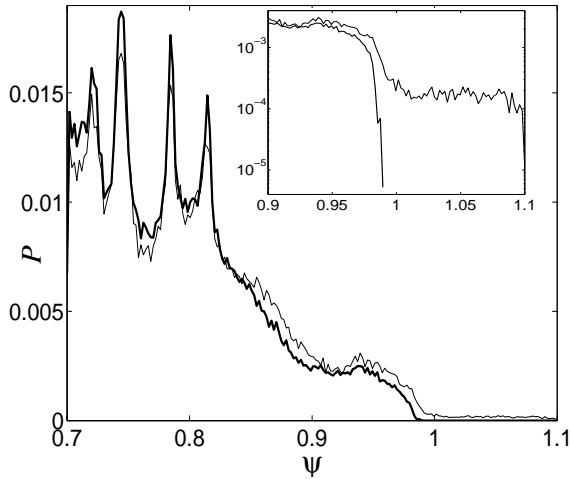


Figure 7. Probability distribution functions of trajectories in the presence of the partial control f_2 (dotted line) and the partial control $f_2 + f_3$ (bold line).

weak ψ dependence. In this case, the control term is roughly :

$$f_2(\theta, \varphi) \approx \left(- \frac{d(1/q(\psi))}{d\psi} \right) \Big|_{\psi=\psi_0} \times \left(\frac{m_1 A_1 \cos(m_1 \theta - n_1 \varphi)}{m_1 \omega - n_1} + \frac{m_2 A_2 \cos(m_2 \theta - n_2 \varphi)}{m_2 \omega - n_2} \right)^2.$$

Such a control term is characterized by four Fourier components, two are corrections to the amplitude of the original tearing modes and two are coupling terms with angle dependence $(m_1 + m_2)\theta - (n_1 + n_2)\varphi$ and $(m_1 - m_2)\theta - (n_1 - n_2)\varphi$. Only the first mode is resonant between the resonant surfaces of the two modes and has therefore the largest weight in the control procedure restoring an invariant torus in its vicinity. This analysis can be transposed to the control of the stochastic boundary addressed here,

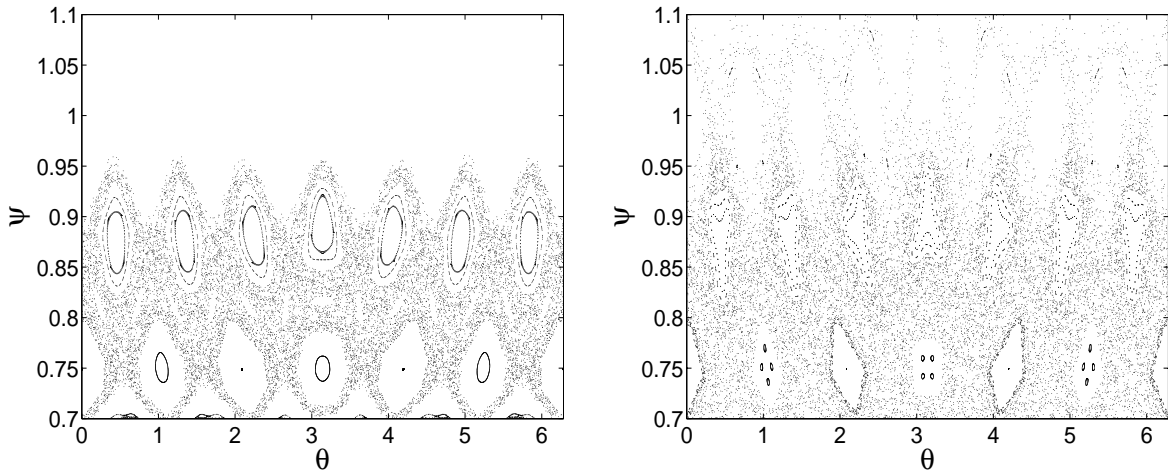


Figure 8. Poincaré surface of section with H_1 given by Eq. (17) with the control term given by Eq. (18) with $\varepsilon = 0.002$ (left panel) and $\varepsilon = 0.003$ (right panel) using $\Omega = \Omega_{loc}$ and $\psi_0 \approx 0.92$.

hence such that $n_1 = n_2 = n$. For instance if one wants to create a magnetic surface between the island with period 7 and the one with period 8, the Fourier mode with wave vector $(15, 4)$ is dominant. In the case of a localization between resonant islands with frequency vector (m_1, n) and (m_2, n) , the approximate control term then reduces to

$$f_{2,1}(\theta, \varphi) = f_{m_1 m_2} \cos[(m_1 + m_2)\theta - 2n\varphi], \quad (18)$$

with

$$f_{m_1 m_2} = (-1)^{m_1 + m_2} \frac{\sin[(m_1 - m_0)\theta_d]}{\pi(m_1 - m_0)} \frac{\sin[(m_2 - m_0)\theta_d]}{\pi(m_2 - m_0)} \psi_0^{(m_1 + m_2)/2} \\ \times \frac{m_1 m_2}{(m_1 \omega - n)(m_2 \omega - n)} \left(\frac{(m_1 + m_2)\omega - 2n}{4\psi_0} - \frac{Q'(\psi_0)}{2} \right).$$

In this expression, the ψ dependence of the perturbation H_1 is taken into account leading to a correction in the magnitude of the control term. Of course, the theorem does no longer ensure the existence of the invariant torus since this control term is approximate. However this simplified control term requires less energy (the ratio between the energy necessary for the control and the one of the magnetic perturbation is 4% for $\varepsilon = 0.003$). Furthermore, the strong simplification of the spectrum of the control term should translate into the design of the dedicated control coil. The effect of the simplified control term of Hamiltonian (17) given by Eq. (18) can be seen on the Poincaré section of Fig. 8. It clearly shows that an invariant torus bounding the motion of magnetic field lines has been created by this simple control term for $\varepsilon = 0.002$. However, when ε is increased to $\varepsilon = 0.003$ there is no longer an invariant torus and field lines leak out toward the $\psi \sim 1$ values. The density of points in the Poincaré section of Fig. 8 is indicative of the existence of a transport barrier that inhibits the transport at the location of the invariant torus observed when the full control term is applied.

In a similar way as in Fig. 6, we have measured the efficiency of the control in the case of Fig. 8 when $\varepsilon = 0.003$ by plotting in Fig. 9 the probability distribution function

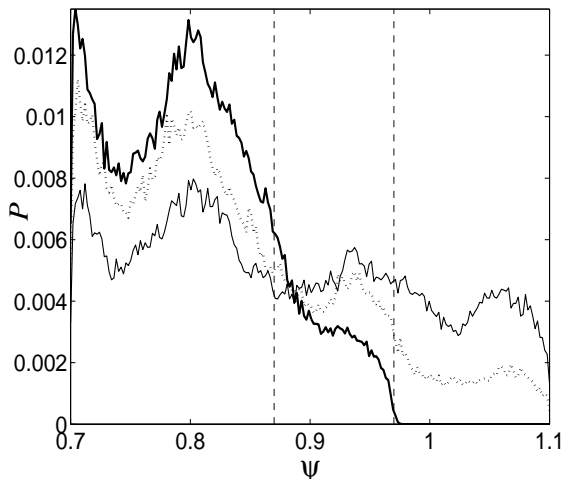


Figure 9. Probability distribution functions of trajectories without control term, with the simplified control term (18) (dotted line) and with the control term f_2 (bold line). The dashed lines indicate the ψ -extension of the created invariant torus. Here $\varepsilon = 0.003$.

(PDF) of trajectories launched from below the barrier (8) for a fixed time. The bold line represents the PDF with the control term f_2 which is sufficient to create a barrier to diffusion when $\varepsilon = 0.003$. The thin line is the PDF without control and the dotted line represents the PDF with the simple control given by Eq. (18). We notice that the simplified control term is still efficient to reduce the diffusion of trajectories (by a factor 2 for the value of the PDF).

Regarding transport issues in a fusion device this remnant loss of magnetic confinement can be comparable to the existence of an invariant torus provided the transport across the barrier along the field lines is smaller than the turbulent plasma transport [39].

In this Section, we have shown that a transport barrier consisting of an invariant torus or a region of reduced field line diffusion can be generated within a stochastic boundary. This possibility can prove to be important not only for ergodic divertor experiments as addressed here but also for other fusion device such as stellarators, where such stochastic boundaries are intrinsic to the magnetic equilibrium for a set of configurations [40, 41, 42] as well as Reverse Field Pinches where restoring the magnetic surfaces in the outermost region is a key to enhanced performance [43]. Finally, the present results can also be used to analyze the ELM control scheme addressed on DIII-D [44]. In the latter experiment a very weak magnetic perturbation is shown to control the large ELMs associated with the H-mode operation. Analysis of the magnetic structure have indicated that the magnitude of the perturbation is comparable to the error field due to toroidal coil misalignment. There is therefore a possibility that a control of stochastic transport of the type analyzed here is at work in these experiments. To investigate such a process one would need to reach a precise modeling of the ELMs, as proposed in Ref. [45], to investigate the impact of such a control on the ELMs and

H-mode pedestal physics. This is clearly beyond the scope of this paper.

5. Conclusion

We have developed a strategy of localized control of Hamiltonian systems. This control scheme is very original insofar that rather than compensating for the perturbation driving the stochastic transport it is based on adding an extra perturbation of lower magnitude with respect to the initial perturbation in order to restore an invariant torus. A simple demonstration of this control scheme is proposed in Sec. 2. The application to magnetic field lines, in Sec. 3, allows one to show that it is possible to create isolated magnetic surfaces within a region of stochastic field lines. This prevents magnetic field lines from diffusing throughout the stochastic region and thus generates two independent regions separated by a transport barrier. It is important to stress that in such cases the perturbation leading to stochastic transport is of order ε while the control term is of order ε^2 . We note that the control we propose can be applied in all generality since the construction is independent of the specific form of the Hamiltonian. Moreover, this method is also applicable if the spectrum of the magnetic perturbation is given numerically. Control for two cases relevant to plasma confinement in magnetic fusion devices are addressed in Sec. 4. A rather generic case of the control of stochastic transport induced by two coupled tearing modes allows one to show that the control term is readily computed and provides control both in the case of a localized control term and in the case where the control term is not restricted to the region of the transport barrier. Such a control can prove to be important to confine the plasma current during the current quench phase that is readily associated with stochastic transport governed by coupled tearing modes. The second example is the control of stochastic boundary layers. In such a situation, it can prove to be important to control the region in contact with the wall via the stochastic boundary. The calculation of such a control scheme is performed in the ergodic divertor framework and can readily be extended to other configurations such as stellarators. In this case, it is shown that a Fourier truncation of the control term still provides a control of transport at the chosen location in the boundary layer. This underlines the robustness of such a control scheme. There is a possibility that such a control procedure be relevant to the ELM control experiments carried out on DIII-D, where several aspects of the observed ELM control remain unexplained. Finally, it is important to underline that the control term appears, in these cases, as a higher order harmonic of the initial perturbation with one order of magnitude lower amplitude. These terms are generally ignored in the analysis of stochastic layers. The present analysis indicates that they can, however, locally strongly modify the transport properties.

Acknowledgments

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Appendix A. Existence and regularity of the control term

In this Appendix, we prove that the control term and the canonical transformation are bounded. It reduces to prove that $\Gamma\partial_{\boldsymbol{\theta}}v$ is bounded. This follows from usual KAM proofs (see for instance Ref. [46]) which also shows the existence of invariant tori. The Fourier expansion of $\Gamma\partial_{\boldsymbol{\theta}}v$ writes :

$$\Gamma\partial_{\boldsymbol{\theta}}v = \sum_{\mathbf{k} \in \mathbb{Z}^L \setminus \{\mathbf{0}\}} \frac{\mathbf{k}v_{\mathbf{k}}}{\boldsymbol{\omega} \cdot \mathbf{k}} e^{i\mathbf{k} \cdot \boldsymbol{\theta}}.$$

We assume that $\boldsymbol{\omega}$ satisfies a Diophantine condition, i.e. there exist $\sigma > 0$ and $\tau > L - 1$ such that

$$|\boldsymbol{\omega} \cdot \mathbf{k}|^{-1} \leq \sigma \|\mathbf{k}\|^\tau, \quad \forall \mathbf{k} \in \mathbb{Z}^L \setminus \{\mathbf{0}\},$$

where $\|\mathbf{k}\| = \sum_i |k_i|$. Moreover, we assume that v is of class $C^{r+\tau+1}$ where $r > 1$, i.e. bounded for the norm of scalar functions $v(\boldsymbol{\theta}) = \sum_{\mathbf{k} \in \mathbb{Z}^L} v_{\mathbf{k}} e^{i\mathbf{k} \cdot \boldsymbol{\theta}}$:

$$\|v\|_{r+\tau+1} = |v_{\mathbf{0}}| + \sum_{\mathbf{k} \in \mathbb{Z}^L \setminus \{\mathbf{0}\}} |v_{\mathbf{k}}| \|\mathbf{k}\|^{r+\tau+1},$$

In the same way we define the norm of vectorial functions of class C^r as $\|\mathbf{g}\|_r = \sum_{i=1}^L \|g_i\|_r$. Thus we have

$$\|\Gamma\partial_{\boldsymbol{\theta}}v\|_r = \sum_{\mathbf{k} \in \mathbb{Z}^L \setminus \{\mathbf{0}\}} \frac{\|\mathbf{k}\| |v_{\mathbf{k}}|}{|\boldsymbol{\omega} \cdot \mathbf{k}|} \|\mathbf{k}\|^r.$$

The Diophantine condition of $\boldsymbol{\omega}$ gives

$$\|\Gamma\partial_{\boldsymbol{\theta}}v\|_r \leq \sigma \sum_{\mathbf{k} \in \mathbb{Z}^L \setminus \{\mathbf{0}\}} \|\mathbf{k}\|^{\tau+1+r} |v_{\mathbf{k}}| \leq \sigma \|v\|_{r+\tau+1}.$$

This norm is bounded and hence $\Gamma\partial_{\boldsymbol{\theta}}v$ is of class C^r . Hence the loss of regularity between v and $\Gamma\partial_{\boldsymbol{\theta}}v$ is the constant $\tau + 1$. We notice that we could have also weakened the hypothesis on $(v, \boldsymbol{\omega})$ into $\|\Gamma\partial_{\boldsymbol{\theta}}v\|_r < \infty$.

In order to have an estimate on the control term, we denote

$$\begin{aligned} \eta_1 &= \sup_{\boldsymbol{\theta} \in \mathbb{T}^L} \|\Gamma\partial_{\boldsymbol{\theta}}v\| \leq \sigma \|v\|_{r+\tau+1}, \\ \eta_2 &= \frac{1}{\varepsilon} \sup_{\|A\| \leq \varepsilon \eta_1} \sup_{\boldsymbol{\theta} \in \mathbb{T}^L} \max_{i=1, \dots, L} |\partial_{A_i} w(\mathbf{A}, \boldsymbol{\theta})|. \end{aligned}$$

If w is given by $w(\mathbf{A}, \boldsymbol{\theta}) = \varepsilon \mathbf{w}_1(\boldsymbol{\theta}) \cdot \mathbf{A} + w_2(\mathbf{A}, \boldsymbol{\theta})$, we see that

$$\eta_2 \leq \|\mathbf{w}_1\|_r + \eta_1 \sup_{\|A\| \leq \varepsilon \eta_1} \sup_{\boldsymbol{\theta} \in \mathbb{T}^L} \max_{i,j} |\partial_{A_i} \partial_{A_j} w_2|.$$

The control term given by Eq. (3) can be rewritten as

$$\varepsilon^2 f = \int_0^1 \partial_{\mathbf{A}} w(-s\varepsilon \Gamma\partial_{\boldsymbol{\theta}}v, \boldsymbol{\theta}) \cdot \varepsilon \Gamma\partial_{\boldsymbol{\theta}}v ds.$$

Thus we have

$$\sup_{\boldsymbol{\theta} \in \mathbb{T}^L} |f(\boldsymbol{\theta})| \leq \eta_1 \eta_2,$$

where η_1 and η_2 are of order 1, and hence the control term $\varepsilon^2 f$ is of order ε^2 . In a very similar way, if we assume that the Hamiltonian is three times differentiable in $(\mathbf{A}, \boldsymbol{\theta})$, then the derivative of the control term is also bounded and of order ε^2 .

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