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Common extensions of semigroup-valued charges.

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§0. INTRODUCTION.

Let $\mathcal{A}$ and $\mathcal{B}$ be fields of subsets of a nonempty set X and let $\mu : \mathcal{A} \to E$ and $\nu : \mathcal{B} \to E$ be finitely additive measures ("charges") taking values in a commutative semigroup E. We assume that μ and ν are *consistent* (*e.g.* $\mu = \nu$ on $\mathcal{A} \cap \mathcal{B}$) and ask whether they have a common extension to a charge $\rho : \mathcal{A} \vee \mathcal{B} \to E$.

Now, we shall see (proposition 2.3) that the most natural consistency condition which we can formulate involves a *partial preordering* (which may not be an ordering) on E. Furthermore, the formulation of our results will be much clearer when expressed with the preordering than without (see for example theorem 3.2); note that this situation is reminiscent of [**12**]. More generally, the consideration of the preordering seems fundamental in the study of homomorphism extension properties of commutative semigroups when these are rather viewed as positive cones (of, say, ordered groups), *see* [**14**] to [**17**]. For all these reasons, we shall consider charges with values in what we will call here a *pp-semigroup* (definition 1.1) rather than just a semigroup. Note that in [**14**] to [**17**], where completeness with respect to the ordering plays an important role, these structures are called P.O.M.'s.

We say that a *pp*-semigroup E has the 2-*charge extension property* (from now on 2-CHEP) when any two consistent E-valued charges μ, ν have a common extension ρ; when one restricts oneself to *finite* Boolean algebras $\mathcal{A}$ and $\mathcal{B}$, then we will say that E has the *grid property*. Finally, by considering only one algebra, one gets the following definition of the 1-*CHEP*: a *pp*-semigroup E has the 1-CHEP when for every Boolean subalgebra $\mathcal{A}$ of a Boolean algebra $\mathcal{B}$, every E-valued charge on $\mathcal{A}$ extends to a E-valued charge on

* This work has been completed while the first author was a Fulbright Stipendiat at the University of Essen (Germany).

$\mathcal{B}$. Note that *a priori*, there is no implication between the 2-CHEP and the 1-CHEP. Our reference [**8**] contains a lot of relevant information about many types of charge extension properties.

All these problems may be reformulated in terms of a system of linear equations and inequalities in E. Such systems and their solutions are studied in chapter 1: proposition 1.7 provides a general existence theorem for them.

In chapter 2 (theorem 2.4), it is shown that the 2-CHEP implies the 1-CHEP.

In theorem 3.2, the class of *pp*-semigroups E with the grid property is fully characterized in terms of a small set of very simple axioms. In particular, the theory of *pp*-semigroups satisfying the grid property is shown to be finitely axiomatizable.

Finally, we provide some converses of results in [**12**] and in this paper, by showing (theorem 4.8) that if G is a directed partially ordered abelian group, then the following holds:
(i) If G_+ satisfies the 2-CHEP, then G is a complete ℓ-group.
(ii) If G is a ℓ-group, then G_+ has the 1-CHEP if and only if G is a complete ℓ-group.

We denote the set of all natural numbers by ω or $\mathbb{N}$ according to the case that we consider it as the least limit ordinal or a *pp*-semigroup.

§1. PROPER SYSTEMS, DIAGONAL PROPERTIES.

We first recall the context in which most of our work takes place. It has been used in several places, *see* [**12**] and [**14**] to [**17**].

1.1. Definition. A *positively preordered semigroup*, from now on a *pp-semigroup*, is a structure $(A, +, 0, \leq)$ where $(A, +, 0)$ is a commutative semigroup with zero and $\leq$ is a preordering of A satisfying both axioms
(i) $(\forall x, y, z)(x \leq y \Rightarrow x + z \leq y + z)$,
(ii) $(\forall x)(0 \leq x)$.

We recall now some definitions used in [**14**] to [**17**]. A *pp*-semigroup A is *minimal* when it satisfies $(\forall x, y)\big(x \leq y \Leftrightarrow (\exists z)(x + z = y)\big)$, *antisymmetric* when its underlying preordering $\leq_A$ is antisymmetric, *cancellative* when it satisfies both statements $(\forall x, y, z)(x + z \ \rho \ y + z \Rightarrow x \ \rho \ y)$ whenever ρ is either $\leq$ or $=$. Throughout this work, we shall always use the abbreviations $x \equiv y \Leftrightarrow (x \leq y$ **and** $y \leq x)$, $x \ll y \Leftrightarrow x + y = y$, or $\equiv_A$ and $\ll_A$ if the context does not make the ground *pp*-semigroup A clear. Furthermore, if X and Y are two subsets of a given preordered set, then we shall write $X \leq Y$ instead

of $(\forall(x,y) \in X \times Y)(x \leq y)$; if X or Y is a finite set, we shall omit the braces in our notations, writing *e.g.* $a_1, \ldots, a_m \leq b_1, \ldots, b_n$ instead of $\{a_1, \ldots, a_m\} \leq \{b_1, \ldots, b_n\}$. The *finite refinement property* (*see e.g.* [**6**], [**13**]) is the statement

$$(\forall a_0, a_1, b_0, b_1)\big(a_0 + a_1 = b_0 + b_1 \Rightarrow (\exists_{i,j<2} c_{ij})(\forall i < 2)(a_i = c_{i0} + c_{i1} \textbf{ and } b_i = c_{0i} + c_{1i})\big).$$

Now, we shall define *strong refinement pp-semigroups* in a slightly different way as in [**14**] or [**16**] (where they are called strong refinement P.O.M.'s) , but it will turn out to be equivalent.

1.2. Definition. A *strong refinement pp-semigroup* is a minimal, antisymmetric *pp*-semigroup satisfying both following axioms:

(i) *Pseudo-cancellation property:*

$$(\forall a, b, c)\big(a + c \leq b + c \Rightarrow (\exists d \ll c)(a \leq b + d)\big).$$

(ii) $(\forall a, b, x, y)\big(b \leq a + x, a + y \Rightarrow (\exists z \leq x, y)(b \leq a + z)\big)$.

One of the useful features of strong refinement *pp*-semigroups is that they satisfy the following

1.3. Lemma. *Let A be a strong refinement pp-semigroup. Then A satisfies the following* finite interpolation property*: for all finite subsets X and Y of A such that $X \leq Y$, there is z in A such that $X \leq z \leq Y$.*

Proof. For $|X| = |Y| = 2$, it is an easy consequence of definition 1.2. The general case is derived by an easy induction. ∎

Now, we shall connect strong refinement *pp*-semigroups and *linear systems.* For all m, n in $\mathbb{N}$, define $m \dot{-} n = \max(m - n, 0)$. Let θ be an atomic formula of the language $(+, \leq, =)$ with parameters in some *pp*-semigroup A, with variables $x_1, \ldots, x_n$; thus θ can be written in the form

$$\sum_{i<k} m_i x_i + a \ \rho \ \sum_{i<k} n_i x_i + b,$$

where ρ is either $\leq$ or $=$. Define the *reduct* $\bar{\theta}$ of θ to be the following formula:

$$\sum_{i<k} (m_i \dot{-} n_i) x_i + a \ \rho \ \sum_{i<k} (n_i \dot{-} m_i) x_i + b.$$

It is obvious that every *pp*-semigroup satisfying $\bar{\theta}$ also satisfies θ, and that the converse holds in *cancellative pp*-semigroups. We shall say that θ is *reduced* when $\theta = \bar{\theta}$. A *linear system* is a set of atomic formulas of $(+, \leq, =)$. The theory of linear systems which we shall present in this chapter bears very close similarities with the theory presented in [**17**] (in particular the notion of *resolvent*). If S is a linear system, the *reduct* of S is $\bar{S} = \{\bar{\theta} : \ \theta \in S\}$; S is *reduced* when $S = \bar{S}$. Throughout this work, we shall consider only *finite* linear systems. If x is an unknown of a linear system S, then we shall say that S is *proper in* x when it can be written under the following form:

$$(1.1) \qquad \begin{cases} a_i + x \leq b_i & (\text{all } i \in I) \\ b_j \leq a_j + x & (\text{all } j \in J) \\ a_k + x = b_k & (\text{all } k \in K) \\ R & [\text{some system where } x \text{ does not appear}], \end{cases}$$

where x does not appear in the a_l, b_l's. If m is in ω, then (1.1) is *m-proper* in x when $|K| \leq m$. The *resolvent* in x of (1.1) is the following linear system:

$$(1.2) \qquad \begin{cases} a_i \leq b_i & (\text{all } i \in I \cup K) \\ a_i + b_k \leq a_k + b_i & (\text{all } (i,k) \in I \times K) \\ a_k + b_j \leq a_j + b_k & (\text{all } (j,k) \in J \times K) \\ a_k + b_l = a_l + b_k & (\text{all } (k,l) \in K \times K) \\ a_i + b_j \leq a_j + b_i & (\text{all } (i,j) \in I \times J) \\ R. \end{cases}$$

Now, let S be a linear system with parameters in some *pp*-semigroup A, let $\sigma = \langle x_1, \ldots, x_n \rangle$ be an enumeration of all unknowns of S; this we shall sometimes indicate by writing $S(x_1, \ldots, x_n)$ instead of just S. Let k be in ω. We construct inductively $S_m^{(k)}(x_m, x_{m+1}, \ldots, x_n)$ by the following rule: $S_1^{(k)}$ is the reduct of S; if $S_m^{(k)}$ is defined and if it is k-proper in x_m, then $S_{m+1}^{(k)}$ is the reduct of the resolvent of $S_m^{(k)}$ in x_m. If $S_n^{(k)}$ is defined, then we will call it the *reduced resolvent* of S along σ and we will say that S is *k-proper*.

1.4. Lemma. *Let E be a pp-semigroup, let S be a proper system with parameters from E and one unknown. If S admits a solution in E, then E satisfies the resolvent of S; furthermore, if E is a strong refinement pp-semigroup, then the converse is true.*

Proof. The first assertion is obvious. Conversely, assume that E satisfies the resolvent of S and that E is a strong refinement *pp*-semigroup. Since E is antisymmetric, every system of the form (1.1) is equivalent to a system of the form (1.1) with $K = \emptyset$ and $R = \emptyset$; furthermore, it is trivial to see that both corresponding resolvents are equivalent. So we

may assume without loss of generality that S is (1.1) with $K = \emptyset$ and $R = \emptyset$. For all i in I, let c_i in E such that $a_i + c_i = b_i$; thus for all (i, j) in $I \times J$, $a_i + b_j \leq a_i + a_j + c_i$, whence $b_j \leq a_j + c_i + d_{ij}$ for some $d_{ij} \ll a_i$; thus $b_j \leq a_j + c_i + d_i$ where $d_i = \sum_{j \in J} d_{ij}$; since $d_i \ll a_i$, changing c_i into $c_i + d_i$ does not affect the definition of c_i, so that we may assume without loss of generality that $b_j \leq a_j + c_i$. Thus, by definition of a strong refinement *pp*-semigroup, for all j in J, there is e_j such that $b_j \leq a_j + e_j$ and $e_j \leq c_i$ for all i in I. By lemma 1.3, there is x such that $(\forall i \in I)(x \leq c_i)$ and $(\forall j \in J)(e_j \leq x)$. For all i in I, $a_i + x \leq a_i + c_i = b_i$ and for all j in J, $b_j \leq a_j + e_j \leq a_j + x$, whence x satisfies the required conditions. ■

Now, say that for every k in ω, the *k-diagonal property* is the following property:

"Let $a_i,\ b_i$ $(i < k)$ such that $a_i \equiv b_i$ (all $i < k$) and $a_i + b_j = a_j + b_i$ (all $i, j < k$). Then there exists $x \equiv 0$ such that $a_i + x = b_i$ (all $i < k$)."

Say that a *pp*-semigroup E is a *k-strong refinement pp-semigroup* when $E/\equiv_E$ is a strong refinement *pp*-semigroup and E satisfies the k-diagonal property.

1.5. Lemma. *Let m in ω, let E be a pp-semigroup, let S be a m-proper linear system with parameters from E. If S admits a solution in E, then E satisfies the resolvent of S; furthermore, if E is a m- strong refinement pp-semigroup, then the converse is true.*

Proof. The first assertion is obvious. Conversely, assume that E satisfies the resolvent of S and that E is a m-strong refinement *pp*-semigroup. Write S in the form (1.1), where $|K| \leq m$; without loss of generality, $R = \emptyset$. Let S' be the result of replacing $=$ by $\equiv$ everywhere in S. Then E satisfies the resolvent of S', thus, applying lemma 1.4 in $E/\equiv$, S' admits a solution c in E. Then, since E satisfies the resolvent of S and by the m-diagonal property, there exists $d \equiv 0$ in E such that $a_k + c + d = b_k$ for all k in K. Hence, $c + d$ is a solution of S. ■

Note that it is easy to prove that in fact, lemma 1.5 *characterizes* m-strong refinement *pp*-semigroups.

Lemma 1.5 yields us one of our main technical results:

1.6. Theorem. *Let k in ω, let A be a sub-pp-semigroup of a cancellative pp-semigroup B, let S be a k-proper linear system with parameters from A; let f be an preordered semigroup-homomorphism from A to a k-strong refinement pp-semigroup E. If S admits a solution in B, then $f(S)$ admits a solution in E.*

Proof. By induction on the number of unknowns in S. With no unknown it is trivial; so suppose that S is k-proper along $\langle x_1, \ldots, x_n \rangle$ where $n \geq 1$. By hypothesis, S admits a solution in B, say $\langle b_1, \ldots, b_n \rangle$. Thus $\langle b_2, \ldots, b_n \rangle$ is a solution in B of the resolvent in x_1 of $S(x_1, x_2, \ldots, x_n)$, thus also of the reduct $S'(x_2, \ldots, x_n)$ of the latter since B is cancellative. But S' is by definition k-proper along $\langle x_2, \ldots, x_n \rangle$, thus, by induction hypothesis, $f(S')$ admits a solution in E, say $\langle c_2, \ldots, c_n \rangle$. Since $f(S')(x_2, \ldots, x_n)$ is the reduct of the resolvent (in x_1) $S''(x_2, \ldots, x_n)$ of $f(S)(x_1, \ldots, x_n)$, E also satisfies $S''(c_2, \ldots, c_n)$. But $f(S)(x_1, c_2, \ldots, c_n)$ is k-proper (in x_1), thus, by lemma 1.5, it admits a solution in E; the conclusion follows. ■

The particular cases of proper systems we shall use are contained in the following

1.7. Proposition.

(i) *Any linear system of the form*

$$(1.3) \qquad \begin{cases} x + z = a; \quad y + z = b; \quad a' + x = b' + y; \\ a_{i_+} + x \leq b_{i_+} \text{ (all } i_+ \in I_+); \quad b_{i_-} \leq a_{i_-} + x \text{ (all } i_- \in I_-); \\ a_{j_+} + y \leq b_{j_+} \text{ (all } j_+ \in J_+); \quad b_{j_-} \leq a_{j_-} + y \text{ (all } j_- \in J_-); \\ a_k + x \leq b_k + y \text{ (all } k \in K); \quad b_l + y \leq a_l + x \text{ (all } l \in L); \end{cases}$$

is 2-proper.

(ii) *Let m in ω. Then any linear system of the form*

$$(1.4) \qquad \begin{cases} a_{i_+} + x \leq b_{i_+} \text{ (all } i_+ \in I_+); \quad b_{i_-} \leq a_{i_-} + x \text{ (all } i_- \in I_-); \\ a_{j_+} + y \leq b_{j_+} \text{ (all } j_+ \in J_+); \quad b_{j_-} \leq a_{j_-} + y \text{ (all } j_- \in J_-); \\ a_k + x \leq b_k + y \text{ (all } k \in K); \quad b_l + y \leq a_l + x \text{ (all } l \in L); \\ a_u + x = b_u + y \text{ (all } u < m); \end{cases}$$

is m-proper.

Proof. It is straightforward to verify that the reduct of the resolvent of (1.3) in z is formed with a system of the form (1.4) together with a system without unknowns, and that the reduct of the resolvent of (1.4) in y is a 0-proper system in x. ■

§2. CHARGE EXTENSION PROPERTIES.

Let us first introduce some notation and terminology. For all x, y in some Boolean algebra $\mathcal{B}$, write $x \setminus y = x \wedge \neg y$. A *Boolean subring* of $\mathcal{B}$ is a subset of $\mathcal{B}$ which contains $\{0\}$ and which is closed under the binary operations $\wedge$, $\vee$ and $\setminus$. A *Boolean ring* is a

Boolean subring of some Boolean algebra; it is easy to prove that this notion is finitely axiomatizable in the language $(0, \wedge, \vee, \setminus)$. In every Boolean ring $\mathcal{R}$, we shall write $z = x+y$ (resp. $z = \sum_{i<n} x_i$) when z is the disjoint join of x and y (resp. of the x_i, $i < n$).

2.1. Definition. Let $\mathcal{R}$ be a Boolean ring, let E be a *pp*-semigroup. Then a *E-valued charge* on $\mathcal{R}$ is a map $\mu : \mathcal{R} \to E$ satisfying $\mu(0_{\mathcal{R}}) = 0_E$ and $\mu(z) = \mu(x) + \mu(y)$ for all x, y, z in $\mathcal{R}$ such that $z = x + y$.

It is obvious that then, μ is an increasing map.

If $\mathcal{R}$ is a Boolean ring, we introduce the *pp*-semigroup $\mathbb{N}[\mathcal{R}]$ of all finite linear combinations with coefficients in $\mathbb{N}$ of elements of $\mathcal{R}$, the zero of $\mathcal{R}$ being identified with zero and the sum of disjoint elements of $\mathcal{R}$ being the same in $\mathcal{R}$ and in $\mathbb{N}[\mathcal{R}]$. If E is a *pp*-semigroup and μ is a E-valued charge on $\mathcal{R}$, then μ extends to a unique preordered semigroup-homomorphism (the 'integral') from $\mathbb{N}[\mathcal{R}]$ to E, which we still denote by μ. If $\mathcal{R}$ is represented as a subring of some $\mathcal{P}(X)$, then $\mathbb{N}[\mathcal{R}]$ is canonically isomorphic to the *pp*-semigroup of functions $f : X \to \mathbb{N}$ with finite range such that $f^{-1}\{n\} \in \mathcal{R}$ for all n in $\mathbb{N} \setminus \{0\}$, and the integral takes a simple form (the so-called *Abel transform*) with a straightforward proof:

2.2. Proposition. *Let X be a set, let $\mathcal{R}$ be a Boolean subring of $\mathcal{P}(X)$, let E be a pp-semigroup, let μ be a E-valued charge on $\mathcal{R}$. Then for all f in $\mathbb{N}[\mathcal{R}]$, we have*

$$\mu(f) = \sum_{n=1}^{+\infty} \mu\big(\{x \in X : \ f(x) \geq n\}\big).$$

■

Of course, with a proper formulation, proposition 2.2 does not need any representation of $\mathcal{R}$ as a ring of sets.

Consider now the previous definition of the 1-CHEP (*see* the introduction). This definition can of course easily be restricted to particular classes of Boolean algebras (*e.g.* finite algebras), or on the contrary widened (*e.g.* to Boolean rings). Sufficient conditions implying the 1-CHEP (for Boolean algebras) have been given in [**12**]. In this paper, we will later on give a few necessary conditions for the 1-CHEP.

Let E be a *pp*-semigroup, let $\mathcal{A}$ and $\mathcal{B}$ be two Boolean subrings of a Boolean ring $\mathcal{C}$, let $\mu : \mathcal{A} \to E$ and $\nu : \mathcal{B} \to E$ be charges. Then μ and ν will be said to be *consistent* when

for all X in $\mathcal{A}$ and Y in $\mathcal{B}$, $X \leq Y$ (resp. $X \geq Y$, $X = Y$) implies $\mu(X) \leq \nu(Y)$ (resp. $\mu(X) \geq \nu(Y)$, $\mu(X) = \nu(Y)$).

2.3. Proposition. *In the context above, μ and ν are consistent if and only if there is an preordered semigroup-homomorphism from $\mathbb{N}[\mathcal{A}] + \mathbb{N}[\mathcal{B}]$ to E extending both μ and ν.*

($\mathbb{N}[\mathcal{A}] + \mathbb{N}[\mathcal{B}]$ is of course the internal sum in $\mathbb{N}[\mathcal{C}]$ of $\mathbb{N}[\mathcal{A}]$ and $\mathbb{N}[\mathcal{B}]$).

Proof. The condition given here is obviously sufficient. Conversely, suppose that μ and ν are consistent.

Claim. *Let $f \in \mathbb{N}[\mathcal{A}]$ and $g \in \mathbb{N}[\mathcal{B}]$. Then $f = g \Rightarrow \mu(f) = \nu(g)$ and $f \leq g \Rightarrow \mu(f) \leq \nu(g)$ and $f \geq g \Rightarrow \mu(f) \geq \nu(g)$.*

Proof of claim. Immediate from proposition 2.2. ■ Claim .

Now, let ρ be either $=$ or $\leq$, let f, f' in $\mathbb{N}[\mathcal{A}]$, g, g' in $\mathbb{N}[\mathcal{B}]$ such that $f + g \ \rho \ f' + g'$; we shall prove that $\mu(f) + \nu(g) \ \rho \ \mu(f') + \nu(g')$. Substracting $f \wedge f'$ from f and f' and $g \wedge g'$ from g and g', we see that without loss of generality, $f \wedge f' = g \wedge g' = 0$. Suppose first that ρ is $=$. Then $f = g'$ and $g = f'$, thus the conclusion follows by the claim. If ρ is $\leq$, then $f \leq g'$ and $g \leq f'$, whence $\mu(f) \leq \nu(g')$ and $\nu(g) \leq \mu(f')$ by the claim, the conclusion follows again. ■

Consider now the previous definition of the 2-CHEP (*see* the introduction). Again, it can be either restricted or widened (to arbitrary classes of Boolean rings). By proposition 2.3, every injective *pp*-semigroup has the (full) 2-CHEP (*see* [**14**]); in particular, $\overline{\mathbb{P}} = ([0, +\infty], +, 0, \leq)$ is, by Tarski's theorem, injective, thus it has the 2-CHEP. Moreover, the terminology used here suggests that the 2-CHEP is stronger than the 1-CHEP, which is far from being obvious in the definition above. Strangely (?), this is always the case (in a sort of 'uniform way'), as shown by the theorem below:

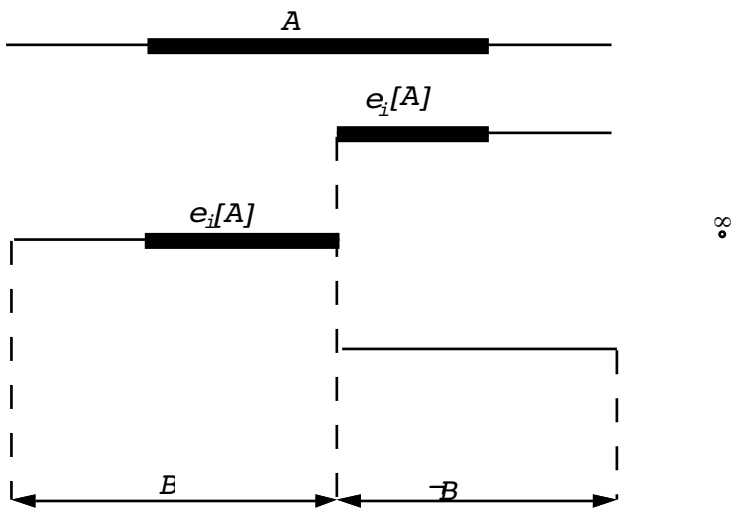
2.4. Theorem. *The 2-CHEP implies the 1-CHEP.*

Note that the analogue of this theorem in the case of abelian groups is immediate in view of the (non-trivial!) result that every abelian group (with the coarse preordering) has the 1-CHEP [**3**].

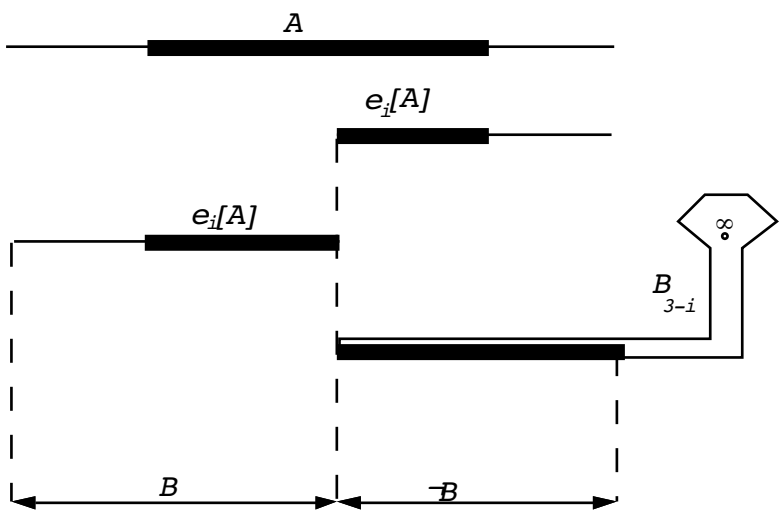
Proof. Let E be a *pp*-semigroup satisfying the 2-CHEP. To prove that E satisfies the 1-CHEP, it suffices, by Zorn's lemma, to prove that for every Boolean subalgebra $\mathcal{A}$ of a Boolean algebra $\mathcal{B}$ such that $\mathcal{B}$ is generated above $\mathcal{A}$ by one element B and every E-valued

charge μ on $\mathcal{A}$, μ extends to a E-valued charge on $\mathcal{B}$. For convenience sake, represent $\mathcal{B}$ (*via* Stone's representation theorem) as an algebra of subsets of some set X. Let ∞ be some object not in $X \times \{0,1,2\}$, let $\Omega = (B \times \{0\}) \cup (\neg B \times \{1,2\}) \cup \{\infty\}$. For $i = 1$ or 2, let $e_i : X \to \Omega,\ x \mapsto (x,0)\ (x \in B),\ x \mapsto (x,i)\ (x \in \neg B)$. Thus e_i is a one-to-one map. Put $B_i = \big((\neg B) \times \{i\}\big) \cup \{\infty\}$, and let $\mathcal{A}_i$ be the Boolean subalgebra of $\mathcal{P}(\Omega)$ generated by $\{e_i[A] :\ A \in \mathcal{A}\} \cup \{B_{3-i}\}$. Thus an element of $\mathcal{A}_i$ can have two forms:

— Either $e_i[A]$, $A \in \mathcal{A}$:



— Or $e_i[A] + B_{3-i}$, $A \in \mathcal{A}$:



Finally, put $c = \mu(X)$. Define E-valued charges $\mu_i : \mathcal{A}_i \to E$ by

$$\begin{cases} \mu_i\big(e_i[A]\big) = \mu(A) & (\text{all } A \in \mathcal{A}), \\ \mu_i(B_{3-i}) = c. \end{cases}$$

Claim. *μ_1 and μ_2 are consistent.*

Proof of claim. First, let Z in $\mathcal{A}_1 \cap \mathcal{A}_2$. If $Z \subseteq B \times \{0\}$, then $Z = e_1[A] = e_2[A]$ for some $A \in \mathcal{A}$, $A \subseteq B$, whence $\mu_1(Z) = \mu_2(Z) = \mu(A)$. Otherwise, it is not difficult to see that $Z = e_1[A] + B_2 = e_2[A] + B_1$ for some $A \in \mathcal{A}$ containing $\neg B$; thus $\mu_1(Z) = \mu(A) + c = \mu_2(Z)$. So we have proved that $\mu_1|_{\mathcal{A}_1 \cap \mathcal{A}_2} = \mu_2|_{\mathcal{A}_1 \cap \mathcal{A}_2}$. Now, let $Z_1 \in \mathcal{A}_1$, $Z_2 \in \mathcal{A}_2$ such that $Z_1 \subseteq Z_2$; we shall prove that $\mu_1(Z_1) \leq \mu_2(Z_2)$. We argue by cases:

Case 1. $Z_1 \subseteq B \times \{0\}$, $\infty \notin Z_1$.

Then $Z_1 = A_1 \times \{0\}$, $Z_2 \supseteq A_2 \times \{0\}$ where A_1 and A_2 are in $\mathcal{A}$ and $A_1 \subseteq A_2$. The conclusion follows.

Case 2. $Z_1 \nsubseteq B \times \{0\}$, $\infty \notin Z_1$.

Thus $Z_1 = e_1[A]$ for some $A \in \mathcal{A}$, and $B_1 \subseteq Z_2$, thus $\mu_1(Z_1) = \mu(A) \leq c \leq \mu_2(Z_2)$.

Case 3. $\infty \in Z_1$.

Then $Z_i = e_i[A_i] + B_{3-i}$ for $i = 1$ or 2, where $A_i \in \mathcal{A}$ and $A_1 \subseteq A_2$. Thus $\mu_1(Z_1) = \mu(A_1) + c \leq \mu(A_2) + c = \mu_2(Z_2)$.

In all three cases, $\mu_1(Z_1) \leq \mu_2(Z_2)$.

Similarly, $Z_1 \in \mathcal{A}_1$, $Z_2 \in \mathcal{A}_2$ and $Z_1 \supseteq Z_2$ implies $\mu_1(Z_1) \geq \mu_2(Z_2)$. So, by definition, the claim holds. ■ Claim .

Now, since E has the 2-CHEP, there exists a charge $\rho : \mathcal{A}_1 \vee \mathcal{A}_2 \to E$ extending both μ_1 and μ_2. For all A in $\mathcal{A}$, $(A \cap B) \times \{0\} = e_1[A] \cap e_2[A] \in \mathcal{A}_1 \vee \mathcal{A}_2$, and $(A \setminus B) \times \{1\} = e_1[A] \setminus e_2[A] \in \mathcal{A}_1 \vee \mathcal{A}_2$; therefore, for all C in $\mathcal{A}_1 \vee \mathcal{A}_2$, $(C \cap B) \times \{0\}$ and $(C \setminus B) \times \{1\}$ belong to $\mathcal{A}_1 \vee \mathcal{A}_2$. This allows us to define a E-valued charge ν on $\mathcal{A}_1 \vee \mathcal{A}_2$ by

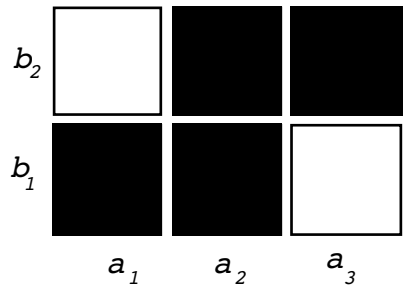
$$\nu(C) = \rho\Big(\big((C \cap B) \times \{0\}\big) \cup \big((C \setminus B) \times \{1\}\big)\Big).$$

Since ρ is a charge, ν is a charge. Furthermore, for all A in $\mathcal{A}$, we have $\nu(A) = \rho\big(e_1[A]\big) = \mu_1\big(e_1[A]\big) = \mu(A)$. So ν satisfies the required conditions. ■

§3. THE GRID PROPERTY.

In this chapter, we shall completely solve the extension problem of two consistent charges on *finite* algebras. We shall first adopt a convenient notation and terminology. Say that a *grid* is an ordered pair $(\mathcal{A}, \mathcal{B})$ where $\mathcal{A}$ and $\mathcal{B}$ are Boolean subalgebras of some $\mathcal{P}(S)$ where S is a finite set. A *charged grid* [on $(\mathcal{A}, \mathcal{B})$] is an ordered pair (μ, ν) where μ (resp. ν) is a E-valued charge (on some *pp*-semigroup E) on $\mathcal{A}$ (resp. on $\mathcal{B}$); it is said

to be *consistent* when μ and ν are consistent. Grids and charged grids can always be represented the following way: let X (resp. Y) be the set of atoms of $\mathcal{A}$ (resp. $\mathcal{B}$), let $I = \{(A, B) \in X \times Y : \ A \cap B \neq \emptyset\}$; then $\mathcal{A}$ (resp. $\mathcal{B}$) is isomorphic to the Boolean algebra of all sections of I by some vertical (resp. horizontal) band. Throughout this chapter, all our examples of grids will be with $X = \{1, \ldots, m\}$ and $Y = \{1, \ldots, n\}$ for some m, n in $\omega \setminus \{0\}$. So to define a charged grid, we shall draw a $X \times Y$-rectangle, the subset I of $X \times Y$ (any subset with full first and second projection can do), and put the values of μ (resp. ν) at the atoms of $\mathcal{A}$ (resp. of $\mathcal{B}$) on the corresponding axis. For example, the following picture



represents the case where $X = \{1, 2, 3\}$, $Y = \{1, 2\}$, $I = \{(1, 1), (2, 1), (2, 2), (3, 2)\}$, $\mu\big(\{i\}\big) = a_i$, $\nu\big(\{j\}\big) = b_j$ (all $i \in X$, $j \in Y$).

The following definition is exactly the restriction of the definition of the 2-CHEP to *finite* Boolean algebras.

3.1. Definition. A *pp*-semigroup E has the *grid property* when it has the 2-CHEP for *finite* Boolean algebras.

Our main theorem is the following (k-strong refinement *pp*-semigroups have been defined just before the statement of lemma 1.5):

3.2. Theorem. *A pp-semigroup has the grid property if and only if it is a 2-strong refinement pp-semigroup.*

Thus the grid property is *finitely axiomatizable*; a possible list of axioms equivalent to the grid property is the following:

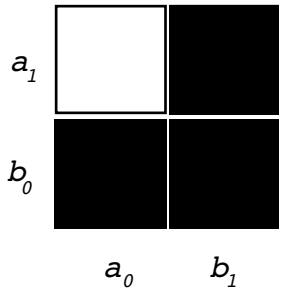
$$(\forall a, b)\big(a \leq b \Rightarrow (\exists x)(a + x = b)\big);$$
$$(\forall a, b, c)\big(a + c \leq b + c \Rightarrow (\exists x \ll c)(a \leq b + x)\big);$$
$$(\forall a, b, c, d)\big(b \leq a + c, a + d \Rightarrow (\exists x \leq c, d)(b \leq a + x)\big);$$
$$(\forall a, a', b, b')\big((a \equiv b \textbf{ and } a' \equiv b' \textbf{ and } a + b' = a' + b) \Rightarrow (\exists x \equiv 0)(a + x = b \textbf{ and } a' + x = b'\big).$$

(We recall here that $a \ll b$ stands for $a+b=b$ and $a \equiv b$ stands for $a \leq b$ and $b \leq a$).

Proof. Assume first that E satisfies the grid property. We prove that E is a 2-strong refinement *pp*-semigroup, in a sequence of claims.

Claim 1. *Let a_i, b_i ($i=0$ or 1) in E such that $a_i \leq b_i$ ($i=0$ or 1) and $a_0+b_1=a_1+b_0$. Then there is x in E such that $a_i+x=b_i$ ($i=0$ or 1).*

Proof of claim. Consider the following E-valued charged grid:

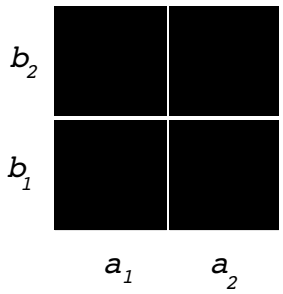


Our hypotheses imply consistency of this grid. Let ρ be an extension charge for this grid, put $x=\rho\big(\{(2,1)\}\big)$. Then x satisfies the required conditions. ■ Claim 1.

Taking $a_0=a_1$ and $b_0=b_1$, we immediately get that E *is minimal.*

Claim 2. *E satisfies the finite refinement property.*

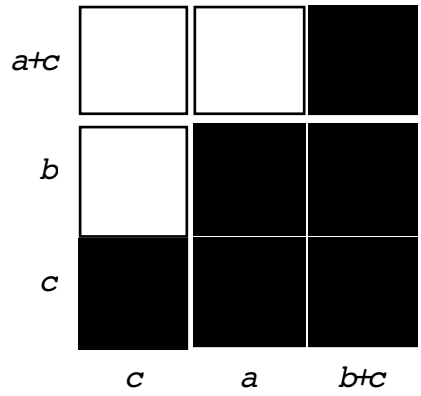
Proof of claim. Let a_1, a_2, b_1, b_2 in E such that $a_1+a_2=b_1+b_2$. Consider the following E-valued charged grid:



Its consistency results immediately from $a_1+a_2=b_1+b_2$. If ρ is an extension charge for this grid, put $c_{ij}=\rho\big(\{(i,j)\}\big)$. Then $a_i=c_{i1}+c_{i2}$ and $b_i=c_{1i}+c_{2i}$ for all i in $\{1,2\}$. ■ Claim 2.

Claim 3. *E satisfies the pseudo-cancellation property.*

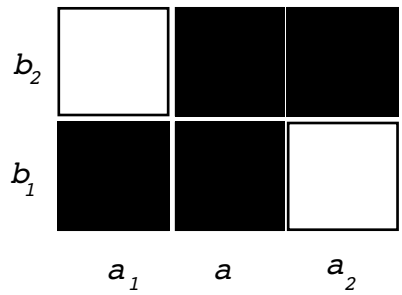
Proof of claim. Let a, b, c in E such that $a+c \leq b+c$. Consider the following E-charged grid:



Our hypothesis implies consistency of this grid. If ρ is an extension charge for this grid, put $x = \rho\big(\{(2,1),(3,1)\}\big)$. Then $x \ll c$ and $a \leq b + x$. ■ Claim 3.

Claim 4. *For all a, a_i, b_i ($i = 1$ or 2) in E such that $a_i \leq b_i \leq a_i + a$ ($i = 1$ or 2) and $a_1 + a_2 + a = b_1 + b_2$, there are x_1, x_2 in E such that $a_i + x_i = b_i$ ($i = 1$ or 2) and $x_1 + x_2 = a$.*

Proof of claim. Consider the following E-charged grid:



Our hypothesis implies consistency of this grid. If ρ is an extension charge for this grid, put $x_i = \rho\big(\{(2,i)\}\big)$ ($i = 1$ or 2). It is immediate that x_1 and x_2 satisfy the required conditions. ■ Claim 4.

Claim 5. *Let a, b, c in E such that $a \leq b \leq a + c$. Then there is $x \leq c$ in E such that $b = a + x$.*

Proof of claim. An immediate application of claim 4, where a_1 becomes a, a_2 becomes b, a becomes c, b_1 becomes b and b_2 becomes $a + c$. ■ Claim 5.

Claim 6. *E satisfies the finite interpolation property.*

Proof of claim. As in [**13**, 2.28] (only minimality, the finite refinement property and the result of claim 5 are used). ■ Claim 6.

Claim 7. *$E/\equiv$ is a strong refinement pp-semigroup.*

Proof of claim. We have already seen that E is minimal and satisfies the pseudo-cancellation property; thus, it suffices to show that E satisfies part (ii) of definition of a strong refinement *pp*-semigroup (definition 1.2). So let a, b, c, d in E such that $b \leq a+c, a+d$. By claim 6, there is h in E such that $a, b \leq a+h \leq a+c, a+d$. Using minimality of E, the pseudo-cancellation property and the finite refinement property, we see that there are v, e such that $h = v+e$ and $v \leq c$ and $e+a \leq a$. Thus $a+v \leq a+h \leq a+d$, whence there are $x \leq d$ and w such that $v = x+w$ and $w+a \leq a$. Hence, $x \leq v \leq c$ and $b \leq a+h = a+v+e \leq a+v = a+x+w \leq a+x$, so that x satisfies the required properties. ■ Claim 7.

Claim 8. *E satisfies the 2-diagonal property.*

Proof of claim. Let a_i, b_i in E ($i = 1$ or 2) such that $a_i \equiv b_i$ and $a_1 + b_2 = a_2 + b_1$. Using claim 4 after having replaced a by 0, a_2 by b_1, b_2 by a_1, we obtain x and y in E such that $x + y = 0$ and $a_1 + x = b_1$ and $a_2 = b_2 + y$, whence $x \equiv 0$ and $b_2 = a_2 + x$, so that x satisfies the required conditions. ■ Claim 8.

We conclude one direction by claims 7 and 8.

Conversely, suppose that E is a 2-strong refinement *pp*-semigroup. We shall prove that E satisfies the grid property. If $\mathcal{G} = (\mu, \nu)$ is a E-charged grid on a grid $(\mathcal{A}, \mathcal{B})$, define the *weight* of $\mathcal{G}$ to be the integer $m+n+s$, where m (resp. n, s) is the number of atoms of $\mathcal{A}$ (resp. $\mathcal{B}$, $\mathcal{A} \vee \mathcal{B}$), so that $s \leq mn$. Furthermore, suppose that $\mathcal{G}$ is consistent; we have to find a E-valued charge on $\mathcal{A} \vee \mathcal{B}$ extending both μ and ν. Without loss of generality, the atoms of $\mathcal{A} \vee \mathcal{B}$ are points; let Ω be their set. Furthermore, it is sufficient to solve the extension problem on each atom of $\mathcal{A} \cap \mathcal{B}$, so that we may restrict ourselves to the case where $\mathcal{A} \cap \mathcal{B} = \{\emptyset, \Omega\}$. Then, we define the *interaction matrix* of $\mathcal{A}$ and $\mathcal{B}$ by

$$M = \big(1_{A_i}(x)\big)_{(i,x) \in [1,\ m+n] \times \Omega} \quad ,$$

where $(A_i)_{i \in [1,\ m]}$ (resp. $(A_i)_{i \in [m+1,\ m+n]}$) is the list of atoms of $\mathcal{A}$ (resp. $\mathcal{B}$). To avoid trivialities, $m \geq 1$ and $n \geq 1$.

Claim 9. *The image space of [the column vectors of] M is the orthocomplement of the vector $\vec{a} = (a_i)_{i \in [1,\ m+n]}$ where $a_i = 1$ for $1 \leq i \leq m$, $a_i = -1$ for $m+1 \leq i \leq m+n$. In particular, M has rank $m+n-1$.*

Proof of claim. For all i, let $R_i = \big(1_{A_i}(x)\big)_{x \in \Omega}$ be the ith row of M; let b_i ($1 \leq i \leq m+n$) in $\mathbb{Z}$ such that $\sum_{i=1}^{m+n} b_i R_i = 0$. Let P (resp. N) be the set of i such that $b_i \geq 0$

(resp. $b_i < 0$), define $f_+,\ f_-,\ g_+,\ g_-$ in $\mathbb{N}[\mathcal{P}(\Omega)]$ by

$$f_+ = \sum_{i\in P\cap[1,\ m]} b_i \cdot 1_{A_i}, \quad f_- = \sum_{i\in N\cap[1,\ m]} (-b_i)\cdot 1_{A_i},$$
$$g_+ = \sum_{i\in P\cap[m+1,\ m+n]} b_i \cdot 1_{A_i}, \quad g_- = \sum_{i\in N\cap[m+1,\ m+n]} (-b_i)\cdot 1_{A_i}.$$

It follows that $f_+ + g_+ = f_- + g_-$ and $f_+ \wedge f_- = g_+ \wedge g_- = 0$, whence $f_+ = g_-$ and $f_- = g_+$; thus all four of them are in $\mathbb{N}[\mathcal{A}] \cap \mathbb{N}[\mathcal{B}] = \mathbb{N}[\mathcal{A}\cap\mathcal{B}] = \mathbb{N}$, *i.e.* they are constant. Let p (resp. q) be the constant value of f_+ (resp. f_-), so that both are in $\mathbb{N}$. If $f_+ = 0$, then $b_i = -q$ for $1 \le i \le m$, $b_i = q$ for $m+1 \le i \le m+n$. If $f_+ \ne 0$, then $b_i = p$ for $1 \le i \le n$, $b_i = -p$ for $m+1 \le i \le m+n$. The conclusion follows. ■ Claim 9.

Now, we prove that for any consistent E-valued grid (μ,ν) defined on the grid $(\mathcal{A},\mathcal{B})$ such that $\mathcal{A}\cap\mathcal{B} = \{\emptyset,\Omega\}$ and $\mathcal{A}\vee\mathcal{B} = \mathcal{P}(\Omega)$, μ and ν extend to a E-valued charge on $\mathcal{P}(\Omega)$, by induction on the weight of (μ,ν). As before, let m (resp. n) be the number of atoms of $\mathcal{A}$ (resp. $\mathcal{B}$), let $s = |\Omega|$. To avoid trivialities, $m,n,s \ge 2$. Note that by claim 9, $m+n-1 \le s \le mn$. By proposition 2.3, there is an preordered semigroup-homomorphism λ from $\mathbb{N}[\mathcal{A}]+\mathbb{N}[\mathcal{B}]$ to E extending both μ and ν. For each $S \subseteq [1,\ m+n]$, put $A_S = \bigcup_{i\in S} A_i$. We consider two cases.

Case 1. $s \ge m+n$.

By claim 9 and the fact that M has $s \ge m+n$ columns, there exists x_0 in Ω such that the matrix $M_0 = \big(1_{A_i}(x)\big)_{(i,x)\in[1,\ m+n]\times(\Omega\setminus\{x_0\})}$ has still rank $m+n-1$; put $\Omega_0 = \Omega\setminus\{x_0\}$. We may assume without loss of generality that $\{x_0\} = A_1 \cap A_{m+1}$. If $A_1 = \{x_0\}$, then the column space of M_0 is orthogonal to the vector $(a'_i)_{i\in[1,\ m+n]}$ such that $a'_0 = 0$, $a'_i = 1$ for $2 \le i \le m$ and $a'_i = -1$ for $m+1 \le i \le m+n$, which contradicts claim 9 and the fact that M and M_0 have the same column space; hence $A_1 \ne \{x_0\}$. Similarly, $A_{m+1} \ne \{x_0\}$. Put $X = A_1 \setminus \{x_0\}$, $Y = A_{m+1}\setminus\{x_0\}$, $Z = \{x_0\}$, and let $\mathcal{A}_0 = \{A\cap\Omega_0 :\ A \in \mathcal{A}\}$, $\mathcal{B}_0 = \{B\cap\Omega_0 :\ B\in\mathcal{B}\}$. Then $\mathcal{A}_0$ has atoms X, A_2,..., A_m and $\mathcal{B}_0$ has atoms Y, A_{m+2},..., A_{m+n}. Put $I = [2,\ m]$ and $J = [m+2,\ m+n]$.

Claim 10. *$\mathcal{A}_0 \cap \mathcal{B}_0 = \{\emptyset,\Omega_0\}$ and $\mathcal{A}_0 \vee \mathcal{B}_0 = \mathcal{P}(\Omega_0)$.*

Proof of claim. For each x in Ω_0, there are $i \in [1,\ m]$ and $j \in [m+1, m+n]$ such that $A_i \cap A_j = \{x\}$, whence $\{x\} = (A_i\setminus\{x_0\})\cap(A_j\setminus\{x_0\})$. Thus $\mathcal{A}_0\vee\mathcal{B}_0 = \mathcal{P}(\Omega_0)$. Now, let C be an element of $\mathcal{A}_0\cap\mathcal{B}_0$ not in $\{\emptyset,\Omega_0\}$. If $C = A_S + X = A_T$ for some $S \subseteq I$ and $T \subseteq J$, then for all x in Ω_0, we have

$$1_{A_1}(x) + \sum_{i\in S} 1_{A_i}(x) = 1_X(x) + \sum_{i\in S} 1_{A_i}(x) = 1_{X+A_S}(x) = 1_{A_T}(x),$$

whence the vector $(a''_i)_{i\in[1,\ m+n]}$ defined by $a''_i = 1$ $(i \in \{1\} \cup S)$, $a''_i = -1$ $(i \in T)$, $a''_i = 0$ $(i \notin S \cup T \cup \{1\})$ is in the orthocomplement of the image space of M_0, whence, by claim 9 and definition of M_0, $S = I$ and $T = J$, a contradiction. Similarly, one cannot have $C = A_T + Y = A_S$ for some $S \subseteq I$, $T \subseteq J$. ■ Claim 10.

Define finite sets $\mathcal{I}_+$, $\mathcal{I}_-$, $\mathcal{J}_+$, $\mathcal{J}_-$, $\mathcal{K}$, $\mathcal{L}$ by

$$\begin{aligned}
\mathcal{I}_+ &= \{(S,T) \in \mathcal{P}(I) \times \mathcal{P}(J) :\ A_S + X \leq A_T\},\\
\mathcal{I}_- &= \{(S,T) \in \mathcal{P}(I) \times \mathcal{P}(J) :\ A_T \leq A_S + X\},\\
\mathcal{J}_+ &= \{(S,T) \in \mathcal{P}(I) \times \mathcal{P}(J) :\ A_T + Y \leq A_S\},\\
\mathcal{J}_- &= \{(S,T) \in \mathcal{P}(I) \times \mathcal{P}(J) :\ A_S \leq A_T + Y\},\\
\mathcal{K} &= \{(S,T) \in \mathcal{P}(I) \times \mathcal{P}(J) :\ A_S + X \leq A_T + Y\},\\
\mathcal{L} &= \{(S,T) \in \mathcal{P}(I) \times \mathcal{P}(J) :\ A_T + Y \leq A_S + X\}.
\end{aligned}$$

Furthermore, note that the only ordered pair (S,T) in $\mathcal{P}(I)\times\mathcal{P}(J)$ such that $A_S+X = A_T + Y$ is (I,J) itself. Then, consider the following linear system, with coefficients in $\mathbb{N}[\mathcal{A}] + \mathbb{N}[\mathcal{B}]$:

$$\begin{cases}
x + z = A_1; & \\
y + z = A_{m+1}; & \\
A_S + x \leq A_T & (\text{all } (S,T) \in \mathcal{I}_+)\\
A_T \leq A_S + x & (\text{all } (S,T) \in \mathcal{I}_-)\\
A_T + y \leq A_S & (\text{all } (S,T) \in \mathcal{J}_+)\\
A_S \leq A_T + y & (\text{all } (S,T) \in \mathcal{J}_-)\\
A_S + x \leq A_T + y & (\text{all } (S,T) \in \mathcal{K})\\
A_T + y \leq A_S + x & (\text{all } (S,T) \in \mathcal{L})\\
A_I + x = A_J + y &
\end{cases} \tag{3.1}$$

Then this linear system admits a solution in $\mathbb{N}[\mathcal{P}(\Omega)]$, namely (X,Y,Z); but this is a 2-proper linear system by proposition 1.7, hence, applying theorem 1.6 to the preordered semigroup-homomorphism λ from $\mathbb{N}[\mathcal{A}]+\mathbb{N}[\mathcal{B}]$ to E, we obtain that the image of (3.1) under λ admits a solution in E; denote by (x,y,z) this solution. One can define E-valued charges μ_0 and ν_0, respectively on $\mathcal{A}_0$ and $\mathcal{B}_0$, by $\mu_0(A_i) = \mu(A_i)$ (all $i \in I$), $\nu_0(A_i) = \nu(A_i)$ (all $i \in J$) and $\mu_0(X) = x$, $\nu_0(Y) = y$. By definition of (x,y,z), μ_0 and ν_0 are consistent. Since (μ_0,ν_0) has weight $m+n+s-1$, it follows from the induction hypothesis that there is a E-valued charge ρ_0 on $\mathcal{P}(\Omega_0)$ extending both μ_0 and ν_0. Define a E-valued charge ρ on $\mathcal{P}(\Omega)$ extending ρ_0 by $\rho(Z) = z$. Then $\rho(A_1) = x+z = \mu(A_1)$ and $\rho(A_{m+1}) = y+z = \nu(A_{m+1})$; it follows that ρ extends both μ and ν.

Case 2. $s = m + n - 1$.

We first prove the

Claim 11. *There is i in $[1,\ m+n]$ such that A_i is a singleton.*

Proof of claim. Assume *e.g.* that $m \geq n$. Suppose that $|A_i| \geq 2$ for all $i \in [1,\ m]$. Then $s = |\Omega| = \sum_{i=1}^{m} |A_i| \geq 2m$, whence $n > m$, a contradiction. ■ Claim 11.

So we can assume without loss of generality that $A_1 = \{x_0\}$ and $x_0 \in A_{m+1}$ for some x_0. Note that $A_{m+1} \neq \{x_0\}$, otherwise $\{x_0\}$ would be in $\mathcal{A} \vee \mathcal{B}$, a contradiction. Put $Y = A_{m+1} \setminus \{x_0\}$; let $\Omega_0 = \Omega \setminus \{x_0\}$, $\mathcal{A}_0 = \{A \cap \Omega_0 :\ A \in \mathcal{A}$, $\mathcal{B}_0 = \{B \cap \Omega_0 :\ B \in \mathcal{B}$. Thus $\mathcal{A}_0$ has atoms A_2,..., A_m and $\mathcal{B}_0$ has atoms Y, A_{m+2},..., A_{m+n}. Put $I = [2,\ m]$ and $J = [m+2,\ m+n]$.

Claim 12. *$\mathcal{A}_0 \cap \mathcal{B}_0 = \{\emptyset, \Omega_0\}$ and $\mathcal{A}_0 \vee \mathcal{B}_0 = \mathcal{P}(\Omega_0)$.*

Proof of claim. The proof of the second fact is as in claim 10. To prove the first fact, we need only to check the case where $A_S = A_T + Y$ for some $S \subseteq I$ and $T \subseteq J$. Thus $A_1 + A_S = A_{m+1} + A_T$, whence their common value is Ω, whence $S = I$ and $T = J$.

■ Claim 12.

Now, let $\mathcal{I}$ and $\mathcal{J}$ be the finite sets defined by

$$\mathcal{I} = \{(S,T) \in \mathcal{P}(I) \times \mathcal{P}(J) :\ A_S \leq A_T + Y\},$$
$$\mathcal{J} = \{(S,T) \in \mathcal{P}(I) \times \mathcal{P}(J) :\ A_T + Y \leq A_S\}.$$

Then, consider the following linear system, with coefficients in $\mathbb{N}[\mathcal{A}] + \mathbb{N}[\mathcal{B}]$:

$$\begin{cases} A_1 + y = A_{m+1}; & \\ A_S \leq A_T + y & (\text{all } (S,T) \in \mathcal{I}); \\ A_T + y \leq A_S & (\text{all } (S,T) \in \mathcal{J}); \\ A_J + y = A_I. & \end{cases} \tag{3.2}$$

Then this linear system admits a solution in $\mathbb{N}[\mathcal{P}(\Omega)]$, namely Y; but this is a 2-proper linear system, hence, applying theorem 1.6 to the preordered semigroup-homomorphism λ from $\mathbb{N}[\mathcal{A}] + \mathbb{N}[\mathcal{B}]$ to E, we obtain that the image of (3.2) under λ admits a solution in E; denote by y this solution. One can define E-valued charges μ_0 and ν_0, respectively on $\mathcal{A}_0$ and $\mathcal{B}_0$, by $\mu_0 = \mu|_{\mathcal{A}_0}$, $\nu_0(A_i) = \nu(A_i)$ for $i \in J$, and $\nu_0(Y) = y$. By definition of y, μ_0 and ν_0 are consistent. Since (μ_0, ν_0) has weight $m + n + s - 2$, it follows from the induction hypothesis that there is a E-valued charge ρ_0 on $\mathcal{P}(\Omega_0)$ extending both μ_0 and ν_0. Define a E-valued charge ρ on $\mathcal{P}(\Omega)$ extending ρ_0 by $\rho(A_1) = \mu(A_1)$. Then $\rho(A_{m+1}) = \rho(A_1 + Y) = \lambda(A_1) + y = \lambda(A_{m+1})$, whence ρ extends both μ and ν.

This concludes the proof of theorem 3.2. ■

Since we deal with finite algebras, the conclusion of theorem 3.1 can even be strengthened to Boolean *rings*:

3.3. Theorem. *The grid property for Boolean algebras implies the grid property for Boolean rings.*

Note that theorem 3.3 does not generalize to the infinite case, since *e.g.* $\mathbb{N}$ has the 2-CHEP for Boolean algebras but not for Boolean rings.

Proof. Let E be a *pp*-semigroup satisfying the grid property [for Boolean algebras]. By theorem 3.2, E is a 2-strong refinement *pp*-semigroup. Let $\mathcal{A}$ and $\mathcal{B}$ be two Boolean subrings of a finite Boolean ring $\mathcal{C}$, let $\mu : \mathcal{A} \to E$ and $\nu : \mathcal{B} \to E$ be consistentcharges. We can suppose that $\mathcal{C} = \mathcal{P}(C)$ for some finite set C. Let $A = \bigcup \mathcal{A}$ and $B = \bigcup \mathcal{B}$, so that $\mathcal{A}$ (resp. $\mathcal{B}$) is a Boolean subalgebra of $\mathcal{P}(A)$ (resp. $\mathcal{P}(B)$). Let c be some object not in C, let $C' = C \cup \{c\}$. Put $X = C' \setminus A$, $Y = C' \setminus B$ (the introduction of c guarantees that X and Y are nonempty), let $\mathcal{A}'$ (resp. $\mathcal{B}'$) be the Boolean subalgebra of $\mathcal{P}(C)$ generated by $\mathcal{A}$ and $\{X\}$ (resp. by $\mathcal{B}$ and $\{Y\}$). We shall extend μ (resp. ν) to a E-valued charge μ' (resp. ν') on $\mathcal{A}'$ (resp. $\mathcal{B}'$) such that μ' and ν' are consistent: then we will be done since E has the grid property for Boolean algebras.

Let $\mathcal{E} = \{(U, V) \in \mathcal{A} \times \mathcal{B} : \ U + X = V + Y\}$. It is obvious that $(A, B) \in \mathcal{E}$ and that $(U, V) \in \mathcal{E}$ and $(U', V') \in \mathcal{E}$ implies $(U \cap U', V \cap V') \in \mathcal{E}$, whence $\mathcal{E}$ admits a least element, say (U_0, V_0). Consider the finite sets $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$, $\mathcal{L}$ defined by

$$\begin{aligned}
\mathcal{I} &= \{(U, V) \in \mathcal{A} \times \mathcal{B} : \ U \leq V + Y\}, \\
\mathcal{J} &= \{(U, V) \in \mathcal{A} \times \mathcal{B} : \ V \leq U + X\}, \\
\mathcal{K} &= \{(U, V) \in \mathcal{A} \times \mathcal{B} : \ U + X \leq V + Y\}, \\
\mathcal{L} &= \{(U, V) \in \mathcal{A} \times \mathcal{B} : \ V + Y \leq U + X\}.
\end{aligned}$$

Then consider the following linear system, with coefficients in $\mathbb{N}[\mathcal{A}] + \mathbb{N}[\mathcal{B}]$:

$$\begin{cases}
U_0 + x = V_0 + y; & \\
U \leq V + y & (\text{all } (U, V) \in \mathcal{I}); \\
V \leq U + x & (\text{all } (U, V) \in \mathcal{J}); \\
U + x \leq V + y & (\text{all } (U, V) \in \mathcal{K}); \\
V + y \leq U + x & (\text{all } (U, V) \in \mathcal{L}).
\end{cases} \tag{3.3}$$

Then this linear system admits a solution in $\mathbb{N}[\mathcal{P}(\Omega)]$, namely (X, Y); but by proposition 1.7, this is a 1-proper (thus 2-proper) linear system, hence, applying theorem 1.6 to the preordered semigroup-homomorphism λ from $\mathbb{N}[\mathcal{A}] + \mathbb{N}[\mathcal{B}]$ to E (given by proposition

2.3), we obtain that the image of (3.3) under λ admits a solution in E; denote by (x, y) this solution. Define E-valued charges $\mu' : \mathcal{A}' \to E$ and $\nu' : \mathcal{B}' \to E$ by $\mu'(X) = x$ and $\nu'(Y) = y$. We show that μ' and ν' are consistent. Since we can never have $U + X \leq V$ (or $V + Y \leq U$) for $(U, V) \in \mathcal{A} \times \mathcal{B}$ (because $c \notin A \cup B$), the part of the consistency dealing with $\leq$ and $\geq$ results immediately from the definition of (x, y). It remains to prove that for all (U, V) in $\mathcal{A} \times \mathcal{B}$ such that $U + X = V + Y$, we have $\lambda(U) + x = \lambda(V) + y$. By definition of (U_0, V_0), there are S in $\mathcal{A}$ and T in $\mathcal{B}$ such that $U = U_0 + S$ and $V = V_0 + T$, whence $S = T$, whence $S, T \in \mathcal{A} \cap \mathcal{B}$. Hence,

$$\begin{aligned}\lambda(U) + x &= \lambda(U_0) + x + \lambda(S)\\ &= \lambda(V_0) + y + \lambda(T)\\ &= \lambda(V) + y.\end{aligned}$$

Thus μ' and ν' are consistent. The conclusion follows. ■

We can now harvest the corollaries of theorems 3.2 and 3.3. First of all, note that theorem 3.2 implies immediately the result obtained by Basile and Rao in [**1**]:

3.4. Proposition. *Any abelian group (equipped with its coarse preordering) satisfies the grid property.* ■

Actually, theorem 3.2 shows immediately the following

3.5. Corollary. *The only coarse pp-semigroups satisfying the grid property are the abelian groups.* ■

At the opposite end of *pp*-semigroups, theorem 3.2 also characterizes the *antisymmetric pp*-semigroups satisfying the grid property:

3.6. Corollary. *An antisymmetric pp-semigroup satisfies the grid property if and only if it is a strong refinement pp-semigroup.* ■

So this yields us an *algebraic* (as opposed to 'arithmetical') characterization of strong refinement *pp*-semigroups.

Note also that the proof of theorem 3.2 (see claim 2) also shows us that strong refinement *pp*-semigroups satisfy the finite refinement property, thus that the definition given

here and the definition given in [**14**] are equivalent. This point could have been established much more directly by using lemma 1.4.

Theorem 3.2 also allows us to characterize immediately all *cancellative pp*-semigroups satisfying the grid property:

3.7. Corollary. *A cancellative pp-semigroup satisfies the grid property if and only if it satisfies the finite interpolation property.* ■

It also shows us that adjoining an infinite element does not harm the grid property:

3.8. Corollary. *A pp-semigroup E has the grid property if and only if $E \cup \{\infty\}$ has the grid property.* ■

Note also that the grid property implies *separativeness* (*see* [**17**]):

3.9. Corollary. *Any pp-semigroup satisfying the grid property is separative.*

Proof. Let E be a *pp*-semigroup satisfying the grid property. Since E is minimal, it is *preminimal, i.e.* it satisfies both statements

$$(\forall a, b, c, d)\big((a + c \leq b + c \textbf{ and } c \leq d) \Rightarrow a + d \leq b + d\big),$$
$$(\forall a, b, c, d)\big((a + c = b + c \textbf{ and } c \leq d) \Rightarrow a + d = b + d\big).$$

Furthermore, the pseudo-cancellation property yields that E satisfies the statement

$$(\forall a, b)(a + b \leq 2b \Rightarrow a \leq b).$$

Finally, let a, b in E such that $2a = a + b = 2b$. By the previous result, $a \equiv b$. By the 2-diagonal property, there is $x \equiv 0$ satisfying the following linear system:

$$\begin{cases} a + x = b; \\ a + x = a. \end{cases}$$

Thus, $a = b$. This checks the definition of separativeness introduced in [**17**]. ■

This implies easily that every *pp*-semigroup satisfying the grid property can be embedded (in a natural way) into a product of 'cones with infinity' $G_+ \cup \{\infty\}$ where G is a preordered abelian group satisfying the finite interpolation property.

Let us conclude this chapter by showing a connection between the grid property and the 2-CHEP. Say that a *pp*-semigroup E is *algebraically compact* when every (infinite) linear system with parameters from E of which every finite subsystem admits a solution in E admits a solution in E.

3.10. Proposition. *Every algebraically compact 2-strong refinement pp-semigroup satisfies the 2-CHEP for Boolean rings.*

Proof. Let $\mathcal{A}$, $\mathcal{B}$ be Boolean subrings of a Boolean ring $\mathcal{C}$, let E be an algebraically compact strong refinement *pp*-semigroup, let $\mu : \mathcal{A} \to E$ and $\nu : \mathcal{B} \to E$ be consistent charges. Let $(x_c)_{c\in\mathcal{C}}$ be a $\mathcal{C}$-indexed family of variables; consider the following linear system:

$$\begin{cases} x_c = x_a + x_b & (\text{all } a,\, b,\, c \text{ in } \mathcal{C} \text{ such that } c = a + b); \\ x_a = \mu(a) & (\text{all } a \in \mathcal{A}); \\ x_b = \nu(b) & (\text{all } b \in \mathcal{B}). \end{cases}$$

Since E has the 2-CHEP for finite Boolean rings (by theorem 3.3), every finite subsystem of this system admits a solution in E; since E is algebraically compact, the whole system admits a solution in E; this solution is obviously a E-valued charge on $\mathcal{C}$ extending both μ and ν. ■

3.11. Corollary. $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ *satisfies the 2-CHEP for Boolean rings.*

Proof. The Alexandroff compactification of the discrete topology on $\mathbb{N}$ yields a compact Hausdorff topology on $\overline{\mathbb{N}}$, for which the addition is continuous. Thus $\overline{\mathbb{N}}$ is *a fortiori* algebraically compact, whence the conclusion follows. ■

Another example of algebraically compact *pp*-semigroup is $G_+ \cup \{\infty\}$, where G is any *divisible* complete ℓ-group. In fact, $G_+ \cup \{\infty\}$ is an *injective pp*-semigroup by the characterization given in [**14**], whence it is algebraically compact.

3.12. Question. Let G be a complete ℓ-group. Is $G_+ \cup \{\infty\}$ algebraically compact?

Let us finally mention the fact that for abelian groups, algebraic compactness is *strictly* stronger that the 2-CHEP: this is a consequence of [**10**], where the authors show that an abelian group satisfies the 2-CHEP if and only if it is cotorsion.

§4. CASE OF POSITIVE CONES OF ORDERED ABELIAN GROUPS.

In this chapter, we shall prove some partial converses of theorems proved in [**12**] and here.

We shall first need some elementary results about ordered sets. Let E be an ordered set, let κ and λ be non zero ordinals, let $s \in E^\kappa$. Then E satisfies the (s, λ)-*interpolation property* (in abbreviation the (s, λ)-IP) when for all $t \in E^\lambda$ such that $s \leq t$ (*i.e.* $(\forall \xi < \kappa)(\forall \eta < \lambda)(s(\xi) \leq t(\eta))$), there exists c in E such that $s \leq c \leq t$. Similarly, E satisfies the $(s, \lambda \downarrow)$-IP when the conclusion above holds for all *decreasing* families $t \in E^\lambda$. One defines similarly the (κ, t)-IP, the $(\kappa \uparrow, t)$-IP, *etc....* E satisfies the (κ, λ)-IP when for all $s \in E^\kappa$ and $t \in E^\lambda$ such that $s \leq t$, there is c in E such that $s \leq c \leq t$. One defines similarly the $(\kappa, \lambda \downarrow)$-IP, *etc....* In 1.1 and 1.2, E is a fixed ordered set.

4.1. Lemma. *Let $\kappa \geq 1$, $\lambda \geq 2$, let $s \in E^\kappa$. Then the following are equivalent:*

(i) *E satisfies the (s, λ)-IP;*

(ii) *E satisfies the $(s, 2)$-IP and the the $(s, \eta \downarrow)$-IP for all nonzero $\eta \leq \lambda$.*

Proof. An easy induction. ■

4.2. Lemma. *Let $\kappa, \lambda \geq 2$. Then the following are equivalent:*

(i) *E satisfies the (κ, λ)-IP;*

(ii) *For all non zero $\xi \leq \kappa$ and $\eta \leq \lambda$, E satisfies the $(\xi \uparrow, 2)$-IP, the $(2, \eta \downarrow)$-IP and the $(2, 2)$-IP.*

Proof. Assuming (ii), one proves, using lemma 4.1 (for $\leq$ and $\geq$) that for all $s \in E^\kappa$, E satisfies the (s, λ)-IP. The proof is straightforward. ■

Now, from 4.3 to 4.7, let E be the positive cone of an abelian ordered group.

4.3. Lemma. *Let $\mathcal{A}$ be a Boolean subring of a Boolean algebra $\mathcal{B}$ such that $1_\mathcal{B} \notin \mathcal{A}$, let μ be a E-valued charge on $\mathcal{A}$ and let c in E such that $(\forall A \in \mathcal{A})(\mu(A) \leq c)$. Let $\mathcal{A}^* = \mathcal{A} \cup \{\neg A : A \in \mathcal{A}\}$ be the Boolean subalgebra of $\mathcal{B}$ generated by $\mathcal{A}$. Define a map μ^* from $\mathcal{A}^*$ to E extending μ by putting $\mu^*(\neg A) = c - \mu(A)$. Then μ^* is a E-valued charge on $\mathcal{A}^*$ extending μ.*

Proof. Straightforward. ■

4.4. Lemma. *Let $\mathcal{A}$, $\mathcal{B}$ be Boolean subrings of a Boolean algebra $\mathcal{C}$ such that $(\forall(A,B)\in\mathcal{A}\times\mathcal{B})(A\vee B\neq 1_{\mathcal{C}})$, let $\mu:\mathcal{A}\to E$ and $\nu:\mathcal{B}\to E$ be* bounded *(say, by $c\in E$) consistent E-valued charges. Then μ^* and ν^* [as they have been defined in lemma 4.3] are consistent.*

Proof. The additional hypothesis on $\mathcal{A}$ and $\mathcal{B}$ allows us to eliminate one case in the verification of consistency of μ^* and ν^*; from then on, the proof is straightforward. ■

4.5. Lemma. *Suppose that E satisfies the 2-CHEP for Boolean algebras. Then E has the 2-CHEP for* bounded *charges on Boolean rings.*

Proof. Let $\mu:\mathcal{A}\to E$ and $\nu:\mathcal{B}\to E$ be bounded (say, by some $c\in E$) consistent charges, $\mathcal{A}$ and $\mathcal{B}$ being Boolean subrings of a Boolean ring $\mathcal{C}$. Without loss of generality, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ satisfy the hypothesis of lemma 4.4. Thus, μ^* and ν^* (their definition depends on c) are consistent charges on the Boolean algebras $\mathcal{A}$ and $\mathcal{B}$. The conclusion follows from the hypothesis. ■

4.6. Lemma. *Suppose that E satisfies the 2-CHEP [for Boolean algebras]. Then for every infinite ordinal κ, E satisfies the $(\kappa\uparrow,2)$-IP and the $(2,\kappa\downarrow)$-IP.*

Proof. Since E is the positive cone of an abelian ordered group, it suffices to prove the first property. So let a, b in E, let $(c_\xi)_{\xi<\kappa}$ be an increasing κ-sequence of elements of E such that $c_\xi\leq a,b$ for all ξ. Let u, v be two distinct objects not in κ, let $\Omega=\kappa\cup\{u,v\}$. Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the Boolean subring of $\mathcal{P}(\Omega)$ generated by all intervals $\eta\setminus\xi$ $(\xi\leq\eta<\kappa)$ and $(\kappa\setminus\xi)\cup\{u\}$ (resp. $(\kappa\setminus\xi)\cup\{v\}$), $\xi<\kappa$. One can define a E-valued charge μ (resp. ν) on $\mathcal{A}$ (resp. $\mathcal{B}$) by putting $\mu(\eta\setminus\xi)=\nu(\eta\setminus\xi)=c_\eta-c_\xi$ in the first case, and $\mu\big((\kappa\setminus\xi)\cup\{u\}\big)=a-c_\xi$, $\nu\big((\kappa\setminus\xi)\cup\{v\}\big)=b-c_\xi$ for all $\xi<\kappa$. It is straightforward that μ and ν are consistent charges. By assumption, there is a charge ρ from the Boolean ring $\mathcal{C}=\mathcal{A}\vee\mathcal{B}$ to E extending both μ and ν. Let $c=\rho(\kappa)$. It is immediate that $c_\xi\leq c$ for all ξ and that $c\leq a,b$. ■

To complete the picture, we prove the

4.7. Lemma. *Suppose that E satisfies the 1-CHEP [for Boolean algebras]. Then for all infinite ordinals κ and λ, E satisfies the $(\kappa\uparrow,\lambda\downarrow)$-IP.*

Proof. Without loss of generality, κ and λ are limit ordinals. Let $(a_\xi)_{\xi<\kappa}$ be an increasing κ-sequence of elements of E and let $(b_\eta)_{\eta<\lambda}$ be a decreasing sequence of elements

of E such that $(\forall(\xi,\eta)\in\kappa\times\lambda)(a_\xi\le b_\eta)$. Let $\Omega=(\kappa\times\{0\})\cup(\lambda\times\{1\})$, equipped with the linear ordering $\le$ defined by

$$(\xi,m)\le(\eta,n)\Leftrightarrow\big(m<n\ \textbf{or}\ (m=n=0\text{ and }\xi\le\eta)\ \textbf{or}\ (m=n=1\ \textbf{and}\ \eta\le\xi)\big).$$

Adjoin a least element, say o, to Ω; let $\Omega^*=\Omega\cup\{o\}$. Define a family $(c_\zeta)_{\zeta\in\Omega^*}$ by $c_o=0$, $c_{(\xi,0)}=a_\xi$ (all $\xi<\kappa$), $c_{(\eta,1)}=b_\eta$ (all $\eta<\lambda$). Then $(c_\zeta)_{\zeta\in\Omega^*}$ is increasing (for the ordering of Ω^*), and one can define a E-valued charge μ on the Boolean subalgebra $\mathcal{A}$ of $\mathcal{P}(\Omega)$ generated by all intervals $(\sigma,\tau]$ by putting $\mu\big((\sigma,\tau]\big)=c_\tau-c_\sigma$ $(\sigma\le\tau)$. By assumption, μ extends to a E-valued charge ν on the Boolean subalgebra $\mathcal{B}$ of $\mathcal{P}(\Omega)$ generated by $\mathcal{A}$ and $\{I\}$ where $I=\bigcup_{\xi<\kappa}[(0,0),(\xi,0)]$. Let $c=\nu(I)$. It is straightforward to verify that $(\forall\xi<\kappa)(a_\xi\le c)$ and $(\forall\eta<\lambda)(c\le b_\eta$. ■

From lemmas 4.2, 4.6 and 4.7 and results in [**12**], we deduce immediately the following result:

4.8. Theorem. *Let G be a directed, abelian ordered group. If G_+ satisfies the 2-CHEP, then G is a* complete *ℓ-group. Furthermore, if G is a ℓ-group, then G_+ satisfies the 1-CHEP if and only if it is a complete ℓ-group.* ■

Certain results of this chapter can be extended (with similar proofs) to the objects called *D.P.O.M.'s* introduced in [**17**]; for example, if G is an abelian ordered group, then $G_+\cup\{\infty\}$ is a D.P.O.M. .

(Added in proof) The second author has answered positively question 3.12; that is, the positive cone of a complete ℓ-group with infinity adjoined is always algebraically compact; thus, by proposition 3.10, it has the 2-CHEP.

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