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Commentary on ‘The Three-Dimensional Current and Surface Wave Equations’ by George Mellor

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ABSTRACT

The lowest order sigma-transformed momentum equation that takes into account a phase-averaged wave forcing based on Airy wave theory is shown to be generally inconsistent due to inadequate approximations in the vertical fluxes of wave momentum. The expression of these fluxes is required to first order in parameters that define the large scale evolution of the wave field, such as the bottom slope. Unfortunately there is no analytical expression for these fluxes, except for fluxes associated with wind-wave generation. Alternative coordinate transforms that allow a separation of wave and mean flow momenta do not suffer from this inconsistency, because the problematic vertical fluxes are part of the wave momentum flux, and are not directly relevant to the mean flow evolution.

1. Introduction

Wave-induced motions are of prime importance in the upper ocean, and in the coastal ocean (e.g. Ardhuin et al. 2005 for a recent review). Thus, common three-dimensional primitive equations must be modified to account for waves, hopefully in a consistent way. Among such modified equations, those based on surface-following coordinates provide physically sound definitions of velocities right up to the free surface, allowing a proper representation of surface shears and mixing on a vertical scale smaller than the wave height (i.e. a few meters). Any change of coordinate adds some complexity in the derivation, but the final equations can be relatively simple because some of the advective fluxes are removed, and boundary conditions may be simplified. A new set of such equations was recently derived by Mellor (2003) using a change of the vertical coordinate only, arguably the simplest possible. Like Lagrangian coordinates, solutions to first order in the wave amplitude in these coordinates contain some contributions that only appear at higher order with fixed cartesian coordinates. Mellor’s (2003) set of equations was originally derived for monochromatic waves, but it is easily extended to random waves (e.g. Ardhuin et al. 2004, eq. 8). However, a correct form of these equations is not so simple in the presence of horizontal gradients in the wave field,

water depth, or current. We shall show that these equations, in the form given by Mellor, are not applicable to the problem of wave shoaling in the absence of dissipation. A modified form of the equations is proposed that verifies the known balance of forces in shoaling waves. However, this form requires a numerical evaluation of the wave forcing terms. This difficulty is due to the choice of averaging, and the same problem arises with the alternative Generalized Lagrangian Mean equations of Andrews and McIntyre (1978). Both Mellor’s and the alternative GLM equations describe the evolution of a momentum quantity that contains the three-dimensional wave (pseudo)-momentum (hereinafter called ‘wave momentum’ for simplicity, see McIntyre 1981 for details). Writing an evolution equation for this quantity requires an explicit description of the complex vertical fluxes of wave momentum that are necessary to maintain the vertical structure of the wave field in the surface gravity waveguide.

2. The problem: wave motions and wave-following vertical coordinates

We discuss here the simple case of monochromatic waves of amplitude a and wavenumber k propagating in the horizontal x direction, with all quantities uniform in the other horizontal y direction. The surface and bottom elevations are $\eta(x, t)$ and $-h(x)$, respectively, so that the local mean water depth is $D(x, t) = h(x) + \overline{\eta(x, t)}$, with the overbar denoting an Eulerian average over the wave

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phase. We shall assume that the maximum surface slope is a small parameter $\varepsilon_1 = ka \ll 1$, that the Eulerian mean current \bar{u} in the x -direction is uniform over the depth. Thus ω will denote the radian wave frequency related to k by the linear wave dispersion relation (e.g. Mei 1989),

$$\omega = k\bar{u} + \sigma = k\bar{u} + [gk \tanh(kD)]^{1/2}. \quad (1)$$

Finally, we assume that the water depth, current and wave amplitude change slowly along the x -axis with a slowness measured by a second small parameter ε_2 taken to be the maximum bottom slope. We thus assume $|(\partial a/\partial x)/(ka)| \leq \varepsilon_2$, $|(\partial \bar{u}/\partial x)/(\sigma)| \leq \varepsilon_2$, $|(\partial D/\partial x)| \leq \varepsilon_2$, $|(\partial a/\partial t)/(\sigma a)| \leq \varepsilon_2$, $|k(\partial \bar{u}/\partial t)/\sigma^2| \leq \varepsilon_2$, $|(\partial D/\partial t)k/\sigma| \leq \varepsilon_2$. The conditions on the bottom slope and current gradients are consistent with the condition on the wave amplitude gradient because in steady conditions the wave amplitude would change due to shoaling over the current and/or bottom.

The vertical coordinate z is implicitly transformed into Mellor's ζ coordinate through

$$z = s(x, \zeta, t) = \bar{\eta} + \zeta D + \tilde{s} \quad (2)$$

with \tilde{s} defined by Mellor's eq. (23b) as

$$\tilde{s} = \tilde{s}_0 = aF_{SS} \cos(kx - \omega t) \quad (3)$$

and the vertical profile function F_{SS} defined by

$$F_{SS} = \frac{\sinh[kD(1 + \zeta)]}{\sinh(kD)} = \frac{\sinh[k(z + h)]}{\sinh(kD)} + O\left(\frac{a}{D}\right). \quad (4)$$

The coordinate transformation from z to ζ has the very nice property of following the vertical wave-induced motion, at least for linear waves on a flat bottom, and to first order in ε_1 . In that case the iso- ζ surfaces are material surfaces, and the fluxes of horizontal momentum through one of these surfaces are simply correlations of pressure p times the slope of that surface $\partial \tilde{s}/\partial x$ (figure 1.c), which replaces the wave-induced advective flux uw in an Eulerian point of view (figure 1.a).

Using this coordinate transform, Mellor obtained a phase-averaged equation for the drift current $U = \hat{u} + u_S$ where u_S is the Stokes drift, i.e. the mean velocity of water particles induced by fast wave-induced motions. U is strictly defined as the phase-average particle drift velocity when following the up-and-down wave motion, and $\hat{u} = U - u_S$ is a quasi-Eulerian mean current. Below the wave crests \hat{u} is equal, to second order in the wave slope, with the Eulerian mean current \bar{u} .

Mellor's horizontal mean momentum equation (34a) is reproduced here for completeness, in our conditions with a flow restricted to the vertical x, z plane, a constant water density, no Coriolis force, and no turbulent fluxes and the atmospheric mean pressure set to zero (the wind-wave generation term due to pressure fluctuations is absorbed in the F_{x3} term),

$$\frac{\partial DU}{\partial t} + \frac{\partial DU^2}{\partial x} + \frac{\partial \Omega U}{\partial \zeta} + gD \frac{\partial \hat{\eta}}{\partial x} = F_{xx} + F_{x3} \quad (5)$$

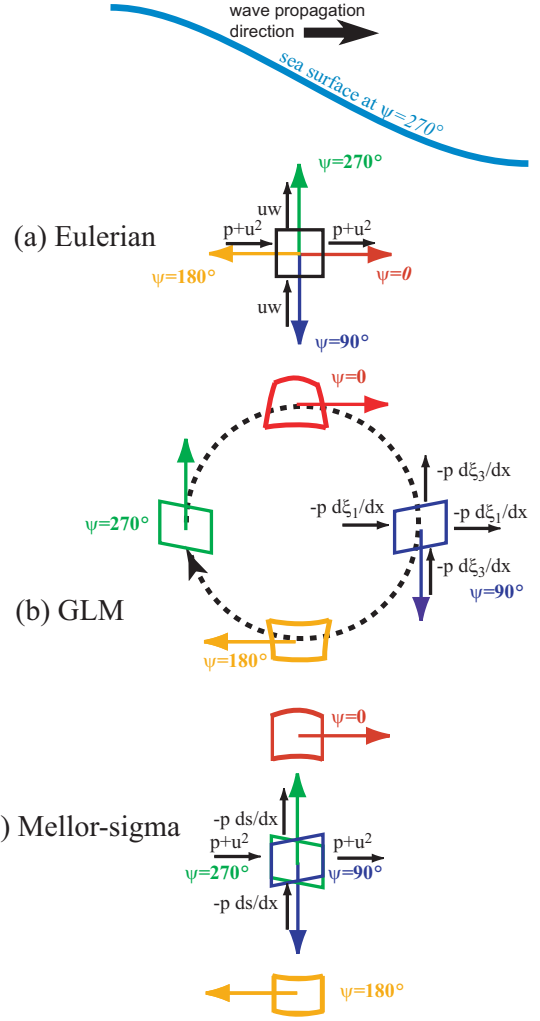


FIG. 1. Wave-induced fluxes of horizontal momentum in Eulerian, Generalized Lagrangian, and Mellor-sigma averages of the flow. Viscous or turbulent fluxes are neglected for simplicity. Distorted squares represent an elementary fluid volume and its position at four phases of the wave cycle, and the large arrow indicate the local wave orbital velocity. The horizontal and vertical fluxes of the horizontal momentum are represented by smaller arrows.

On the right-hand side the term

$$F_{x3} = -\frac{\partial S_{x3}}{\partial \zeta} = \frac{\partial}{\partial \zeta} \left(\overline{\tilde{p} \partial \tilde{s} / \partial x} \right) \quad (6)$$

represents the convergence of a vertical flux of horizontal momentum that accelerates the mean drift velocity U . When averaging is performed following water particles over their trajectory (Lagrangian) or over their vertical displacement (Mellor-sigma), the corresponding advective flux of momentum $u_i u_j$ is replaced by a modified pressure force. For the Generalized Lagrangian Mean (GLM) only the contributions to lowest order in ε_1 are

indicated. Indeed, in GLM the wave-induced advective flux is not strictly zero, but of higher order, since the average only follows a zero-mean displacement, contrary to a truly Lagrangian mean with zero advection (e.g. Jenkins 1986).

Defining g as the acceleration due to the apparent gravity, \tilde{p} and \tilde{s} are of the order of ga and a respectively. In the case where \tilde{p} and \tilde{s} are in phase, F_{x3} is of the order of $gDka \partial a / \partial x$, with the Dk factor coming from the vertical divergence of the flux. Thus F_{x3} is of the order of $gD\varepsilon_1^2\varepsilon_2$.

The other term

$$F_{xx} = -\frac{\partial S_{xx}}{\partial x} = -D \frac{\partial}{\partial x} \left(\overline{\tilde{u}^2 + \tilde{p} \frac{\partial \tilde{s}}{\partial x}} \right) \quad (7)$$

represents the convergence of a horizontal flux of horizontal momentum that accelerates the mean drift velocity U .

Mellor estimated the vertical momentum flux S_{x3} from (3) and the corresponding lowest order wave-induced kinematic pressure on iso- ζ levels (i.e. including a hydrostatic correction due to the vertical displacement),

$$\tilde{p} = \tilde{p}_0 = ga (F_{CC} - F_{SS}) \cos(kx - \omega t), \quad (8)$$

g is the acceleration due to the apparent gravity, and the vertical profile function F_{CC} is defined by

$$F_{CC} = \frac{\cosh[kD(1 + \zeta)]}{\cosh(kD)} = \frac{\cosh[k(z + h)]}{\cosh(kD)} + O\left(\frac{a}{D}\right). \quad (9)$$

The terms on the right hand side of eq. (5) are of order $gD\varepsilon_1^2\varepsilon_2$. The estimation of F_{x3} thus requires the knowledge of $\tilde{p}/(gD)$ and $k\tilde{s}$ to order $\varepsilon_1\varepsilon_2$, for which Airy theory is insufficient. In particular, this estimation demands a formal definition of \tilde{s} .

For a general surface ζ defined implicitly by $z = s(x, \zeta, t)$, we define the (nearly vertical) wave-induced velocity through an iso- ζ level by (e.g. Mellor 2003 eq. 20),

$$\begin{aligned} \omega &= \frac{d(z - s)}{dt} - \frac{d(z - s)}{dt} \\ &= \tilde{w} - \tilde{u} \frac{\partial \tilde{s}}{\partial x} - \tilde{u} \frac{\partial \tilde{s}}{\partial x} - \frac{\partial \tilde{s}}{\partial t}. \end{aligned} \quad (10)$$

In the spirit of Mellor (2003), the iso- ζ levels should be material surfaces for wave-only motions, thus \tilde{w} should be small. Otherwise the mean Jacobian-weighted vertical velocity $\Omega = \overline{\tilde{w}/(1 + \partial s/\partial \zeta/D)}$ would carry a large part of the vertical flux of wave momentum. Using the wave-induced vertical and horizontal displacements, $\xi_3(x, \zeta, t)$ and $\xi_1(x, \zeta, t)$, defined by $\partial \xi_i / \partial t = \tilde{u}_i(x_1 + \xi_1, z + \xi_3, t)$, one may take

$$\tilde{s}' = \xi_3 - \xi_1 \frac{\partial}{\partial x} (\bar{\eta} + \zeta D). \quad (11)$$

The first term corresponds to Mellor's definition while the second is a $O(\varepsilon_2)$ relative correction. This definition yields a wave-induced vertical velocity $\tilde{w} = -\tilde{u} \partial \tilde{s}' / \partial x$ through the iso- ζ surfaces redefined by $z = s(x, \zeta, t) = \bar{\eta} + \zeta D + \tilde{s}'$. This vertical velocity is of a higher order compared to that given by \tilde{s} defined by eq. (2) as in Mellor (2003).

a. Wave-induced vertical displacements and pressure over a sloping bottom

A sloping bottom induces a modification of ξ_3 and ξ_1 and p , that enter the definition of \tilde{s}' as well as that of the vertical momentum flux S_{x3} . A major problem in this situation is that the wave velocity potential becomes a non-local function of the water depth (this effect may be described in terms of evanescent modes, see e.g. Massel 1993). The velocity potential and pressure fields may only be investigated analytically for plane beds (e.g. Ehrenmark 2005) or specific bottom profiles for which a conformal mapping to a constant depth can be defined (e.g. Roseau 1976).

For small bottom slopes, the modifications \tilde{s}'_1, p_1 of the displacement and pressure due are expected to be of the order of solutions without the slope (as given by Mellor) times the slope, i.e. of order $a\varepsilon_2$ and $ga\varepsilon_1$, respectively.

The horizontal gradient of \tilde{s}'_1 is of order $ka\varepsilon_2 = \varepsilon_1\varepsilon_2$ and is expected to be in phase with the wave-induced pressure (8), of order ga , giving another term of order $gD\varepsilon_1^2\varepsilon_2$ omitted by Mellor in his estimation of $\partial(\tilde{p} \partial \tilde{s}' / \partial x) / \partial \zeta$. The modification of the pressure may be obtained from the modification of the velocity potential, and it may be in phase with $\partial \tilde{s}'_0 / \partial x$, thus also contributing at the same order to F_{x3} . However, a WBKB approximation using Airy's theory is sufficient for estimating F_{xx} because the horizontal gradient of any wave-averaged quantity ϕ is of order $\varepsilon_2 k \phi$.

Mellor's (2003) approximate expression for F_{x3} is thus not consistent in general, in particular in the case exposed here of waves shoaling on a gentle slope without wind forcing or dissipation. Similar contributions to the momentum fluxes should also arise due to horizontal gradients in the wave amplitude or other effects.

In order to be convinced of the problem, one may consider the case of steady monochromatic shoaling waves over a slope without bottom friction, viscosity or any kind of surface stress, and in the case of no net depth-integrated mass transport. We also neglect the Coriolis force. In this mathematical experiment, the flow is purely irrotational and can be solved for numerically. We consider that the non-dimensional depth kH is of order 1, and that there is no net mass flux across any vertical section. In that case the mean current and the Stokes drift are of the same order, i.e. of the order $C\varepsilon_1^2$ with C the phase speed. The mean current exactly compensates the divergence of the wave-induced mass transport, and the mean sea level must be lower in the area where the wave

height is increased, (Longuet-Higgins 1967)

$$\hat{\eta}(x) = -\frac{kE}{\sinh(2kD)} + \frac{k_0 E_0}{\sinh(2k_0 D_0)} \quad (12)$$

where the 0 subscript correspond to quantities evaluated at the offshore boundary of the domain, but the exact location where these are evaluated is irrelevant to the estimation of horizontal gradients of $\hat{\eta}$.

Because of the steady condition, the first term in (5) is zero and the second is of order $DC^2\varepsilon_1^4\varepsilon_2/D \simeq gD\varepsilon_1^4\varepsilon_2$. The vertical velocity Ω can be estimated from the steady mass conservation equation,

$$\frac{\partial DU}{\partial x} + \frac{\partial \Omega}{\partial \zeta} = 0 \quad (13)$$

where the first term is of order $DC\varepsilon_1^2\varepsilon_2/D$ and the second is of order Ω . Thus the third term in (5) is of order $C^2\varepsilon_1^4\varepsilon_2 \simeq gD\varepsilon_1^4\varepsilon_2$. The remaining terms in (5) are of order $\varepsilon_1^2\varepsilon_2$ giving the lowest order momentum balance

$$-D \frac{\partial}{\partial x} (g\bar{\eta}) - \frac{\partial S_{xx}}{\partial x} + \frac{\partial \tilde{p}\tilde{s}/\partial x}{\partial \zeta} = 0. \quad (14)$$

For reference this corresponds to the lowest order Eulerian mean balance

$$-\frac{\partial}{\partial x} (g\bar{\eta} - \bar{w}^2) - \frac{\partial \bar{u}^2}{\partial x} - \frac{\partial \bar{u}\bar{w}}{\partial z} = 0, \quad (15)$$

which is known to be verified (e.g. Rivero and Sanchez-Arcilla 1994, McWilliams et al. 2004, Lane et al. 2006).

The balance (14) will now be tested numerically. We take a Roseau-type bottom profile (1976) by x and z coordinates given by the real and imaginary part of the complex function

$$Z(x') = \frac{h_1(x' - i\alpha) + (h_2 - h_1)\ln(1 + e^{x' - i\alpha})}{\alpha}. \quad (16)$$

With $\alpha = 15\pi/180$, $h_1 = 6$ m and $h_2 = 4$ m (figure 1), and a radian frequency $\omega = 1.2$ rad s^{-1} (i.e. a frequency $f = 0.2$ Hz), the non-dimensional water depth varies between $0.85 < kH < 1.1$. The reflection coefficient for the wave amplitude is 1.4×10^{-9} (Roseau 1976), so that reflected waves may be neglected in the momentum balance. We illustrate the force balance obtained for waves, and offshore amplitude $a_0 = 0.12$ m, which corresponds to a maximum steepness $\varepsilon_1 = ka$ equal to the maximum bottom slope $\varepsilon_2 = 2.6 \times 10^{-2}$. The change in wave amplitude is given by the conservation of the wave energy flux (see Ardhuin 2006 for a thorough discussion), and the wave phase ψ is taken as the integral over x of the local wavenumber, so that $\partial\psi/\partial x = k$. The various terms are then estimated using second order finite differences on a regular grid in ζ coordinates, with 201 by 401

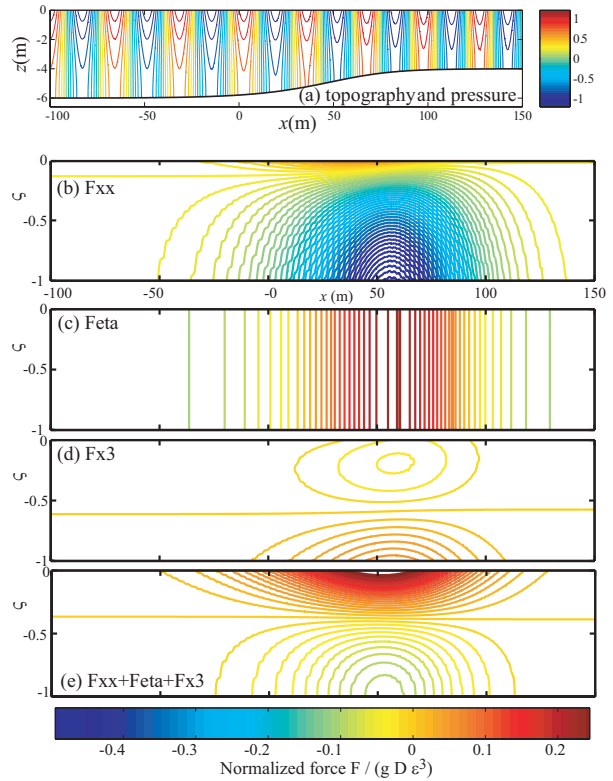


FIG. 2. (a) Snapshot of the pressure field for a slowly varying Airy wave over a the bottom topography given by eq. (16). The forces in the balance (14) are shown in panels b, c and d, with their sum in panel e, all estimated from Mellor's analytical expressions. All forces have been normalized by $gD\varepsilon^3$. N.B. in the case shown here $\varepsilon_1 = \varepsilon_2 = \varepsilon$.

points covering the domain shown in figure 1. The three terms in eq. (14) are shown in figure 2.

We have verified that, within 0.1%, the depth-integrated forces are in balance. However, at any given depth there is a large imbalance (of the order of the individual forces, i.e. $gD\varepsilon_1^2\varepsilon_2$), showing that the three-dimensional equations of motion of Mellor (2003) are not consistent to the dominant order $gD\varepsilon_1^2\varepsilon_2$, at least for the case of shoaling waves without breaking. This conclusion holds for any relative magnitude of the wave and bottom slopes ε_1 and ε_2 .

b. Wind-forced waves

Clearly, any deviation of the wave-induced fields s' , p , u from Airy-wave theory may have strong effects on momentum flux terms. Another example of such a situation is correctly described by Mellor, this is the case of wind-wave generation. We briefly address it here because here an analytical solution can be found, and it has not been fully given previously. Mellor focused on the wind-wave generation contribution to the vertical momentum flux term $\tilde{p}\tilde{s}/\partial x$ term. As he very well expressed it, it

equals wave-supported wind stress at the sea surface, and, below, it explains the growth of the wave momentum profile with the same profile as that of the Stokes drift.

We define the wave-induced non-hydrostatic kinematic Eulerian pressure $\tilde{p}_E = \tilde{p}_{E0} + \tilde{p}_{Ew}$, elevation $\eta = \zeta_0 + \zeta_w$ and velocity potential $\phi = \phi_0 + \phi_w$, in which the 0 subscript refers to the primary waves, and the w subscript refers to the added components in the presence of wind forcing. Taking a primary surface elevation of the form $\zeta_0 = a \cos \psi$ with the phase $\psi = kx - \omega t$, Mellor considered an atmospheric kinematic pressure fluctuation in quadrature with the primary waves

$$\tilde{p}_a = -g\beta \frac{\rho_w}{\rho_a} a \sin \psi, \quad (17)$$

with β a small non-dimensional wave growth factor, and ρ_w and ρ_a the densities of water and air respectively. He then assumed that the water-side wave-induced pressure was of the form

$$\tilde{p}_{\text{Mellor}} = -g\beta a \frac{\cosh[k(z+h)]}{\cosh(kD)} \sin \psi. \quad (18)$$

Implicitly \tilde{s}'_w is zero, and for his purpose ϕ_w was irrelevant. We shall now also determine ϕ_w . The continuity of dynamic pressures at the surface is¹

$$\tilde{p}_{Ew} + g\tilde{s}'_w = -g\beta a \sin \psi \quad \text{at} \quad z = \zeta. \quad (19)$$

These solutions are obtained by solving Laplace's equation with proper boundary conditions, to first order in β . The wave amplitude is now also a function of time, and for the sake of simplicity we shall obtain the solution in the absence of current, which is equivalent to working in the frame of reference moving at the velocity at which the wave phase is advected by the current. The boundary conditions include the Bernoulli equation,

$$\frac{\partial \phi}{\partial t} = -g\zeta - p_a, \quad \text{at} \quad z = \bar{\zeta}, \quad (20)$$

in which non-linear terms have been neglected because they are the sum of products of the form $\nabla \phi_0 \cdot \nabla \zeta_0$, unchanged from the case without wind, and terms of the form $\nabla \phi_w \cdot \nabla \zeta_0$, which are negligible compared to the left-hand side terms for primary waves of small slope. Similarly, the surface kinematic boundary condition is linearized as

$$\frac{\partial \phi}{\partial z} = \frac{\partial \zeta}{\partial t} \quad \text{at} \quad z = \bar{\zeta}. \quad (21)$$

The combination of both yields

$$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = -\frac{\partial \tilde{p}_a}{\partial t} \quad \text{at} \quad z = \bar{\zeta}. \quad (22)$$

¹Here the pressure is Eulerian. For correspondence to Mellor's pressures on ζ levels, one should take $\tilde{p} = \tilde{p}_E - g\tilde{s}'$.

ϕ_w is also a solution of Laplace's equation with the bottom boundary condition $\partial \phi_w / \partial z = 0$ at $z = -h$. With the fully resonant atmospheric pressure (17) envisaged by Mellor, one has

$$\zeta_0 = a(t) \cos \psi, \quad (23)$$

$$\phi_0 = \frac{ga(t)}{\sigma} F_{CC} \sin \psi, \quad (24)$$

$$\tilde{p}_0 = ga(t) F_{CC} \cos \psi, \quad (25)$$

$$\frac{da(t)}{dt} = \frac{\beta \sigma a(t)}{2}, \quad (26)$$

$$\phi_w = \beta g \frac{a}{2\sigma} F_{CC} [A \cos \psi + B \cos \psi'] \quad (27)$$

with $\psi' = (kx + \omega t)$. The elevation and under-water non-hydrostatic pressure corresponding to ϕ_w are given by (21) and the linearized Bernoulli equation

$$\frac{\partial \phi_w}{\partial t} = -\tilde{p}_{Ew} \quad (28)$$

yielding

$$\zeta_w = \beta \frac{a}{2} [(1-A) \sin \psi + B \sin \psi'] \quad (29)$$

$$\tilde{p}_{Ew} = g\beta \frac{a}{2} F_{CC} [-(1+A) \sin \psi + B \sin \psi']. \quad (30)$$

Mellor's expression for \tilde{p}_w given by eq. (18) is obtained by replacing ζ_w and \tilde{p}_{Ew} in (19), giving $A = 1$. One may take $B = 1$ to have $\phi_w = 0$ at $t = 0$, or more simply $B = 0$, which gives $\zeta_w = 0$, and $p_{Ew} = F_{CC} \tilde{p}_a \rho_a / \rho_w$. The choice of B has no dynamical effect. In the present case ϕ_w should give a contribution to $S_{\alpha\beta}$ because it is in phase with ϕ_0 , but this is a relative correction of order β , thus negligible. The contribution of \tilde{p}_w to $(\tilde{p} \partial \bar{s} / \partial x)$ is quite important because for uniform horizontal conditions this flux is otherwise zero.

3. A fix to the problem ?

Contrary to that particular wind-forcing term, there is no simple asymptotically analytical correction for \tilde{p} and \tilde{s}' that can account for the bottom slope and wave field gradient. Numerical solutions for the three-dimensional wave motion are generally found as infinite series of modes (e.g. Massel 1993). The velocity potential for any of these modes satisfies Laplace's equation with a local vertical profile F_n proportional to $\cos(k_n z + k_n h)$ and a dispersion relation $\sigma^2 = g k_n \tan(k_n D)$. The local amplitudes of these modes are non-local functions of the water depth, and may be obtained numerically with a coupled-mode model (Massel 1993). This non-local dependence of the wave amplitude on the water depth arises from the elliptic nature of Laplace's equation, satisfied by the velocity potential in irrotational conditions, but it only becomes important over steep topography. The series of

modes can be made to converge faster by adding a ‘sloping bottom mode’ that often accounts for a large part of the correction and is a local function of the depth and bottom slope. It is thus of interest to see if that correction only, without the infinite series, may provide a first order analytical correction to Mellor’s momentum flux S_{x3} .

Following Athanassoulis and Belibassakis (1999), one may define the velocity potential for that mode as

$$\phi_1 = -\frac{dh}{dx} a \sigma D F(z) \cos \psi, \quad (31)$$

In order to satisfy the bottom boundary condition $w = \partial\phi_1/\partial z = -\frac{dh}{dx}\partial\phi_0/\partial x$, the function F should verify $DdF(-h)/dz = 1/\sinh(kH)$ and the satisfaction of the surface boundary condition may be obtained with $F(0) = dF(0)/dz = 0$. Athanassoulis and Belibassakis (1999) have used

$$F = F_{AB} \equiv \frac{1}{\sinh(kD)} \left[\left(\frac{z-\bar{\zeta}}{D} \right)^3 + \left(\frac{z-\bar{\zeta}}{D} \right)^2 \right], \quad (32)$$

and Chandrasekara and Cheung (2001) have used

$$F = F_{Ch} \equiv \frac{1}{kD \sinh^2(kD)} [1 - \cosh(kz - k\bar{\zeta})]. \quad (33)$$

With these choices ϕ_1 does not satisfy exactly Laplace’s equation, and thus requires further corrections in the form of evanescent modes. An infinite number of other choices is available, either satisfying Laplace’s equation or the surface boundary conditions, but never both, so that each of these solutions is only approximate, and the exact solution is given by the infinite series of modes, which can be computed numerically for any bottom topography (e.g. Athanassoulis and Belibassakis 1999, Belibassakis et al. 2001, Magne et al. 2006).

The vertical displacement and Eulerian pressure corrections are given by time integration of the vertical velocity and the linearized Bernoulli equation,

$$\xi_{31} = \frac{dh}{dx} a D \frac{dF}{dz} \sin \psi \quad (34)$$

$$\tilde{p}_{E1} = \frac{dh}{dx} a D F(z) \sin \psi \quad (35)$$

Thus, in absence of wind forcing but taking into account the ‘sloping bottom mode’ to first order in the bottom slope, the wave-induced flux of momentum through iso- ζ surfaces is

$$\begin{aligned} \tilde{p} \frac{\partial \tilde{s}'}{\partial x} = & (F_{CC} - F_{SS}) \left[\frac{ga}{2} \frac{\partial (a F_{SS})}{\partial x} \right. \\ & \left. + \frac{gka^2}{2} \frac{dh}{dx} \left(D \frac{dF}{dz} + \zeta F_{CS} \right) \right] \\ + \frac{gka^2}{2} \frac{dh}{dx} & \left[-F_{SC} F + F_{SS} \left(D \frac{dF}{dz} + \zeta F_{CS} \right) \right], \end{aligned} \quad (36)$$

with $F_{CS} = \cosh[kD(1+\zeta)]/\sinh(kD)$. The first line is the term given by Mellor (2003). The second line arises from the correction due to the difference between \tilde{s}' and \tilde{s} , and the third line arises due to corrections $\tilde{p}_1 = \tilde{p}_{E1} - g s'_1$ to the pressure on ζ levels. These additional term are of the same order as the first term, and have no flux at the bottom and surface. Thus the depth-integrated equations including that term also comply with known depth-integrated equations (e.g. Smith 2006).

In the case chosen here F_{Ch} gives a net momentum balance closer to zero than Mellor’s (2003) original expression (figure 3). However, the remaining error is significant. Thus one cannot use only that mode, and the amplitude of the evanescent modes have to be computed, which can only be done numerically with a coupled mode model.

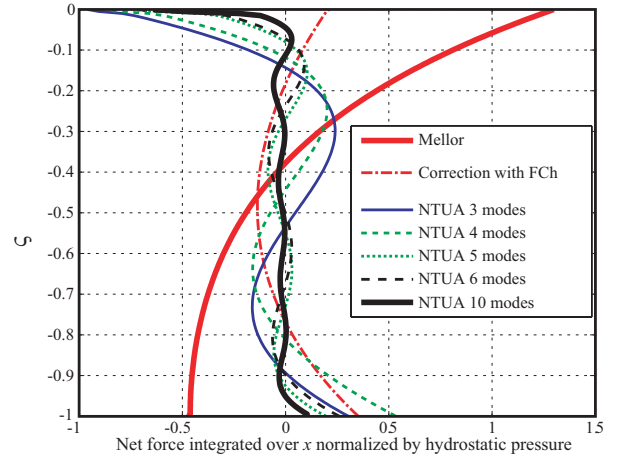


FIG. 3. Net forces in the momentum balance (14) for steady shoaling waves over a smooth bottom profile. The net force has been integrated over x and normalized by a similar integration of the hydrostatic pressure force F_{eta} . Several solutions are obtained. One corresponds to Mellor’s original expression, one possible analytical correction using F_{Ch} , and numerical estimations using the NTUA model, with various numbers of modes.

A numerical evaluation of the forces was performed using the NTUA coupled mode model (Athanassoulis and Belibassakis 1999). The NTUA solution was obtained in a domain with 401 points in the horizontal dimension. For the small bottom slope used here, the model contains a numerical reflection $R = 0.002$ much larger than the analytical value given by Roseau (1976). However, this only introduces a modulation in the x direction of the estimated forces. This modulation is significant but still relatively smaller than the average. The net force estimated from NTUA results is found to converge to the expected force balance described by eq. (14) as the number of evanescent modes is increased (figure 3). In this calculation the values of F_{xx} does not differ significantly from the values estimated using Mellor’s analytical expressions, as expected. The only significant difference

between the NTUA numerical result with 10 modes, and Mellor's analytical expression is found in F_{x3} , with a much stronger vertical flux of momentum near the surface in the numerical result, allowing a balance with the strongly sheared F_{xx} .

4. Conclusions

The momentum flux $\tilde{p}^{\xi} \partial \tilde{s}' / \partial x$ is the surface-following coordinate analog of the Eulerian vertical momentum flux term \overline{uw} discussed by Rivero and Arcilla (1995), with \tilde{p}^{ξ} the wave-induced pressure at the displaced position (in the surface-following coordinates). Just like the Eulerian momentum flux \overline{uw} is modified by the bottom slope, wave amplitude gradients, wind-wave generation, boundary layers, or vertical current shears, these effects also modify the vertical wave-induced displacement and pressure. In particular the coordinate change defined by Mellor (2003) through the analytical function \tilde{s} must be modified. Here we propose the modified expression given by eq. (11), providing consistent equations for the mean drift velocity U at the lowest order. These equations involve a vertical gradient of a wave-induced term, requiring an estimate of that term to first order in the bottom slope. Unfortunately there is no analytical expression for the wave motion at first order in the wave slope. Thus Mellor's equations, even when corrected, require a computer-intensive solution that is generally not feasible. For example, Magne et al. (2006) only included a total of five modes in their calculation of wave propagation over a submarine canyon. In an example shown here, this small number of modes is insufficient for an accurate estimation of wave-forcing terms.

At the same time, consistent phase-averaged equations of motion for the quasi-Eulerian velocity $\hat{u} = U - u_S$ do not suffer from this problem, at least at the lowest order of approximation. Such equations may be obtained in the limit of vanishing wave amplitude using an analytical continuation (e.g. using a Taylor expansion) of the current profile across the surface (McWilliams et al. 2004). A more general and explicit solution can also be obtained from the exact Generalized Lagrangian Mean (GLM) equations of Andrews and McIntyre (1978a) expanded to second order in the surface slope ε . The resulting equation for the horizontal momentum involves no flux term like $\overline{p \partial \tilde{s} / \partial x}$ because this corresponds to the flux $\overline{p^{\xi} \partial \xi_3 / \partial x} [1 + O(\varepsilon)]$ of wave momentum u_S (Andrews and McIntyre 1978b, eq. 2.7b), not directly relevant to the problem of mean flow evolution (see also Jenkins and Arduin 2004). Such a flux of wave momentum can be found in evolution equations for the total momentum given by Mellor (2003) or the 'alternative' form of the GLM equations (Andrews and McIntyre 1978, eq. 8.7a), but it is absent from the MRL04 evolution equations for the quasi-Eulerian momentum (McWilliams et al. 2004) and the original GLM momentum equation (Andrews and McIntyre 1978, eq. 3.8a).

The simplicity and consistency of order $g\varepsilon_1^2\varepsilon_2$ approximations to MRL04 equations and the GLM equations, in their adiabatic form (without wave dissipation), arises from the fact that wave forcing terms in the horizontal momentum equation are only present in horizontal gradient terms. Thus in momentum equations for the quasi-Eulerian velocity $U - u_S$, the wave motion is needed to zeroth order in small parameters such as the bottom slope. This quasi-Eulerian velocity was defined and used by Jenkins (1987) in horizontally homogenous situations. Further details on the relationships between all these equations, and further validation against numerical solutions of Laplace's equation can be found in Arduin et al. (2007, manuscript submitted to Ocean Modelling).

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