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# Crystal graphs of higher level $q$ -deformed Fock spaces, Lusztig $a$ -values and Ariki-Koike algebras

Nicolas JACON <sup>\*†</sup>

## Abstract

We show that the different labelings of the crystal graph for irreducible highest weight  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$ -modules lead to different labelings of the simple modules for non semisimple Ariki-Koike algebras by using Lusztig  $a$ -values.

## 1 Introduction

The Ariki-Koike algebras have been introduced by Ariki and Koike in [AK]. They can be seen as natural generalizations of Iwahori-Hecke algebras of type  $A_{n-1}$  and  $B_n$  and as special cases of cyclotomic Hecke algebras introduced by Broué and Malle in [BM]. Let  $R$  be a commutative ring, let  $l \in \mathbb{N}$ ,  $n \in \mathbb{N}$  and let  $v, x_1, x_2, \dots, x_l$  be  $l + 1$  parameters in  $R$ . The Ariki-Koike algebra  $\mathcal{H}_{R,n} := \mathcal{H}_{R,n}(v; x_1, \dots, x_l)$  (or cyclotomic Hecke algebra of type  $G(l, 1, n)$ ) over  $R$  is the unital associative  $R$ -algebra presented by:

- generators:  $T_0, T_1, \dots, T_{n-1}$ ,
- relations:

$$\begin{aligned} T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \quad (i = 1, \dots, n-2), \\ T_i T_j &= T_j T_i \quad (|j - i| > 1), \\ (T_0 - x_1)(T_0 - x_2) \dots (T_0 - x_l) &= 0, \\ (T_i - v)(T_i + 1) &= 0 \quad (i = 1, \dots, n-1). \end{aligned}$$

Assume that  $R$  is a field of characteristic 0. When  $\mathcal{H}_{R,n}$  is a semisimple algebra, it is known that the simple  $\mathcal{H}_{R,n}$ -modules are given by the set of Specht modules  $S_R^\lambda$  parametrized by the  $l$ -partitions of rank  $n$ . Using results of Dipper and Mathas [DM], the study of the representation theory of  $\mathcal{H}_{R,n}$  in the non semisimple case can be reduced to the case where  $R = \mathbb{C}$  and :

$$v = \eta_e, \quad x_i = \eta_e^{s_j}, \quad j = 1, \dots, l,$$

where  $\eta_e := \exp\left(\frac{2i\pi}{e}\right) \in \mathbb{C}$  and where  $s_j \in \mathbb{Z}$  for  $j = 1, 2, \dots, l$ .

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In [Ac], Ariki has first provided a description of the simple modules in this modular case. Proving and generalizing a conjecture by Lascoux, Leclerc and Thibon, he has shown in [Ad] that the associated decomposition matrices can be identified with the Kashiwara-Lusztig canonical basis elements of the irreducible  $\mathcal{U}(\widehat{\mathfrak{sl}}_e)$ -module of highest weight  $\Lambda_{s_1(\text{mod } e)} + \Lambda_{s_2(\text{mod } e)} + \dots + \Lambda_{s_l(\text{mod } e)}$  (where the  $\Lambda_i$  denote the fundamental weights). It is known that these elements can be labeled by using the “crystal graph theory” and there are several ways to do that. Each of these ways corresponds to a certain realization of the Fock space as a module over the quantum group  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$ . In particular, one of these realizations leads to a labeling of the canonical basis elements by the “Kleshchev  $l$ -partitions” and Ariki has given a parametrization of the  $\mathcal{H}_{\mathbb{C},n}$ -simple modules by using this class of  $l$ -partitions.

This paper is a continuation of the works done by Geck [G], Geck and Rouquier [GR] and the author [Jp]. The aim is to give natural parametrizations of the simple modules for non semisimple Ariki-Koike algebras. Let  $\mathbf{s}_l = (s_1, s_2, \dots, s_l) \in \mathbb{Z}^l$  and let  $y$  be an indeterminate. We consider the Ariki-Koike algebra  $\mathcal{H}_{A,n}$  over  $A := \mathbb{C}[y, y^{-1}]$  with the following choice of parameters:

$$\begin{aligned} u_j &= y^{lm^{(j)}} \eta_l^{j-1} \text{ for } j = 1, \dots, l, \\ v &= y^l. \end{aligned}$$

where  $\eta_l := \exp(\frac{2i\pi}{l})$  and where for  $j = 1, \dots, l$ , we have  $m^{(j)} = s_j - \frac{(j-1)e}{l} + \alpha e$  ( $\alpha$  is a positive integer such that  $m^{(j)} \geq 0$  for  $j = 1, \dots, l$ ).

If we specialize  $y$  to  $\eta_{le} := \exp(\frac{2i\pi}{le})$  via a ring homomorphism  $\theta$ , we obtain the above Ariki-Koike algebra  $\mathcal{H}_{\mathbb{C},n} := \mathcal{H}_{\mathbb{C},n}(\eta_e; \eta_e^{s_1}, \dots, \eta_e^{s_l})$ . Let  $\mathcal{H}_{\mathbb{C}(y),n} := \mathbb{C}(y) \otimes_A \mathcal{H}_{A,n}$ . Then  $\mathcal{H}_{\mathbb{C}(y),n}$  is a split semisimple algebra and the simple  $\mathcal{H}_{\mathbb{C}(y),n}$ -modules are the Specht modules  $S_{\mathbb{C}(y)}^\Delta$  defined over  $\mathbb{C}(y)$ . We obtain a well-defined decomposition map  $d$  between the Grothendieck groups of finitely generated  $\mathcal{H}_{\mathbb{C}(y),n}$ -modules and  $\mathcal{H}_{\mathbb{C},n}$ -modules. For  $V \in \text{Irr}(\mathcal{H}_{\mathbb{C}(y),n})$ , we have equations:

$$d([V]) = \sum_{M \in \text{Irr}(\mathcal{H}_{\mathbb{C},n})} d_{V,M} [M].$$

Using the symmetric algebra structure of  $\mathcal{H}_{A,n}$ , we can attach an integer  $a_{\mathbf{s}_l}(V)$  to each simple  $\mathcal{H}_{\mathbb{C}(y),n}$ -module  $V$ . This is called the  $a$ -value of  $V$ . Note that for Hecke algebras of type  $A_{n-1}$  and  $B_n$ , this value can also be defined using the Kazhdan-Lusztig theory but such a theory is not available for the wider case of Ariki-Koike algebras. The aim of this paper is to show the following theorem (see Theorem 4.7).

**Main Theorem.** *For each  $M \in \text{Irr}(\mathcal{H}_{\mathbb{C},n})$ , there exists a unique simple  $\mathcal{H}_{\mathbb{C}(y),n}$ -module  $V_M$  such that the following two conditions hold:*

- $d_{V_M, M} = 1$ .
- if there exists  $W \in \text{Irr}(\mathcal{H}_{\mathbb{C}(y),n})$  such that  $d_{W, M} \neq 0$  then  $a_{\mathbf{s}_l}(W) > a_{\mathbf{s}_l}(V_M)$ .

*The function which sends  $M$  to  $V_M$  is injective. As a consequence the associated decomposition matrix is unitriangular and the following set is in natural bijection*

with  $\text{Irr}(\mathcal{H}_{\mathbb{C},n})$ :

$$\mathcal{B} = \{V_M \mid M \in \text{Irr}(\mathcal{H}_{\mathbb{C},n})\}.$$

Moreover  $\mathcal{B}$  is parametrized by the crystal of the associated  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$ -module in one of the realizations of the Fock space mentioned above.

This result both generalizes and gives a new proof of the main result of [Jp] where the case  $0 \leq s_1 \leq \dots \leq s_l < e$  was considered. As the Ariki-Koike algebra  $\mathcal{H}_{\mathbb{C},n}$  only depends on the classes modulo  $e$  of the integers  $s_j$  ( $j = 1, \dots, l$ ), the theorem provides several natural parametrizations for the simple modules of  $\mathcal{H}_{\mathbb{C},n}$ . Note also that in the context of Hecke algebras, the existence of such “basic sets”  $\mathcal{B}$  is linked with the existence of a Kazhdan-Lusztig theory (see [Gs]). Hence, this result gives indices for the existence of such a theory for Ariki-Koike algebras.

The paper is organized as follows. In section 2 and 3, we summarize known results on the representation theory of Ariki-Koike algebras. We introduce combinatoric objects and we recall works of Uglov on higher level  $q$ -deformed Fock space which will be crucial for the proof of the main result. Section 4 contains the main theorem of the paper, Theorem 4.7. The proof of this Theorem requires a combinatorial study of the deep results of Uglov [U].

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## 2 Decomposition maps for Ariki-Koike algebras

Let  $R$  be a commutative associative ring with unit and let  $v, x_1, \dots, x_l$  be  $l + 1$  invertible elements in  $R$ . Let  $n \in \mathbb{N}$ . Let  $\mathcal{H}_{R,n} := \mathcal{H}_{R,n}(v; x_1, \dots, x_l)$  be the associated Ariki-Koike algebra as it is defined in the introduction. For a complete survey of the representation theory of  $\mathcal{H}_{R,n}$ , see [Ma].

It is known that this algebra is a “cellular” algebra in the sense of Graham and Lehrer [GL] and thus has “Specht modules” which are parametrized by the  $l$ -partitions of rank  $n$ . A  $l$ -partition  $\underline{\lambda}$  of rank  $n$  is a sequence of  $l$  partitions  $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(l)})$  such that  $\sum_{k=1}^l |\lambda^{(k)}| = n$ . We denote by  $\Pi_l^n$  the set of  $l$ -partitions of rank  $n$ .

For each  $l$ -partition  $\underline{\lambda}$  of rank  $n$ , we can associate a remarkable  $\mathcal{H}_{R,n}$ -module  $S_R^{\underline{\lambda}}$  which is free over  $R$ . This is called a Specht module (see the definition of “dual” Specht modules in [DJM]). Assume that  $R$  is a field. In general, the Specht modules are reducible and each of these modules can be endowed with a natural bilinear form. Let  $\text{rad}(\cdot)$  denotes the radical of this form. For  $\underline{\lambda} \in \Pi_l^n$ , we denote :

$$D_R^{\underline{\lambda}} := S_R^{\underline{\lambda}} / \text{rad}(S_R^{\underline{\lambda}}).$$

Then the theory of cellular algebras gives the following result:

**Theorem 2.1** (*Graham-Lehrer [GL], Dipper-James-Mathas [DJM]*) *Assume that  $R$  is a field then:*

1. *Non zero  $D_R^{\underline{\lambda}}$  form a complete set of non-isomorphic simple  $\mathcal{H}_{R,n}$ -modules. Moreover, all of these modules are absolutely irreducible.*

2. If  $\mathcal{H}_{R,n}$  is a semisimple algebra then it is split semisimple and for all  $\underline{\lambda} \in \Pi_l^n$  we have  $\text{rad}(S_R^{\underline{\lambda}}) = 0$ . Thus, the  $S_R^{\underline{\lambda}}$  form a complete set of non isomorphic simple modules.

Using this theorem, when  $\mathcal{H}_{R,n}$  is semisimple, the simple modules are explicitly known and are given by the Specht modules. The following theorem gives a criterion of semisimplicity for the algebra  $\mathcal{H}_{R,n}$ .

**Theorem 2.2** (Ariki [As])  $\mathcal{H}_{R,n}$  is split semisimple if and only if:

- for all  $i \neq j$  and for all  $d \in \mathbb{Z}$  such that  $|d| < n$ , we have:

$$v^d x_i \neq x_j,$$

- $\prod_{i=1}^n (1 + v + \dots + v^{i-1}) \neq 0$ .

Hence we are reduced to understand the representations of  $\mathcal{H}_{R,n}$  in the modular case. Assume that  $R$  is the field of complex numbers  $\mathbb{C}$ . Then, using the results of Dipper and Mathas ([DM]) and Mathas ([Ms]), the non semisimple case can be reduced to the case where all the  $u_i$  are powers of the same root of unity  $\eta_e := \exp(\frac{2i\pi}{e})$  with  $e \geq 2$ . Let  $\mathcal{H}_{\mathbb{C},n}$  be the Ariki-Koike algebra over  $\mathbb{C}$  with the following choice of parameters:

$$\begin{aligned} x_j &= \eta_e^{v_j} \text{ for } j = 1, \dots, l, \\ x &= \eta_e, \end{aligned}$$

where  $0 \leq v_l \leq \dots \leq v_1 < e$ . The problem of describing the simple modules of  $\mathcal{H}_{\mathbb{C},n}$  is linked with the problem of determining the decomposition map which we now define.

Let  $\mathcal{H}_{A,n}$  be an Ariki-Koike defined over a commutative ring  $A$  with unit. Let  $K$  be the field of fractions of  $A$ . We assume that:

- (1)  $A$  is integrally closed in  $K$ ,
- (2)  $\mathcal{H}_{K,n} := K \otimes_A \mathcal{H}_{A,n}$  is split semisimple,
- (3) we have a ring homomorphism  $\theta : A \rightarrow \mathbb{C}$  such that  $\mathbb{C} = \text{Frac}(\theta(A))$  and such that the specialized algebra  $\mathbb{C} \otimes_A \mathcal{H}_{A,n}$  is the Ariki-Koike algebra  $\mathcal{H}_{\mathbb{C},n}$  with the above choice of parameters.

Then, by [Gm], we have a well-defined decomposition map  $d_\theta$  between the Grothendieck groups of finitely generated  $\mathcal{H}_{K,n}$  and  $\mathcal{H}_{\mathbb{C},n}$ -modules. In the context of cellular algebras, this function can be easily defined. Let:

$$\Phi_{\{e;v_1,\dots,v_l\}}^n := \{\underline{\lambda} \in \Pi_l^n \mid D_{\mathbb{C}}^{\underline{\lambda}} \neq 0\}.$$

Let  $R_0(\mathcal{H}_{\mathbb{C},n})$  be the Grothendieck group of finitely generated  $\mathcal{H}_{\mathbb{C},n}$ -modules. This is generated by the classes of simple  $\mathcal{H}_{\mathbb{C},n}$ -modules  $[D_{\mathbb{C}}^{\underline{\lambda}}]$  with  $\underline{\lambda} \in \Pi_l^n$ . Hence, for all  $\underline{\lambda} \in \Pi_l^n$ , there exist numbers  $d_{\underline{\lambda},\underline{\mu}}$  with  $\underline{\mu} \in \Phi_{\{e;v_1,\dots,v_l\}}^n$  such that:

$$[S_{\mathbb{C}}^{\underline{\lambda}}] = \sum_{\underline{\mu} \in \Phi_{\{e;v_1,\dots,v_l\}}^n} d_{\underline{\lambda},\underline{\mu}} [D_{\mathbb{C}}^{\underline{\mu}}].$$

The matrix  $(d_{\underline{\lambda}, \underline{\mu}})_{\underline{\lambda} \in \Pi_l^n, \underline{\mu} \in \Phi^n}$  is called the decomposition matrix of  $\mathcal{H}_{\mathbb{C}, n}$  (where we denote  $\Phi^n := \Phi_{\{e; v_1, \dots, v_l\}}^n$ ). Let  $\mathcal{F}^n$  be the  $\mathbb{C}$ -vector space which is generated by the symbols  $[[S^{\underline{\lambda}}]]$  with  $\underline{\lambda} \in \Pi_l^n$ . We obtain a homomorphism:

$$\begin{aligned} \mathcal{F}^n &\rightarrow R_0(\mathcal{H}_{\mathbb{C}, n}) \\ [[S^{\underline{\lambda}}]] &\mapsto [S_{\mathbb{C}}^{\underline{\lambda}}] = \sum_{\underline{\mu} \in \Phi^n} d_{\underline{\lambda}, \underline{\mu}} [D_{\mathbb{C}}^{\underline{\mu}}]. \end{aligned}$$

Now,  $\mathcal{F}^n$  can be naturally identified with the Grothendieck group of finitely generated modules over a semisimple algebra  $\mathcal{H}_{K, n}$  verifying (1) – (3), by identifying the classes of simple  $\mathcal{H}_{K, n}$ -modules  $[S_K^{\underline{\lambda}}]$  with the symbols  $[[S^{\underline{\lambda}}]]$ . Hence, the decomposition map is defined as follows:

$$\begin{aligned} d: R_0(\mathcal{H}_{K, n}) &\rightarrow R_0(\mathcal{H}_{\mathbb{C}, n}) \\ [S_K^{\underline{\lambda}}] &\mapsto [S_{\mathbb{C}}^{\underline{\lambda}}] = \sum_{\underline{\mu} \in \Phi^n} d_{\underline{\lambda}, \underline{\mu}} [D_{\mathbb{C}}^{\underline{\mu}}]. \end{aligned}$$

By results of Ariki and Uglov, the problem of determining the decomposition matrices for Ariki-Koike algebras can be translated to the problem of computing the canonical basis of  $q$ -deformed Fock spaces. In the next section, we recall these results.

### 3 Canonical basis of higher level $q$ -deformed Fock spaces

The higher  $q$ -deformed Fock spaces have been introduced in [JM] (see also [TU]). These spaces which are spanned by the set of “multipartitions” can be endowed with a structure of integrable  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$ -module. In [U], generalizing works by Leclerc and Thibon [LT], Uglov has given a construction of canonical bases for these spaces. In this part, we review this construction following [U] (see also [Us] and [Yt]). This will be used for the proof of the main result of this paper. Then, we explain the links with the representation theory of Ariki-Koike algebras which are given by Ariki’s theorem.

#### 3.A $q$ -wedge products and $q$ -deformed Fock spaces

Let  $q$  and  $z$  be indeterminates and let  $l$  and  $e$  be positive integers. Let  $V_e$  be an  $e$ -dimensional vector space over  $\mathbb{Q}(q)$  with basis  $v_1^{(e)}, v_2^{(e)}, \dots, v_e^{(e)}$ . We put  $V_{e, l} := (V_e \otimes V_l)[z, z^{-1}]$ . For  $a \in \{1, \dots, e\}$ ,  $b \in \{1, \dots, l\}$  and  $m \in \mathbb{Z}$ , we put  $k := a + e(l - b) - elm$  and  $u_k := v_a^{(e)} z^m v_b^{(l)}$ . Then  $V_{e, l}$  is a  $\mathbb{Q}(q)$  vector space with basis  $\{u_k \mid k \in \mathbb{Z}\}$ .

The  $q$ -wedge square  $\bigwedge^2 V_{e, l}$  of  $V_{e, l}$  can be viewed as a  $q$ -deformation of the exterior square of  $V_{e, l}$ . This is a  $\mathbb{Q}(q)$ -vector space generated by the monomials  $u_{k_1} \wedge u_{k_2}$  with  $(k_1, k_2) \in \mathbb{Z}^2$ . A basis of this space is given by the “ordered” monomials that is the monomials  $u_{k_1} \wedge u_{k_2}$  such that  $k_1 > k_2$ . Any monomial  $u_{k_1} \wedge u_{k_2}$  can be expressed as a linear combination of ordered monomials using the following rules (R1), (R2), (R3) and (R4).

Let  $k_1 \leq k_2$  and for  $i = 1, 2$ , put  $k_i = a_i + e(l - b_i) - elm_i$  where  $a_i \in \{1, \dots, e\}$ ,  $b_i \in \{1, \dots, l\}$  and  $m_i \in \mathbb{Z}$ . We define  $\alpha$  and  $\beta$  to be the unique integers in  $[0, el - 1]$  such that  $\alpha \equiv (a_2 - a_1) \pmod{el}$  and  $\beta \equiv (e(b_1 - b_2)) \pmod{el}$ . Then

the relations (R1), (R2), (R3) and (R4) are given as follows:

(R1) if  $\alpha = 0$  and  $\beta = 0$ :

$$u_{k_1} \wedge u_{k_2} = -u_{k_2} \wedge u_{k_1};$$

(R2) if  $\alpha \neq 0$  and  $\beta = 0$ :

$$\begin{aligned} u_{k_1} \wedge u_{k_2} = & -q^{-1}u_{k_2} \wedge u_{k_1} + \\ & +(q^{-2} - 1) \sum_{m \geq 0} q^{-2m} u_{k_2 - \alpha - elm} \wedge u_{k_1 + \alpha + elm} - \\ & -(q^{-2} - 1) \sum_{m \geq 1} q^{-2m+1} u_{k_2 - elm} \wedge u_{k_1 + elm}; \end{aligned}$$

(R3) if  $\alpha = 0$  and  $\beta \neq 0$ :

$$\begin{aligned} u_{k_1} \wedge u_{k_2} = & qu_{k_2} \wedge u_{k_1} + \\ & +(q^2 - 1) \sum_{m \geq 0} q^{2m} u_{k_2 - \beta - elm} \wedge u_{k_1 + \beta + elm} + \\ & +(q^2 - 1) \sum_{m \geq 1} q^{2m-1} u_{k_2 - elm} \wedge u_{k_1 + elm}; \end{aligned}$$

(R4) if  $\alpha \neq 0$  and  $\beta \neq 0$ :

$$\begin{aligned} u_{k_1} \wedge u_{k_2} = & u_{k_2} \wedge u_{k_1} + \\ & +(q - q^{-1}) \sum_{m \geq 0} \frac{q^{2m+1} + q^{-2m-1}}{q + q^{-1}} u_{k_2 - \beta - elm} \wedge u_{k_1 + \beta + elm} + \\ & +(q - q^{-1}) \sum_{m \geq 0} \frac{q^{2m+1} + q^{-2m-1}}{q + q^{-1}} u_{k_2 - \alpha - elm} \wedge u_{k_1 + \alpha + elm} + \\ & +(q - q^{-1}) \sum_{m \geq 0} \frac{q^{2m} - q^{-2m}}{q + q^{-1}} u_{k_2 - \beta - \alpha - elm} \wedge u_{k_1 + \beta + \alpha + elm} + \\ & +(q - q^{-1}) \sum_{m \geq 1} \frac{q^{2m} - q^{-2m}}{q + q^{-1}} u_{k_2 - elm} \wedge u_{k_1 + elm}; \end{aligned}$$

where the summations continue as long as the monomials appearing under the sums are ordered.

For any integer  $r \geq 2$ , we can now define the  $r$ -fold  $q$ -wedge product  $\bigwedge^r V_{e,l}$ . This is the  $\mathbb{Q}(q)$ -vector space generated by the elements  $u_{k_1} \wedge u_{k_2} \wedge \dots \wedge u_{k_r}$  with  $k_i \in \mathbb{Z}$ . Again the ordered monomials, that is the monomials  $u_{k_1} \wedge u_{k_2} \wedge \dots \wedge u_{k_r}$  with  $k_1 > k_2 > \dots > k_r$ , form a basis of  $\bigwedge^r V_{e,l}$ . Moreover, an arbitrary monomial can be expressed as a linear combination of ordered monomials using the relations (R1), (R2), (R3) and (R4) in every adjacent pair of the factors.

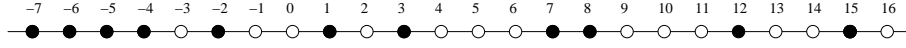
Finally, for  $s \in \mathbb{Z}$ , the semi-infinite  $q$ -wedge product  $\bigwedge^{s+\frac{\infty}{2}} V_{e,l}$  of charge  $s$  is the inductive limit of  $\bigwedge^r V_{e,l}$  where the maps  $\bigwedge^r V_{e,l} \rightarrow \bigwedge^{r+1} V_{e,l}$  are given by  $v \mapsto v \wedge u_{s-r}$ . Hence,  $\bigwedge^{s+\frac{\infty}{2}} V_{e,l}$  is the  $\mathbb{Q}(q)$ -vector space generated by the semi-infinite monomials  $u_{k_1} \wedge u_{k_2} \wedge \dots$  with  $k_i \in \mathbb{Z}$  and such that  $k_i = s - i + 1$  for  $i \gg 0$ . A basis is given by the ordered monomials that is the monomials  $u_{k_1} \wedge u_{k_2} \wedge \dots$  with  $k_1 > k_2 > \dots$  and the relations (R1)-(R4) provide a way to express an arbitrary monomial as a linear combination of ordered monomials.

Now, each semi-infinite ordered monomial can be labeled by a pair  $(\underline{\lambda}, \mathbf{s}_l)$  where  $\underline{\lambda} \in \Pi_l^n$  is a  $l$ -partition of rank  $n$  and  $\mathbf{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l$  is such that  $\sum_{i=1}^l s_i = s$ . This labeling coincides with that of [U] up to transformation  $(s_1, \dots, s_l) \mapsto (s_l, \dots, s_1)$  and  $(\lambda^{(1)}, \dots, \lambda^{(l)}) \mapsto (\lambda^{(l)}, \dots, \lambda^{(1)})$ . Let  $u_{k_1} \wedge u_{k_2} \wedge \dots \in \bigwedge^{s+\frac{\infty}{2}} V_{e,l}$  be a semi-infinite ordered monomial. For  $i = 1, 2, \dots$ , we put  $k_i = a_i + e(l - b_i) - elm_i$  where  $a_i \in \{1, \dots, e\}$ ,  $b_i \in \{1, \dots, l\}$  and  $m_i \in \mathbb{Z}$ . For  $b = 1, \dots, l$ , let  $k_1^{(b)} > k_2^{(b)} > \dots$  be the semi-infinite sequence obtained by ordering the elements of the set  $\{a_i - em_i \mid b_i = b\}$  in strictly decreasing order. Then, there is a unique  $s_b \in \mathbb{Z}$  such that  $k_i^{(b)} = s_b - i + 1$  for  $i \gg 1$ . We put  $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(l)})$  where for  $i > 0$ ,  $\lambda_i^{(b)} = k_i^{(b)} - s_b + i - 1$  and  $\mathbf{s}_l = (s_1, \dots, s_l)$ . This defines a bijection between the set of semi-infinite ordered monomials and the set of pairs  $(\underline{\lambda}, \mathbf{s}_l)$  where  $\underline{\lambda} \in \Pi_l^n$  and  $\mathbf{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l$  is such that  $\sum_{i=1}^l s_i = s$ .

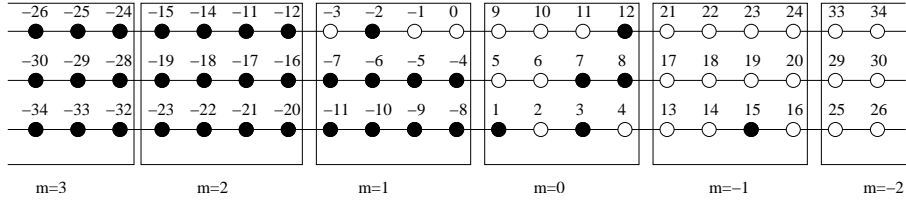
This bijection can be described as follows (see [Y] for much details). First, the infinite decreasing sequence  $\underline{k} = (k_1, k_2, \dots)$  can be pictured as a set of black beads on an infinite runner. For example, the following sequence :

$$\underline{k} = (15, 12, 8, 7, 3, 1, -2, -4, -5, -6, \dots)$$

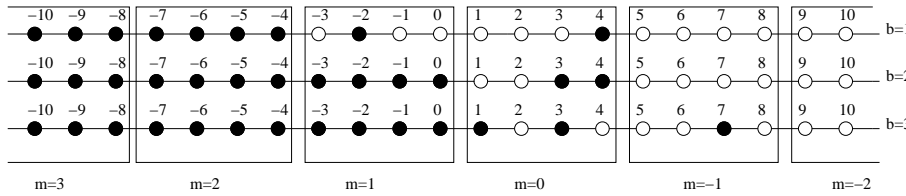
is represented by the following abacus:



On the other hand, using the decomposition  $k_i = a_i + e(l - b_i) - elm_i$ , the same sequence can be represented by an  $l$ -abacus, that is, as a set of black beads on  $l$  infinite runners. For  $e = 4$  and  $l = 3$ , we obtain:



In the above representation, the beads are labeled by the integers  $k_i$  (with  $i = 1, 2, \dots$ ). We can alternatively label them by the integers  $k_i^{(b)}$  (with  $i = 1, 2, \dots$  and  $b = 1, 2, \dots, l$ ) as follows:



Let  $\underline{\lambda}$  be the  $l$ -partition and  $\mathbf{s}_l = (s_1, \dots, s_l)$  be the  $l$ -tuple of integers associated to  $\underline{k}$ . For  $i = 1, 2, \dots$  and  $b = 1, 2, \dots, l$ , we have  $\lambda_i^{(b)} = k_i^{(b)} - s_b + i - 1$ . Thus, one can now easily determine  $\underline{\lambda}$  by counting the number of white beads at the left of each black bead on each runner. Continuing the above example, we obtain  $\underline{\lambda} = ((6, 1), (2, 2), (4, 1))$  and  $\mathbf{s}_l = (-2, 2, 3)$ .



Now, the higher level  $q$ -deformed Fock space of charge  $\mathbf{s}_l$  is defined to be the  $\mathbb{C}(q)$ -vector space generated by the symbols  $|\underline{\lambda}, \mathbf{s}_l\rangle$  with  $\underline{\lambda} \in \Pi_l^n$  :

$$\mathcal{F}_{e, \mathbf{s}_l} := \bigoplus_{\substack{\underline{\lambda} \in \Pi_l^n \\ n \in \mathbb{N}}} \mathbb{C}(q)|\underline{\lambda}, \mathbf{s}_l\rangle.$$

Thus, if we identify the semi-infinite ordered monomial  $u_{\underline{k}} \in \bigwedge^{s+\frac{\infty}{2}} V_{e,l}$  with the pair  $|\underline{\lambda}, \mathbf{s}_l\rangle$  defined by the above bijection, we get:

$$\bigwedge^{s+\frac{\infty}{2}} V_{e,l} = \bigoplus_{s_1+\dots+s_l=s} \mathcal{F}_{e, \mathbf{s}_l}.$$

### 3.B Canonical basis and Ariki's theorem

Let  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$  be the quantum group associated to the Lie algebra  $\widehat{\mathfrak{sl}}_e$ . We denote by  $e_i, f_i$  and  $k_i$  with  $i = 0, \dots, e-1$ , the Chevalley generators and by  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$  the subalgebra of  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$  generated by  $e_i, f_i, k_i, k_i^{-1}$ . We can construct a structure of  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -module on  $\bigwedge^{s+\frac{\infty}{2}} V_{e,l}$  (see [U, §2.1, §3.5, §4.2]) such that each of the subspace  $\mathcal{F}_{e, \mathbf{s}_l}$  is stable with respect to this action. Before describing the action of the  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -module  $\mathcal{F}_{e, \mathbf{s}_l}$ , we need some combinatorial definitions.

Let  $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(l)})$  be a  $l$ -partition of rank  $n$ . The diagram of  $\underline{\lambda}$  is the following set:

$$[\underline{\lambda}] = \left\{ (a, b, c) \mid 1 \leq c \leq l, 1 \leq b \leq \lambda_a^{(c)} \right\}.$$

The elements of this diagram are called the nodes of  $\underline{\lambda}$ . Let  $\gamma = (a, b, c)$  be a node of  $\underline{\lambda}$ . The residue of  $\gamma$  associated to the set  $\{e; s_1, \dots, s_l\}$  is the unique element of  $[0, e[$  such that:

$$\text{res}(\gamma) \equiv (b - a + s_c) \pmod{e}.$$

If  $\gamma$  is a node with residue  $i$ , we say that  $\gamma$  is an  $i$ -node. Let  $\underline{\lambda}$  and  $\underline{\mu}$  be two  $l$ -partitions of rank  $n$  and  $n+1$  such that  $[\underline{\lambda}] \subset [\underline{\mu}]$ . There exists a node  $\gamma$  such that  $[\underline{\mu}] = [\underline{\lambda}] \cup \{\gamma\}$ . Then, we denote  $[\underline{\mu}]/[\underline{\lambda}] = \gamma$  and if  $\text{res}(\gamma) = i$ , we say that  $\gamma$  is an addable  $i$ -node for  $\underline{\lambda}$  and a removable  $i$ -node for  $\underline{\mu}$ . Now, we introduce an order on the set of nodes of a  $l$ -partition. We say that  $\gamma = (a, b, c)$  is *above*  $\gamma' = (a', b', c')$  if:

$$b - a + s_c < b' - a' + s_{c'} \text{ or if } b - a + s_c = b' - a' + s_{c'} \text{ and } c' < c.$$

Let  $\underline{\lambda}$  and  $\underline{\mu}$  be two  $l$ -partitions of rank  $n$  and  $n+1$  such that there exists an  $i$ -node  $\gamma$  such that  $[\underline{\mu}] = [\underline{\lambda}] \cup \{\gamma\}$ . We define the following numbers:

$$\begin{aligned} N_i^a(\underline{\lambda}, \underline{\mu}) &= \#\{\text{addable } i\text{-nodes of } \underline{\lambda} \text{ above } \gamma\} \\ &\quad - \#\{\text{removable } i\text{-nodes of } \underline{\mu} \text{ above } \gamma\}, \\ N_i^b(\underline{\lambda}, \underline{\mu}) &= \#\{\text{addable } i\text{-nodes of } \underline{\lambda} \text{ below } \gamma\} \\ &\quad - \#\{\text{removable } i\text{-nodes of } \underline{\mu} \text{ below } \gamma\}, \\ N_i(\underline{\lambda}) &= \#\{\text{addable } i\text{-nodes of } \underline{\lambda}\} \\ &\quad - \#\{\text{removable } i\text{-nodes of } \underline{\lambda}\}. \end{aligned}$$

Now,  $\mathcal{F}_{e, \mathbf{s}_l}$  is an integrable  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -module with action:

$$\begin{aligned} e_i |\underline{\lambda}, \mathbf{s}_l\rangle &= \sum_{\text{res}([\underline{\lambda}]/[\underline{\mu}]) \equiv i} q^{-N_i^a(\underline{\mu}, \underline{\lambda})} |\underline{\mu}, \mathbf{s}_l\rangle, \\ f_i |\underline{\lambda}, \mathbf{s}_l\rangle &= \sum_{\text{res}([\underline{\mu}]/[\underline{\lambda}]) \equiv i} q^{N_i^b(\underline{\lambda}, \underline{\mu})} |\underline{\mu}, \mathbf{s}_l\rangle, \\ k_i |\underline{\lambda}, \mathbf{s}_l\rangle &= q^{N_i(\underline{\lambda})} |\underline{\lambda}, \mathbf{s}_l\rangle. \end{aligned}$$

where  $0 \leq i \leq e-1$ . Note that this action coincides with that of [U] up to transformation  $(s_1, \dots, s_l) \mapsto (s_l, \dots, s_1)$ .

We now introduce another basis for  $\mathcal{F}_{e, \mathbf{s}_l}$  namely the Leclerc-Thibon canonical basis. This basis is defined by using an involution on the wedge space  $\bigwedge^{s+\frac{\infty}{2}} V_{e,l}$  which has been introduced in [LT] for  $l=1$  and generalized to any  $l$  in [U]. Let  $u_{\underline{k}} := u_{k_1} \wedge u_{k_2} \wedge \dots$  be a semi infinite monomial (ordered or not). For  $i=1, 2, \dots$ , we put  $k_i = a_i + e(l - b_i) - elm_i$  where  $a_i \in \{1, \dots, e\}$ ,  $b_i \in \{1, \dots, l\}$  and  $m_i \in \mathbb{Z}$ . For  $r \in \mathbb{N}$ , we put:

$$\begin{aligned} \omega(u_{\underline{k}}) &= \#\{i < j \mid a_i = a_j\}, \\ \omega'(u_{\underline{k}}) &= \#\{i < j \mid b_i = b_j\}. \end{aligned}$$

Then, for  $r \geq \sum_{i=1}^{\infty} k_i - (s - i + 1)$  we set:

$$\overline{u_{\underline{k}}} := (-q)^{\omega'(u_{\underline{k}})} q^{-\omega(u_{\underline{k}})} u_{k_r} \wedge u_{k_{r-1}} \wedge \dots \wedge u_{k_1} \wedge u_{k_{r+1}} \wedge u_{k_{r+2}} \wedge \dots$$

One can prove that this monomial is independent of  $r$  and that  $u \mapsto \overline{u}$  defines an involution on  $\bigwedge^{s+\frac{\infty}{2}} V_{e,l}$ . The canonical basis is now defined as follows:

**Theorem 3.1** (Leclerc-Thibon [LT, Theorem 4.1], Uglov [U, §4.4]) *Let  $s \in \mathbb{Z}$ . There exists a unique basis:*

$$\{G(\underline{\lambda}, \mathbf{s}_l) \mid \sum_{i=1}^l s_i = s, \underline{\lambda} \in \Pi_l^n, n \in \mathbb{N}\}$$

of  $\bigwedge^{s+\frac{\infty}{2}} V_{e,l}$  such that:

- $\overline{G(\underline{\lambda}, \mathbf{s}_l)} = G(\underline{\lambda}, \mathbf{s}_l)$ ,
- $G(\underline{\lambda}, \mathbf{s}_l) - |\underline{\lambda}, \mathbf{s}_l\rangle \in \bigoplus_{\underline{\mu}} q\mathbb{C}[q] |\underline{\mu}, \mathbf{s}_l\rangle$ .

This is called the canonical basis of  $\bigwedge^{s+\frac{\infty}{2}} V_{e,l}$ .

In particular the set  $\{G(\underline{\lambda}, \mathbf{s}_l) \mid \underline{\lambda} \in \Pi_l^n, n \in \mathbb{N}\}$  gives a basis of the Fock space  $\mathcal{F}_{e, \mathbf{s}_l}$ . Now, we consider the subspace  $M_{\mathbf{s}_l} := \mathcal{U}'_q(\widehat{\mathfrak{sl}}_e) \cdot |\underline{0}, \mathbf{s}_l\rangle$ . It is well known that this is isomorphic to the irreducible  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ -module  $V(\Lambda)$  with highest weight  $\Lambda := \Lambda_{s_1(\text{mod } e)} + \Lambda_{s_2(\text{mod } e)} + \dots + \Lambda_{s_l(\text{mod } e)}$ . Note that if  $\mathbf{s}'_l = (s'_1, \dots, s'_l) \in \mathbb{Z}^l$  is such that  $s_i \equiv s'_i \pmod{e}$ , then the modules  $M_{\mathbf{s}_l}$  and  $M_{\mathbf{s}'_l}$  are isomorphic (but the action of  $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$  on the elements of the standard basis  $|\underline{\lambda}, \mathbf{s}_l\rangle$  and  $|\underline{\lambda}, \mathbf{s}'_l\rangle$  are different in general).

A basis of  $M_{\mathbf{s}_l}$  can be given by using the canonical basis of  $\mathcal{F}_{e, \mathbf{s}_l}$  and by studying the associated crystal graph. This graph can be described combinatorially as follows.

Let  $\underline{\lambda}$  be a  $l$ -partition and let  $\gamma$  be an  $i$ -node of  $\underline{\lambda}$ , we say that  $\gamma$  is a normal  $i$ -node of  $\underline{\lambda}$  if, whenever  $\eta$  is an  $i$ -node of  $\underline{\lambda}$  below  $\gamma$ , there are more removable  $i$ -nodes between  $\eta$  and  $\gamma$  than addable  $i$ -nodes between  $\eta$  and  $\gamma$ . If  $\gamma$  is the highest normal  $i$ -node of  $\underline{\lambda}$ , we say that  $\gamma$  is a good  $i$ -node. Note that this notion depends on the choice of  $\mathbf{s}_l$ .

Then, the crystal graph of  $\mathcal{F}_{e, \mathbf{s}_l}$  is given by:

- vertices: the  $l$ -partitions,
- edges:  $\underline{\lambda} \xrightarrow{i} \underline{\mu}$  if and only if  $[\underline{\mu}]/[\underline{\lambda}]$  is a good  $i$ -node.

By using properties of crystal bases, we can obtain the crystal graph of  $M_{\mathbf{s}_l}$ : this is the connected components of that of  $\mathcal{F}_{e, \mathbf{s}_l}$  which contain the vacuum vector  $|\underline{0}, \mathbf{s}_l\rangle$ . The vertices of this graph are given by the following class of  $l$ -partitions.

**Definition 3.2** Let  $\mathbf{s}_l \in \mathbb{Z}^n$ . The set of Uglov  $l$ -partitions  $\Lambda_{e; \mathbf{s}_l}^n$  is defined recursively as follows.

- We have  $\underline{0} := (\emptyset, \emptyset, \dots, \emptyset) \in \Lambda_{e; \mathbf{s}_l}^n$ .
- If  $\underline{\lambda} \in \Lambda_{e; \mathbf{s}_l}^n$ , there exist  $i \in \{0, \dots, e-1\}$  and a good  $i$ -node  $\gamma$  such that if we remove  $\gamma$  from  $\underline{\lambda}$ , the resulting  $l$ -partition is in  $\Lambda_{e; \mathbf{s}_l}^n$ .

**Remark 3.3** Assume that  $\mathbf{s}_l \in \mathbb{Z}^n$  is such that  $0 \leq s_1 \leq s_2 \leq \dots \leq s_l < e$  then it is shown in [FL, Proposition 2.11] that the set of Uglov  $l$ -partitions are the  $l$ -partitions  $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(l)})$  such that:

1. for all  $1 \leq j \leq l-1$  and  $i = 1, 2, \dots$ , we have:

$$\begin{aligned} \lambda_i^{(j)} &\geq \lambda_{i+s_{j+1}-s_j}^{(j+1)}, \\ \lambda_i^{(l)} &\geq \lambda_{i+e+s_1-s_l}^{(1)}; \end{aligned}$$

2. for all  $k > 0$ , among the residues appearing at the right ends of the length  $k$  rows of  $\underline{\lambda}$ , at least one element of  $\{0, 1, \dots, e-1\}$  does not occur.

Such  $l$ -partitions are called FLOTW  $l$ -partitions in [Jp].

Assume that  $\mathbf{s}_l \in \mathbb{Z}^n$  is such that  $s_1 \gg s_2 \gg \dots \gg s_l$ , then the set Uglov  $l$ -partitions  $\Lambda_{e; \mathbf{s}_l}^n$  coincides with the set of Kleshchev  $l$ -partitions by [Ac, §3].

Now, the canonical basis of  $M_{\mathbf{s}_l}$  is the following set:

$$\{G(\underline{\lambda}, \mathbf{s}_l) \mid \underline{\lambda} \in \Lambda_{e; \mathbf{s}_l}^n, n \in \mathbb{N}\}.$$

The following theorem gives a link between the canonical basis elements of  $M_{\mathbf{s}_l}$  and the decomposition matrices of Ariki-Koike algebras.

**Theorem 3.4** (Ariki, see [Ab, Theorem 4.4.9]) Let  $\mathcal{H}_{\mathbb{C},n}$  be the Ariki-Koike algebra over  $\mathbb{C}$  with the following choice of parameters:

$$\begin{aligned} x_j &= \eta_e^{v_j} \text{ for } j = 1, \dots, l, \\ v &= \eta_e, \end{aligned}$$

where  $0 \leq v_1 \leq \dots \leq v_l < e$ . Let  $\Phi^n := \Phi_{\{e; v_1, \dots, v_l\}}^n$  and  $(d_{\underline{\lambda}, \underline{\mu}})_{\underline{\lambda} \in \Pi_l^n, \underline{\mu} \in \Phi^n}$  be the associated decomposition matrix.

Let  $\mathbf{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l$  be such that  $v_j \equiv s_j \pmod{e}$  for  $j = 1, \dots, l$ . Then, for each  $\underline{\nu} \in \Lambda_{e; \mathbf{s}_l}^n$ , write

$$G(\underline{\nu}, \mathbf{s}_l) = \sum_{\underline{\lambda} \in \Pi_l^n} d_{\underline{\lambda}, \underline{\nu}}(q) |_{\underline{\lambda}, \mathbf{s}_l},$$

where  $d_{\underline{\lambda}, \underline{\nu}}(q) \in \mathbb{C}[q]$ . Then there exists a bijection  $\kappa : \Phi^n \rightarrow \Lambda_{e; \mathbf{s}_l}^n$  such that for all  $\underline{\lambda} \in \Pi_l^n$  and  $\underline{\mu} \in \Phi^n$ , we have:

$$d_{\underline{\lambda}, \underline{\mu}} = d_{\underline{\lambda}, \kappa(\underline{\mu})}(1).$$

In other words, Ariki's theorem asserts that the columns of the decomposition matrix of  $\mathcal{H}_{\mathbb{C},n}$  with the above choice of parameters coincides with the canonical basis elements of  $M_{\mathbf{s}_l}$  evaluated at  $q = 1$  whenever  $v_j \equiv s_j \pmod{e}$  for  $j = 1, \dots, l$ . Note that this decomposition matrix only depends on  $\{e; v_1, \dots, v_l\}$  whereas the canonical basis elements of  $M_{\mathbf{s}_l}$  depends on  $\{e; s_1, \dots, s_l\}$ . In particular, if  $s_j \equiv s'_j \pmod{e}$  for  $j = 1, \dots, l$ , the labelings of the canonical basis elements of  $M_{\mathbf{s}_l}$  and  $M_{\mathbf{s}'_l}$  by  $\Lambda_{e; \mathbf{s}_l}^n$  and  $\Lambda_{e; \mathbf{s}'_l}^n$  are different in general as we can see in the following example.

**Example 3.5** Assume that  $l = 2$ ,  $e = 4$ ,  $v_1 = 0$ ,  $v_2 = 1$ . Then different values for  $\mathbf{s}_l$  lead to different labelings of the same crystal graph which are given in Example 4.8.

In [Ac], Ariki has given an explicit description of  $\Phi_{\{e; v_1, \dots, v_l\}}^n$  by showing that this set coincides with the set  $\Lambda_{e; \mathbf{s}_l}^n$  with  $s_1 \gg s_2 \gg \dots \gg s_l$  and  $v_j \equiv s_j \pmod{e}$  for  $j = 1, \dots, l$ . In [Jp], another parametrization of the simple  $\mathcal{H}_{\mathbb{C},n}$ -modules has been given by using the set  $\Lambda_{e; \mathbf{s}_l}^n$  with  $\mathbf{s}_l = (v_1, \dots, v_l)$  (namely the set of FLOTW  $l$ -partitions) and an ordering of the rows of the decomposition matrices by Lusztig  $a$ -values. In the next section, we will show that each of the sets  $\Lambda_{e; \mathbf{s}_l}^n$  with  $v_j \equiv s_j \pmod{e}$  has a natural interpretation in the representation theory of  $\mathcal{H}_{\mathbb{C},n}$ .

## 4 Unitriangularity of the decomposition matrices of Ariki-Koike algebras

### 4.A Specialisations and Lusztig $a$ -values

Let  $e$  be an integer and let  $\mathbf{s}_l = (s_1, s_2, \dots, s_l) \in \mathbb{Z}^l$  be a sequence of integers. The aim of this part is to study the Ariki-Koike algebra  $\mathcal{H}_{\mathbb{C},n}$  with the following choice of parameters:

$$\begin{aligned} u_j &= \eta_e^{s_j \pmod{e}} \text{ for } j = 1, \dots, l, \\ v &= \eta_e. \end{aligned}$$

For  $j = 1, \dots, l$ , we define rational numbers:

$$m^{(j)} = s_j - \frac{(j-1)e}{l} + \alpha e,$$

where  $\alpha$  is a positive integer such that  $m^{(j)} \geq 0$  for  $j = 1, \dots, l$ . Let  $y$  be an indeterminate and put  $A := \mathbb{C}[y, y^{-1}]$ . We consider the Ariki-Koike algebra  $\mathcal{H}_{A,n}^{s_l}$  over  $A$  with the following parameters:

$$\begin{aligned} u_j &= y^{lm^{(j)}} \eta_l^{j-1} \text{ for } j = 1, \dots, l, \\ v &= y^l \end{aligned}$$

(this algebra is denoted by  $\mathcal{H}_{A,n}$  in the introduction but here, we use the notation  $\mathcal{H}_{A,n}^{s_l}$  to insist on the dependance on  $s_l$ ).

By Theorem 2.2,  $\mathcal{H}_{\mathbb{C}(y),n}^{s_l} := \mathbb{C}(y) \otimes_A \mathcal{H}_{A,n}^{s_l}$  is split semisimple. Moreover, if we specialize the parameter  $y$  to  $\eta_{le} := \exp(\frac{2i\pi}{le})$ , we obtain the above Ariki-Koike algebra  $\mathcal{H}_{\mathbb{C},n}$ . Hence, we have a well-defined decomposition map between  $R_0(\mathcal{H}_{\mathbb{C}(y),n}^{s_l})$  and  $R_0(\mathcal{H}_{\mathbb{C},n})$ .

We now associate to each simple  $\mathcal{H}_{\mathbb{C}(y),n}^{s_l}$ -module  $S_{\mathbb{C}(y)}^{\underline{\lambda}}$  an  $a$ -value following [Jp]. Put  $\mathbf{m}_l = (m^{(1)}, \dots, m^{(l)})$  where the  $m^{(j)}$  are defined above. We need to define the notion of “ $\mathbf{m}_l$ -translated symbols” associated to  $l$ -compositions.

Let  $n \in \mathbb{N}$  and  $l \in \mathbb{N}$ . An  $l$ -composition  $\underline{\lambda}$  of rank  $n$  is an  $l$ -tuple  $(\lambda^{(1)}, \dots, \lambda^{(l)})$  where :

- for all  $i = 1, \dots, l$ , we have  $\lambda^{(i)} = (\lambda_1^{(i)}, \dots, \lambda_{h^{(i)}}^{(i)})$  for  $h^{(i)} \in \mathbb{N}$  and  $\lambda_j^{(i)} \in \mathbb{N}_{>0}$  ( $j = 1, \dots, h^{(i)}$ ),  $h^{(i)}$  is called the height of  $\lambda^{(i)}$ ,
- $\sum_{i=1}^l \sum_{j=1}^{h^{(i)}} \lambda_j^{(i)} = n$ .

Let  $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(l)})$  be a  $l$ -composition and let  $h^{(i)}$  be the heights of the compositions  $\lambda^{(i)}$ . Then the height of  $\underline{\lambda}$  is the following positive integer:

$$h_{\underline{\lambda}} := \max\{h^{(1)}, \dots, h^{(l)}\}.$$

Let  $k$  be a positive integer. The translated symbol  $\mathbf{B}[\mathbf{m}_l]'$  associated to  $\mathbf{m}_l$ ,  $k$  and  $\underline{\lambda}$  is:

$$\mathbf{B}[\mathbf{m}_l]' := (B'^{(1)}, \dots, B'^{(l)}),$$

where  $B'^{(i)}$ , for  $i = 1, \dots, l$ , is given by:

$$B'^{(i)} := (B_1'^{(i)}, \dots, B_{h_{\underline{\lambda}}+k}^{\prime(i)}),$$

in which:

$$B_j^{\prime(i)} := \lambda_j^{(i)} - j + h_{\underline{\lambda}} + k + m^{(i)} \quad (1 \leq j \leq h_{\underline{\lambda}} + k).$$

The integer  $h_{\underline{\lambda}} + k$  is called the height of  $\mathbf{B}[\mathbf{m}_l]'$ .

Now, the  $a$ -value on the  $\mathcal{H}_{\mathbb{C}(y),n}^{s_l}$ -module  $S_{\mathbb{C}(y)}^{\underline{\lambda}}$  is defined to be the lower degree of the associated the Schur element. These elements have been explicitly computed by Geck, Iancu and Malle in [GIM] (see [Jp, Proposition 3.1]). We obtain the following definition:

**Definition 4.1** Let  $\underline{\lambda}$  be a  $l$ -partition of rank  $n$  and let  $S_{\mathbb{C}(y)}^{\underline{\lambda}}$  be the simple  $\mathcal{H}_{\mathbb{C}(y),n}^{s_l}$ -module. Let  $k$  be a positive integer and  $\mathbf{B}[\mathbf{m}_l]'$  be the  $\mathbf{m}_l$ -translated symbol associated to  $\mathbf{m}_l$  and  $k$ . Then, if  $h$  is the height of  $\mathbf{B}[\mathbf{m}_l]'$ , the  $a$ -value of  $S_{\mathbb{C}(y)}^{\underline{\lambda}}$  is the following rational number:

$$a_{s_l}(\underline{\lambda}) = f(n, h, \mathbf{m}_l) + \sum_{\substack{1 \leq i \leq j < l+1 \\ (a,b) \in B'^{(i)} \times B'^{(j)} \\ a > b \text{ if } i=j}} \min\{a, b\} - \sum_{\substack{1 \leq i, j < l+1 \\ a \in B'^{(i)} \\ 1 \leq k \leq a}} \min\{k, m^{(j)}\},$$

where  $f(n, h, \mathbf{m}_l)$  is a rational number which only depends on the parameters  $\{e; s_1, \dots, s_l\}$ , on  $h$  and on  $n$  (the expression of  $f$  is given in [Jp, Proposition 3.2]).

We need to introduce the following preorder on the set of  $l$ -compositions.

**Definition 4.2** Let  $\underline{\mu}$  and  $\underline{\nu}$  be  $l$ -compositions of rank  $n$ . Let  $k$  and  $k'$  be such that  $h_{\underline{\mu}} + k = h_{\underline{\nu}} + k'$ . Let  $\mathbf{B}_{\underline{\mu}}[\mathbf{m}_l]'$ , (resp.  $\mathbf{B}_{\underline{\nu}}[\mathbf{m}_l]'$ ) be the  $\mathbf{m}_l$ -translated symbol associated to  $\underline{\mu}$ ,  $\mathbf{m}_l$  and  $k$  (resp.  $\underline{\nu}$ ,  $\mathbf{m}_l$  and  $k'$ ). Then we write:

$$\underline{\mu} \prec \underline{\nu},$$

if:

$$\sum_{\substack{1 \leq i \leq j < l+1 \\ (a,b) \in B_{\underline{\mu}}'^{(i)} \times B_{\underline{\mu}}'^{(j)} \\ a > b \text{ if } i=j}} \min\{a, b\} - \sum_{\substack{1 \leq i, j < l+1 \\ a \in B_{\underline{\mu}}'^{(i)} \\ 1 \leq k \leq a}} \min\{k, m^{(j)}\} < \\ \sum_{\substack{1 \leq i \leq j < l+1 \\ (a,b) \in B_{\underline{\nu}}'^{(i)} \times B_{\underline{\nu}}'^{(j)} \\ a > b \text{ if } i=j}} \min\{a, b\} - \sum_{\substack{1 \leq i, j < l+1 \\ a \in B_{\underline{\nu}}'^{(i)} \\ 1 \leq k \leq a}} \min\{k, m^{(j)}\}.$$

In particular, if  $\underline{\mu}$  and  $\underline{\nu}$  are  $l$ -partitions, as the translated symbol of  $\underline{\mu}$  and  $\underline{\nu}$  have the same height  $h$ , we have:

$$\underline{\mu} \prec \underline{\nu} \iff a_{s_l}(\underline{\mu}) < a_{s_l}(\underline{\nu}).$$

Before beginning the proof of the main result, we need to give the following useful proposition which has been shown in [Jp, Lemma 4.4] in the case  $0 \leq s_1 \leq \dots \leq s_l < e$ . One can easily check that the proof holds in the general case.

**Proposition 4.3** Let  $\underline{\lambda}$  be a  $l$ -composition of rank  $n$ , let  $\mathbf{B} := (B^{(1)}, \dots, B^{(l)})$  be an ordinary symbol of  $\underline{\lambda}$ , let  $\beta_1$  and  $\beta_2$  be two elements of  $\mathbf{B}[\mathbf{m}_l]'$ , we assume that:

$$\beta_1 < \beta_2.$$

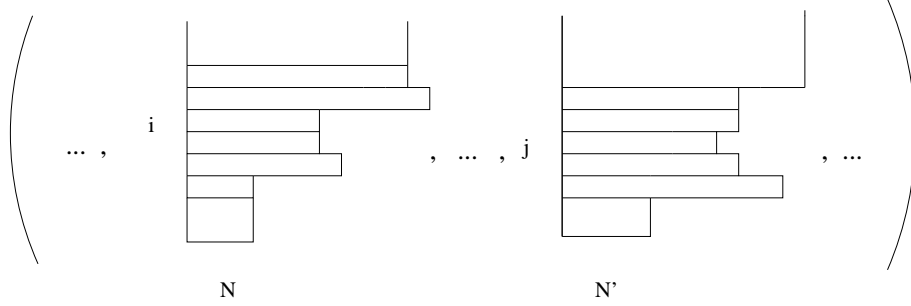
Let  $r \in \mathbb{N}$ . We add  $r$  nodes to  $\underline{\lambda}$  on the part associated to  $\beta_1$ . Let  $\underline{\mu}$  be the resulting  $l$ -composition of rank  $n+r$ . We add  $r$  nodes to  $\underline{\lambda}$  on the part associated to  $\beta_2$ . Let  $\underline{\nu}$  be the resulting  $l$ -composition of rank  $n+r$ . Then, we have:

$$\underline{\nu} \prec \underline{\mu}.$$

In particular, if  $\underline{\mu}$  and  $\underline{\nu}$  are some  $l$ -partitions, as the translated symbols of  $\underline{\mu}$  and  $\underline{\nu}$  have the same height, we have:

$$a_{s_l}(\underline{\mu}) > a_{s_l}(\underline{\nu}).$$

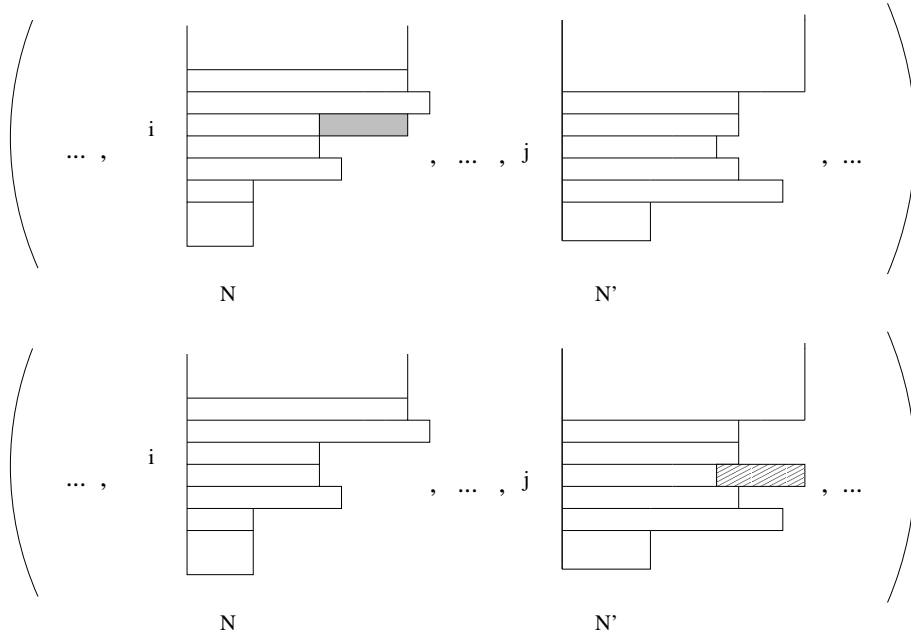
Assume that  $\underline{\lambda}$  is a  $l$ -composition of rank  $n$ .



Assume that  $\underline{\mu}$  and  $\underline{\mu}'$  are two  $l$ -compositions which are obtained from  $\underline{\lambda}$  by adding  $r$  nodes on the parts  $\lambda_i^{(N)}$  and  $\lambda_j^{(N')}$  respectively where  $1 \leq N, N' \leq l$ . Assume in addition that we have:

$$\lambda_i^{(N)} - i + s_N - \frac{(N-1)e}{l} > \lambda_j^{(N')} - j + s_{N'} - \frac{(N'-1)e}{l}.$$

Keeping the above representation of  $\underline{\lambda}$ ,  $\underline{\mu}$  and  $\underline{\mu}'$  are respectively given as follows:



Then Proposition 4.3 asserts that we have:

$$\underline{\mu} \prec \underline{\mu}'.$$

The aim of the following part is to show that the matrix associated to the canonical basis elements of the irreducible modules  $M_{\mathbf{s}_l}$  is unitriangular with respect to  $a$ -values.

## 4.B Study of the involution on the semi-infinite $q$ -wedge product

Let  $s \in \mathbb{Z}$ . We keep the notations of the previous sections. We will work with the semi-infinite  $q$ -wedge product  $\bigwedge^{s+\frac{\infty}{2}} V_{e,l}$ . First, we attach to each semi-infinite monomial an  $a$ -value. Let  $u_{\underline{k}} \in \bigwedge^{s+\frac{\infty}{2}} V_{e,l}$  be a semi-infinite  $q$ -wedge product and let  $u_{\tilde{\underline{k}}}$  be the monomial obtained by reordering the  $k_i$  in strictly decreasing order. Then, the bijection described in section 3.A shows that we can associate to  $u_{\tilde{\underline{k}}}$  a symbol  $|\underline{\lambda}, \mathbf{s}_l\rangle$  with  $\underline{\lambda} \in \Pi_l^n$  and  $\mathbf{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l$  verifying  $\sum_{i=1}^l s_i = s$ . We put:

$$\begin{aligned}\pi(u_{\tilde{\underline{k}}}) &:= |\underline{\lambda}, \mathbf{s}_l\rangle, \\ a(u_{\tilde{\underline{k}}}) &:= a_{\mathbf{s}_l}(\underline{\lambda}).\end{aligned}$$

In this part, we show the following proposition:

**Proposition 4.4** *Let  $u_{\underline{k}} \in \bigwedge^{s+\frac{\infty}{2}} V_{e,l}$  be a semi-infinite ordered  $q$ -wedge product. Then, we have:*

$$\overline{u_{\underline{k}}} = u_{\underline{k}} + \text{sum of ordered monomials } u_{\underline{r}} \text{ with } a(u_{\underline{r}}) > a(u_{\underline{k}}),$$

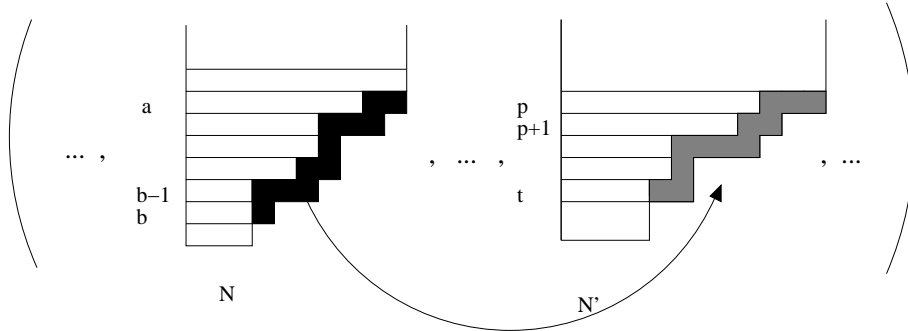
where the involution is defined in section 3.B.

We note that a similar property is shown in [U] but for a (partial) order which is not compatible with the preorder induced by the  $a$ -values.

Let  $u_{\underline{k}}$  be an arbitrary semi-infinite monomial such that there exists  $i \in \mathbb{N}$  such that  $k_i < k_{i+1}$ . Then the relations (R1) – (R4) in section 3.A show how to express  $u_{\underline{k}}$  in terms of semi-infinite monomials  $u_{\underline{k}'}$  such that  $k'_i > k'_{i+1}$ . Let  $u_{\underline{k}'}$  be such a semi-infinite monomial such that  $k'_i \neq k_{i+1}$  and  $k'_{i+1} \neq k_i$ . Then the  $l$ -partitions associated to  $u_{\underline{k}}$  and  $u_{\underline{k}'}$  have the same charge  $\mathbf{s}_l$ . We put :

$$\begin{aligned}|\underline{\lambda}, \mathbf{s}_l\rangle &:= \pi(u_{\underline{k}}), \\ |\underline{\lambda}', \mathbf{s}_l\rangle &:= \pi(u_{\underline{k}'}).\end{aligned}$$

From the relations (R1) – (R4), using the representations of  $u_{\underline{k}}$  and  $u_{\underline{k}'}$  by  $l$ -abacus, we can see that  $u_{\underline{k}'}$  is obtained from  $u_{\underline{k}}$  by moving two black beads representing  $k_i$  and  $k_{i+1}$  to two black beads representing  $k'_i$  and  $k'_{i+1}$  and lying in the same runners as  $k_i$  and  $k_{i+1}$ . As a consequence, the  $l$ -partition  $\underline{\lambda}'$  is obtained from  $\underline{\lambda}$  by removing a ribbon  $R$  of size  $x$  from a component  $b$  and adding a ribbon  $R'$  of size  $x$  to a component  $b'$  as in the following figure.





Hence, there exist positive integers  $r, s, p$  and  $t$  such that:

$$\begin{aligned}\lambda^{(b)} &= (\lambda_1^{(b)}, \dots, \lambda_{r-1}^{(b)}, \lambda_{r+1}^{(b)} - 1, \dots, \lambda_s^{(b)} - 1, \lambda_r^{(b)} - (x - (r + s)), \lambda_{s+1}^{(b)}, \dots, \lambda_{h^{(b)}}^{(s)}), \\ \lambda^{(b')} &= (\lambda_1^{(b')}, \dots, \lambda_{p-1}^{(b')}, \lambda_t^{(b')} + (x - (p - k)), \lambda_p^{(b')} + 1, \dots, \lambda_{t-1}^{(b')} + 1, \lambda_{t+1}^{(b')}, \dots, \lambda_{h^{(b')}}^{(b')}).\end{aligned}$$

In fact, the two  $l$ -partitions  $\underline{\lambda}'$  and  $\underline{\lambda}$  are both obtained by adding  $R$  and  $R'$  to the components  $b$  and  $b'$  of the same  $l$ -partition  $\underline{\nu} = \pi(u_{\underline{r}})$  where

$$\underline{r} = (k_1, \dots, k_{i-1}, \hat{k}_i, \hat{k}_{i+1}, k_{i+2}, \dots).$$

Here,  $\hat{k}_i$  and  $\hat{k}_{i+1}$  are the minimal integers of  $\{k_i, k_{i+1}, k'_i, k'_{i+1}\}$  such that the black beads associated to  $\hat{k}_i$  and  $\hat{k}_{i+1}$  lie at the components  $b$  and  $b'$  of the  $l$ -abacus of  $u_{\underline{r}}$  and  $\hat{k}_{i+1} < \hat{k}_i$ .

**Lemma 4.5** *We keep the above notations. Assume that  $\underline{\lambda} \neq \underline{\lambda}'$ . Put  $\hat{k}_i = a + e(l - b) - elm$  and  $\hat{k}_{i+1} = a' + e(l - b') - elm'$  where  $a, a' \in \{1, \dots, e\}$ ,  $b, b' \in \{1, \dots, l\}$  and,  $m, m' \in \mathbb{Z}$ . Then we have:*

$$a - em - \frac{(b-1)e}{l} > a' - em' - \frac{(b'-1)e}{l}.$$

*Proof.* For  $j = i, i + 1$ , put  $k_j = a_j + e(l - b_j) - elm_j$  where  $a_j \in \{1, \dots, e\}$ ,  $b_j \in \{1, \dots, l\}$  and  $m_j \in \mathbb{Z}$ . As in section 3.A, we define  $\alpha$  and  $\beta$  to be the unique integers in  $[0, el - 1]$  such that  $\alpha \equiv (a_{i+1} - a_i) \pmod{el}$  and  $\beta \equiv e(b_i - b_{i+1}) \pmod{el}$ . As  $\underline{\lambda} \neq \underline{\lambda}'$ , we can assume that  $\alpha \neq 0$  or  $\beta \neq 0$ .

Now, regarding the relations (R1) – (R4) in section 3.A, we have 5 cases to consider.

- Assume that  $k'_i = k_{i+1} - elm$  and  $k'_{i+1} = k_i + elm$  with  $m > 0$ . If  $\beta = 0$ , we have  $\hat{k}_i = k'_{i+1}$  and  $\hat{k}_{i+1} = k_i$  and since  $k'_{i+1} > k_i$ , the result is clear. If  $\beta \neq 0$ , we have  $\hat{k}_i = k'_i$  and  $\hat{k}_{i+1} = k_i$ .

As  $k'_i > k'_{i+1}$ , we have:

$$a_{i+1} + e(l - b_{i+1}) - el(m_{i+1} + m) > a_i + e(l - b_i) - el(m_i - m).$$

Now, as  $m > 0$  and  $a_{i+1} - a_i > \frac{a_{i+1} - a_i}{l} - e$ , we conclude that:

$$a_{i+1} - e(m_{i+1} + m) - (b_{i+1} - 1)\frac{e}{l} > a_i - em_i - (b_i - 1)\frac{e}{l}.$$

- Assume that  $k'_i = k_{i+1} - \beta - elm$  and  $k'_{i+1} = k_i + \beta + elm$  with  $m \geq 0$  and  $\beta \neq 0$ . Then, we have  $\hat{k}_i = k'_{i+1}$  and  $\hat{k}_{i+1} = k_i$ .

– If  $b_{i+1} < b_i$  then  $\beta = e(b_i - b_{i+1})$  and we have:

$$\begin{aligned}\hat{k}_i &= a_i + e(l - b_i) - elm_i + e(b_i - b_{i+1}) + elm \\ &= a_i + e(l - b_{i+1}) - el(m_i - m).\end{aligned}$$

We conclude that:

$$a_i - e(m_i - m) - (b_{i+1} - 1)\frac{e}{l} > a_i - em_i - (b_i - 1)\frac{e}{l}.$$

– If  $b_i < b_{i+1}$  then  $\beta = e(b_i - b_{i+1}) + el$  and we have:

$$\begin{aligned}\hat{k}_i &= a_i + e(l - b_i) - elm_i + e(b_i - b_{i+1}) + elm + el \\ &= a_i + e(l - b_{i+1}) - el(m_i - m - 1).\end{aligned}$$

We conclude that:

$$a_i - e(m_i - m - 1) - (b_{i+1} - 1)\frac{e}{l} > a_i - em_i - (b_i - 1)\frac{e}{l}.$$

- Assume that  $k'_{i+1} = k_i + \alpha + elm$  and  $k'_i = k_{i+1} - \alpha - elm$  with  $m \geq 0$  and  $\alpha \neq 0$ . If  $\beta = 0$ , we have  $\hat{k}_i = k'_{i+1}$  and  $\hat{k}_{i+1} = k_i$  and since  $k'_{i+1} > k_i$ , the result is clear. If  $\beta \neq 0$ , we have  $\hat{k}_i = k'_i$  and  $\hat{k}_{i+1} = k_i$ .

– If  $a_{i+1} - a_i > 0$  then  $\alpha = a_{i+1} - a_i$  and we have:

$$\begin{aligned}\hat{k}_i &= a_{i+1} + e(l - b_{i+1}) - elm_{i+1} - a_{i+1} + a_i + elm \\ &= a_i + e(l - b_{i+1}) - el(m_{i+1} - m).\end{aligned}$$

We conclude that:

$$a_i - e(m_{i+1} - m) - (b_{i+1} - 1)\frac{e}{l} > a_i - em_i - (b_i - 1)\frac{e}{l}.$$

Indeed, if  $m_{i+1} - m < m_i$ , the result is clear. If otherwise, as  $m_{i+1} \leq m_i$ , we have  $m = 0$  and  $m_{i+1} = m_i$ . Hence, as  $k'_i > k'_{i+1}$ , we have  $b_{i+1} < b_i$ .

– If  $a_{i+1} - a_i < 0$  then  $\alpha = a_{i+1} - a_i + el$  and we have:

$$\begin{aligned}\hat{k}_i &= a_{i+1} + e(l - b_{i+1}) - elm_{i+1} - a_{i+1} + a_i + el(m + 1) \\ &= a_i + e(l - b_{i+1}) - el(m_{i+1} - m - 1).\end{aligned}$$

We conclude that:

$$a_i - e(m_{i+1} - m - 1) - (b_{i+1} - 1)\frac{e}{l} > a_i - em_i - (b_i - 1)\frac{e}{l}.$$

- Assume that  $k'_{i+1} = k_i + \alpha + \beta + elm$  and  $k'_i = k_{i+1} - \alpha - \beta - elm$  with  $m \geq 0$ ,  $\beta \neq 0$  and  $\alpha \neq 0$ . Then, we have  $\hat{k}_i = k'_{i+1}$  and  $\hat{k}_{i+1} = k_i$ .

– if  $a_{i+1} > a_i$  and  $b_i > b_{i+1}$  then  $\alpha = a_{i+1} - a_i$  and  $\beta = e(b_i - b_{i+1})$ . We have:

$$\begin{aligned}\hat{k}_i &= a_i + e(l - b_i) - elm_i + e(b_i - b_{i+1}) + (a_{i+1} - a_i) + elm \\ &= a_{i+1} + e(l - b_{i+1}) - el(m_i - m).\end{aligned}$$

We conclude that:

$$a_{i+1} - e(m_i - m) - (b_{i+1} - 1)\frac{e}{l} > a_i - em_i - (b_i - 1)\frac{e}{l}.$$

– if  $a_{i+1} < a_i$  and  $b_i > b_{i+1}$  then  $\alpha = a_{i+1} - a_i + el$  and  $\beta = e(b_i - b_{i+1})$ . We have:

$$\begin{aligned}\hat{k}_i &= a_i + e(l - b_i) - elm_i + e(b_i - b_{i+1}) + (a_{i+1} - a_i) + el(m + 1) \\ &= a_{i+1} + e(l - b_{i+1}) - el(m_i - m - 1).\end{aligned}$$

We conclude that:

$$a_{i+1} - e(m_i - m - 1) - (b_{i+1} - 1)\frac{e}{l} > a_i - em_i - (b_i - 1)\frac{e}{l}.$$

- if  $a_{i+1} > a_i$  and  $b_i < b_{i+1}$  then  $\alpha = a_{i+1} - a_i$  and  $\beta = e(b_i - b_{i+1}) + el$ .  
We have:

$$\begin{aligned}\hat{k}_i &= a_i + e(l - b_i) - elm_i + e(b_i - b_{i+1}) + (a_{i+1} - a_i) + elm + el \\ &= a_{i+1} + e(l - b_{i+1}) - el(m_i - m - 1).\end{aligned}$$

We conclude that:

$$a_{i+1} - e(m_i - m - 1) - (b_{i+1} - 1)\frac{e}{l} > a_i - em_i - (b_i - 1)\frac{e}{l}.$$

- if  $a_{i+1} > a_i$  and  $b_i < b_{i+1}$  then  $\alpha = a_{i+1} - a_i + el$  and  $\beta = e(b_i - b_{i+1}) + el$ . We have:

$$\begin{aligned}\hat{k}_i &= a_i + e(l - b_i) - elm_i + e(b_i - b_{i+1}) + (a_{i+1} - a_i) + elm + 2el \\ &= a_{i+1} + e(l - b_{i+1}) - el(m_i - m - 2).\end{aligned}$$

We conclude that:

$$a_{i+1} - e(m_i - m - 2) - (b_{i+1} - 1)\frac{e}{l} > a_i - em_i - (b_i - 1)\frac{e}{l}.$$

□

We are now able to prove Proposition 4.4. Keeping the notations of the beginning of this section, Lemma 4.5 shows that we have:

$$\nu_s^{(b)} - s + s_b - \frac{(b-1)e}{l} > \nu_t^{(b')} - t + s_{b'} - \frac{(b'-1)e}{l}. \quad (*)$$

Assume that we have  $\lambda_i^{(b')} = \nu_i^{(b')} + y_i$  for integers  $y_i$  with  $i = p, p+1, \dots, t$  and assume that we have  $\lambda_j^{(b)} = \nu_j^{(b)} + x_j$  for integers  $x_j$  with  $j = r, r+1, \dots, s$ . We define a  $l$ -composition  $\underline{\Delta}[1]$  as follows.

- If  $y_t \leq x_s$ , then we put  $j_1 := s$  and we define:

$$\underline{\Delta}[1]_i^{(k)} = \begin{cases} \lambda_i^{(k)} & \text{if } (i, k) \neq (s, b) \text{ and } (i, k) \neq (t, b'), \\ \lambda_i^{(k)} + y_t & \text{if } (i, k) = (s, b), \\ \lambda_i^{(k)} - y_t & \text{if } (i, k) = (t, b'). \end{cases}$$

In this case,  $\underline{\Delta}[1]$  and  $\underline{\lambda}'$  are both obtained by adding the same number of nodes on a  $l$ -composition. Then, by (\*) and by Proposition 4.3, we get:

$$\underline{\Delta}[1] \prec \underline{\lambda}'.$$

- If  $y_t > x_s$ , there exists  $j_1 \in \{r, r+1, \dots, s-1\}$  such that:

$$x_{j_1} + x_{j_1+1} + \dots + x_s \geq y_t \geq x_{j_1+1} + \dots + x_{s-1} + x_s.$$

Then, we define:

$$\underline{\Delta}[1]_i^{(k)} = \begin{cases} \lambda_i^{(k)} + x_i & \text{if } k = b \text{ and } \\ & s \geq i \geq j_1 + 1, \\ \lambda_i^{(k)} + y_t - (x_{j_1+1} + \dots + x_{s-1} + x_s) & \text{if } (i, k) = (j_1, b), \\ \lambda_i^{(k)} - y_t & \text{if } (i, k) = (t, b'), \\ \lambda_i^{(k)} & \text{if otherwise.} \end{cases}$$

For all  $i$  such that  $r \leq i \leq s$ , observe that  $\nu_s^{(b)} + x_{i+1} + \dots + x_{s-1} + x_s = \nu_i^{(b)} + (s - i)$ . Then, by (\*), for all  $i$  such that  $j_1 \leq i \leq s$ , we have:

$$\nu_i^{(b)} - i + s_b - \frac{(b-1)e}{l} > \nu_t^{(b')} - t + x_{i+1} + \dots + x_s + s_{b'} - \frac{(b'-1)e}{l}.$$

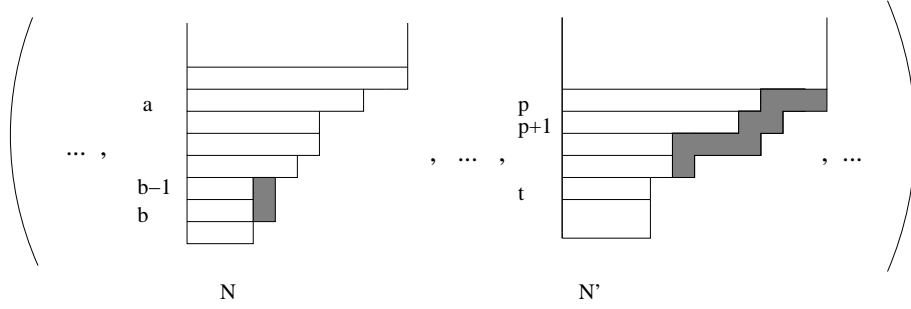
Thus, for all  $i$  such that  $j_1 \leq i \leq s$ , we obtain:

$$\lambda_i^{(b)} - i + s_b - \frac{(b-1)e}{l} > \lambda_t^{(b')} - t + x_{i+1} + \dots + x_{s-1} + x_s - y_t + s_{b'} - \frac{(b'-1)e}{l}.$$

Thus, using Proposition 4.3, we get:

$$\underline{\lambda}[1] \prec \underline{\lambda}'.$$

Keeping the above figure,  $\underline{\lambda}[1]$  is as follows.



As we have  $\nu_t^{(b')} - 1 + y_t = \nu_{t-1}^{(b')}$ , note that:

$$\lambda_{j_1}^{(b)} - j_1 + s_b - \frac{(b-1)e}{l} > \nu_{t-1}^{(b')} - (t-1) + s_{b'} - \frac{(b'-1)e}{l}. \quad (**)$$

We now define a  $l$ -composition  $\underline{\lambda}[2]$  as follows. Put :

$$x'_{j_1} := x_{j_1} + x_{j_1+1} + \dots + x_s - y_t.$$

- If  $y_{t-1} \leq x'_{j_1}$ , we put  $j_2 := j_1$  and we define:

$$\underline{\lambda}[2]_i^{(k)} = \begin{cases} \lambda[1]_i^{(k)} & \text{if } (i, k) \neq (j_1, b) \text{ and } (i, k) \neq (t-1, b'), \\ \lambda[1]_i^{(k)} + y_{t-1} & \text{if } (i, k) = (j_1, b), \\ \lambda[1]_i^{(k)} - y_{t-1} & \text{if } (i, k) = (t-1, b'). \end{cases}$$

By (\*\*), we have:

$$\lambda_{j_1}^{(b)} - j_1 + s_b - \frac{(b-1)e}{l} > \lambda_{t-1}^{(b')} - y_{t-1} - (t-1) + s_b - \frac{(b'-1)e}{l}.$$

By Proposition 4.3, we get:

$$\underline{\lambda}[2] \prec \underline{\lambda}[1].$$

- If  $y_{t-1} > x'_{j_1}$ , there exists  $j_2 \in \{r, r+1, \dots, j_1-1\}$  such that:

$$x_{j_2} + x_{j_2+1} + \dots + x_{j_1-1} + x'_{j_1} \geq y_{t-1} > x_{j_2+1} + \dots + x_{j_1-1} + x'_{j_1}.$$

Then, we define:

$$\underline{\lambda}[2]_i^{(k)} = \begin{cases} \lambda[1]_i^{(k)} + x'_{j_1} & \text{if } (i, k) = (j_1, b), \\ \lambda[1]_i^{(k)} + x_i & \text{if } k = b \text{ and} \\ & j_1 - 1 \geq i > j_2, \\ \lambda[1]_i^{(k)} - y_{t-1} & \text{if } (i, k) = (t-1, b'), \\ \lambda[1]_i^{(k)} + y_{t-1} - (x_{j_2+1} + \dots + x_{j_1-1} + x'_{j_1}) & \text{if } (i, k) = (j_2, b), \\ \lambda[1]_i^{(k)} & \text{if otherwise.} \end{cases}$$

By (\*\*), we have:

$$\lambda[1]_{j_1}^{(b)} - i + s_b - \frac{(b-1)e}{l} > \lambda[1]_{t-1}^{(b')} - (t-1) - y_{t-1} + s_{b'} - \frac{(b'-1)e}{l}.$$

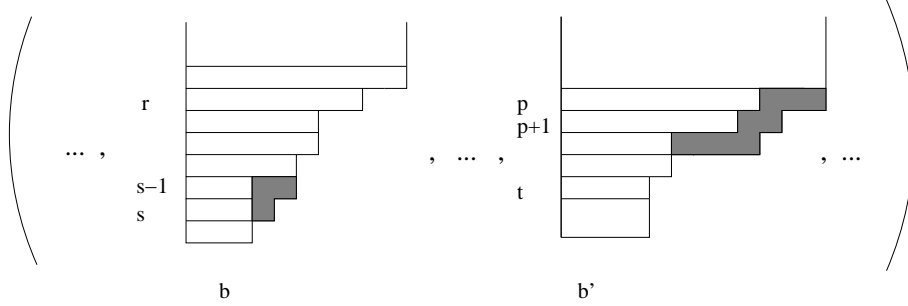
Moreover, we have  $\lambda[1]_{j_1}^{(b)} + x'_{j_1} = \lambda[1]_{j_1-1}^{(b)} + 1$  and  $\lambda[1]_i^{(b)} + x_i = \lambda[1]_{i-1}^{(b)} + 1$  for all  $i$  such that  $j_1 - 1 \geq i \geq r$ . Thus, for all  $i$  such that  $j_1 - 1 \geq i \geq j_2$ , by (\*\*), we obtain:

$$\lambda[1]_i^{(b)} - i + s_b - \frac{(b-1)e}{l} > \lambda[1]_{i-1}^{(b')} - (t-1) + x_{i+1} + \dots + x_{j_1-1} + x'_{j_1} - y_{t-1} + s_{b'} - \frac{(b'-1)e}{l}.$$

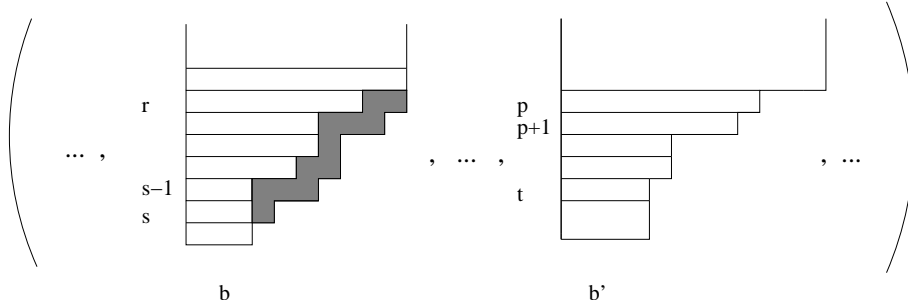
Hence, by Proposition 4.3, we conclude that:

$$\underline{\lambda}[2] \prec \underline{\lambda}[1].$$

Keeping the above figure,  $\underline{\lambda}[2]$  is as follows.



Continuing in this way, we obtain the  $l$ -composition  $\underline{\lambda}[p-t+1] = \underline{\lambda}$ :



We have :

$$\underline{\lambda} \prec \underline{\lambda}'.$$

Thus, we conclude that:

$$a_{\mathbf{s}_l}(\underline{\lambda}) < a_{\mathbf{s}_l}(\underline{\lambda}').$$

Now all the ordered monomials  $u_{\underline{p}}$  which appear in  $\overline{u_{\underline{k}}}$  are obtained recursively by using the relations (R1) – (R4). Hence, by induction, there exists  $\beta$  such that:

$$\overline{u_{\underline{k}}} = \beta u_{\underline{k}} + \text{sum of ordered monomials } u_{\underline{r}} \text{ with } a(u_{\underline{r}}) > a(u_{\underline{k}}),$$

By [U, Remark 3.24], we have  $\beta = 1$ . This concludes the proof of the proposition.

## 4.C Consequences

The result of the previous section leads to the following theorem:

**Theorem 4.6** *Let  $\mathbf{s}_l \in \mathbb{Z}^n$  and let  $M_{\mathbf{s}_l} := U'_q(\widehat{\mathfrak{sl}}_e) \cdot |\underline{\emptyset}, \mathbf{s}_l\rangle \subset \mathcal{F}_{e, \mathbf{s}_l}$ . Let*

$$\{G(\underline{\mu}, \mathbf{s}_l) \mid \underline{\mu} \in \Lambda_{e; \mathbf{s}_l}^n, n \in \mathbb{N}\}$$

*be the canonical basis elements. Then, for all  $n \in \mathbb{N}$  and  $\underline{\mu} \in \Lambda_{e; \mathbf{s}_l}^n$ , we have:*

$$G(\underline{\mu}, \mathbf{s}_l) = |\underline{\mu}, \mathbf{s}_l\rangle + \sum_{\underline{\lambda} \in \Pi_l^n, a_{\mathbf{s}_l}(\underline{\lambda}) > a_{\mathbf{s}_l}(\underline{\mu})} d_{\underline{\lambda}, \underline{\mu}}(q) |\underline{\lambda}, \mathbf{s}_l\rangle,$$

where  $d_{\underline{\lambda}, \underline{\mu}}(q) \in q\mathbb{C}[q]$ .

*Proof.* By Proposition 4.4, we have for all  $\underline{\lambda} \in \Pi_l^n$ :

$$\overline{|\underline{\lambda}, \mathbf{s}_l\rangle} = |\underline{\lambda}, \mathbf{s}_l\rangle + \text{sum of } |\underline{\mu}, \mathbf{s}_l\rangle \text{ with } a_{\mathbf{s}_l}(\underline{\mu}) > a_{\mathbf{s}_l}(\underline{\lambda}).$$

Assume that  $\underline{\lambda} \in \Lambda_{e; \mathbf{s}_l}^n$  and that  $\underline{\mu}$  is one of the minimal  $l$ -partitions with respect to  $a_{\mathbf{s}_l}$  which appears in  $G(\underline{\lambda}, \mathbf{s}_l)$ . Then the characterization of the canonical basis in Theorem 3.1 immediately implies that  $\underline{\lambda} = \underline{\mu}$ . □

Now, we give the consequences on the decomposition matrices of Ariki-Koike algebras. By Ariki's theorem, we obtain the following result. Note that this generalizes and gives a new proof of the main result of [Jp] where the case  $0 \leq s_1 \leq \dots \leq s_l < e$  was considered (but the proof of [Jp] gave an explicit construction of the canonical basis elements). Note also that it provides an interpretation of the parametrization by the Kleshchev  $l$ -partitions by using  $a$ -values.

Let  $\mathbf{s}_l = (s_1, \dots, s_l) \in \mathbb{Z}^l$ . Let  $\mathcal{H}_{\mathbb{C}, n}$  be the Ariki-Koike algebra over  $\mathbb{C}$  with the following choice of parameters:

$$\begin{aligned} x_j &= \eta_e^{s_j} \text{ for } j = 1, \dots, l, \\ x &= \eta_e, \end{aligned}$$

Let  $\mathcal{H}_{\mathbb{C}(y), n}^{\mathbf{s}_l}$  be the Ariki-Koike algebra over  $\mathbb{C}(y)$  with the following parameters:

$$\begin{aligned} u_j &= y^{lm^{(j)}} \eta_l^{j-1} \text{ for } j = 1, \dots, l, \\ v &= y^l, \end{aligned}$$

where  $m^{(j)} = s_j - \frac{(j-1)e}{l} + \alpha e$  for  $j = 1, \dots, l$  and where  $\alpha$  is a positive integer such that  $m^{(j)} > 0$  for  $j = 1, \dots, l$ . Then the specialisation  $\theta : \mathbb{C}[y, y^{-1}] \rightarrow \mathbb{C}$  which sends  $y$  to  $\exp(\frac{2i\pi}{le})$  induces a decomposition map:

$$d : R_0(\mathcal{H}_{\mathbb{C}(y),n}^{\mathbf{s}_l}) \rightarrow R_0(\mathcal{H}_{\mathbb{C},n}).$$

**Theorem 4.7** *For each  $M \in \text{Irr}(\mathcal{H}_{\mathbb{C},n})$ , there exists a unique simple  $\mathcal{H}_{\mathbb{C}(y),n}^{\mathbf{s}_l}$ -module  $V_M$  such that the following two conditions hold:*

- $d_{V_M, M} = 1$ ,
- if there exists  $W \in \mathcal{H}_{\mathbb{C}(y),n}^{\mathbf{s}_l}$  such that  $d_{W, M} \neq 0$  then  $a(W) > a(V_M)$ .

Moreover the function which sends  $M$  to  $V_M$  is injective. As a consequence the associated decomposition matrix is unitriangular and the following set is in natural bijection with  $\text{Irr}(\mathcal{H}_{\mathbb{C},n})$ :

$$\mathcal{B}_{\mathbf{s}_l} = \{V_M \mid M \in \text{Irr}(\mathcal{H}_{\mathbb{C},n})\}.$$

Finally, we have:

$$\mathcal{B}_{\mathbf{s}_l} = \{S_{\mathbb{C}(y)}^\lambda \mid \lambda \in \Lambda_{e; \mathbf{s}_l}^n\}.$$

**Example 4.8** *The decomposition matrices of Ariki-Koike algebras can be computed using the algorithm described in [Ja]. We put  $l = 2$ ,  $x = \eta_4$ ,  $x_1 = \eta_4^0$  and  $x_2 = \eta_4^1$ . Then Theorem 4.7 shows how we can order the rows of the associated decomposition matrix to have a unitriangular shape with 1 along the diagonal. To do this, we must choose  $(s_1, s_2)$  such that  $s_1 \equiv 0 \pmod{4}$  and  $s_1 \equiv 1 \pmod{4}$  and order the rows with respect to the  $a$ -values. These rows are indexed by the 2-partitions of rank 4, we also give the associated  $a$ -values. We obtain the explicit bijection between  $\Lambda_{4, (s_1, s_2)}^4$  and  $\text{Irr}(\mathcal{H}_{\mathbb{C},n})$ .*

- When  $s_1 = 0$  and  $s_2 = 1$ , the simple  $\mathcal{H}_{\mathbb{C},n}$ -modules may be labeled by  $\Lambda_{4, (0,1)}^4$ .

$$\begin{aligned} \Lambda_{4, (0,1)}^4 = \{ & (\emptyset, (4)); ((1), (2, 1)); ((1, 1), (1, 1)); ((1), (3)); ((1, 1), (2)); \\ & ((2), (1, 1)); ((2), (2)); ((2, 1), (1)); ((2, 1, 1), \emptyset); ((2, 2), \emptyset); ((3), (1)); \\ & ((3, 1), \emptyset); ((4), \emptyset)\}; \end{aligned}$$







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