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# BAIRE PARADOXICAL DECOMPOSITIONS NEED AT LEAST 6 PIECES.

Friedrich WEHRUNG

**Abstract.** We show that in certain cases, paradoxical decompositions of compact metric spaces using sets (or even  $[0, 1]$ -valued functions) with the property of Baire modulo meager sets need more pieces than paradoxical decompositions with unrestricted pieces. In particular, any Baire paradoxical decomposition of the sphere  $S^2$  using isometries needs at least 6 pieces.

Let  $G$  be a group of isometries of a compact metric space  $(X, d)$ . Denote by  $\mathcal{B}(X)$  the additive semigroup of real-valued bounded positive functions on  $X$  that have the Baire property (see [2] for this and related terminology — *e.g.* first category, Baire category theorem...). If  $\varphi$  and  $\psi$  are two elements of  $\mathcal{B}(X)$ ,  $\varphi \geq \psi$  (B – a.e.) will mean that the set of all  $x$  such that  $\varphi(x) < \psi(x)$  is of first category; similarly for  $\varphi = \psi$  (B – a.e.), *etc.* There is a canonical action of  $G$  on  $\mathcal{B}(X)$ , defined by  $(g\varphi)(x) = \varphi(g^{-1}x)$ . Two elements  $\varphi$  and  $\psi$  of  $\mathcal{B}(X)$  are said to be (*continuously*)  $G$ -*equidecomposable* (see [4]) when there are  $n$  in  $\mathbb{N}$  and  $\varphi_1, \dots, \varphi_n$  in  $\mathcal{B}(X)$  and  $g_1, \dots, g_n$  in  $G$  such that

$$\varphi = \sum_{i=1}^n \varphi_i \text{ (B – a.e.) and } \psi = \sum_{i=1}^n g_i \varphi_i \text{ (B – a.e.)}.$$

When the  $\varphi_i$  are restricted to be  $\{0, 1\}$ -valued functions, this is the usual notion of Baire equidecomposability, which we shall call *discrete*, as opposed to continuous (see [4]). In general, continuous equidecomposability is weaker than discrete equidecomposability. We will say that  $\varphi$  is *Baire paradoxical* (with respect to the group  $G$ ) when  $\varphi$  and  $2\varphi$  are equidecomposable in  $\mathcal{B}(X)$ ; we will emphasize “continuous” or “discrete” if the context does not make it clear. If  $Y \subseteq X$ , we will identify  $Y$  with its characteristic function  $\mathbf{1}_Y$ .

**Lemma.** *Let  $\varphi$  in  $\mathcal{B}(X)$  and  $g$  in  $G$  such that  $g\varphi \geq \varphi$  (B – a.e.). Then  $g\varphi = \varphi$  (B – a.e.).*

**Proof.** There is a comeager subset  $Y$  of  $X$  such that the restriction  $\varphi|_Y$  is continuous (see [2]); furthermore, one can suppose without loss of generality that  $gY = Y$  and that  $(\forall x \in Y)(\varphi(g^{-1}x) \geq \varphi(x))$ . We prove that for all  $x$  in  $Y$ , we have in fact  $\varphi(g^{-1}x) = \varphi(x)$ . For, let  $(g^{-n_k}x)_{k \in \mathbb{N}}$  be a convergent subsequence of  $(g^{-n}x)_{n \in \mathbb{N}}$ ; put  $m_k = n_{k+1} - n_k$ , so that  $m_k > 0$ . Since  $g$  is an isometry, we have  $x = \lim_{k \rightarrow \infty} g^{-m_k}x$ ; since  $\varphi|_Y$  is continuous, it follows that  $\varphi(x) = \lim_{k \rightarrow \infty} \varphi(g^{-m_k}x)$ . But for all  $k$ , we have  $\varphi(g^{-m_k}x) \geq \varphi(g^{-1}x) \geq \varphi(x)$ , whence  $\varphi(g^{-1}x) = \varphi(x)$ . ■

As a possible application, we get for example the following

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**Corollary.** Let  $m, n$  in  $\mathbb{N}$  and  $g_i \in G$ ,  $\varphi_i \in \mathcal{B}(X)$  ( $1 \leq i \leq m+n$ ) such that

$$1_X = \sum_{i=1}^{m+n} \varphi_i = \sum_{i=1}^m g_i \varphi_i = \sum_{i=m+1}^{m+n} g_i \varphi_i \quad (\mathbb{B} - \text{a.e.}).$$

Then  $m \geq 3$  and  $n \geq 3$ .

Thus, if  $X$  is Baire-paradoxical, then any Baire paradoxical decomposition of  $X$  uses at least 6 pieces; this works of course as well for continuous as for discrete equidecomposability. Furthermore, this has been recently shown to be possible in a very general context by R. Dougherty and M. Foreman [1]; in particular,  $S^2$  is (discretely) Baire paradoxical using six pieces with respect to the group  $SO_3$  of rotations of  $\mathbb{R}^3$  leaving the origin fixed. Thus, six is the optimal number of pieces required to realize a Baire paradoxical decomposition of  $S^2$ .

**Proof.** Suppose *e.g.* that  $m = 2$ . Furthermore, without loss of generality,  $g_1 = 1$ . Thus  $g_2 \varphi_2 = 1_X - \varphi_1 = \sum_{i=2}^{m+n} \varphi_i \geq \varphi_2$  ( $\mathbb{B} - \text{a.e.}$ ), whence  $g_2 \varphi_2 = \varphi_2$  ( $\mathbb{B} - \text{a.e.}$ ) by the lemma. Thus  $\varphi_i = 0$  ( $\mathbb{B} - \text{a.e.}$ ) for all  $i \geq 3$ , which contradicts the fact that  $X$  is not of first category in itself. ■

Note that the same argument generalizes (with the same proof) to the case where  $X$  is a metric space which is *precompact* (*i.e.* for each  $r > 0$ , it can be covered by finitely many balls of radius  $r$ ) — so that convergent sequences are replaced by Cauchy sequences in the proof of the lemma — and which is not of first category in itself.

This result has to be put in contrast with the fact that  $S^2$  is paradoxical using 4, and not fewer than 4, pieces (which do not necessarily satisfy the Baire property), *see* [3]. Thus, it is (much!) more difficult to realize paradoxes with pieces satisfying the property of Baire (even, as in this paper, modulo meager sets) than paradoxes with unrestricted pieces.

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