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A practical three dimensional formalism for the ocean, coupling random waves, mean flow, and turbulence

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Wave-mean flow equations for mass and momentum conservation in three dimensions, previously obtained with a vertical coordinate transform, are generalized here to random waves, and re-derived from the Generalized Lagrangian Mean equations. Combining the momentum equation with a three-dimensional equation for wave pseudo-momentum yields a set of equations describing the interactions of waves, mean flow and turbulence. Consistent boundary conditions and parameterizations are described. These include a Generalized Lagrangian Mean equation for the turbulent kinetic energy that describes the stretching of turbulence by the shear of the wave-induced Stokes drift. The equations derived here are readily integrated by coupling a spectral wave model to an ocean circulation model, with practical applications to the forecasting of surface drift, upper ocean mixing, coastal ocean circulation, and wave forecasting.

1. Introduction

The influence of waves on the mean flow has been well recognized in specific situations, and forms the basis of models of the nearshore circulation, in and around the surf zone. In the deep ocean the role of surface waves in shaping the ocean mixed layer through air-sea fluxes and mixing is now well established, but ocean circulation models are still not accounting for these important effects. Putting current theories into practice requires a consistent, accurate, and easy-to-use formalism for coupling waves, mean flow and turbulence. The present paper proposes such a formalism, based on first principles and amenable to further extensions.

Although depth-integrated equations are well established (Phillips 1977) and have been extended to account for internal waves (Kudryavtsev 1994) and rotation of the Earth (e.g. Ardhuin, Chapron & Elfouhaily 2004), the basic three-dimensional (3D) equations with phase-averaged wave effects are not yet well established for general situations. Several sets of 3D equations have been proposed to describe the interactions of waves with the mean flow, with potential applications to oceanographic problems. In particular the popular Craik-Leibovich equations (Craik & Leibovich 1976) were successfully applied to the formation of Langmuir circulations (LCs), as demonstrated by Large Eddy Simulations (LES, e.g. McWilliams, Sullivan & Moeng 1997). Nevertheless these equations were derived for a uniform wave field (Leibovich 1980, Holm 1996) which is a priori incompatible with the existence of the vortex force that is responsible for driving LCs. Although

wave field gradients may be weak on the scale of LCs, momentum conservation is satisfied only when refraction of waves over LCs modifies the waves pseudo-momentum. This refraction compensates for the change of mean flow momentum represented by the vortex force (Garrett 1976). This effect may be termed ‘local recoil’ by analogy with the similar remote recoil described by Bühler & McIntyre (2003). The Craik-Leibovich equations are therefore inapplicable to regions of strong gradients such as the surf zone, and attempts at a parameterization of wave breaking and the induced enhancement of turbulence have been only schematic (Noh, Min & Raasch 2004). Recently McWilliams *et al.* (2004) proposed a new formalism separating waves, long (infragravity) waves, and the mean flow, based on expansions in the wave slope in cartesian coordinates, including effects of gradients in the wave field, but equations do not yet include turbulence in a realistic way.

One essential difficulty for obtaining a realistic description of the ocean are the large gradients at the air-sea interface. These strong variations impose that 3D models use either very high resolution and interface tracking schemes, that will not be considered here, or surface-following coordinates. The simplest coordinate transform was proposed recently by Mellor (2003), with a transformation of the vertical coordinate only. Mellor’s equations are not so easy to use because they give a rate of change of the Lagrangian-mean velocity although the Eulerian-mean is also needed to evaluate boundary conditions and turbulent fluxes. Besides, the original formulation of Mellor’s equations and the general use of Craik-Leibovich equations is for monochromatic waves that cannot represent both the scale of the Stokes layer and the full production of Turbulent Kinetic Energy (TKE) associated with wave-induced shear. Finally, a very high resolution would be required in order to resolve this shear at the surface.

Another type of coordinate transform gives the Generalized Lagrangian Mean (GLM) equations of Andrews & McIntyre (1978a). Averages in GLM follow the mean displacement of particles, so that each mean quantity can be expressed as an Eulerian mean plus a Stokes correction. GLM is exact even for non-linear and rotational waves and has been applied to wave-current interactions using approximations in the form of low-order series expansions in the wave slope (e.g. Groeneweg 1999, Groeneweg & Battjes 2003). GLM was also used to derive the Craik-Leibovich equations (Leibovich 1980). However, the original GLM is not so easy to use because GLM-transformed equations involve a mean vertical displacement that is second-order in the wave slope (McIntyre 1988).

Further complications arise with the separation of the flow into ‘mean flow’, ‘waves’ and ‘turbulence’. Although some patterns will be clearly understood as waves or mean flow, it may be advantageous to describe structures such as LCs as a mean flow over which waves refract, typically in LES models, or as turbulence stretched by the Stokes drift, in large scale oceanic models. The separation of waves from the rest of the flow is itself problematic. A long tradition has used the property that waves are mostly irrotational. Yet, in the interaction of waves with turbulence Phillips (1958) demonstrated that wave motions have a significant vorticity, which was recently confirmed by Magnaudet & Masbernat (1990, see also also Thais & Magnaudet 1996). Wave vorticity also appears as a result of the Coriolis force (e.g. Backus 1962, Hasselmann 1970, Pollard 1970, Xu & Bowen 1994). Although we shall not consider vorticity in the wave motion except for the effect of the Coriolis force and the bottom boundary layer, we present here a formalism that is designed to be easily extended to include wave vorticity in the upper ocean. We thus define turbulence by the difference between the full fields and averages of those fields over flow realizations with fixed wave phases. This allows for a modulation of turbulent properties over a wave period, which could be considered part of the wave motion if we had chosen a different definition. Wave effects on the mean flow are obtained with

a further GLM average, which is an average on the wave phases, and can be computed from a wave spectrum. Another simple solution, not detailed here, is to define the GLM as a low-pass time average, allowing a variation of wave properties on the scale of wave groups as done by Reniers & al (2004). That would be a narrow spectrum approximation with amplitude modulations.

In order to obtain a practical and consistent set of equations we thus generalize Mellor's equations for random waves in § 2, with consistent boundary conditions and mixing parameterizations, and give a more general derivation of these equations from the GLM formalism, giving GLM-Reynolds Averaged Navier Stokes Equations at second order in wave slope, later referred to as GLM2-RANS. A GLM2 equation for the Turbulent Kinetic Energy is derived in § 3, with a description of its possible use for turbulent closure of the GLM2-RANS equations. Using a 3D equation for the wave pseudo-momentum we obtain a corresponding Mellor-type equation for the Eulerian mean momentum in § 4. Processes related to wave evolution are discussed in § 5, and conclusions in § 6.

Notations used below are, as much as possible, consistent with both Mellor (2003) and Andrews & McIntyre (1978a), with additions from Ardhuin & Herbers (2002) for spectral wave aspects.

2. 3D equations for the mean flow

2.1. Sigma-coordinate transform for random waves

Practical applications require the treatment of random waves, and common sense recommends the use of already-existing wave models, even if they need to be improved. An outline of how this can be done was given by Jenkins (1989), who coupled a one-vertical-dimension hydrodynamic model with a single-point version of the WAM spectral wave model (WAMDI 1988). Mellor's (2003) equations were derived by defining implicitly the vertical coordinate ς (ζ in his notation), so that the Cartesian vertical coordinate z is

$$z = s(\mathbf{x}, \varsigma, t) = \hat{\eta} + \varsigma D + \tilde{s}, \quad (2.1)$$

with \mathbf{x} the Cartesian horizontal position vector, $\hat{\eta}$ the mean (i.e. over flow realizations) sea surface elevation, D the total water depth from the bottom to the mean sea surface, and \tilde{s} a wave contribution so that $s(\varsigma = 0) = \eta$, the actual position of the sea surface, including wave motions, and, below the surface, \tilde{s} corresponds to the vertical displacement of water particles due to the wave motion.

A natural generalization of Mellor's (2003) equations to random waves is to let \tilde{s} be the sum of the contributions of each wave train, as illustrated by figure 1. In doing this we still impose that $s(\varsigma = 0) = \eta$ at the surface, and that \tilde{s} be the wave-induced vertical displacement in the water column. However, it is impossible to write down an exact expression for surface elevation because of the non-linearities and the presence of bound-wave terms.

Using the small parameter ε_1 to represent the maximum wave steepness $a_0 k_0$, the vertical displacement \tilde{s} is expanded to first order in ε_1 ,

$$\tilde{s} = \sum_{\mathbf{k}, s_1} F_k^{SS} Z_{1, \mathbf{k}}^{s_1} e^{i\psi_{1, \mathbf{k}}^{s_1}} \quad (2.2)$$

where the wavenumbers \mathbf{k} take values in the entire horizontal plane while s_1 is a sign index with values $+$ and $-$. The phase functions $\psi_{1, \mathbf{k}}^{s_1}$ are locally approximated by plane waves, that is,

$$\psi_{1, \mathbf{k}}^{s_1} = \mathbf{k} \cdot \mathbf{x} - s_1 \sigma t + O(\varepsilon_2), \quad (2.3)$$

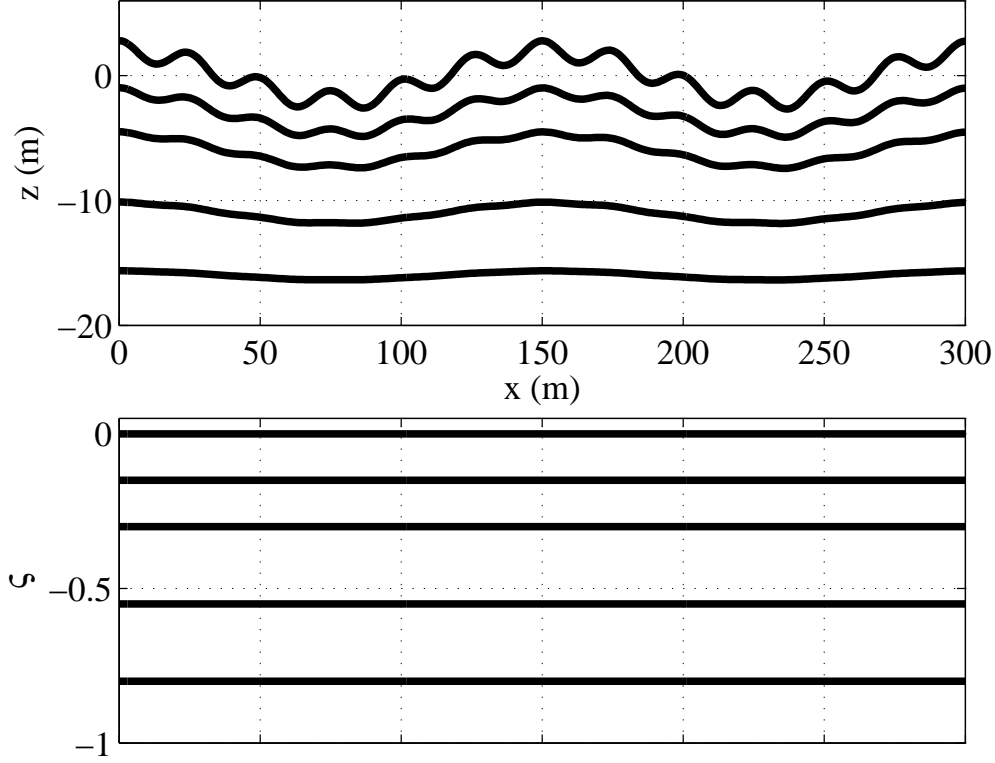


FIGURE 1. Example of the vertical coordinate transformation for a superposition of two linear waves of periods 4 and 12 s. Top: non-stationary streamlines of wave motion in cartesian coordinates. Bottom: The same streamlines transformed to fixed horizontal lines in ζ coordinates.

where ε_2 is a small parameter representing medium variations that affect wave propagation, e.g. water depth, current, or wave amplitude (Willebrand 1975), and $Z_{1,\mathbf{k}}^{s_1}$ is the complex amplitude slowly varying of the surface elevation component (\mathbf{k}, s_1) , and the vertical profile function F_k^{SS} is given by

$$F_k^{SS} = \frac{\sinh[kD(1 + \zeta/D)]}{\sinh(kD)}. \quad (2.4)$$

Formally we may define

$$\varepsilon_2 = \max_{\mathbf{k}, s_1} \left\{ \nabla(\tanh(kD)) / k; U/C; \nabla Z_{1,\mathbf{k}}^{s_1} / (k Z_{1,\mathbf{k}}^{s_1}); \left(\partial Z_{1,\mathbf{k}}^{s_1} / \partial t \right) / \left(\sigma Z_{1,\mathbf{k}}^{s_1} \right) \right\} \quad (2.5)$$

Any first order quantity A associated with the surface wave component of wave number vector \mathbf{k} propagating in the direction of the vector \mathbf{k} , is the sum of two complex components, with opposite imaginary parts, denoted $A_{1,\mathbf{k}}^+$ and $A_{1,-\mathbf{k}}^-$.

With this generalized change of the vertical coordinate the first order wave quantities are straightforward superpositions of linear wave components, e.g., to first order in ε_1 and zeroth order in ε_2

$$\tilde{u}_\alpha = \sum_{\mathbf{k}, s_1} k_\alpha Z_{1,\mathbf{k}}^{s_1} \frac{\sigma}{k} F_k^{CS} e^{i\psi_{1,\mathbf{k}}^{s_1}} \quad (2.6)$$

with the vertical profile function

$$F_k^{CS} = \frac{\cosh[kD(1 + \varsigma)]}{\sinh(kD)} \quad (2.7)$$

This result can be obtained either as a solution of the sigma-transformed equations or as a sigma-transformed solution of the usual cartesian equations.

The wave elevation spectrum to lowest order is obtained by taking the limit

$$E(\mathbf{k}) = \lim_{\Delta \mathbf{k} \rightarrow 0} \frac{2Z_{1,\mathbf{k}}^+ Z_{1,-\mathbf{k}}^-}{\Delta \mathbf{k}}, \quad (2.8)$$

or, more formally, by a Fourier-Stieltjes transform extended to evolutionary spectra (Priestley 1981). The equation for the evolution of $E(\mathbf{k})$ can be obtained by considering, as in § 3.2 (see also Ardhuin & Herbers 2002), the higher order terms in ε_1 and ε_2 given by wind forcing, wave dissipation near the surface, bottom friction and wave scattering processes.

2.2. 3D momentum and mass equations

The basic momentum equation was derived for monochromatic waves by Mellor (2003, equation 50). Although the effect of air pressure-surface slope correlations was not properly justified (see Appendix A), the final form of Mellor's equation 50 is correct. Because Mellor's equations are accurate to second order in ε_1 and first order in ε_2 the effect of random waves is the linear sum of the effects of each spectral component (see e.g. Kenyon 1969). These equations can alternatively be derived from Andrews and McIntyre's (1978a) Generalized Lagrangian Mean formalism (see Appendix B), using the spectral extension outlined above, giving,

$$\begin{aligned} & \frac{\partial}{\partial t}(\rho_w D U_\alpha) + \frac{\partial}{\partial x_\beta}(\rho_w D U_\alpha U_\beta) + \frac{\partial}{\partial \varsigma}(\rho_w \Omega U_\alpha) + \epsilon_{\alpha\beta\gamma} f_3 D U_\beta \\ & + D \frac{\partial}{\partial x_\alpha}(\rho_w g \hat{\eta} + \hat{p}^a) + \rho_w D^2 \int \left(\frac{\partial b}{\partial x_\alpha} - \varsigma \frac{\partial D}{\partial x_\alpha} \frac{\partial b}{\partial \varsigma} \right) \\ & = -\frac{\partial S_{\alpha\beta}^{\text{rad}}}{\partial x_\beta} + D T_\alpha^{\text{in}} - \frac{\partial S_\alpha^3}{\partial \varsigma} - (\overline{X}_\alpha^L)^\star, \end{aligned} \quad (2.9)$$

where the hat denotes Eulerian means. $U_\alpha = \hat{u}_\alpha + U_{s,\alpha}$ is the GLM-mean velocity in the horizontal direction α , sum of the Eulerian mean and the Stokes drift component $U_{s,\alpha}$. Ω is a mean vertical velocity (Mellor 2003, Appendix B). In the next term, $\epsilon_{ijk} A_j B_k$ is the i component of the vector product of \mathbf{A} and \mathbf{B} , so that term is the Coriolis force applied to the total (Eulerian plus Stokes drift) momentum, with the fraction applied to the Stokes drift first described by Hasselmann (1970) and thus called the 'Hasselmann force'. The terms on the second line represent the hydrostatic pressure gradient due to mean sea level $\hat{\eta}$, mean atmospheric pressure \hat{p}^a and the buoyancy b , which is the relative difference between the actual density and the mean density ρ_w . S_α^3 is defined by (B 28) and corresponds to Mellor's terms $\partial(\overline{s_\alpha} + \overline{p})/\partial \varsigma$. The volume force \mathbf{T}^{in} is due to the transfer of momentum from the wind to the wave field, it is noted $\overline{s_\alpha} + \overline{p}^w$ by Mellor. Finally, $(\overline{X}_\alpha^L)^\star$ represents viscous and turbulent momentum diffusion. Note that we have assumed that $\hat{b} = b$ so that these equations neglect surface wave—internal wave coupling (e.g. Kudryavtsev 1994).

The continuity equation (Mellor's equation 50) is

$$\frac{\partial}{\partial x_\alpha}[D(\hat{u}_\alpha + U_{s,\alpha})] + \frac{\partial \Omega}{\partial \varsigma} + \frac{\partial \hat{\eta}}{\partial t} = 0, \quad (2.10)$$

and the advection of passive tracers with concentration $c = \widehat{c} + c'$ is described by (Mellor, equation 54),

$$\frac{\partial(D\widehat{c})}{\partial t} + \frac{\partial}{\partial x_\alpha} [D\widehat{c}(\widehat{u}_\alpha + U_{s,\alpha})] + \frac{\partial(\Omega\widehat{c})}{\partial \varsigma} + \frac{\partial}{\partial x_\alpha} (Dc'u'_\alpha) + \frac{\partial\overline{c'w'}}{\partial \varsigma} = 0. \quad (2.11)$$

That equation neglects wave-induced diffusion (see Herterich & Hasselmann 1982, Balk 2002). It should be noted that an explicit parameterization of the last two turbulent mixing terms with non-isotropic eddy viscosities may yield a mean transport of the tracer (Middleton & Loder 1989).

2.3. Surface boundary conditions

Assuming that the air-sea fluxes are known, we impose the continuity of velocity and the normal and shear stresses P_{nn} and P_{ns} across the air-sea interface, and velocity only on the bottom. Assuming small Froude numbers for turbulent motions, turbulence is blocked by the surface so that the surface position and vertical velocity are equal to their Reynolds averages. Thus continuity of velocity across the interface reads,

$$\widehat{u}_\alpha^+ + \widetilde{u}_\alpha^+ = \widehat{u}_\alpha^- + \widetilde{u}_\alpha^- \quad \text{at} \quad \varsigma = 0. \quad (2.12)$$

For the horizontal stress, we use the notations and results of Xu & Bowen (1994),

$$\tau_{1\alpha} = P_{nn}n_1 + P_{ns}n_3 \quad \text{at} \quad \varsigma = 0 \quad (2.13)$$

with \mathbf{P} the stress tensor,

$$P_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (2.14)$$

with μ the dynamic viscosity, and the local unit vector normal to the surface, to first order in ε_1 ,

$$\mathbf{n} = (0, 0, 1) - \sum_{\mathbf{k}, s_1} i(k_1, k_2, 0) Z_{1,\mathbf{k}}^{s_1} e^{i\psi_{1,\mathbf{k}}^{s_1}}. \quad (2.15)$$

The second order expansion of the local shear stress P_{ns} was derived by Xu & Bowen (1994, equations 25 and 35), with a small error in the expression of the second order expression for \mathbf{n} (they omitted the second order Stokes surface elevation), but that will not affect our results here since this omitted term has a zero mean. We only need the normal stress P_{nn} to first order because the air pressure correlated with the surface slope is a first order quantity (see Appendix A), that is,

$$P_{nn} = -p + 2\mu \frac{\partial \widetilde{w}}{\partial z} \quad (2.16)$$

Taking the Lagrangian mean of (2.12) and (2.13) one obtains,

$$\widehat{u}_\alpha^+ + U_{s\alpha}^+ = \widehat{u}_\alpha^- + U_{s\alpha}^- \quad \text{at} \quad \varsigma = 0, \quad (2.17)$$

and, assuming a constant mean vertical momentum flux τ_a near the surface (the so-called wind stress),

$$\tau_a = \overline{\tau_{x\alpha}}^L = \tau_{a\alpha}^{\text{in}} + \rho\nu \frac{1}{D} \frac{\partial \widehat{u}_\alpha}{\partial \varsigma} + \rho\nu \frac{1}{D} \frac{\partial U_{s\alpha}}{\partial \varsigma} \quad \text{at} \quad \varsigma = 0, \quad (2.18)$$

where the density ρ and kinematic viscosity ν correspond to either air or water, depending on the side of the interface where the stresses are evaluated. In this partition of the wind stress the first term τ^{in} is total wave-supported stress, thus the momentum flux from the

atmosphere to the wave field. Neglecting variations of the shear stress over the phase of the waves (Jenkins 1992) τ^{in} is only due to correlations of the surface air pressure with the surface slope (see Appendix A). The second term is the viscous stress applied to the Eulerian velocity that is transferred beyond the surface viscous layer as a turbulent flux, and the last term τ_{vw} is usually called the ‘virtual wave stress’ (e.g. Xu & Bowen 1994) and may be a significant fraction of surface viscous stress as measured by Banner & Peirson (1998).

The wave-supported stress τ^{in} can be extended in the water column as the correlation of wave-induced pressure and streamline slope. In (x, y, ς) coordinates this is easily interpreted as a pseudo-shear stress, supported by the wave field, so that, the vertical divergence of this flux,

$$\frac{1}{D} \frac{\partial \tau^{\text{in}}}{\partial \varsigma} = \mathbf{T}^{\text{in}} \quad (2.19)$$

is the force that contributes to the growth of wave pseudo-momentum at depth $z = \varsigma D$, with a vertical profile identical to that of the Stokes drift.

In the same way, τ_{vw} can be generalized to all elevations on either side of the interface as $\tau_{vw} = \rho \nu (\partial U_{s\alpha} / \partial \varsigma) / D$. The vertical divergence of τ_{vw} is thus the momentum source for the mean flow associated to the viscous dissipation of wave energy into heat (see Phillips 1977 and the Appendix in Xu & Bowen 1994). A small fraction $(\rho_a \nu_a) / (\rho_w \nu_w)$ of the water-side wave pseudo momentum is transferred to the atmosphere, but most of the momentum is given up in the water very near the surface because $\partial \tau_{vw} / \partial \varsigma$ is proportional to the seventh moment of the frequency spectrum which is determined by capillary waves with wavelengths of a few centimetres (Phillips 1977).

We can thus express (2.18) at the base of the surface viscous layer, $\varsigma = -\delta_s$, away from the large shear in the Eulerian velocity (confined to the top millimetre, see McLeish & Putland 1975, Peirson & Banner 1998), in an area where Eulerian velocity gradients should be resolved by a numerical model,

$$\tau_a = \overline{\tau_{x\alpha}}^L = \tau_{\alpha}^{\text{in}} + \rho_w K_z \frac{\partial \hat{u}_{\alpha}}{\partial \varsigma} + \frac{\rho_w}{D} \nu_w \frac{\partial U_{s\alpha}}{\partial \varsigma} \quad \text{at} \quad \varsigma = -\delta_s. \quad (2.20)$$

2.4. Bottom boundary conditions

All wave effects were derived so far using linear wave theory with a free-slip condition on the bottom. This is essentially valid at a distance of a few δ_b from the actual water-sediment interface, with the wave boundary layer thickness $\delta_b \approx u_{*w} / \sigma$ where u_{*w} is the friction velocity in the wave boundary layer. δ is generally less than 10 cm. A proper representation of the wave boundary layer therefore does not affect previous results, and we may therefore adopt a state of the art representation of bottom friction including effects of moveable bed roughness (e.g. Trowbridge & Madsen 1984a, Madsen & al. 1990) and near bed mass transport (known as ‘streaming’, see e.g. Trowbridge & Madsen 1984b, Marin 2004). As ocean circulation models typically do not resolve the WBBL, we may represent the loss of wave energy and momentum in the WBBL as a modification of the bottom boundary condition that apply at the top of the WBBL, with a parameterized stress and non-zero Eulerian and Stokes drift velocities.

We thus have

$$\hat{u}_{\alpha} = \kappa u_{*c} \ln \left[\frac{D(1 + \varsigma)}{z_{0a'}} \right], \quad \text{for} \quad \frac{\delta_f}{D} < 1 + \varsigma \ll 1 \quad (2.21)$$

with von Kármán’s constant $\kappa = 0.41$, and $z_{0a'}$ a roughness for the Eulerian current that takes into account wave mixing and mass transport (e.g. Mathisen & Madsen 1996). The

$$\tau_b = -\rho_w u_{*c}^2 \frac{\widehat{\mathbf{u}}}{\widehat{u}} + \tau^{\text{bfri}}, \quad (2.22)$$

3. Mixing

We represent mixing as the sum of vertical mixing and horizontal mixing,

Various parameterizations have been proposed for the horizontal mixing tensor $R_{\alpha\beta}^h$ (e.g. Smagorinsky 1965) and these will not be discussed here, we only note that they are highly resolution-dependent, with specific parameterizations proposed for the surf zone (e.g. Svendsen & Putrevu 1994). We further expect that high-resolution simulations may need to incorporate the diffusion effects of Langmuir circulations and waves, that could be verified by existing observations.

We rather insist on the parameterization of K_z . If the popular KPP scheme is used, one should be careful to modify the original profile of K_z (Large, McWilliams & Doney 1994, see also Jézéquel, Pichon & Mazé 2004) so that K_z goes to realistically large values at the surface. This can be done by modifying the polynomial expression of the profile function G , that should go to z_{0w} at the surface, with the waterside roughness length z_{0w} determined by the wave field (see e.g. Mellor & Blumberg 2004 for a parameterization of z_{0w}).

Another option is to use a so-called Mellor-Yamada scheme (Mellor & Yamada 1982, Mellor & Blumberg 2004) or any other scheme that is based on at least one equation for the Turbulent Kinetic Energy (TKE). Such a description of TKE evolution will be explored here because it gives insight into the coupling of waves, mean flow and turbulence.

Taking an average over the flow realization for given wave phases we have the usual equation for the TKE,

$$q^2 = \overline{u'_i u'_i} / 2$$

in Cartesian coordinates (e.g. Phillips 1977, with his mean flow $\bar{\mathbf{u}}$ taken to be the Eulerian mean flow $\hat{\mathbf{u}}$ plus the wave velocity $\tilde{\mathbf{u}}$),

$$\left[\frac{\partial}{\partial t} + (\hat{u}_i + \tilde{u}_i) \frac{\partial}{\partial x_i} \right] q^2 + \frac{\partial}{\partial x_i} [\overline{u'_i (p'/\rho_w + q^2)}] = -\overline{u'_i u'_j} \frac{\partial (\hat{u}_i + \tilde{u}_i)}{\partial x_j} + \overline{b' w'} - \varepsilon, \quad (3.2)$$

with ε the usual viscous dissipation of TKE into heat. Taking a Lagrangian mean of (3.2) one obtains,

$$\left[\frac{\partial}{\partial t} + (\hat{u}_i + U_{si}) \frac{\partial}{\partial x_i} \right] \overline{q^2}^L + \frac{\partial}{\partial x_i} [\overline{u'_i (p'/\rho_w + q^2)}]^L = -\overline{u'_i u'_j} \frac{\partial (\hat{u}_i + \tilde{u}_i)}{\partial x_j}^L + \overline{b' w'}^L - \bar{\varepsilon}^L, \quad (3.3)$$

in which equation (2.15) of Andrews and McIntyre (1978a) has been used.

For each term Y the Lagrangian mean \bar{Y}^L may be computed as the Eulerian mean \hat{Y} plus a Stokes correction, \bar{Y}^S , given by Andrews and McIntyre (1978a, equation 2.27), at second order in ε_1 ,

$$\bar{Y}^S = \overline{\tilde{\xi}_j \frac{\partial \tilde{Y}}{\partial x_j}}^\phi, \quad (3.4)$$

in which \tilde{Y} is the wave induced perturbation of Y and $\tilde{\xi}_j$ is the wave-associated displacement in direction j .

3.2. TKE production and Stokes drift shear

The first term on the right hand side of (3.1) is the shear production P_s of TKE. As a first approximation we may assume that Reynolds stresses $\overline{u'_i u'_j}$ are not correlated with the wave phase,

$$\overline{u'_i u'_j}^L = \overline{u'_i u'_j} \quad (3.5)$$

$$\frac{\overline{u'_i u'_j \frac{\partial (\hat{u}_i + \tilde{u}_i)}{\partial x_j}}^L}{\overline{u'_i u'_j}} = \frac{\overline{u'_i u'_j \frac{\partial (\hat{u}_i + \tilde{u}_i)}{\partial x_j}}}{\overline{u'_i u'_j}} \quad \text{for all } i \text{ and all } j \quad (3.6)$$

This last equation implies that the Reynolds stresses that interact with one wave component to produce TKE are not themselves produced by the wave shear of that component. This is clearly not applicable to the WBBL. Near the surface, fluctuations of turbulence with the wave phase were observed by Thais & Magnaudet (1996) but they should only give higher order corrections to P_s .

If we chose not to resolve the Langmuir circulations we may neglect horizontal gradients of \hat{u}_α . This classical ‘boundary layer approximation’ yields

$$P_s = -\overline{u'_\alpha w'} \frac{\partial \hat{u}_\alpha}{\partial z} - \overline{u'_\alpha w'} \left(\frac{\partial \tilde{u}_\alpha}{\partial z}^S + \frac{\partial \tilde{w}_\alpha}{\partial x_\alpha}^S \right) - \overline{u'_\alpha u'_\beta} \frac{\partial \tilde{u}_\alpha}{\partial x_\beta}^S - (\overline{w'})^2 \frac{\partial \tilde{w}}{\partial z}^S, \quad (3.7)$$

where P_s is clearly the sum of a current shear production P_{cs} , the first right hand side term in (3.7), and a wave shear production P_{ws} , the other terms. At second order in ε_1 , application of (3.4) shows that the last two terms of (3.7) are zero.

Ardhuin & Jenkins (On the interaction of waves and upper ocean turbulence, manuscript submitted to the Journal of Physical Oceanography) considered the case TKE production due to long period swell for which they assumed that the turbulent flux is vertically uniform and carries all the air-sea momentum flux,

$$-\rho_w \overline{u'_\alpha w'} = \tau_\alpha, \quad (3.8)$$

because most of the air-sea flux is carried by short waves that lose their energy to turbulence very close to the surface (relative to the wavelength of longer waves). They obtained

$$P_s = \frac{\tau_\alpha}{\rho_w} \left(\frac{\partial \hat{u}_\alpha}{\partial z} + \frac{\overline{\partial \tilde{u}_\alpha}^S}{\partial z} + \frac{\overline{\partial \tilde{w}_\alpha}^S}{\partial x_\alpha} \right). \quad (3.9)$$

Using (3.4) one gets P_{ws} , the conversion of wave energy to TKE,

$$P_{ws} = \frac{\tau_\alpha}{\rho_w} \frac{\partial U_{s\alpha}}{\partial z}. \quad (3.10)$$

This production can be either positive, if waves propagate in the direction of the turbulent stress, or negative in the other case. The vertical profile of P_{ws} is identical to the Stokes drift shear, and is thus concentrated near the surface.

In order to compute P_{ws} due to all wave components, the assumed turbulent flux in (3.8) should be corrected by the ratio $\overline{u'_\alpha w'}/\tau_{a\alpha}$, as the turbulent flux is dwarfed by the wave-supported flux near the surface, and dominates below the depth at which waves have transferred most of their momentum to the mean flow. Assuming that the total flux is the turbulent flux plus the wave-supported flux due to wave generation by the wind, that ratio can be estimated by using the spectral shape of the wind input source term,

$$\overline{u'_\alpha w'}/\tau_{a\alpha} = 1 - \frac{1}{\tau_{a\alpha}} \int_\varsigma^0 T_\alpha^{\text{in}} d\varsigma'. \quad (3.11)$$

Because $U_{s\alpha}(\varsigma = 0)$ is proportional to the third moment of the frequency spectrum of the surface elevation variance (the ‘wave spectrum’), it is sensitive to that variance at high frequencies. The order of magnitude of $P_{ws} \approx \tau_{a\alpha} U_{s\alpha}(\varsigma = 0)/\rho_w$ may be obtained by using a properly defined wave spectrum that matches observations of wave energy (the zeroth moment) and mean square surface slope (the fourth moment), as a function of wind speed and fetch, such as proposed by Kudryavtsev *et al.* (1999). This gives $U_s(\varsigma = 0) = 0.012U_{10}$ for unlimited fetch, U_{10} being the wind speed at 10 m height. In comparison, the momentum flux generally associated with wave breaking is expected to be one order of magnitude larger for active wind-wave generation, $P_{wb} = A\tau_{a\alpha}U_{10}/\rho_w$ with A of the order of 0.1–0.2 (e.g. Craig & Banner 1994, Mellor & Blumberg 2004).

3.3. TKE production in the WBBL

In the WBBL where the turbulent flux is actually due to the wave shear, one may use (3.1) and a parameterization of the form,

$$\overline{u'_i u'_j} = -N_{ijlm} \delta_{il} \delta_{jm} \frac{\partial \tilde{u}_l}{\partial u_m}, \quad (3.12)$$

where N_{ijlm} is a viscosity tensor, giving

$$P_{ws} = N_{ij} \overline{\left(\frac{\partial \tilde{u}_i}{\partial x_j} \right)^2}. \quad (3.13)$$

Such a parameterization, including the time-variation of N , is given by Davies & Villaret (1999, see also Marin 2004). However, the determination of the roughness length for the wave orbital motion poses another problem that can be avoided by with a direct parameterization of the wave dissipation (e.g. Ardhuin *et al.* 2003 for parameterization over sand-dominated sediments with weak currents).

Similar arguments can be developed for the ‘inner layer’ layer at the ocean surface (see

e.g. Makin & Kudryavtsev 1999 for a description of the atmospheric inner layer), but we will neglect this effect on the ground that this layer is very thin.

4. A practical 3D momentum equation

In order to estimate shear stresses, the Eulerian velocity must be estimated. Besides, practical applications may not be able to afford the necessary vertical resolution to resolve the vertical shear of the Lagrangian Mean velocity near the surface. We therefore seek an equation in ς coordinates, similar the first form of the GLM equations of Andrews and McIntyre (1978a), for their variable $\bar{\mathbf{u}}^L - \mathbf{p}$ which corresponds to our $\hat{\mathbf{u}}$. Such an equation could possibly be derived directly from that GLM equation following the method used in Appendix B. However, now that it has been properly derived, we shall use (2.9).

Splitting the momentum in the first rate of change term of (2.9), into an Eulerian velocity plus a wave pseudo-momentum that is then moved to the right hand side, one gets,

$$\begin{aligned} & \frac{\partial}{\partial t}(\rho_w D \hat{u}_\alpha) + \frac{\partial}{\partial x_\beta}(\rho_w D U_\alpha U_\beta) + \frac{\partial}{\partial \varsigma}(\rho_w \Omega U_\alpha) + \varepsilon_{\alpha\beta 3} f_3 D U_\beta \\ & + D \frac{\partial}{\partial x_\alpha}(\rho_w g \hat{\eta} + \hat{p}^a) + \rho_w D^2 \int \left(\frac{\partial b}{\partial x_\alpha} - \varsigma \frac{\partial D}{\partial x_\alpha} \frac{\partial b}{\partial \varsigma} \right) \\ & = -\frac{\partial S_{\alpha\beta}^{\text{rad}}}{\partial x_\beta} + D T_\alpha^{\text{in}} - \frac{\partial S_\alpha^3}{\partial \varsigma} - (\bar{X}_\alpha^L)^\star - \frac{\partial}{\partial t}(\rho_w D U_{s\alpha}). \end{aligned} \quad (4.1)$$

A naïve implementation of (4.1) in a numerical model would consist of estimating $\partial \mathbf{u}^{\text{st}} / \partial t$ by applying a Stokes drift profile to the increments of the wave energy E over one time step. This approach may lead to errors as the gains or losses of wave pseudo-momentum $\partial \mathbf{U}_s / \partial t$ may have vertical profiles quite different from the wave pseudo-momentum $\mathbf{p} = \mathbf{U}_s$. In particular, it must be kept in mind that the waves support a significant fraction of the momentum flux at the surface and bottom boundaries. We have seen in § 2.3 that momentum lost due to viscous dissipation of the waves is partially given to the atmosphere and bottom sediments. It can thus be wrong to make waves exchange momentum with the water column only, as done by Dolata & Rosenthal (1984). They considered the case of waves attenuated by bottom friction (which can be significant, see e.g. Ardhuin & al. 2003), and assumed that the momentum source for the mean flow had the vertical profile of the Stokes drift. However, it is not clear why dissipation in the wave bottom boundary layer would lead to a production of mean flow momentum over the entire water column.

Waves redistribute their pseudo-momentum over the vertical due to the fact that they propagate in a waveguide that imposes the vertical profiles of \mathbf{p} for free waves. The associated vertical fluxes of \mathbf{p} are unknown, so that the only practical solution at hand is to evaluate the vertical profiles of momentum exchanges between waves and the mean flow, i.e. we need a 3D equation for \mathbf{p} .

4.1. 3D wave pseudo-momentum

GLM theory provides a 3D equation for the wave action A in the form (Andrews and McIntyre 1978b, equation (2.15) and (4.12)),

$$\frac{\partial}{\partial t}(\rho_w J A) + \frac{\partial}{\partial x_\beta} [(\bar{u}_\beta^L + C_{g\beta}) \rho_w J A] + \frac{\partial}{\partial z} (\bar{w}^L J A) = \rho_w J \mathcal{F}, \quad (4.2)$$

It can be transformed to our ς coordinates (rewriting A^* as A and \mathcal{F}^* as \mathcal{F} see Appendix B),

$$\begin{aligned} \frac{\partial}{\partial t}(\rho_w DA) + \frac{\partial}{\partial x_\beta} [(U_\beta + C_{g\beta})\rho_w DA] + \frac{\partial}{\partial \varsigma}(\rho_w \Omega A) \\ = \rho_w D\mathcal{F} + \frac{\partial}{\partial x_\beta} [(\hat{u}_{A\beta} - U_\beta)\rho_w DA] - \frac{\partial}{\partial \varsigma}(\rho_w C_{g\beta} \frac{s_\beta}{s_\varsigma} DA), \end{aligned} \quad (4.3)$$

where $\hat{u}_{A\beta}$, the advection speed of waves by a vertically sheared current, is a depth-weighted velocity (Kirby & Chen 1989), so that the second term on the right hand side of (4.3) gives a zero vertical integral at second order, provided that vertical variations in \hat{u} are much less than σ/k , the intrinsic phase speed.

The last term on the right-hand-side of (4.3) arises from the change of vertical coordinate and the fact that the vertical velocity Ω is defined by (B 13) to include the horizontal advection velocity through iso- ς surfaces. Because A is also transported by propagation with the group speed C_g , the last term represents this non-advective flux of A through the iso- ς surfaces, which is a ‘vertical’ flux in our new coordinates.

Combining (4.3) with the two-dimensional equation for the wave number k_α (e.g. Mellor 2003, in between equations 29 and 30),

$$\frac{\partial k_\alpha}{\partial t} + (\hat{u}_{A\beta} + C_{g\beta}) \frac{\partial k_\alpha}{\partial x_\beta} = -k_\beta \frac{\partial \hat{u}_{A\beta}}{\partial x_\alpha} - \frac{k\sigma}{\sinh 2kD} \frac{\partial D}{\partial x_\alpha}, \quad (4.4)$$

we obtain finally an equation for the 3D wave pseudo-momentum (or the Stokes drift), $U_{s\alpha} = k_\alpha A$,

$$\begin{aligned} \frac{\partial}{\partial t}(\rho_w DU_{s\alpha}) + \frac{\partial}{\partial x_\beta} [(U_\beta + C_{g\beta})\rho_w DU_{s\alpha}] + \frac{\partial}{\partial \varsigma}(\rho_w \Omega U_{s\alpha}) \\ = \rho_w k_\alpha D\mathcal{F} + \rho_w DA (\hat{u}_{A\beta} - U_\beta) \frac{\partial k_\alpha}{\partial x_\beta} - \rho_w k_\alpha \frac{\partial}{\partial \varsigma} (C_{g\beta} \frac{s_\beta}{s_\varsigma} DA) \\ - \rho_w DU_{s\beta} \frac{\partial \hat{u}_\beta}{\partial x_\alpha} - \rho_w DA \frac{k\sigma}{\sinh 2kD} \frac{\partial D}{\partial x_\alpha}. \end{aligned} \quad (4.5)$$

4.2. Pseudo-momentum source terms

The total momentum source term $\rho_w k_\alpha \mathcal{F}$ may be decomposed as follows,

$$\rho_w k_\alpha \mathcal{F} = T_\alpha^{\text{in}} + T_\alpha^{\text{nl}} + T_\alpha^{\text{ds}} + T_\alpha^{\text{bfric}} + T_\alpha^{\text{bscat}} + T_\alpha^{\text{wg}}. \quad (4.6)$$

These are source terms for the wind input, wave-wave interactions, wave dissipation, bottom friction, wave-bottom scattering, and wave guide effects, respectively. \mathbf{T}^{wg} integrates vertically to zero, just like the term for wave-wave interactions \mathbf{T}^{nl} .

We can further split \mathbf{T}^{ds} into effect of whitecapping, viscous dissipation, and wave-turbulence interaction (outside of the wave bottom boundary layer represented in $\mathbf{T}^{\text{bfric}}$),

$$T_\alpha^{\text{ds}} = T_\alpha^{\text{wc}} + T_\alpha^{\text{turb}} + T_\alpha^{\text{visc}}. \quad (4.7)$$

Since the vertical distribution of wave guide effects represented by \mathbf{T}^{wg} are completely unknown and will not be explored here, the present effort of establishing a 3D equation for the wave pseudo-momentum may look pointless. The following two assumptions are now necessary, they will hopefully be verified by observations:

When waves and turbulence are present, the conversion of wave energy to TKE for each spectral component corresponds to a conversion of wave pseudo-momentum to mean flow momentum, with the same vertical profile. Further, wave-wave interactions and wave guide effects does not directly contribute to exchanges with the mean flow.

The first hypothesis is clearly different from that of Dolata & Rosenthal (1984), who took a Stokes drift profile for the source of mean flow momentum. In particular the present assumption will yield a stronger enhancement of bottom streaming in the case of bottom friction, and weaker acceleration of the flow in the entire water column. Such a choice is even more important for the loss of wave pseudomomentum due to whitecapping that is generally much stronger and expected to be concentrated near the surface (e.g. Melville *et al.* 2002). We now briefly describe the parameterizations to be used for each source term.

wind input: as shown by Mellor (2003, see also Appendix A) the momentum source due to wave generation by the wind is given by

$$\mathbf{T}^{\text{in}} = \frac{\rho_w g}{D} v \int 2kD \frac{\cosh[2kD(1+\varsigma)]}{\sinh(2KD)} \mathbf{k} S^{\text{in}} / \sigma d\mathbf{k} \quad (4.8)$$

with S^{in} the *energy* source term as used in a numerical wave model. Among the existing parameterization for S^{in} the one by Janssen (1989, with extensions described in Komen & al. 1994) seems to give good results. It is based on quasi-linear theory and adjusted to observations. However it does not take into account the presence of swell that may slightly modify the wind input. It can also be refined by using the sum of linear and quadratic functions of u_* (e.g. Banner & Young 1994) in order to better represent wind input observations for both long (Snyder *et al.* 1981, Hasselmann & Bösenberg 1991) and short waves (Plant 1982). The quadratic term may also be reduced at very high frequency to account for the sheltering of short waves by long waves (Hara & Belcher 2002).

wave-wave interactions: we may assume that

$$\mathbf{T}^{\text{nl}} = \frac{\rho_w g}{D} \int 2kD \frac{\cosh[2kD(1+\varsigma)]}{\sinh(2KD)} \mathbf{k} S^{\text{nl}} / \sigma d\mathbf{k}, \quad (4.9)$$

with S^{nl} given by Herterich & Hasselmann (1980), possibly simplified as, for example, the Discrete Interaction Approximation (Hasselmann & Hasselmann 1985). Although we know that \mathbf{T}^{nl} integrates to zero over the vertical, there is no theory for the vertical redistribution of wave pseudo-momentum due to wave-wave interactions that could justify the profile assumed here.

bottom friction: A quantitatively good approximation of the decay of wave energy over a sandy bottom without significant currents can be obtained by taking into account the formation of sand ripples (see Ardhuin *et al.* 2003 for details of the parameterization and a validation). Further refinements for the effects of currents on waves can be found in Myrhaug *et al.* (2001). Muddy and rocky areas need special treatment.

wave-bottom scattering: wave reflection over the bottom topography results in an exchange of momentum between waves and the bottom due to correlations of pressure and bottom slope on the scale of the wavelength. We may thus include this source term in the bottom boundary layer, writing

$$\mathbf{T}^{\text{bscat}} = \delta_{(\varsigma, -1)} \frac{\rho_w g}{D} \int \frac{\mathbf{k}}{\sigma} S^{\text{bscat}} d\mathbf{k}. \quad (4.10)$$

Partial reflections over localized abrupt features in the bottom topography can also be represented by this source term, as long as they have small amplitudes relative to the water depth (Magne *et al.* 2004).

water viscosity: Since the local loss of wave pseudo-momentum is concentrated within a few centimetres of the air-sea interface (see § 2.3), the source term can be regarded as

part of the surface boundary condition,

$$\mathbf{T}^{\text{visc}} = \delta_{(\varsigma,0)} \frac{\rho_w}{D^2} \nu_w \frac{\partial \mathbf{U}_s}{\partial \varsigma}. \quad (4.11)$$

We include the contribution of viscous stresses in the highly sheared WBBL into the bottom friction term.

wave-turbulence interactions outside the WBBL: As wave energy is converted to TKE according to (3.10) and (3.11), momentum is exchanged with the mean flow. We will assume that this momentum exchange \mathbf{T}^{turb} has the same profile as P_{ws} .

whitecapping: There are many debates on the spectral distribution of wave energy losses due to whitecapping. In particular, wave models that do not separate wave-turbulence interaction from whitecapping have used a rather broad distribution of wave dissipation over the entire spectrum (see Komen et al. 1994). We will here take the view that wave energy is lost essentially in the saturated part of the wave spectrum and is well predicted by a saturation threshold, as described by Alves & Banner (2003). We now need to describe at what depth the wave momentum is given up to the mean flow, with a vertical profile that should be given by a detailed description of a breaking waves. Although some measurements of the detailed flow below breaking waves have now been made (Melville *et al.* 2002), it is still fairly arbitrary to decide at what depth the momentum and turbulence are injected by breakers before further diffusion with turbulent mixing. We have at most an upper value for this depth for a given breaker size and type, that may be used with a distribution (e.g. Phillips 1985, Melville & Matusov 2002) of self-similar breakers (e.g. Reul & Chapron 2003). This matter is further discussed in § 5.

4.3. Final equations

The non-linear advection term $\partial(\rho_w D U_\alpha U_\beta) / \partial x_\beta$ in (4.1) can be expanded as

$$\frac{\partial}{\partial x_\beta} (\rho_w D U_\alpha U_\beta) = \frac{\partial}{\partial x_\beta} [\rho_w D (\hat{u}_\alpha \hat{u}_\beta + U_{s,\alpha} U_\beta + U_{s,\beta} \hat{u}_\alpha)]. \quad (4.12)$$

The last term in that expansion can now be combined with the last but one term in (4.5), using the same algebraic combination as Garrett (1976 equations 3.10 and 3.11), obtaining the so-called ‘vortex force’,

$$\begin{aligned} D U_{s\beta} \frac{\partial \hat{u}_\beta}{\partial x_\alpha} - \frac{\partial}{\partial x_\beta} (\rho_w D U_{s\beta} \hat{u}_\alpha) &= -\rho_w \epsilon_{\alpha 3 \beta} D U_{s\beta} \epsilon_{3 i j} \left(\frac{\partial \hat{u}_j}{\partial x_i} - \frac{\partial \hat{u}_i}{\partial x_j} \right) - \hat{u}_\alpha \frac{\partial}{\partial x_\beta} (D U_{s\beta}) \\ &= -\rho_w \epsilon_{\alpha 3 \beta} \omega_3 D U_{s\beta} - \hat{u}_\alpha \frac{\partial}{\partial x_\beta} (\rho_w D U_{s\beta}) \end{aligned} \quad (4.13)$$

where $\omega_3 = \epsilon_{3 i j} (\partial \hat{u}_j / \partial x_i - \partial \hat{u}_i / \partial x_j)$ is the vertical component of the vorticity of the velocity field $\hat{\mathbf{u}}$.

One obtains,

$$\begin{aligned} \frac{\partial}{\partial t} (\rho_w D \hat{u}_\alpha) &+ \frac{\partial}{\partial x_\beta} (\rho_w D \hat{u}_\alpha \hat{u}_\beta) + \frac{\partial}{\partial \varsigma} (\rho_w \Omega \hat{u}_\alpha) + \rho_w D \epsilon_{\alpha 3 \beta} [f_3 \hat{u}_\beta + (f_3 + \omega_3) U_{s\beta}] \\ &+ D \frac{\partial}{\partial x_\alpha} (\rho_w g \hat{\eta} + \hat{p}^a) + \rho_w D^2 \int \left(\frac{\partial b}{\partial x_\alpha} - \varsigma \frac{\partial D}{\partial x_\alpha} \frac{\partial b}{\partial \varsigma} \right) \\ &= \frac{\partial R_{\alpha\beta}^h}{\partial x_\beta} + \frac{1}{D} \frac{\partial}{\partial \varsigma} \left(K_z \frac{\partial \rho_w \hat{u}_\alpha}{\partial \varsigma} \right) - \frac{\partial}{\partial x_\beta} (S_{\alpha\beta}^{\text{rad}} - \rho_w D C_{g\beta} U_{s\alpha}) \\ &\quad - \hat{u}_\alpha \frac{\partial}{\partial x_\beta} (\rho_w D U_{s\beta}) + \frac{\partial}{\partial \varsigma} \left(\rho_w D U_{s\alpha} C_{g\beta} \frac{s_\beta}{s_\varsigma} - S_\alpha^3 \right) \end{aligned}$$

$$+ \rho_w D A \frac{k\sigma}{\sinh 2kD} \frac{\partial D}{\partial x_\alpha} - D (T_\alpha^{\text{wc}} + T_\alpha^{\text{turb}}) \quad \text{for} \quad \left(-1 + \frac{\delta}{D}\right) < \varsigma < 0. \quad (4.14)$$

we repeat the mass conservation equation (B 12)

$$\nabla \cdot [\rho_w D(\hat{\mathbf{u}} + \mathbf{U}_s)] + \frac{\partial \rho_w \Omega}{\partial \varsigma} + \frac{\partial \rho_w \hat{\eta}}{\partial t} = 0. \quad (4.15)$$

Using (4.15), we may simplify (4.14) as

$$\begin{aligned} & \frac{\partial}{\partial t}(\rho_w D \hat{u}_\alpha) + \rho_w D \hat{u}_\beta \frac{\partial \hat{u}_\alpha}{\partial x_\beta} + \rho_w \Omega \frac{\partial \hat{u}_\alpha}{\partial \varsigma} - \frac{\partial}{\partial t}(\rho_w \hat{\eta}) + \rho_w D \epsilon_{\alpha 3 \beta} [f_3 \hat{u}_\beta + (f_3 + \omega_3) U_{s\beta}] \\ & + D \frac{\partial}{\partial x_\alpha}(\rho_w g \hat{\eta} + \hat{p}^a) + \rho_w D^2 \int \left(\frac{\partial b}{\partial x_\alpha} - \varsigma \frac{\partial D}{\partial x_\alpha} \frac{\partial b}{\partial \varsigma} \right) \\ & = \frac{\partial R_{\alpha\beta}^h}{\partial x_\beta} + \frac{1}{D} \frac{\partial}{\partial \varsigma} \left(K_z \frac{\partial \rho_w \hat{u}_\alpha}{\partial \varsigma} \right) \\ & - \frac{\partial}{\partial x_\beta} (S_{\alpha\beta}^{\text{rad}} - \rho_w D C_{g\beta} U_{s\alpha}) + \frac{\partial}{\partial \varsigma} \left(\rho_w D U_{s\alpha} C_{g\beta} \frac{s_\beta}{s_\varsigma} - S_\alpha^3 \right) \\ & + \rho_w D A \frac{k\sigma}{\sinh 2kD} \frac{\partial D}{\partial x_\alpha} - D (T_\alpha^{\text{wc}} + T_\alpha^{\text{turb}}) \quad \text{for} \quad \left(-1 + \frac{\delta}{D}\right) < \varsigma < 0. \end{aligned} \quad (4.16)$$

In (4.14) and (4.16) the term $\partial (S_{\alpha\beta}^{\text{rad}} - \rho_w D C_{g\beta} U_{s\alpha}) / \partial x_\beta$ is clearly Hasselmann's (1970) interaction stresses with a vertical profile given by the Stokes drift profile. However, as found by Mellor for the radiation stresses, the next term $\partial (D U_{s\alpha} C_{g\beta} \frac{s_\beta}{s_\varsigma} - S_\alpha^3) / \partial \varsigma$ modifies the interaction stresses so that their vertical profile is not exactly given by the Stokes drift. Our equation further suggest that the force $\rho_w D A k \sigma / \sinh 2kD \partial D / \partial x_\alpha$, may set up an Eulerian current, flowing towards the deeper waters as waves shoal.

Subtracting wave effects from (2.12) the surface boundary conditions for $\hat{\mathbf{u}}$ are,

$$\hat{u}_\alpha^+ = \hat{u}_\alpha^- + (U_{s,\alpha}^- - U_{s,\alpha}^+) \quad \text{at} \quad \varsigma = 0, \quad (4.17)$$

with $U_{s,\alpha}^-$ and $U_{s,\alpha}^+$ the x_α components of the Stokes drift in the water and the air respectively. The difference of these two is positive for waves in shallow water.

The direct surface momentum source $\hat{\tau}_{a\alpha}$ for the Eulerian current is,

$$\hat{\tau}_{a\alpha} = \tau_{a\alpha} - \tau_{a\alpha}^{\text{in}} - \rho_w \nu \frac{1}{D} \frac{\partial U_{s\alpha}}{\partial \varsigma} \quad \text{at} \quad \varsigma = 0. \quad (4.18)$$

Below the top millimeter, we may rewrite this quantity as

$$\hat{\tau}_{a\alpha} = \rho_w K_z \frac{\partial \hat{u}_\alpha}{\partial \varsigma} \quad \text{at} \quad \varsigma = -\delta_s. \quad (4.19)$$

In the current bottom boundary layer, at the outer edge of the WBBL, one gets,

$$\hat{u}_\alpha = \kappa u_{\star c} \ln \left[\frac{D(1+\varsigma)}{z_{0a'}} \right], \quad \text{for} \quad \frac{\delta_f}{D} < 1 + \varsigma \ll 1 \quad (4.20)$$

with a bottom stress for the Eulerian momentum,

$$\hat{\tau}_{b\alpha} = -\rho_w u_{\star c}^2 \frac{\hat{u}_\alpha}{\hat{u}}, \quad \text{for} \quad \frac{\delta_f}{D} < 1 + \varsigma \ll 1. \quad (4.21)$$

Finally the mixing parameterization should include the production of TKE by the Stokes shear, as proposed in §3 for example.

5. Discussion

Results and performances of the model outlined here will be investigated elsewhere because they may vary with choices in parameterization (A coherent wave-current-turbulence parameterization of upper ocean processes, by Rascle, Ardhuin & Marchesiello, manuscript in preparation for the Journal of Physical Oceanography). However, it is important to discuss the impact of a few parameterization choices.

5.1. Wave breaking

It is obvious from observations that the upper ocean is highly mixed relative to the boundary layer below a rigid boundary (e. g. Agrawal *et al.* 1992, Terray *et al.* 2000), and the high level of turbulence very near the surface is apparently due to wave breaking. In a numerical model that does not intend to resolve breaking events, one may probably obtain good profiles of TKE and velocities either with a strong injection of TKE below the surface, and relatively little mixing, or with a strong injection of TKE at the surface and a more vigorous mixing.

Some applications may require a detailed description of this process that has been done with the parameterization of coherent motions (*rollers*) acting as a momentum or energy buffers between the wave field and the mean flow or turbulence. This may be justified by the difference density the aerated fluid that is called roller in applications to the nearshore circulation (e.g. Govender *et al.* 2002). This roller is not too different from the entraining plume of Longuet-Higgins & Turner (1974), that takes its momentum from the breaking wave and gives momentum to the underlying fluid by a horizontal drag force at the plume interface.

5.2. Long (infragravity) waves

Although wave effects were explicitly formulated from a wave spectrum, the spectrum may alternatively be replaced by energy modulated in time on the scale of wave groups with slowly varying carrier frequency and wave direction, with a corresponding modulation of all second-order wave quantities (Stokes drift, action ...). This latter option allows the time and space resolution of infragravity (IG) motions as part of the ‘mean flow’. These motions are most important in the nearshore, in particular when sediment transport is considered (e.g. Reniers *et al.* 2004). IG motions may also be relevant to future ocean altimetry missions or other applications. Obtaining IG bound wave properties in the spectral formulation as given in present paper would require using higher order spectral statistics (e.g. Herbers & Burton 1997). Using either one of these options avoids the more complex separation in waves, IG waves and mean flow performed by McWilliams *et al.* (2004).

6. Conclusions

We have derived a consistent set of equations and boundary conditions that generalizes Mellor’s (2003) equations to spectral waves. Because this derivation uses the Generalized Lagrangian Mean of Andrews & McIntyre (1978), it is amenable to further extensions, including the interaction of surface and internal waves, that will be added in future work, or the inclusion of nonlinear effects that might appear at higher order in the wave slope. The present formalism, once the proper parameterizations are fully defined, should

be applicable for describing nearshore flows, surface wave evolution and upper ocean processes. Results obtained by Mellor & Blumberg (2004) and Lewis & Belcher (2003), already validate parts of the present formalism, i.e. the parameterization of enhanced turbulence near the surface and its impact on the surface temperature cycle, and the effects of the Hasselmann force (the combined effect of Coriolis and Stokes drift) on the velocity profile below the depth of enhanced turbulence, respectively. A general validation of the full model will be presented later, including surface drift velocities.

Open discussions with George Mellor, Bertrand Chapron, Tanos Elfouhaily and Stephen Henderson contributed significantly to the advancement of the present work.

Appendix A. Effects of surface pressure fluctuations

A.1. Wind-wave growth

Waves are generated by pressure and tangential stress variations on the scale of the wavelength. We solve here the problem with the usual cartesian coordinate system before transforming the solution to sigma coordinates. The variation of tangential stresses is neglected (see Lamb 1932 p. 629, Jenkins 1992). Atmospheric pressure at the surface can be described as,

$$p^a(\mathbf{x}, t) = \hat{p}^a(\mathbf{x}, t) + \sum_{\mathbf{k}, \sigma_p} P_{\mathbf{k}, \sigma_p}^a e^{i\mathbf{k} \cdot \mathbf{x} - \sigma_p t}. \quad (\text{A } 1)$$

with $P_{\mathbf{k}, \sigma_p}^a$ the Fourier component of the air pressure at the surface, with wavenumber \mathbf{k} and angular frequency σ_p . Following the general procedure for solving second-order differential equations, the wave field can now be obtained by adding the general solution in absence of forcing and a particular solution of the wave equations that satisfies this forcing. Neglecting terms that are second order in ε_1 , the surface equation for the wave potential ϕ is

$$\frac{\partial \phi}{\partial t} = -g\eta - \frac{1}{\rho} p_a, \quad \text{at} \quad z = \eta, \quad (\text{A } 2)$$

which can be combined with the kinematic boundary condition to give,

$$\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = -\frac{1}{\rho} \frac{\partial p_a}{\partial t} \quad \text{at} \quad z = 0. \quad (\text{A } 3)$$

A particular solution ϕ_p , that also must satisfy the Laplace equation and the bottom boundary condition, is given by,

$$\phi_p = \sum_{\mathbf{k}} \frac{\cosh[k(z + D)]}{\cosh(kD)} \Phi_{\mathbf{k}}^p(t) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (\text{A } 4)$$

where $\Phi_{\mathbf{k}}^p(t)$ is the solution of (A 3). This solution can be written as a resonant term plus some bound terms, with resonance obtained for $\sigma_p = \sigma \equiv gk \tanh(kH)$

$$\Phi_{\mathbf{k}}^p(t) = i \sum_{\sigma_p \neq \pm \sigma} \frac{\sigma_p P_{\mathbf{k}, \sigma_p}^a}{\rho [\sigma^2 - \sigma_p^2]} e^{-i\sigma_p t} - \sum_{s_1} t \frac{P_{\mathbf{k}, s_1}^a}{2\rho} e^{-is_1 t} \quad (\text{A } 5)$$

Pressure in the water is given by the (linearized) Bernoulli equation that we may write

$$p = -\rho g z - \rho \frac{\partial \phi}{\partial t} + O(a_0 g \varepsilon_1), \quad (\text{A } 6)$$

with the non-hydrostatic part p^p given by taking the derivative of (A 4),

$$p^p = \sum_{\mathbf{k}} \frac{\cosh[k(z + D)]}{\cosh(kD)} P_{\mathbf{k}}^p(t) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (\text{A } 7)$$

with

$$P_{\mathbf{k}}^p(t) = - \sum_{\sigma_p \neq \pm \sigma} \frac{\sigma_p^2 P_{\mathbf{k}, \sigma_p}^a}{[\sigma^2 - \sigma_p^2]} e^{-i\sigma_p t} + \sum_{s_1} \frac{P_{\mathbf{k}, s_1 \sigma}^a}{2} e^{-is_1 \sigma t} - i\sigma t \sum_{s_1} s \frac{P_{\mathbf{k}, s_1 \sigma}^a}{2} e^{-is_1 \sigma t} \quad (\text{A } 8)$$

so that the pressure response under the water is not entirely in phase with the pressure forcing, which was mistakenly suggested by Mellor (2003). Although it is an apparent paradox that the wave pressure p^p at the surface is not equal to the atmospheric pressure, the difference is explained by the partial adjustment of the surface elevation and the resulting hydrostatic pressure: Again, p^p is the non-hydrostatic pressure only.

To obtain the surface elevation amplitudes $Z_{\mathbf{k}}^p(t)$ at first order, we subtract (A 2) from (A 6) at $z = 0$,

$$\eta = \frac{1}{\rho g} [p^p|_{z=0} - p^a].$$

Hence

$$Z_{\mathbf{k}}^p(t) = - \sum_{\sigma_p \neq \pm \sigma} \left[\frac{\sigma_p^2}{\sigma^2 - \sigma_p^2} + 1 \right] \frac{P_{\mathbf{k}, \sigma}^a}{\rho g} e^{-i\sigma_p t} - \sum_{s_1} \frac{P_{\mathbf{k}, s_1 \sigma}^a}{2\rho g} e^{-is_1 \sigma t} - i\sigma t \sum_{s_1} s_1 \frac{P_{\mathbf{k}, s_1 \sigma}^a}{2\rho g} e^{-is_1 \sigma t} \quad (\text{A } 9)$$

Atmospheric pressure is generally influenced by the waves, say, to first order, proportional with a complex coefficient $\beta_C = (-\beta_R - i\beta_I)$ to the elevation,

$$P_{\mathbf{k}, \sigma_p}^a = \delta(\sigma_p, s_1 \sigma) \rho g \beta_C Z_{\mathbf{k}}^{s_1} \quad (\text{A } 10)$$

where $\delta(x, y)$ equals 0 unless $x = y$, and with β_I positive for growing waves, and β_R positive also due to the Bernoulli equation in the air: for winds faster than the waves, the flow accelerates over the wave crests due to streamline convergence, and thus the pressure decreases.

Thus the wave energy will be augmented at first order in β_I by the following term, E^p ,

$$E^p(\mathbf{k}) = t\sigma\beta_I E_2(\mathbf{k}) \quad (\text{A } 11)$$

This equation is only valid for short time scales since we have assumed a constant spectrum, it is thus better written as a time derivative (over long times), following the method of Hasselman (1962),

$$\frac{\partial E(\mathbf{k})}{\partial t} = \sigma\beta_I E_2(\mathbf{k}) = S_{\text{in}}(\mathbf{k}) \quad (\text{A } 12)$$

A.2. Bound waves and momentum equation

We have thus computed waves that are induced by air pressure fluctuations. These waves are characterized by η^p , p^p , ϕ^p . They have a free wave structure propagating at the speed of the air pressure perturbation: the polarization relations between all variables are identical to those of free waves, except for one extra term in the pressure and elevation represented by the second terms in (A 8) and (A 9), respectively.

The bound wave terms (p^{pb} , η^{pb}) can be written as

$$p^{pb} = \sum_{\mathbf{k}, s_1} -Z_{\mathbf{k}}^{pb, s_1}(t) e^{i(\mathbf{k} \cdot \mathbf{x} - s_1 \sigma t)} \quad (\text{A } 13)$$

$$\eta^{pb} = \sum_{\mathbf{k}, s_1} Z_{\mathbf{k}}^{pb, s_1}(t) e^{i(\mathbf{k} \cdot \mathbf{x} - s_1 \sigma t)} \quad (\text{A } 14)$$

with

$$Z_{\mathbf{k}}^{pb, s_1}(t) = -\frac{\beta_C}{2} Z_{\mathbf{k}}^{s_1} = \frac{\beta_R + i\beta_I}{2} Z_{\mathbf{k}}^{s_1}. \quad (\text{A } 15)$$

The β_I component of the pressure fluctuations, in quadrature with the free wave elevation, clearly drives bound waves with a surface elevation in quadrature ahead of the free waves.

The β_R component of the pressure fluctuations, in anti-phase with the elevation, tends to increase the wave height since the resulting surface elevation is in phase with the free waves.

It is striking that this ‘bound wave’ has no corresponding velocity and a pressure opposite to the corresponding pressure if it were a free wave. Indeed the associated velocity is part of the rate of change of the free wave velocity. It should be noted that integration of the vertical velocity does yield the vertical displacements of the bound wave. In terms of the coordinate transform (2.1), keeping $s(\zeta = 0) = \eta$ requires a modification of s due to the bound wave. In order to change s we can add to it a term of amplitude $S_{\mathbf{k}}^{pb, s_1}$, with a vertical profile given by bound terms in the vertical displacement, that happen to have the same vertical profile as the free waves,

$$S_{\mathbf{k}}^{pb, s_1}(t) = -\beta_C S_{\mathbf{k}}^{s_1}. \quad (\text{A } 16)$$

This part of the change of variable s induces extra terms in the equations of motion, including a vertical velocity \tilde{w}^{pb} , which now has a component in quadrature with the velocity and pressure. Considering only the solution driven by the pressure component in quadrature with the elevation, and evaluating all modified terms in the equations of motion, one gets exactly the same term as in Mellor’s (2003) equation (51a), that is,

$$-\frac{\partial}{\partial \zeta} \left(\overline{\tilde{s}_{\alpha}^{pb} \tilde{p} + \tilde{s}_{\alpha} \tilde{p}^{pb} + \tilde{w}^{pb} \tilde{u}_{\alpha}} \right) = \frac{\partial F^{SS} F^{CC}}{\partial \zeta} \rho g \int_{\mathbf{k}} \beta_I k_{\alpha} E(\mathbf{k}) d\mathbf{k}, \quad (\text{A } 17)$$

that we may rewrite as

$$T_{\alpha}^{\text{in}} = \overline{\tilde{p}_{w\eta} \frac{\partial \tilde{\eta}}{\partial x_{\alpha}}} \frac{\partial F^{SS} F^{CC}}{\partial \zeta}. \quad (\text{A } 18)$$

Indeed, if one considers the hypothetical case of a uniform wave field with no current and no dissipation we see that the wind to wave momentum flux is distributed over depth in the same way as the Stokes drift.

The part of the pressure that is in anti-phase with the surface elevation modifies slightly the term

$$\overline{\tilde{p} \tilde{s}_{\alpha}}, \quad (\text{A } 19)$$

in the momentum equation, which is already a second order correction. We may therefore neglect this effect, and obtain Mellor’s (2003) momentum equation with the effect of random waves accurate to second order in ε_1 .

Appendix B. Transformation of the GLM equation to sigma-coordinates

For simplicity equations are derived considering a single wave train of wavenumber vector \mathbf{k} and intrinsic frequency σ . Results from random waves are obtained by replacing the surface elevation variance var_{η} by the spectral energy density. All second order quantities are simply the sums of the following monochromatic solution for all wavenumber

vectors. Another option is to use a narrow spectrum approximation and resolve explicitly the variations in wave properties over the scale of wave groups.

We shall apply results obtained by Andrews and McIntyre (1978a, 1978b), replacing their equations (3.2) and (4.1) by Reynolds-Averaged Navier Stokes equations (RANS), which means that their dissipative forces X represent both viscous forces and turbulent Reynolds stresses. We shall retain all wave effects up to second order in the wave slope and first order in the wave. The resulting equations are therefore second order Generalized Lagrangian Mean RANS equation, abbreviated as GLM2-RANS.

B.1. Mass conservation

The Jacobian J of the GLM coordinate transformation (from Eulerian coordinates) can be shown to be equal to 1 plus a second order quantity. Using the 3D wave action A (see Andrews and McIntyre 1978b), one has,

$$J = 1 + J_2 + O(\varepsilon_1^3) \quad (\text{B1})$$

$$J_2 = -\frac{k^2 A}{\sigma} = -k^2 \text{var}_\eta \frac{\cosh[2k(z+h)]}{\sinh^2(kD)}, \quad (\text{B2})$$

where var_η is the surface elevation variance due to the waves. Because there is no mean stretching of the horizontal coordinates, a vertical distance $dz' = Jdz$ in GLM corresponds to a Cartesian distance dz . As $J < 1$ over the entire water column one has $dz' > dz$. Thus the vertical GLM position is everywhere larger than the mean Eulerian elevation of the same water particles. In a sense this is because at any time there are more particles per horizontal length of crest than of trough (McIntyre 1988).

Integrating over depth we define

$$s^G(x, z, t) = -\int_{-h}^z J_2(z') dz' = k \text{var}_\eta \frac{\sinh[2k(z+h)]}{2 \sinh^2(kD)}. \quad (\text{B3})$$

Using the second order expression for $\bar{\eta}^L$ (e.g. Jenkins & Ardhuin 2004)

$$\bar{\eta}^L = \text{var}_\eta \frac{k}{\tanh kD}, \quad (\text{B4})$$

we can see that

$$\int_{-h}^{\bar{\eta}^L} J dz = \bar{\eta}^L + D - s^G(0) = D, \quad (\text{B5})$$

which is a further verification of the vertical stretching induced by GLM.

By analogy to 2.1 we thus define

$$s = \varsigma D + s^G + \hat{\eta} \quad (\text{B6})$$

for which we can use the chain rules given by Mellor (2003) to go from (x_α, z, t) to $(x_\alpha^*, \varsigma, t^*)$, i.e. for any variable ϕ

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi^*}{\partial t^*} - \frac{s_t}{s_\varsigma} \frac{\partial \phi^*}{\partial \varsigma} \quad (\text{B7})$$

$$\frac{\partial \phi}{\partial x_\alpha} = \frac{\partial \phi^*}{\partial x_\alpha^*} - \frac{s_\alpha}{s_\varsigma} \frac{\partial \phi^*}{\partial \varsigma} \quad (\text{B8})$$

$$\frac{\partial \phi}{\partial z} = \frac{1}{s_\varsigma} \frac{\partial \phi^*}{\partial \varsigma} \quad (\text{B9})$$

with s_t , s_ς and s_α the partial derivatives of s with respect to t , ς and x_α , respectively.

In our case we have the remarkable identity

$$s_\zeta J = D. \quad (\text{B } 10)$$

Dropping the star superscripts just like Mellor (2003) going from his equation (14) to (22), we can transform the GLM mass conservation equation

$$\frac{\partial \rho_w J}{\partial t} + \frac{\partial \rho_w J \bar{u}_\alpha^L}{\partial x_\alpha} + \frac{\partial J \bar{w}^L}{\partial z} = 0 \quad (\text{B } 11)$$

to the following

$$\frac{\partial \rho_w \hat{\eta}}{\partial t} + \frac{\partial D \rho_w U_\alpha}{\partial x_\alpha} + \frac{\partial \rho_w \Omega}{\partial \zeta} = 0, \quad (\text{B } 12)$$

by defining

$$\Omega = [J \bar{w}^L - J (\bar{u}_\alpha^L s_\alpha + s_t)]^*, \quad (\text{B } 13)$$

and $(\bar{u}_\alpha^L)^* = U_\alpha$, that is, the Lagrangian drift velocity in Mellor's coordinate is indeed the transformed GLM velocity. For constant ρ_w (B 11) is clearly Mellor's equation (51).

B.2. 3D momentum in Mellor's coordinates

We start here from the 'alternative form' of the GLM equations, (Andrews and McIntyre 1978, equation 8.7a), considering only the vertical component of the Earth's rotation,

$$\bar{D}^L \bar{u}_\alpha^L + \epsilon_{\alpha 3 \beta} f_3 \bar{u}_\beta^L + \frac{1}{\rho_w J} \frac{\partial \bar{p}^L}{\partial x_\alpha} + \bar{X}_\alpha^L = \frac{1}{\rho_w J} \left(\frac{\partial R_{\alpha \beta}}{\partial x_\beta} + \frac{\partial R_{\alpha 3}}{\partial z} \right). \quad (\text{B } 14)$$

First looking at the Lagrangian-mean pressure we have to second order in ϵ_1 ,

$$\bar{p}^L = \bar{p} + \xi_j \overline{\frac{\partial p'}{\partial x_j}}. \quad (\text{B } 15)$$

The mean vertical momentum equation for steady mean motions gives,

$$-\frac{\partial \bar{p}}{\partial z} = \rho_w g + \frac{\partial \bar{w}^2}{\partial z} + \frac{\partial \bar{u} \bar{w}}{\partial x}. \quad (\text{B } 16)$$

Over a gently sloping bottom uw is third order in ϵ_1 and (B 16) integrates vertically to

$$\bar{p} = \bar{p}^H - \rho_w \sigma^2 F^{SS} F^{SS} \text{var}_\eta, \quad (\text{B } 17)$$

where \bar{p}^H is the mean hydrostatic pressure and the second term is the wave-induced Eulerian mean pressure.

Assuming that we have only free waves, the Stokes correction in (B 15) can be rewritten as

$$\xi_j \overline{\frac{\partial p'}{\partial x_j}} = \rho_w \text{var}_\eta (\sigma^2 F^{SS} F^{SS} - g k F^{CC} F^{CS}), \quad (\text{B } 18)$$

so that

$$\bar{p}^L = \bar{p}^H - \rho_w g \text{var}_\eta k F^{CC} F^{CS}. \quad (\text{B } 19)$$

Finally the radiation stress is defined by Andrews and McIntyre (1978a, equation 8.6) as

$$\begin{aligned} R_{\alpha j} &= \overline{p^\xi (1 - J) \delta_{\alpha j}} + p^\xi \overline{\frac{\partial \xi_i}{\partial x_\alpha} K_{ij}}, \\ &= \overline{p^\xi (1 - J) \delta_{\alpha j}} + B_{\alpha j} \end{aligned} \quad (\text{B } 20)$$

where the K_{ij} are the cofactors of the coordinate transform matrix $\mathbf{x} \rightarrow (\mathbf{x} + \xi)$, and $B_{\alpha j}$ is the flux in direction j of a generalized wave action in direction opposite to x_α (Andrews and McIntyre 1978b, equation 2.7b, see also Jenkins & Ardhuin 2004),

$$B_{\alpha j} = \overline{p' \xi_{j,\alpha}} + \overline{\bar{p} \xi_{m,\alpha} \xi_{j,\alpha}}. \quad (\text{B } 21)$$

To second order we thus have

$$R_{\alpha 3} = \overline{p' \xi_{3,1}} + \overline{\rho g \xi_{3,1} \xi_{3,\alpha}} = \overline{p' \xi_{3,1}} \quad (\text{B } 22)$$

It should be noted that this term does not integrate to zero over the vertical as pointed out by Andrews and McIntyre (1978b, see corrigendum). We can now consider the contribution of bound waves due to air pressure fluctuations over the waves, as considered in § 2. These clearly contribute to $\partial R_{\alpha 3} / \partial z$, giving the extra term found in Appendix A, with the same vertical profiles as the Stokes drift,

$$T_\alpha^{\text{in}}(z) = \overline{p^a \frac{\partial \eta}{\partial x_\alpha}} k D (F^{CC} F^{CS} + F^{SC} F^{SS}). \quad (\text{B } 23)$$

For $R_{\alpha\beta}$, simple algebra shows that for either $\alpha = \beta$ or $\alpha \neq \beta$ we find at second order,

$$R_{\alpha\beta} = \overline{p' \frac{\partial \xi_\beta}{\partial x_\alpha}}. \quad (\text{B } 24)$$

We may now transform (B 14), to the new coordinates, using the GLM mass conservation equation. We first consider the Lagrangian mean derivative. Using (B 13) and (B 10) we get

$$\begin{aligned} s_\varsigma \rho_w J \overline{D^L} \overline{u}_\alpha^L &= s_\varsigma \left[\frac{\partial}{\partial t} (\rho_w J \overline{u}_\alpha^L) + \frac{\partial}{\partial x_\beta} (\rho_w J \overline{u}_\alpha^L \overline{u}_\beta^L) + \frac{\partial}{\partial z} (\rho_w J \overline{w}^L) \right] \\ &= \frac{\partial}{\partial t} (\rho_w D U_\alpha) + \frac{\partial}{\partial x_\beta} (\rho_w D U_\alpha U_\beta) + \frac{\partial}{\partial \varsigma} (\rho_w J \overline{w}^L U_\alpha) \\ &\quad - \frac{\partial s}{\partial t} \frac{\partial}{\partial \varsigma} (\rho_w J U_\alpha) - \frac{\partial s}{\partial x_\beta} \frac{\partial}{\partial \varsigma} (\rho_w J U_\alpha U_\beta) - \rho_w J U_\alpha \left(\frac{\partial^2 s}{\partial t \partial \varsigma} + U_\beta \frac{\partial^2 s}{\partial \varsigma \partial x_\beta} \right) \\ &= \frac{\partial}{\partial t} (\rho_w D U_\alpha) + \frac{\partial}{\partial x_\beta} (\rho_w D U_\alpha U_\beta) + \frac{\partial}{\partial \varsigma} (\rho_w \Omega U_\alpha), \end{aligned} \quad (\text{B } 25)$$

with the $*$ superscripts omitted on the right hand side. Transforming the pressure gradient term, one gets, to second order in ε_1 ,

$$\begin{aligned} s_\varsigma \frac{\partial \overline{p}^L}{\partial x_\alpha} &= s_\varsigma \frac{\partial}{\partial x_\alpha} \overline{p}^H - \frac{\partial s}{\partial x_\alpha} \frac{\partial}{\partial \varsigma} \overline{p}^H + s_\varsigma \frac{\partial}{\partial x_\alpha} (\rho_w g \text{var}_\eta k F^{CC} F^{CS}) \\ &= D \frac{\partial}{\partial x_\alpha} (g \hat{\eta} + \hat{p}^a) + D^2 \int \left(\frac{\partial b}{\partial x_\alpha} - \varsigma \frac{\partial D}{\partial x_\alpha} \frac{\partial b}{\partial \varsigma} \right) + \frac{\partial S_{\alpha\beta}^p}{\partial x_\beta}, \end{aligned} \quad (\text{B } 26)$$

with $S_{\alpha\beta}^p$ the isotropic part of the radiation stress $S_{\alpha\beta}$.

We finally transform the GLM radiation stress R ,

$$\begin{aligned} s_\varsigma \left(\frac{\partial R_{\alpha\beta}}{\partial x_\beta} + \frac{\partial R_{\alpha 3}}{\partial z} \right) &= \frac{\partial}{\partial x_\beta} \left(s_\varsigma \overline{p' \frac{\partial \xi_\beta}{\partial x_\alpha}} \right) + \frac{\partial}{\partial \varsigma} \left(\overline{p' \frac{\partial \xi_3}{\partial x_\alpha}} - s_\beta \overline{p' \frac{\partial \xi_\beta}{\partial x_\alpha}} \right) \\ &= \frac{\partial}{\partial x_\beta} S_{\alpha\beta}^u + T_\alpha^{\text{in}} + \frac{\partial}{\partial \varsigma} S_\alpha^3, \end{aligned} \quad (\text{B } 27)$$

with $S_{\alpha\beta}^u$ the non-isotropic part of the radiation stresses $S_{\alpha\beta}$, and S_α^3 a wave-induced

vertical flux of momentum defined by

$$S_{\alpha}^3 = \left[2\rho_w g \text{var}_{\eta} F^{CC} \frac{\partial F^{SS}}{\partial x_{\alpha}} + \rho_w g F^{CC} F^{SS} \frac{\partial}{\partial x_{\alpha}} \text{var}_{\eta} - s_{\beta} \rho_w g \text{var}_{\eta} F^{CC} F^{CS} \frac{k_{\alpha} k_{\beta}}{k^2} \right]. \quad (\text{B } 28)$$

Putting all the pieces of this puzzle together (B14) can be rewritten in the new coordinate system to obtain a generalization of Mellor's (2003) equation (51a)

$$\begin{aligned} & \frac{\partial}{\partial t}(\rho_w D U_{\alpha}) + \frac{\partial}{\partial x_{\beta}}(\rho_w D U_{\alpha} U_{\beta}) + \frac{\partial}{\partial \zeta}(\rho_w \Omega U_{\alpha}) + \epsilon_{\alpha 3 \beta} f_3 D U_{\beta} \\ & + D \frac{\partial}{\partial x_{\alpha}}(\rho_w g \hat{\eta} + \hat{p}^a) + \rho_w D^2 \int \left(\frac{\partial b}{\partial x_{\alpha}} - \varsigma \frac{\partial D}{\partial x_{\alpha}} \frac{\partial b}{\partial \zeta} \right) \\ & = - \frac{\partial S_{\alpha \beta}}{\partial x_{\beta}} + T_{\alpha}^{\text{in}} + \frac{\partial S_{\alpha}^3}{\partial \zeta} - (\overline{X}_{\alpha}^L)^{\star}, \end{aligned} \quad (\text{B } 29)$$

with $(\overline{X}_{\alpha}^L)^{\star}$ the transformed viscous and turbulent stresses.

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