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Optimal quantizers for Radon random vectors in a Banach space

Siegfried Graf * Harald Luschgy† and Gilles Pagès ‡

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Abstract

For $n \in \mathbb{N}, r \in (0,\infty)$ and a Radon random vector X with values in a Banach space E let $e_{n,r}(X,E) = \inf(\mathbb{E}\min_{a\in\alpha}\|X-a\|^r)^{1/r}$, where the infimum is taken over all subsets α of E with $\mathrm{card}(\alpha) \le n$ (n-quantizers). We investigate the existence of optimal n-quantizers for this L^r-quantization propblem, derive their stationarity properties and establish for L^p-spaces E the pathwise regularity of stationary quantizers.

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1 Introduction

We investigate optimal quantizers and the quantization error in the functional L^r-quantization problem for stochastic processes viewed as random variables in a Banach (function) space. So let $(E, \|\cdot\|)$ be a real Banach space and consider a Radon random variable $X : (\Omega, \mathcal{A}, \mathbb{P}) \to E$ which means that X is Borel measurable and its distribution $\mathbb{P}_X$ is a Radon probability measure on E. For $n \in \mathbb{N}$ and $r \in (0,\infty)$, the L^r-quantization problem for X of level n consists in minimizing

$$(\mathbb{E}\min_{a\in\alpha}\|X-a\|^r)^{1/r} = \|\min_{a\in\alpha}\|X-a\|\|_{L^r(\mathbb{P})}$$

over all subsets $\alpha \subset E$ with $\mathrm{card}(\alpha) \le n$. Such a set α is called n-codebook or n-quantizer. The minimal nth quantization error is then defined by

$$e_{n,r}(X,E) := \inf\{(\mathbb{E}\min_{a\in\alpha}\|X-a\|^r)^{1/r} : \alpha \subset E,\ \mathrm{card}(\alpha) \le n\}. \tag{1.1}$$

Under the integrability condition

$$\mathbb{E}\|X\|^r < \infty \tag{1.2}$$

the quantity $e_{n,r}(X,E)$ is finite.

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For a given n-codebook α one defines an associated closest neighbour projection

$$\pi_\alpha := \sum_{a\in\alpha} a 1_{C_a(\alpha)}$$

and the induced $\alpha - quantized\ version$ (or $\alpha - quantization$) of X by

$$\hat{X}^\alpha := \pi_\alpha(X), \tag{1.3}$$

where $\{C_a(\alpha) : a \in \alpha\}$ is a Voronoi partition induced by α, that is a Borel partition of E satisfying

$$C_a(\alpha) \subset \{x \in E : \|x - a\| = \min_{b\in\alpha} \|x - b\|\}$$

for every $a \in \alpha$. Then one easily checks that, for any measurable random variable $X' : \Omega \to \alpha \subset E$,

$$\mathbb{E}\|X - X'\|^r \geq \mathbb{E}\|X - \hat{X}^\alpha\|^r = \mathbb{E}\min_{a\in\alpha} \|X - a\|^r$$

so that finally

$$\begin{aligned} e_{n,r}(X,E) &= \inf\{(\mathbb{E}\|X - \hat{X}\|^r)^{1/r} : \hat{X} = f(X), f : E \to E \text{ Borel measurable}, \text{card } (f(E)) \leq n\} \\ &= \inf\{(\mathbb{E}\|X - \hat{X}\|^r)^{1/r} : \hat{X} : \Omega \to E \text{ measurable, card } (\hat{X}(\Omega)) \leq n\}. \end{aligned} \tag{1.4}$$

Functional quantization of stochastic processes can thus be seen as a discretization of the path-space E of a process and the approximation (coding) of a stochastic process by finitely many deterministic functions from its path-space. Typical settings are $E = L^p([0,1], dt)$ and $E = C([0,1])$. Functional quantization is the natural extension to stochastic processes or Banach space valued random vectors of the so-called optimal vector quantization of random vectors in $E = \mathbb{R}^d$ which has been extensively investigated since the late 1940's in Signal processing and Information Theory (see [9], [15]). For the mathematical aspects of vector quantization in $\mathbb{R}^d$, one may consult [13] and for algorithmic aspects see [25].

Recently, the extension of optimal vector quantization to stochastic processes has given raise to many theoretical developments including the rate of convergence of the quantization errors $e_{n,r}(X)$ to zero as $n \to \infty$ and the construction of good or even rate optimal quantizers (see e.g. [6], [7], [8] [14], [21], [22], [23]). For a first promising application to the pricing of financial derivatives through numerical integration on path-spaces see [26]. In this paper we aim to develop general results on the existence of optimal quantizers and their properties.

The paper is organized as follows. In Section 2, a theorem about the existence of optimal n-quantizers for E-valued Radon random vectors lying in E or in some suitable superspace $G \supset E$ is established under some very general assumptions. It relates existence to intersection properties of closed balls. This problem is connected with its bidual counterpart and enlightened by counterexamples. Furthermore, bounds of the quantization errors $e_{n,r}(X,E)$ in terms of $e_{n,r}(X,G)$ for superspaces G and in terms of marginals of X for vector valued processes are derived. In Section 3 the stationarity property of optimal n-quantizers is investigated. This turns out to be an essential key for the functional quantization of 1-dimensional diffusion processes (see [23]). For smooth Banach spaces stationary quantizers are defined as the critical points of the distortion function. In the case of L^p-spaces E which are natural path-spaces of processes some pathwise regularity for these stationary quantizers is established. The result applies e.g. to Gaussian processes, d-dimensional diffusion processes and certain Lévy processes.

2 Optimal quantizers and quantization errors

Let X be a Radon $(E, \|\cdot\|)$-valued random variable with distribution $\mathbb{P}_X$. The Radon property of $\mathbb{P}_X$ means inner regularity w.r.t. compact sets and on Banach spaces it is the same as tightness which in turn is equivalent to the existence of a separable Borel measurable set with $\mathbb{P}_X$-probability 1. It is to be noticed that if $\mathbb{P}(X \in F) = 1$ for some Banach subspace F of E, X is Radon when viewed as F-valued random variable. On the other hand, if E is a Banach subspace of some Banach space G then X is also Radon as G-valued random variable.

We will assume throughout this section that X satisfies the integrability condition (1.2) for some $r \in (0, \infty)$. Then

$$\lim_{n\to\infty} e_{n,r}(X, E) = 0. \tag{2.1}$$

As a matter of fact, the support of $\mathbb{P}_X$ being separable there exists a countable subset $\{a_n, n \geq 1\}$ everywhere dense in $\mathrm{supp}(\mathbb{P}_X)$. It is clear that

$$0 \leq e^r_{n,r}(X, E) \leq \mathbb{E} \min_{1\leq i\leq n} \|X - a_i\|^r \to 0 \text{ as } n \to \infty$$

by the Lebesgue dominated convergence Theorem. On the other hand, the existence of optimal quantizers, i.e. the fact that $e_{n,r}(X, E)$ actually stands as a minimum needs much more care.

2.1 Existence of optimal quantizers

A set $\alpha \subset E$ with $1 \leq$ card $(\alpha) \leq n$ is called an L^r*-optimal* n*-quantizer* for X if

$$(\mathbb{E} \min_{a\in\alpha} \|X - a\|^r)^{1/r} = e_{n,r}(X, E). \tag{2.2}$$

Let $\mathcal{C}_{n,r}(X, E)$ denote the set of all L^r-optimal n-quantizers for X in E.

We first provide some interesting properties of n-optimal quantizers (they can be seen as necessary conditions for n-optimality). Their proofs are literally the same as those (established in finite-dimension) of Theorem 4.1 and Theorem 4.2 in [13] respectively. They are related with the *Voronoi partitions* induced by a n-quantizer α: these are the Borel partitions $\{C_a(\alpha) : a \in \alpha\}$ of E which satisfy

$$C_a(\alpha) \subset V_a(\alpha) := \left\{ x \in E : \|x - a\| = \min_{b\in\alpha} \|x - b\| \right\}. \tag{2.3}$$

Let us note that $V_a(\alpha)$ is closed and star-shaped relative for a and for every $a \in \alpha$,

$$\left\{ x \in E : \ \|x - a\| < \min_{b\in\alpha\setminus\{a\}} \|x - a\| \right\} \subset \overset{\circ}{C}_a(\alpha) \subset \overline{C_a(\alpha)} \subset V_a(\alpha).$$

Furthermore, as soon as $(E, \|\,.\,\|)$ is *strictly convex* ([1]), any Voronoi partition satisfies for every $a \in \alpha$

$$\overline{C_a(\alpha)} = V_a(\alpha) \tag{2.4}$$

and

$$\overset{\circ}{C}_a(\alpha) = \overset{\circ}{V}_a(\alpha) = \left\{ x \in E : \ \|x - a\| < \min_{b\in\alpha\setminus\{a\}} \|x - b\| \right\}.$$

[1] *i.e.* $B_E(0, 1)$ is a strictly convex set: $\forall x, y \in S_E(0, 1)$, $x \neq y$, $\forall \lambda \in (0, 1)$, $\|\lambda x + (1 - \lambda)y\| < 1$.

Proposition 1 *Assume that card*$(\mathrm{supp}(\mathbb{P}_X)) \geq n$.
(*a*) *Let* $\alpha \in \mathcal{C}_{n,r}(X,E)$. *Then card*$(\alpha) = n$ *and for every* $a \in \alpha$,

$$\mathbb{P}_X(C_a(\alpha)) > 0 \quad \text{and} \quad \{a\} \in \mathcal{C}_{1,r}(\mathbb{P}_X(\,\cdot\,|C_a(\alpha)), E).$$

(*b*) *Assume that* E *is smooth* ([2]) *and strictly convex. If* $\alpha \in \mathcal{C}_{n,r}(X,E)$ *and*

$$(r > 1) \qquad \text{or} \qquad (r = 1 \ \text{ and } \ \mathbb{P}(X \in \alpha) = 0),$$

then

$$\mathbb{P}_X(V_a(\alpha) \cap V_b(\alpha)) = 0 \quad \text{for every } \ a, b \in \alpha, a \neq b. \tag{2.5}$$

Note that, under the strict convexity assumption, (2.5) is then equivalent to both

$$(\forall\, a \in \alpha, \quad \mathbb{P}_X(\partial C_a(\alpha)) = 0) \ \text{ and } \ (\forall\, a \in \alpha, \quad \mathbb{P}_X(\partial V_a(\alpha)) = 0)\,.$$

The first results of existence for optimal quantizers are due to Cuesta-Albertos and Matràn [5] and Pärna [24]) for uniformly convex and reflexive Banach spaces, respectively. We provide an extension to Banach spaces having the property that the closed balls form a *compact system.* A system $\mathcal{K}$ of subsets of E is called compact if each subsystem $\mathcal{K}_0$ of $\mathcal{K}$ which has the finite intersection property (*i.e.* the intersection of each finite subsystem of $\mathcal{K}_0$ is not empty) has a nonempty intersection. Let $B(s,\rho) = B_E(x,\rho) := \{y \in E : \|y - x\| \leq \rho\}$ be the closed ball of radius ρ centered at x.

Definition 1 *A pair* (F,G) *consisting of a Banach space* G *and a Banach subspace* F *of* G *is called admissible if* $\{B_G(x,\rho) : x \in F, \rho > 0\}$ *is a compact system in* G. G *is called admissible if* (G,G) *is admissible.*

The level n L^r-distortion function is defined by

$$D^X_{n,r} : E^n \to \mathbb{R}_+, D^X_{n,r}(a) := \mathbb{E} \min_{1 \leq i \leq n} \|X - a_i\|^r. \tag{2.6}$$

Theorem 1 *Assume that* $\mathbb{P}_X(F) = 1$ *for some Banach subspace* F *of* E *and that* (F,E) *is admissible. Then, for every* $n \in \mathbb{N}$,

$$\mathcal{C}_{n,r}(X,E) \neq \emptyset.$$

Proof. Fix $n \in \mathbb{N}$. Let τ_0 denote the topology on E generated by the system $\{B(x,\rho)^c : x \in F,\, \rho > 0\}$ and let τ be the product topology on E^n (these topologies usually do not satisfy the Hausdorff axiom). The family $\{B(x,\rho) : x \in F,\, \rho > 0\}$ being a compact system in E, one checks that E is τ_0-quasi-compact([3]). Consequently, E^n is τ-quasi-compact. It is obvious that any lower semi-continuous (l.s.c.) function defined on E^n then reaches a minimum. Hence, the proof amounts to showing that the distortion function $D^X_{n,r} : E^n \to \mathbb{R}_+$ is τ-lower semi-continuous.

For every $x \in F$ and $a \in E^n$, set $d(x,a) := \min_{1 \leq i \leq n} \|x - a_i\|$. Then

$$\{a \in E^n : d(x,\cdot)^r \leq c\} = \bigcup_{i=1}^n \{a \in E^n : a_i \in B(x, c^{1/r})\}$$

is τ-closed for every $c \geq 0$. Hence, $a \mapsto d(x,a)^r$ is τ-lower semi-continuous. In turn any convex combination of such functions are τ-l.s.c. as well. This implies that $D^X_{n,r}$ (and $(D^X_{n,r})^{1/r}$) are τ-lower semi-continuous provided card$(\mathrm{supp}(\mathbb{P}_X)) < \infty$.

[2] i.e. the norm is Gateaux-differentiable at every $x \neq 0$.

[3] *i.e.* satisfies the Borel-Lebesgue axiom – from any open covering one may extract a finite open covering – but possibly not the Hausdorff axiom.

For general X we will show that for every $c \geq 0, \{D^X_{n,r} > c\}$ is τ-open. First note that from (1.4) and (2.1), there exists a sequence of quantizations $\widehat{X}_m : \Omega \to F$, $\mathrm{card}(\widehat{X}_m(\Omega)) \leq m$, such that

$$\lim_m \|X - \widehat{X}_m\|_{L^r_E(\mathbb{P})} = 0.$$

Consider first the case $r \geq 1$. It follows from Minkowski's inequality that, for every $a \in E^n$, $X \mapsto (D^X_{n,r}(a))^{1/r}$ is 1-Lipschitz on $L^r_E(\mathbb{P})$:

$$\begin{aligned} |D^X_{n,r}(a)^{1/r} - D^Y_{n,r}(a)^{1/r}| &= \Big| \|d(X,a)\|_{L^r(\mathbb{P})} - \|d(Y,a)\|_{L^r(\mathbb{P})} \Big| \\ &\leq \|d(X,a) - d(Y,a)\|_{L^r(\mathbb{P})} \\ &\leq \|X - Y\|_{L^r_E(\mathbb{P})}. \end{aligned} \tag{2.7}$$

Let $a \in \{(D^X_{n,r})^{1/r} > c\}$. It follows from (2.7) that, the τ-open set $\{(D^{\widehat{X}_m}_{n,r})^{1/r} > c + \|X - \widehat{X}_m\|_{L^r_E(\mathbb{P})}\}$ is always contained in $\{(D^X_{n,r})^{1/r} > c\}$. Furthermore, it contains a for large enough m, still by (2.7). Hence $\{(D^X_{n,r})^{1/r} > c\}$ is τ-open and $D^X_{n,r}$ is τ-l.s.c.

When $0 < r < 1$, one concludes the same way round, using now that $|u^r - v^r| \leq |u - v|^r$ for every $u, v \in \mathbb{R}_+$, one derives that for every $a \in E^n$,

$$|D^X_{n,r}(a) - D^Y_{n,r}(a)| \leq \mathbb{E}\,|d(X,a) - d(Y,a)|^r \leq \|X - Y\|^r_{L^r_E(\mathbb{P})}.$$

□

In the non-quantization setting $n = 1$, Theorem 1 with $F = E$ is due to Herrndorf (see [16]).

One easily checks that if E is a 1-complemented closed subspace of some Banach space G and (E, G) is admissible, then E is admissible. Here E is said to be c-complemented in $G (c \geq 1)$ if there is a linear projection S from G onto E with $\|S\| \leq c$. An interesting case is $G = E^{**}$. One simply notes that the closed balls in the bidual E^{**} of E are weak*-compact and thus E^{**} is admissible. The following characterization is a slight generalization of Theorem 5.9 in [19].

Proposition 2 *(F, E) is admissible if and only if*

$$\bigcap_{x \in F} B_E(x, \|z - x\|) \neq \emptyset \text{ for every } z \in E^{**}.$$

In particular, if E is 1-complemented in its bidual E^{**}, then E is admissible.

An investigation of the admissibility feature of Banach spaces E and the ball topology τ_0 (with $F = E$) used in the proof of Theorem 1 can be found in [10], [11].

One derives for three main classes of Banach spaces the following corollary.

Corollary 1 *In any of the following cases E is 1-complemented in E^{**} and hence, for every $n \in \mathbb{N}$, $\mathcal{C}_{n,r}(X, E) \neq \emptyset$.*

(i) E is a KB (Kantorovich-Banach)-space.

(ii) E is a dual space.

(iii) E is an order complete AM-space with unit.

Proof. (i) By definition, a Banach lattice that is a band in its bidual is a KB-space. Since E^{**} is an order complete Banach lattice, E is a projection band in E^{**} and the band projection from E^{**} onto E has norm 1. (cf. [28], Chap. II.5).

(ii) Dual spaces are clearly 1-complemented in their bidual.

(iii) See [28], Chap. II.7. □

The order complete AM-space without unit $c_0(\mathbb{N})$ and the AM-space with unit $C([0,1])$ which is not order complete admit random variables X without optimal n-quantizers even for $n=1$ (see the subsequent counterexamples) In particular, both spaces are not admissible.

Example. $L^p_{\mathbb{R}^d}$-spaces are equipped with the norm $\|f\|_p = (\int |f(t)|_p^p d\mu(t))^{1/p}$ if $p \in [1,\infty)$ and $\|f\|_\infty = \mu\text{-ess sup}\, |f(t)|_\infty$ if $p=\infty$, where $|\cdot|_p$ denotes the ℓ^p-norm on $\mathbb{R}^d$. $L^1_{\mathbb{R}^d}$-spaces with respect to arbitrary measure spaces and are AL-spaces and hence KB-spaces. $L^p_{\mathbb{R}^d}$-spaces, $1<p<\infty$, with respect to arbitrary measure spaces are reflective and hence dual spaces. $L^\infty_{\mathbb{R}^d}$-spaces with respect to σ-finite measure spaces are dual spaces and also order complete AM-spaces with unit (cf. [28], Chap. IV 7).

Remarks. • Concerning the Banach spaces $E = L^p_{\mathbb{R}^d}$, the above theorem provides new existence results for the L^r-optimal quantizers in the cases $p=1$ and $p=\infty$.
• Any pathwise continuous process $(X_t)_{t\in[0,1]}$ is an $L^\infty([0,1],dt)$-Radon random variable since $(\mathcal{C}([0,1]), \|\,.\,\|_\infty)$ is a Polish subspace of $E = L^\infty([0,1],dt)$ (any probability on a Polish space is tight *i.e.* Radon). The above existence theorem shows that if $\|X\|_\infty \in L^r(\mathbb{P})$ for some $r>0$, then, for every $n\geq 1$, X has at least one L^r-optimal n-quantizer for the $\|\,.\,\|_\infty$-norm. However, nothing is known about the pathwise regularity of these optimal quantizers. Surprisingly, we will see in Section 3 that, for the same process, $(L^r, \|\,.\,\|_p)$-optimal n-quantizers with $p<\infty$ have much more regular paths (*i.e.* considering $E=L^p$ and $r\geq p$).

Optimal 1-quantizers may not exist in $c_0(\mathbb{N})$ Let $(E,\|\,.\,\|) = (c_0(\mathbb{N}), \|\,.\,\|_\infty)$ where $c_0(\mathbb{N})$ denotes the set of real valued sequences $x=(x_k)_{k\geq 1}$ such that $\lim_k x_k = 0$ and $\|x\|_\infty = \sup_k |x_k|$. Let $(u^{(n)})_{n\geq 1}$ denote the canonical basis of $c_0(\mathbb{N})$ defined by $u_k^{(n)} = \delta_{n,k}$ where $\delta_{i,j}$ is for the Kronecker symbol. One considers an E-valued random vector X supported by $\{u^{(n)},\ n\geq 1\}$ with a distribution $p_n = \mathbb{P}(X=u^{(n)})$, $n\geq 1$ satisfying $p_n \in (0,1/2)$ for every $n\geq 1$. Now $E^* = l^1(\mathbb{N})$ so that $E^{**} = \ell^\infty(\mathbb{N})$. One checks that the assumption of Theorem 1 is not fulfilled either since the system $\{B(u^{(n)},1/2),\ n\geq 1\}$ has an empty intersection whereas any finite subsystem has a nonempty intersection.

So let $n=1$ and $r=1$. We will show that

$$e_{1,1}(X, c_0(\mathbb{N})) = 1/2 \qquad \text{and} \qquad \mathcal{C}_{1,1}(X, c_0(\mathbb{N})) = \emptyset.$$

More precisely we will show that the corresponding level 1 quantization problem extended to the Banach space $\ell^\infty(\mathbb{N})$ does have a unique solution a in $\ell^\infty(\mathbb{N})$ given by $a_k = 1/2$, $k\geq 1$, that is $\mathcal{C}_{1,1}(X,\ell^\infty(\mathbb{N})) = \{a\}$ which in turn implies that it admits no solution in $c_0(\mathbb{N})$. In fact,

$$\mathbb{E}\,\|X-a\|_\infty = \sum_{n=1}^\infty p_n \|u^{(n)} - a\|_\infty = 1/2.$$

For an arbitrary $b\in \ell^\infty(\mathbb{N})$ one gets the following: if $\|u^{(n_0)} - b\|_\infty < 1/2$ for some $n_0\geq 1$, then, for every $n\neq n_0$,

$$\|u^{(n)} - b\|_\infty \geq \|u^{(n)} - u^{(n_0)}\|_\infty - \|u^{(n_0)} - b\|_\infty = 1 - \|u^{(n_0)} - b\|_\infty.$$

Hence

$$\begin{aligned} \mathbb{E}\,\|X-b\|_\infty &= \sum_{n\geq 0} p_n \|u^{(n)} - b\|_\infty \\ &\geq \sum_{n\neq n_0} p_n (1 - \|u^{(n_0)} - b\|_\infty) + p_{n_0}\|u^{(n_0)} - b\|_\infty \end{aligned} \tag{2.8}$$

$$
\begin{aligned}
&= 1 - p_{n_0} - (1 - 2p_{n_0})\|u^{(n_0)} - b\|_\infty \\
&> 1 - p_{n_0} - \frac{1}{2}(1 - 2p_{n_0}) \\
&= 1/2.
\end{aligned}
$$

In case $\|u^{(n)} - b\|_\infty \geq 1/2$ for every $n \geq 1$, one clearly obtains

$$\mathbb{E}\,\|X - b\|_\infty = \sum_{n\geq 1} p_n \|u^{(n)} - b\|_\infty \geq 1/2.$$

According to the above reasoning, any $b \in \ell^\infty(\mathbb{N})$ that achieves the infimum must satisfy $\|u^{(n)} - b\|_\infty = 1/2$ for every $n \geq 1$ which clearly implies $b = a$. Finally

$$e_{1,1}(X, \ell^\infty(\mathbb{N})) = \mathbb{E}\|X - a\|_\infty = 1/2 \;\text{ and }\; \mathbb{E}\|X - b\|_\infty > 1/2,\; a \neq b,\; b \in \ell^\infty(\mathbb{N}).$$

On the other hand, as a minimizing sequence from $c_0(\mathbb{N})$ one may choose $a^{(m)} = \frac{1}{2}\sum\limits_{n=1}^{m} u^{(n)},\; m \geq 1$. Then

$$\mathbb{E}\|X - a^{(m)}\|_\infty = \sum_{n\geq 1} p_n \|u^{(n)} - a^{(m)}\|_\infty = \frac{1}{2}\sum_{n=1}^{m} p_n + \sum_{n=m+1}^{\infty} p_n \overset{m\to+\infty}{\longrightarrow} 1/2.$$

Consequently,

$$e_{1,1}(X, c_0(\mathbb{N})) = 1/2$$

and since $a \notin c_0(\mathbb{N})$, it follows that $\mathcal{C}_{1,1}(X, c_0(\mathbb{N}))$ is empty.

This example is enlightened by the general Theorem 2. This theorem solves the correspondence between the quantization problem in E and in E^{**}. It shows that the quantization error does not decrease when X is seen as random vector in the bidual E^{**} of E and that the set of its optimal n-quantizers as an E-valued random vector is made up with those of its optimal n-quantizers as an E^{**}-valued random vector *that lie in* E. In particular, $\mathcal{C}_{n,r}(X, E) = \emptyset$ corresponds to the phenomenon that any optimal n-quantizer of $\mathcal{C}_{n,r}(X, E^{**})$ has at least one element in $E^{**} \setminus E$: this is precisely what happens in the above example.

Theorem 2 (a) *We have for every* $n \in \mathbb{N}$,

$$e_{n,r}(X, E) = e_{n,r}(X, E^{**}).$$

In particular,

$$\mathcal{C}_{n,r}(X, E) = \{\alpha \in \mathcal{C}_{n,r}(X, E^{**}) : \alpha \subset E\}.$$

If card$(\mathrm{supp}(\mathbb{P}_X)) \geq n$, *then* $e_{1,r}(X, E) > \cdots > e_{n,r}(X, E)$.
(b) *Assume that* E *is admissible. Further assume* $\mathrm{supp}(\mathbb{P}_X) = E$. *Then*

$$\mathcal{C}_{n,r}(X, E) = \mathcal{C}_{n,r}(X, E^{**}).$$

We first need the following equivariance properties contained in the lemma below.

Lemma 1 *Let* E_1 *and* E_2 *be Banach spaces and let* X *be a Radon* E_1*-valued random vector satisfying* $\mathbb{E}\|X\|^r < \infty$. *If* $S : E_1 \to E_2$ *is a bounded linear operator, then*

$$e_{n,r}(S(X), E_2) \leq \|S\| e_{n,r}(X, E_1).$$

If $S : E_1 \to E_2$ *is a bijective linear isometry,* $c > 0$ *and* $u_2 \in E_2$, *then*

$$e_{n,r}(c\,S(X) + u_2, E_2) = c\,e_{n,r}(X, E_1) \;\text{ and }\; \mathcal{C}_{n,r}(c\,S(X) + u_2, E_2) = c\,S\left(\mathcal{C}_{n,r}(X, E_1)\right) + u_2.$$

Proof. Let us prove *e.g.* the first assertion. For any $\alpha \subset E_1$ with $1 \leq$ card $(\alpha) \leq n$,

$$\begin{aligned} e_{n,r}(S(X), E_2) &\leq (\mathbb{E} \min_{a\in\alpha} \|S(X)) - S(a)\|^r)^{1/r} \\ &\leq \|S\|(\mathbb{E} \min_{a\in\alpha} \|X - a\|^r)^{1/r} \end{aligned}$$

and thus the assertion. □

Proof of Theorem 2. (a) The inequality

$$e_{n,r}(X, E) \geq e_{n,r}(X, E^{**})$$

is obvious. To prove the converse inequality assume first that $\mathrm{supp}(\mathbb{P}_X)$ is finite. Let $\alpha \in \mathcal{C}_{n,r}(X, E^{**})$ and let G denote the linear subspace of E^{**} spanned by $\mathrm{supp}(\mathbb{P}_X) \cup \alpha$. Since G is finite-dimensional, there exists by local reflexivity of E, for every $\varepsilon > 0$, a bounded linear operator $S : G \to E$ satisfying $\|S\| \leq 1 + \varepsilon$ and $S(x) = x$ for every $x \in G \cap E$. (cf. [20] Lemma 1.e.6). Using Lemma 1, one derives

$$\begin{aligned} e_{n,r}(X, E)^r &\leq \mathbb{E} \min_{b\in S(\alpha)} \|X - b\|^r = \mathbb{E} \min_{a\in\alpha} \|S(X) - S(a)\|^r \\ &\leq (1 + \varepsilon)^r e_{n,r}(X, E^{**})^r. \end{aligned}$$

Hence

$$e_{n,r}(X, E) \leq e_{n,r}(X, E^{**}).$$

For general X and $\varepsilon > 0$, choose a quantization $\widehat{X}_m : \Omega \to E$ of X, $\mathrm{card}(\widehat{X}_m(\Omega)) \leq m$, for sufficiently large m such that

$$\|X - \widehat{X}_m\|^{1\wedge r}_{L^r_E(\mathbb{P})} \leq \varepsilon.$$

Then,

$$|(e_{n,r}(X, E))^{r\wedge 1} - (e_{n,r}(\widehat{X}_m, E))^{r\wedge 1}| \leq \varepsilon$$

and

$$|(e_{n,r}(X, E^{**}))^{r\wedge 1} - (e_{n,r}(\widehat{X}_m, E^{**}))^{r\wedge 1}| \leq \|X - \widehat{X}_m\|^{1\wedge r}_{L^r_E(\mathbb{P})} \leq \varepsilon.$$

Since $\mathrm{card}(\mathrm{supp}(\mathbb{P}_{\widehat{X}_m})) \leq m < \infty$, we have $e_{n,r}(\widehat{X}_m, E) = e_{n,r}(\widehat{X}_m, E^{**})$. This yields

$$|(e_{n,r}(X, E))^{r\wedge 1} - (e_{n,r}(X, E^{**}))^{r\wedge 1}| \leq 2\varepsilon.$$

Hence $e_{n,r}(X, E) = e_{n,r}(X, E^{**})$. Furthermore, since $\mathcal{C}_{n,r}(X, E^{**}) \neq \emptyset$ by Corollary 1, it follows from Proposition 1(a) that $(e_{j,r}(X, E^{**}))_{1\leq j\leq n}$ is strictly decreasing provided $\mathrm{card}(\mathrm{supp}(\mathbb{P}_X)) \geq n$.
(b) The inclusion $\mathcal{C}_{n,r}(X, E) \subset \mathcal{C}_{n,r}(X, E^{**})$ follows from (a). To prove the converse inclusion, we may assume $\dim E \geq 1$. Let $\alpha \in \mathcal{C}_{n,r}(X, E^{**})$. By Proposition 2, for every $a \in \alpha$ there exists $b_a \in E$ such that for every $x \in E$,

$$\|b_a - x\| \leq \|a - x\|.$$

Setting $\beta = \{b_a : a \in \alpha\}$ this implies $\beta \in \mathcal{C}_{n,r}(X, E)$ and that the closed set

$$A := \{x \in E : \min_{b\in\beta} \|x - b\| = \min_{a\in\alpha} \|x - a\|\}$$

satisfies $\mathbb{P}_X(A) = 1$. Therefore, $A = E$ and in particular, $\beta \subset A$. One obtains $\min_{a\in\alpha} \|b - a\| = 0$ for every $b \in \beta$ and hence, $\beta \subset \alpha$. By Proposition 1(a), we have $\mathrm{card}(\alpha) = \mathrm{card}(\beta) = n$ which yields $\beta = \alpha$. Hence $\alpha \in \mathcal{C}_{n,r}(X, E)$. □

Remark. It is to be noticed that the situation $\mathcal{C}_{1,r}(X,E)=\emptyset$ never occurs for Gaussian (Radon) random vectors X. In view of Lemma 1, we may assume without loss of generality that X is centered. Let $r>0$. It follows from the Anderson inequality ([18]) that, for every $a\in E$,

$$\mathbb{E}\|X-a\|^r=\int_0^{+\infty}\mathbb{P}(\|X-a\|^r\geq t)dt\geq\int_0^{+\infty}\mathbb{P}(\|X\|^r\geq t)dt=\mathbb{E}\|X\|^r$$

so that $\{0\}\in\mathcal{C}_{1,r}(X,E)\neq\emptyset$. However, it remains an open question whether $\mathcal{C}_{n,r}(X,E)$ may be empty for $n\geq 2$ or not.

An immediate consequence of Theorem 2(a) is as follows. Let us call a Banach subspace F of E *locally c-complemented* ($c\geq 1$) if there is a linear operator $S:E\to F^{**}$ of norm $\|S\|\leq c$ satisfying $S(x)=x$ for every $x\in F$. Notice that local 1-complementation coincides with the notion of an ideal introduced in [12].

Corollary 2 *Assume that $\mathbb{P}_X(F)=1$ for some Banach subspace F of E and that F is locally 1-complemented in E. Then, for every $n\in\mathbb{N}$,*

$$e_{n,r}(X,F)=e_{n,r}(X,E).$$

In particular, $\mathcal{C}_{n,r}(X,F)\neq\emptyset$ implies $\mathcal{C}_{n,r}(X,E)\neq\emptyset$.

Proof. It follows from Theorem 2(a) and Lemma 1 that

$$\begin{aligned} e_{n,r}(X,F) &= e_{n,r}(X,F^{**})=e_{n,r}(S(X),F^{**}) \\ &\leq \|S\|e_{n,r}(X,E)=e_{n,r}(X,E)\leq e_{n,r}(X,F). \end{aligned}$$

□

One observes that the preceding corollary contains Theorem 2(a) since E is obviously locally 1-complemented in E^{**}.

Example • AM-spaces F are locally 1-complemented as Banach subspace in any Banach space E. In fact, since F^{**} is an order complete AM-space with unit, this feature follows from Theorem II.7.10 in [28]. For instance, if $E=C(T)$ for some compact metric space T and

$$F=\{f\in C(T):f(t)=0 \text{ for all } t\in T_0\}$$

for some closed subset T_0 of T, then F is a closed vector sublattice of the AM-space $C(T)$ and thus an AM-space.
• AL-spaces F are 1-complemented as Banach sublattice in any Banach lattice E (see [28], II.8).

Finite dimensional subspaces of dimension $d\geq 2$ are admissible but not necessarily (locally) 1-complemented. In fact, it may happen that $\mathbb{P}_X(F)=1$ for some 2-dimensional subspace F of E and $\mathcal{C}_{n,r}(X,E)=\emptyset$ even for $n=1$. In particular, (F,E) is not admissible. The following example is taken from Herrndorf [16].

A counterexample when dim $F=2$ Let $\ell^1(\mathbb{N})$ be equipped with the ℓ^1-norm $\|x\|=\sum_{j=1}^{\infty}|x_j|$. Let $(u^{(n)})_{n\geq 1}$ be the canonical basis of $\ell^1(\mathbb{N})$ and set $v^{(1)}:=0, v^{(2)}:=u^{(1)}-u^{(2)}$ and $v^{(3)}:=u^{(1)}-u^{(3)}$.

Consider the $\ell^1(\mathbb{N})$-valued random variable X supported by $\{v^{(1)}, v^{(2)}, v^{(3)}\}$ with $\mathbb{P}(X = v^{(i)}) = 1/3$. Let F denote the linear span of $\{v^{(2)}, v^{(3)}\}$ in $\ell^1(\mathbb{N})$. So $\mathbb{P}(X \in F) = 1$ and $\dim F = 2$.

Let $n = 1$ and $r = 1$. First will show that

$$e_{1,1}(X, F) = 4/3, \; e_{1,1}(X, \ell^1(\mathbb{N})) = 1$$

and

$$\mathcal{C}_{1,1}(X, \ell^1(\mathbb{N})) = \{\{u^{(1)}\}\}.$$

In fact,

$$\mathbb{E}\|X - u^{(1)}\| = \frac{1}{3}\sum_{i=1}^{3} \|v^{(i)} - u^{(1)}\| = 1.$$

On the other hand, once noticed that $\|v^{(i)} - v^{(j)}\| = 2$ for $i \neq j$, one shows like in the previous counterexample that for every $a \in \ell^1(\mathbb{N}), \mathbb{E}\|X - a\| = \frac{1}{3}\sum_{i=1}^{3} \|v^{(i)} - a\| \geq 1$ and that any L^1-optimal 1-quantizer $a \in \ell^1(\mathbb{N})$ must satisfy $\|v^{(i)} - a\| = 1$ for every $i \in \{1, 2, 3\}$ which implies $a = u^{(1)}$. As for $e_{1,1}(X, F)$, observe that

$$\mathbb{E}\|X - v^{(i)}\| = 4/3, \; i \in \{1, 2, 3\}.$$

Any $a \in F$ can be written as $a = (s + t)u^{(1)} - su^{(2)} - tu^{(3)}, s, t \in \mathbb{R}$, so that

$$\begin{aligned} \sum_{i=1}^{3} \|v^{(i)} - a\| &= |s + t| + |s| + |t| + |1 - s - t| + |1 - s| + |t| \\ &\quad + |1 - s - t| + |s| + |1 - t| \\ &\geq 4 \end{aligned}$$

since $|1 - t| + |t| \geq 1, t \in \mathbb{R}$. This yields $e_{1,1}(X, F) = 4/3$.

Now we construct a Banach subspace E of $\ell^1(\mathbb{N})$ such that $F \subset E$ and

$$\mathcal{C}_{1,1}(X, E) = \emptyset.$$

Choose $c = (c_j)_{j\geq 1} \in \ell^\infty(\mathbb{N})$ such that $c_1 = c_2 = c_3 = 1$ and $(c_j)_{j\geq 3}$ is strictly increasing with $\|c\|_\infty = \sup_{j\geq 1} |c_j| > 3$. Define E as the hyperplane

$$E := \{x \in \ell^1(\mathbb{N}) : \sum_{j=1}^{\infty} x_j c_j = 0\}.$$

Then $F \subset E$. For $k \geq 4$, set $a^{(k)} := u^{(1)} - \frac{1}{c_k} u^{(k)}$. One obtains $a^{(k)} \in E$ and

$$\mathbb{E}\|X - a^{(k)}\| = \frac{1}{3}\sum_{i=1}^{3} \|v^{(i)} - a^{(k)}\| = 1 + 1/c_k.$$

Consequently,

$$e_{1,1}(X, E) \leq 1 + 1/\|c\|_\infty < 4/3 = e_{1,1}(X, F).$$

For an arbitray $a \in E$ one gets the following: if $a_j = 0$ for $j \geq 4$, then $a \in F$ and hence $\mathbb{E}\|X - a\| \geq 4/3 > e_{1,1}(X, E)$. If $a_j \neq 0$ for some $j \geq 4$, a can be strictly improved. Set

$$b := a - a_j u^{(j)} + a_j c_j c_{j+1}^{-1} u^{(j+1)}.$$

One checks that $b \in E$ and for every $i \in \{1,2,3\}$,

$$\|v^{(i)} - b\| = \sum_{k=1}^{3} |v_k^{(i)} - a_k| + \sum_{\substack{k\geq 4\\ k\neq j,j+1}} |a_k| + |b_j| + |b_{j+1}| < \|v^{(i)} - a\|.$$

This implies

$$\mathbb{E}\|X - b\| < \mathbb{E}\|X - a\|.$$

Consequently, $\mathcal{C}_{1,1}(X,E) = \emptyset$.

2.2 Optimal quantizers for continuous stochastic processes

Now we turn to $\mathbb{R}^d$-valued pathwise continuous processes $X = (X_t)_{t\in T}$ indexed by a compact metric space T. The space $E := C_{\mathbb{R}^d}(T)$ of $\mathbb{R}^d$-valued continuous functions on T and the space $M^b_{\mathbb{R}^d}(T)$ of bounded, $\mathbb{R}^d$-valued, Borel measurable functions on T are Banach spaces under the norm

$$\|f\|_{\sup} := \sup_{t\in T} |f(t)|_\infty \tag{2.9}$$

where $|\cdot|_\infty$ denotes the ℓ^∞-norm on $\mathbb{R}^d$. Since $C_{\mathbb{R}^d}(T)$ is separable, X is Radon when viewed as $C_{\mathbb{R}^d}(T)$-valued random variable. Consequently, X is Radon as $M^b_{\mathbb{R}^d}(T)$-random variable.

Theorem 3 *Let T be compact metric space. Then the pair $(C_{\mathbb{R}^d}(T), M^b_{\mathbb{R}^d}(T))$ is admissible under the norm (2.9). In particular, if $X = (X_t)_{t\in T}$ is a $\mathbb{R}^d$-valued pathwise continuous process with $\mathbb{E}\|X\|^r_{sup} < \infty$, then for every $n \in \mathbb{N}$,*

$$\mathcal{C}_{n,r}(X, M^b_{\mathbb{R}^d}(T)) \neq \emptyset.$$

The proof of Theorem 3 is based on the admissibility of $L^\infty_{\mathbb{R}^d}$-spaces and the following "lifting property".

Lemma 2 *Let μ be a finite Borel measurable on the compact metric space T with $supp(\mu) = T$. Then for every $h \in M^b_{\mathbb{R}^d}(T)$ there exists $g \in M^b_{\mathbb{R}^d}(T)$ such that $g = h$ μ-a.e. and*

$$\|f - g\|_{\sup} = \|f - h\|_\infty \text{ for every } f \in C_{\mathbb{R}^d}(T)$$

where

$$\|h\|_\infty := \mu\text{-esssup } |h|_\infty. \tag{2.10}$$

Proof. One notes that for $h = (h_1, \ldots, h_d) \in M^b_{\mathbb{R}^d}(T)$,

$$\|h\|_{\sup} = \max_{1\leq i\leq d} \|h_i\|_{\sup}$$

and

$$\|h\|_\infty = \max_{1\leq i\leq d} \|h_i\|_\infty.$$

Therefore, it is enough to consider the case $d = 1$. Set $C(T) = C_{\mathbb{R}}(T)$ and $M^b(T) = M^b_{\mathbb{R}}(T)$. Let D be a countable dense subset of $C(T)$. Observe that the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{\sup}$ coincide on $C(T)$. This is a consequence of the assumption $\text{supp}(\mu) = T$. Let $h \in M^b(T)$. For $f \in C(T)$, set

$$c_f := \|f - h\|_\infty.$$

Then

$$N_f^+ := \{t \in T : f(t) - h(t) > c_f\}$$

and

$$N_f^- := \{t \in T : h(t) - f(t) > c_f\}$$

are Borel subsets of T with μ-measure zero. Consequently,

$$N := \bigcup_{f \in D} (N_f^+ \cup N_f^-)$$

satisfies $\mu(N) = 0$. Since for every $t \in T \setminus N$ and $f \in D$,

$$f(t) - h(t) \leq c_f \quad \text{and} \quad h(t) - f(t) \leq c_f$$

one obtains

$$\sup_{t \in K \setminus N} |f(t) - h(t)| \leq c_f, f \in D. \tag{2.11}$$

The construction of the function g is given in two steps.

Step 1. For $\varepsilon > 0$ and $t \in T$, let

$$d(t, \varepsilon) := \mu\text{-esssup } h_{|U(t,\varepsilon)},$$

where $U(t, \varepsilon)$ denotes the open ball in T of radius ε centered at t. Define the "upper limit function" $\hat{h} : T \to \mathbb{R}$ of h by

$$\hat{h}(t) := \lim_{\varepsilon \downarrow 0} d(t, \varepsilon).$$

One easily checks that for any Borel subset A of T, the function $T \to \mathbb{R}, t \mapsto \mu(U(t, \varepsilon) \cap A)$ is Borel. Therefore, for every $a \in \mathbb{R}$,

$$\begin{aligned} \{t \in T : \hat{h}(t) < a\} &= \left\{t \in T : \exists\, n \in \mathbb{N}, \exists\, m \in \mathbb{N} \text{ such that } h|U(t, \frac{1}{n}) \leq a - \frac{1}{m} \ \mu\text{-a.e.}\right\} \\ &= \bigcup_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \left\{t \in T : \mu(U(t, \frac{1}{n}) \cap \{h > a - \frac{1}{m}\}) = 0\right\} \end{aligned}$$

is a Borel set and thus $\hat{h}$ is Borel measurable. The function $\hat{h}$ has the following property: for every $t \in N$ there exists a sequence (t_n) in $T \setminus N$ such that $\lim_{n \to \infty} t_n = t$ and $\lim_{n \to \infty} h(t_n) = \hat{h}(t)$. In fact, let $t \in N$ and let $\varepsilon_n \downarrow 0$ so that $\hat{h}(t) = \lim_{n \to \infty} d(t, \varepsilon_n)$. For every $n \in \mathbb{N}$, there exists $t_n \in U(t, \varepsilon_n) \setminus N$ such that

$$d(t, \varepsilon_n) - \frac{1}{n} < h(t_n) \leq d(t, \varepsilon_n).$$

This implies

$$\lim_{n \to \infty} h(t_n) = \hat{h}(t) \text{ and } \lim_{n \to \infty} t_n = t.$$

Step 2. Define $g : T \to \mathbb{R}$ by

$$g(t) := \begin{cases} \hat{h}(t) & , t \in N \\ h(t) & , t \in T \setminus N. \end{cases}$$

We show that g has the required properties. Observe that g is Borel measurable, $g = h$ μ-a.e. and $\|g\|_{\sup} \leq \|h\|_{sup} < \infty$. Let $f \in D$ If $t \in T \setminus N$, then $g(t) = h(t)$ and hence by (2.4), $|f(t) - g(t)| \leq c_f$.

By step 1, if $t \in N$, there exists a sequence (t_n) in $T \setminus N$ such that $\lim t_n = t$ and $\lim h(t_n) = \hat{h}(t)$. Therefore,

$$\begin{aligned} |f(t) - g(t)| &= |f(t) - \hat{h}(t)| = |\lim_{n\to\infty} f(t_n) - \lim_{n\to\infty} h(t_n)| \\ &= \lim_{n\to\infty} |f(t_n) - h(t_n)| \\ &\leq \sup_{s\in T\setminus N} |f(s) - h(s)| \leq c_f. \end{aligned}$$

Consequently,

$$\|f - g\|_{\sup} \leq c_f, f \in D. \tag{2.12}$$

Now let $f \in C(T)$. There exists a sequence (f_n) in D such that $\lim_{n\to\infty} \|f - f_n\|_{\sup} = 0$. For every $t \in T$,

$$\begin{aligned} |f(t) - g(t)| &= |\lim_{n\to\infty} f_n(t) - g(t)| = \lim_{n\to\infty} |f_n(t) - g(t)| \\ &\leq \limsup_{n\to\infty} c_{f_n}. \end{aligned}$$

Since

$$c_{f_n} = \|f_n - h\|_\infty \leq \|f_n - f\|_{\sup} + c_f$$

one obtains

$$\|f - g\|_{\sup} \leq \limsup_{n\to\infty} c_{f_n} \leq c_f. \tag{2.13}$$

Conversely, we clearly have

$$c_f = \|f - h\|_\infty = \|f - g\|_\infty \leq \|f - g\|_{\sup}.$$

□

Proof of Theorem 3. Let $\mathcal{K} = \{B_{M^b}(f_i, \rho_i) : i \in I\}$ be a system of closed balls in $M^b_{\mathbb{R}^d}(T)$ with centers $f_i \in C_{\mathbb{R}^d}(T)$ satisfying the finite intersection property. Choose a finite Borel measure μ on T such that $\text{supp}(\mu) = T$ and consider the system $\tilde{\mathcal{K}} = \{B_{L^\infty}(Sf_i, \rho_i) : i \in I\}$ of corresponding closed balls in $L^\infty_{\mathbb{R}^d}(\mu)$ under the norm $\|\cdot\|_\infty$ (see (2.10)) where $S : M^b_{\mathbb{R}^d}(T) \to L^\infty_{\mathbb{R}^d}(\mu)$ denotes the quotient map. It is obvious that $\tilde{\mathcal{K}}$ also has the finite intersection property. Since $L^\infty_{\mathbb{R}^d}(\mu)$ is admissible by Proposition 2 and Corollary 1, $\tilde{\mathcal{K}}$ has a nonempty intersection. Let $S(h)$ be a member of this intersection. Lemma 2 implies that there is a function $g \in M^b_{\mathbb{R}^d}(T)$ such that $g = h$ μ-a.e. and

$$\|f_i - g\|_{\sup} = \|f_i - h\|_\infty = \| Sf_i - Sh \|_\infty \text{ for every } i \in I.$$

Consequently, g belongs to the intersection of $\mathcal{K}$. This yields the required admissibility. □

One derives from Corollary 2 that the quantization error does not decrease when X is seen as $M^b_{\mathbb{R}^d}(T)$-or even $L^\infty_{\mathbb{R}^d}(\mu)$-valued random variable.

Theorem 4 *Assume that $X = (X_t)_{t\in T}$ is a $\mathbb{R}^d$-valued pathwise continuous process indexed by a compact metric space T with $\mathbb{E}\|X\|^r_{sup} < \infty$. . Let μ be a finite Borel measure on T with $supp(\mu) = T$. Then for every $n \in \mathbb{N}$,*

$$e_{n,r}(X, C_{\mathbb{R}^d}(T)) = e_{n,r}(X, M^b_{\mathbb{R}^d}(T)) = e_{n,r}(X, L^\infty_{\mathbb{R}^d}(\mu)),$$

where $C_{\mathbb{R}^d}(T)$ and $M^b_{\mathbb{R}^d}(T)$ are equipped with the sup-norm (2.9) and $L^\infty_{\mathbb{R}^d}(\mu)$ is equipped with the norm (2.10). In particular,

$$\begin{aligned}\mathcal{C}_{n,r}(X, C_{\mathbb{R}^d}(T)) &= \{\alpha \in \mathcal{C}_{n,r}(X, M^b_{\mathbb{R}^d}(T)) : \alpha \subset C_{\mathbb{R}^d}(T)\} \\ &= \{\alpha \in \mathcal{C}_{n,r}(X, L^\infty_{\mathbb{R}^d}(\mu)) : (\text{a } \mu - \text{version of })\alpha \subset C_{\mathbb{R}^d}(T)\}.\end{aligned}$$

Proof. $C_{\mathbb{R}^d}(T)$ is an AM-space so that Corollary 2 applies. We obtain

$$e_{n,r}(X, C_{\mathbb{R}^d}(T)) = e_{n,r}(X, M^b_{\mathbb{R}^d}(T)).$$

Since $C_{\mathbb{R}^d}(T)$ can be considered as a subspace of $L^\infty_{\mathbb{R}^d}(\mu)$, the same argument yields

$$e_{n,r}(X, C_{\mathbb{R}^d}(T)) = e_{n,r}(X, L^\infty_{\mathbb{R}^d}(\mu)).$$

(The latter equality is also an immediate consequence of Lemma 2.) □

We will exhibit a pathwise continuous process $X = (X_t)_{t\in[0,1]}$ having no L^1-optimal 1-quantizer in $C([0,1])$. In particular, due to the lack of order completeness, $C([0,1])$ is not admissible.

Optimal 1-quantizer may not exist in $C([0,1])$ Let $(E, \|\cdot\|) = (C([0,1]), \|\cdot\|_{\sup})$. Define, for every $n \in \mathbb{N}$, a continuous function $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(t) := \begin{cases} 0 & \text{if } \ t \in [0, \frac{1}{2} - 2^{-n}] \cup [\frac{1}{2} - 2^{-(n+1)}, \frac{1}{2}] \\ 2^{n+1}(2t-1) + 4 & \text{if } \ t \in [\frac{1}{2} - 2^{-n}, \frac{1}{2} - 3 \cdot 2^{-(n+2)}] \\ 2^{n+1}(1-2t) - 2 & \text{if } \ t \in [\frac{1}{2} - 3 \cdot 2^{-(n+2)}, \frac{1}{2} - 2^{-(n+1)}] \\ -f_n(1-t) & \text{if } \ t \in [\frac{1}{2}, 1]. \end{cases}$$

One considers an E-valued random variable X supported by $\{f_n : n \geq 1\}$ with $p_n := \mathbb{P}(X = f_n)$ satisfying $p_n \in (0, 1/2)$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} p_n = 1$. The assumption of Theorem 1 is not fulfilled since the system $\{B_E(f_n, \frac{1}{2}) : n \geq 1\}$ has the finite intersection property whereas it has an empty intersection (see below).

Let $n = 1$ and $r = 1$. We will show that

$$e_{1,1}(X, E) = 1/2 \text{ and } \mathcal{C}_{1,1}(X, E) = \emptyset.$$

Recall that by Theorem 4, $e_{1,1}(X, E) = e_{1,1}(X, G)$ where $G = M^b([0,1])$ equipped with $\|\cdot\|_{\sup}$. Set $h := \frac{1}{2}(1_{[0,1/2]} - 1_{(1/2,1]})$. One checks that, for every $n \geq 1$,

$$\|f_n - h\|_{\sup} = 1/2$$

so that

$$\mathbb{E}\|X - h\|_{\sup} = \sum_{n=1}^{\infty} p_n \|f_n - h\|_{\sup} = 1/2.$$

On the other hand, one shows like in the $c_0(\mathbb{N})$-counterexample preceding Theorem 2 that for every $g \in G, \mathbb{E}\|X - g\|_{\sup} \geq 1/2$ and that any L^1-optimal 1-quantizer $\{g\}$ must satisfy $\|f_n - g\|_{\sup} = 1/2$ for every $n \in \mathbb{N}$: one reproduces the string of inequalities starting at (2.8) once noticed that $\|f_n - f_m\|_{\sup} = 1$ for every $n \neq m$. This implies $e_{1,1}(X, G) = 1/2$ and $\{h\} \in \mathcal{C}_{1,1}(X, G)$. Furthermore, no $g \in E$ can satisfy the condition $\|f_n - g\|_{\sup} = 1/2$ for every $n \in \mathbb{N}$. In fact, if $g(1/2) < 1/2$, then $g(t_n) < 1/2$ with $t_n = \frac{1}{2} - 3 \cdot 2^{-(n+2)}$ and n large enough so that

$$|f_n(t_n) - g(t_n)| = 1 - g(t_n) > 1/2.$$

If $g(1/2) \geq 1/2$, then $g(1 - t_n) \geq 0$ for n large enough so that

$$|g(1 - t_n) - f_n(1 - t_n)| = g(1 - t_n) + 1 \geq 1.$$

Consequently, $\mathcal{C}_{1,1}(X, E) = \emptyset$.

2.3 Bounds for quantization errors

As before let X be a Radon random variable in $(E, \|\cdot\|)$ satisfying the integrability condition (1.2). The following observation (a) is already contained in [4].

Proposition 3 *Assume that $\mathbb{P}_X(F) = 1$ for some Banach subspace F of E.*
(a) For every $n \in \mathbb{N}$,

$$e_{n,r}(X, E) \leq e_{n,r}(X, F) \leq 2e_{n,r}(X, E).$$

(b) If F is locally c-complemented in E, then for every $n \in \mathbb{N}$,

$$e_{n,r}(X, F) \leq ce_{n,r}(X, E).$$

Proof. (a) We have to prove only the second inequality. Let $\alpha = \{a_1, \ldots, a_n\} \subset E$ and $\varepsilon > 0$. Choose $b_i \in F$ such that $\|a_i - b_i\| \leq (1 + \varepsilon)\mathrm{dist}(a_i, F)$. This implies that

$$\|a_i - b_i\| \leq (1 + \varepsilon)\|X - a_i\| \text{ a.e.}$$

for every $i \in \{1, \ldots, n\}$ and hence

$$\min_{1 \leq i \leq n} \|X - b_i\| \leq (2 + \varepsilon) \min_{1 \leq i \leq n} \|X - a_i\| \text{ a.e.}$$

Consequently,

$$e_{n,r}(X, F) \leq (2 + \varepsilon)(\mathbb{E} \min_{a \in \alpha} \|X - a\|^r)^{1/r}.$$

This yields the assertion.
(b) is an immediate consequence of Theorem 2(a) and Lemma 1. □

It is to be noticed that the factor 2 in part (a) of the preceding proposition is sharp. It cannot be improved as universal constant. This is demonstrated in the subsequent Example. In view of (a), the cases of interest in part (b) are $c < 2$.

The constant 2 is sharp We modify the setting of the counterexample following Corollary 2. Let $E = \ell^1(\mathbb{N}), \|x\| = \sum_{j=1}^{\infty} |x_j|$ and let $(u^{(n)})_{n \geq 1}$ denote the canonical basis of E. Fix $m \in \mathbb{N}, m \geq 2$ and set $v^{(i)} := u^{(1)} - u^{(i)}, i \in \{1, \ldots, m\}$. One considers the E-valued random variable X supported by $\{v^{(1)}, \ldots, v^{(m)}\}$ with $\mathbb{P}(X = v^{(i)}) = 1/m$. Let F denote the linear span of $\{v^{(1)}, \ldots, v^{(m)}\}$. So $\mathbb{P}(X \in F) = 1$.

Let $n = 1$ and $r = 1$. One checks like in the above mentioned counterexample that

$$e_{1,1}(X, E) = 1.$$

We will show that

$$e_{1,1}(X, F) = 2(m - 1)/m.$$

In fact, for $j \in \{1, \ldots, m\}$,

$$\mathbb{E}\|X - v^{(j)}\| = \frac{1}{m} \sum_{i=1}^{m} \|v^{(i)} - v^{(j)}\| = 2(m - 1)/m.$$

Any $a \in F$ can be written as $a = \sum_{j=2}^{m} s_j u^{(1)} - \sum_{j=2}^{m} s_j u^{(j)}, s_j \in \mathbb{R}$ and hence

$$\begin{aligned} \|v^{(1)} - a\| &= \|a\| = |\sum_{j=2}^{m} s_j| + \sum_{j=2}^{m} |s_j|, \\ \|v^{(i)} - a\| &= |1 - \sum_{j=2}^{m} s_j| + |1 - s_i| + \sum_{\substack{j=2 \\ j \neq i}}^{m} |s_j|, i \in \{2, \ldots, m\}. \end{aligned}$$

Using the elementary inequalities $|1-t|+|t| \geq 1$ and $|1-s-t|+|s|+|t| \geq 1, s,t \in \mathbb{R}$, one obtains

$$\sum_{i=1}^{m} \|v^{(i)} - a\| \geq 2(m-1).$$

Consequently,

$$\mathbb{E}\|X-a\| \geq 2(m-1)/m.$$

Next we describe marginal bounds for $\mathbb{R}^d$-valued stochastic processes. For $p \in [1,\infty)$, let $E = L^p_{\mathbb{R}^d}(T, \mathcal{B}, \mu)$, μ finite measure, equipped with the norm

$$\|f\|_p := (\int |f(t)|_p^p d\mu(t))^{1/p} = (\sum_{i=1}^{d} \int |f_i(t)|^p d\mu(t))^{1/p}. \tag{2.14}$$

Assume that E is separable. Let $X = (X_t)_{t\in T} = (X_{1,t}, \dots X_{d,t})_{t\in T}$ be a bi-measurable $\mathbb{R}^d$-valued process such that

$$\mathbb{E}\|X\|_p^p < \infty. \tag{2.15}$$

Then the process X can be seen as a (Radon) random vector taking its values in $L^p_{\mathbb{R}^d}$. For the sake of simplicity, we consider the case $r = p$. As for bounds when constants are not important there will be no loss of generality since usual inequalities on L^p-norms imply for $r \in [1,\infty)$

$$\mu(T)^{\frac{1}{p}-\frac{1}{p\wedge r}} e_{n,p\wedge r}(X, L^{p\wedge r}_{\mathbb{R}^d}) \leq e_{n,r}(X, L^p_{\mathbb{R}^d}) \leq \mu(T)^{\frac{1}{p}-\frac{1}{p\wedge r}} e_{n,p\vee r}(X, L^{p\vee r}_{\mathbb{R}^d}).$$

Proposition 4 *Let $p \in [1,\infty)$. For every $n, n_1, \dots, n_d \in \mathbb{N}$ such that $\prod_{i=1}^{d} n_i \leq n$,*

$$\sum_{i=1}^{d} e_{n,p}(X_i, L^p)^p \leq e_{n,p}(X, L^p_{\mathbb{R}^d})^p \leq \sum_{i=1}^{d} e_{n_i,p}(X_i, L^p)^p.$$

Proof. As for the upper estimate, let $\alpha_i \subset L^p$ be a L^p-optimal n_i-quantizer for $X_i, i \in \{1,\dots,d\}$ (see Corollary 1). Set $\alpha := \times_{i=1}^{d} \alpha_i$. Thus α consists of functions $a = (a_1, \dots, a_d) \in L^p_{\mathbb{R}^d}$ with $a_i \in \alpha_i$ and $\mathrm{card}(\alpha) \leq n$. One obtains

$$\begin{aligned} e_{n,p}(X, L^p_{\mathbb{R}^d})^p &\leq \mathbb{E}\min_{a\in\alpha} \|X-a\|_p^p \\ &= \mathbb{E}\min_{a\in\alpha} \sum_{i=1}^{d} \int |X_{i,t} - a_i(t)|^p d\mu(t) \\ &= \mathbb{E}\sum_{i=1}^{d} \min_{b\in\alpha_i} \int |X_{i,t} - b(t)|^p d\mu(t) \\ &= \sum_{i=1}^{d} e_{n_i,p}(X_i, L^p)^p. \end{aligned}$$

As for the lower estimate, let $\alpha \subset L^p_{\mathbb{R}^d}$ with $\mathrm{card}(\alpha) \leq n$. Then

$$\begin{aligned} \mathbb{E}\min_{a\in\alpha} \|X-a\|_p^p &\geq \mathbb{E}\sum_{i=1}^{d} \min_{a\in\alpha} \int |X_{i,t} - a_i(t)|^p d\mu(t) \\ &\geq \sum_{i=1}^{d} e_{n,p}(X_i, L^p)^p. \end{aligned}$$

This yields the lower estimate. □

Now let T be a compact metric space and assume that $X = (X_t)_{t\in T}$ is a $\mathbb{R}^d$-valued continuous process. Let $E = C_{\mathbb{R}^d}(T)$ equipped with the sup-norm (2.9). Assume

$$\mathbb{E}\|X\|_{\sup}^r < \infty. \tag{2.16}$$

Proposition 5 *Let $r \in (0,\infty)$. Let $c \in (0,\infty)$ such that $|\cdot|_\infty \le c|\cdot|_r$. Then for every $n, n_1, \dots, n_d \in \mathbb{N}$ such that $\Pi_{i=1}^d n_i \le n$,*

$$\max_{1\le i\le d} e_{n,r}(X_i, C(T))^r \le e_{n,r}(X, C_{\mathbb{R}^d}(T))^r \le c^r \sum_{i=1}^d e_{n_i,r}(X_i, C(T))^r.$$

Proof. For $i \in \{1,\dots,d\}$ and $\varepsilon > 0$, choose $\alpha_i \subset C(T)$ such that $\mathrm{card}(\alpha_i) \le n_i$ and

$$\mathbb{E}\min_{b\in\alpha_i}\|X_i - b\|_{\sup}^r \le e_{n_i,r}(X_i, C(T))^r + \varepsilon.$$

Set $\alpha := \times_{i=1}^d \alpha_i$. Then $\alpha \subset C_{\mathbb{R}^d}(T)$, $\mathrm{card}(\alpha) \le n$ and

$$\begin{aligned}
\mathbb{E}\min_{a\in\alpha}\|X-a\|_{\sup}^r &\le c^r\mathbb{E}\min_{a\in\alpha}\sup_{t\in T}\sum_{i=1}^d |X_{i,t} - a_i(t)|^r \\
&\le c^r\mathbb{E}\min_{a\in\alpha}\sum_{i=1}^d \|X_i - a_i\|_{\sup}^r \\
&= c^r\mathbb{E}\sum_{i=1}^d \min_{b\in\alpha_i}\|X_i - b\|_{\sup}^r \\
&\le c^r\sum_{i=1}^d e_{n_i,r}(X_i, C(T))^r + c^r d\varepsilon.
\end{aligned}$$

This yields the upper esxtimate. As for the lower estimate, let $\alpha \subset C_{\mathbb{R}^d}(T)$ with $\mathrm{card}(\alpha) \le n$. Then for every i,

$$\mathbb{E}\min_{a\in\alpha}\|X-a\|_{\sup}^r \ge \mathbb{E}\min_{a\in\alpha}\|X_i - a_i\|_{\sup}^r \ge e_{n,r}(X_i, C(T))^r$$

which gives the lower estimate. □

In the preceding proposition one may replace $C_{\mathbb{R}^d}(T)$ and $C(T)$ by $L^\infty_{\mathbb{R}^d}(\mu)$ and $L^\infty(\mu)$ respectively for any finite Borel measure μ on T with $\mathrm{supp}(\mu) = T$. This follows from Theorem 4.

3 Stationary quantizers

Let X be a Radon $(E, \|\cdot\|)$-valued random variable satisfying condition (1.2). We will introduce a notion of L^r-stationary quantizer as the critical points of level n L^r-distortion function $D^X_{n,r}$ formerly defined by Equation (2.6). For a quantizer $\alpha = \{a_1,\dots,a_n\}$ let $V_i(\alpha) = V_{a_i}(\alpha)$ and $C_i(\alpha) = C_{a_i}(\alpha)$.

Definition 2 *A n-quantizer $\alpha = \{a_1,\dots,a_n\} \subset E$ of size n is called admissible for X if*

$$\begin{cases} (i) & \mathbb{P}_X(V_i(\alpha)) > 0, \quad i = 1,\dots,n, \\ (ii) & \mathbb{P}_X(V_i(\alpha)\cap V_j(\alpha)) = 0, \quad i,\, j = 1,\dots,n,\ i \ne j. \end{cases}$$

A n-tuple $(a_1,\dots,a_n) \in E^n$ is admissible if its associated n-quantizer is.

Proposition 6 *Assume that E is smooth. Let $r > 1$. Then the L^r-distortion function $D^X_{n,r}$ is Gateaux-differentiable at every admissible n-tuple $(a_1, \ldots, a_n)$ with a Gateaux differential given by*

$$\nabla D^X_{n,r}(a_1, \ldots, a_n) = r\left(\mathbb{E}\left(\mathbf{1}_{C_i(\alpha)\setminus\{a_i\}}(X)\|X - a_i\|^{r-1}\nabla\|\,.\,\|(a_i - X)\right)\right)_{1\le i\le n} \in (E^*)^n$$

where $\{C_i(\alpha) : 1 \le i \le n\}$ denotes any Voronoi partition induced by $\alpha = \{a_1, \ldots, a_n\}$. If the norm is Fréchet-differentiable at every $x \neq 0$, then $\nabla D^X_{n,r}(a_1, \ldots, a_n)$ is the Fréchet derivative. Furthermore, if E is uniformly smooth, then $(a_1, \ldots, a_n) \mapsto \nabla D^X_{n,r}(a_1, \ldots, a_n)$ is continuous on the set of admissible n-tuples (where E^* is endowed with its norm).

When $r = 1$, the above results extend to admissible n-tuples with $\mathbb{P}_X(\{a_1, \ldots, a_n\}) = 0$.

Remark. In case $E = L^1$, the above proposition as well as Proposition 1(b) do not apply since the $\|\,.\,\|_1$-norm is neither smooth nor strictly convex.

Proof. A straightforward adaptation of Lemma 4.10 in [13] yields both differentiability properties. Then, if E is uniformly smooth, the mapping $x \mapsto \nabla\|\,.\,\|(x)$ is continuous (see [2]). One derives the continuity of $\nabla D^X_{n,r}$ by the Lebesgue dominated convergence theorem using that $\nabla\|\,.\,\|$ takes its values in the unit ball of E^*. □

Definition 3 *Let E be a Banach space and let $r \ge 1$. A n-quantizer $a = \{a_1, \ldots, \alpha_n\} \subset E$ of size n is called L^r-stationary for X if $\mathbb{P}_X(C_i(\alpha)) > 0$ and*

$$\mathbb{E}\left(\mathbf{1}_{C_i(\alpha)\setminus\{a_i\}}(X)\|X - a_i\|^{r-1}\nabla\|\,.\,\|(a_i - X)\right) = 0, \qquad i = 1, \ldots, n, \tag{3.1}$$

where $\{C_i(\alpha) : 1 \le i \le n\}$ denotes any Voronoi partition induced by α. (This requires that the Gateaux-differential $\nabla\|\,.\,\|(a_i - x)$ is defined $\mathbb{P}_X(dx)$-a.e. on $C_i(\alpha) \setminus \{a_i\}$ and, furthermore, that $\mathbb{P}(X \in \alpha) = 0$ when $r = 1$).

This finally leads to the following proposition which makes the (expected) connection between optimality and stationarity.

Proposition 7 *Assume that E is smooth and strictly convex. Let $r > 1$. Assume that card$(\operatorname{supp}\mathbb{P}_X) \ge n$. Then any L^r-optimal n-quantizer α is L^r-stationary (and admissible) for X. This extends to $r = 1$ if $\mathbb{P}_X(\alpha) = 0$.*

Proof. Any L^r-optimal n-quantizer $\alpha = \{a_1, \ldots, a_n\}$ is admissible by Proposition 1(b), hence the Gateaux-differential $\nabla D^X_{n,r}(a_1, \ldots, a_n)$ does exist and is 0 which exactly means stationarity. □

3.1 Stationarity for stochastic processes

Let $(T, \mathcal{B}, \mu)$ be a finite measure space, let $X = (X_t)_{t\in T}$ be a bi-measurable $\mathbb{R}^d$-valued process defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $p, r \in [1, +\infty)$. Assume that $L^p_{\mathbb{R}^d}(\mu)$ is separable and that $\|X\|_p \in L^r(\mathbb{P})$ i.e.

$$\mathbb{E}\left(\int_T |X_t|_p^p d\mu(t)\right)^{r/p} < +\infty. \tag{3.2}$$

Then, the process X can be seen as a (Radon) random vector taking its values in the Banach space $(E, \|\,.\,\|) = (L^p_{\mathbb{R}^d}(\mu), \|\,.\,\|_p)$ satisfying an L^r-integrability property, that is $X \in L^r_{L^p_{\mathbb{R}^d}}(\mathbb{P})$. When $p \neq 1$, the $L^p_{\mathbb{R}^d}$-spaces are uniformly smooth and strictly convex, so the above abstract results apply.

Furthermore, if q denotes the conjugate Hölder exponent of p, for every $f = (f_1, \dots, f_d) \in L^p_{\mathbb{R}^d}$, $f \not\equiv 0$,

$$\nabla \| \cdot \|_p(f) = \left(\left(\frac{|f_j|}{\|f\|_p} \right)^{p-1} \mathrm{sign} f_j \right)_{1 \le j \le d} \in E^* = L^q_{\mathbb{R}^d}$$

so that the $(L^r, \| \cdot \|_p)$-stationarity condition reads for any Voronoi partition $\{C_i(\alpha) : 1 \le i \le n\}$ with $\mathbb{P}_X(C_i(\alpha)) > 0$, for every i,

$$\mathbb{E}\left(\mathbf{1}_{C_i(\alpha)}(X) \|X - a_i\|_p^{r-p} |a_{ij} - X_j|^{p-1} \mathrm{sign}(a_{ij} - X_j) \right) \overset{L^q}{=} 0, \qquad i = 1, \dots, n, j = j, \dots, d \tag{3.3}$$

with the convention $\frac{0}{\|0\|} = 0$, where $a_i = (a_{i1}, \dots a_{id})$. When $p = 1$, the condition is formally the same. This may be written in a more synthetic way by introducing the α-quantization $\widehat{X} := \widehat{X}^\alpha$ of X defined by (1.3), namely:

$$\mathbb{E}\left(\|X - \widehat{X}\|_p^{r-p} |X_j - \widehat{X}_j|^{p-1} \mathrm{sign}(\widehat{X}_j - X_j) \,|\, \widehat{X} \right) \overset{L^q}{=} 0. \tag{3.4}$$

When $p = 2$, $r \ge 2$ (and $\mathbb{P}(X \in \alpha) = 0$ if $r > 2$), Equation (3.3) looks simpler and reads

$$a_i \overset{L^2_{\mathbb{R}^d}}{=} \frac{\mathbb{E}(X \mathbf{1}_{C_i(\alpha)}(X) \|X - a_i\|_2^{r-2})}{\mathbb{E}(\mathbf{1}_{C_i(\alpha)}(X) \|X - a_i\|_2^{r-2})}, \qquad 1 \le i \le n. \tag{3.5}$$

One derives from Proposition 7 and Proposition 1 the following corollary.

Corollary 3 *Let $p, r \in [1, +\infty)$, let $n \ge 1$. If*

$$\begin{cases} p,\, r > 1 & \textit{and} \quad \textit{card}(\mathrm{supp}\mathbb{P}_X) \ge n, \\ p > 1,\ r = 1 & \textit{and} \quad \mathbb{P}_X \ \textit{is continuous}, \\ p = 1,\, r \ge 1 & \textit{and} \quad \mathbb{P}_{X_{j,t}} \ \textit{is } \mu(dt)\textit{-a.e. continuous for every } j \in \{1, \dots, d\}, \end{cases} \tag{3.6}$$

then, any $(L^r, \| \cdot \|_p)$-optimal n-quantizer is $(L^r, \| \cdot \|_p)$-stationary in the sense of (3.3).

Proof. It remains to consider the case $p = 1$. The space $L^1_{\mathbb{R}^d}$ is not smooth. However, $\|.\|_1$ is Gateaux-differentiable at every f such that $f_j(t) \neq 0$ $\mu(dt)$ - a.e. for every j. Now, by the Fubini Theorem, one has for every $g \in L^1(\mu)$

$$\int_\Omega \mu(t : X_{j,t}(\omega) = g(t)) \mathbb{P}(d\omega) = \int_T \mathbb{P}(X_{j,t} = g(t)) \mu(dt) = 0$$

i.e. $(X_{j,t} - g(t) \neq 0$ $\mu(dt)$-a.e.) $\mathbb{P}$-a.s. Let $\alpha = \{a_1, \dots, a_n\}$ be an $(L^r, \| \cdot \|_1)$-optimal n-quantizer and $\mathbb{P}_i := \mathbb{P}(\cdot | \{X \in C_i(\alpha)\})$. This definition is consistent since $\mathbb{P}(X \in C_i(\alpha)) > 0$ by Proposition 1 (a). It follows easily that $\Psi_i : f \mapsto \int \|X - f\|_1^r d\mathbb{P}_i$, $f \in L1_{\mathbb{R}^d}$, is Gateaux differentiable with a Gateaux-differential given by

$$\bigtriangledown \Psi_i(f) = \left(r \int \|X - f\|_1^{r-1} \,\mathrm{sign}\,(f_j - X_j) d\mathbb{P}_i \right)_{1 \le j \le d} \in L^\infty_{\mathbb{R}^d}.$$

Now, still following Proposition $1(a)$, a_i is a minimum for Ψ_i so that its Gateaux differential is zero. Hence, for every $i \in \{1, \dots, n\}, j \in \{1, \dots, d\}$,

$$\int 1_{C_i(\alpha)}(X) \|X - a_i\|_1^{r-1} \,\mathrm{sign}(a_{ij} - X_j) d\mathbb{P} = 0.$$

□

Remark. Continuity of $\mathbb{P}_{X_j,t}$ $\mu(dt)$ - a.e. for some j implies continuity of $\mathbb{P}_X$.

3.2 Pathwise regularity of stationary quantizers ($1 \le p \le r < +\infty$)

As before, let $E = L^p_{\mathbb{R}^d}(\mu)$ for some finite measure space $(T, \mathcal{B}, \mu)$ such that E is separable. We will derive from Equations (3.3) (and (3.5)) some pathwise continuity result for the $(L^r, \|\,.\,\|_p)$-stationary quantizers (which extends a result established in [21] in the purely quadratic case $p = r = 2$). For $q \in (0, \infty)$, if $X_t \in L^q_{\mathbb{R}^d}(\mathbb{P})$ for every $t \in T$, define the *"intrinsic" semimetric* ρ^q_X on T by

$$\rho^q_X(s,t) := (\mathbb{E}|X_s - X_t|^q_q)^{1/(q\vee 1)} = \|X_s - X_t\|^{q/(q\vee 1)}_{L^q_{\mathbb{R}^d}(\mathbb{P})},\ s,t \in T.$$

Theorem 5 *Let $p, r \in [1, +\infty)$, $r \ge p$. Let X be a bi-measurable $\mathbb{R}^d$-valued process satisfying (3.2) and*

$$\forall\, t \in T, \qquad X_t \in L^{r-1}_{\mathbb{R}^d}(\mathbb{P}).$$

Let $\alpha = \{a_1, \ldots, a_n\}$ be an $(L^r, \|\,.\,\|_p)$-stationary n-quantizer (in the sense of (3.3)). Set $I_r(\alpha) := \{i \in \{1, \ldots, n\} :\ \mathbb{P}(X = a_i) = 0\}$ if $r > p$ and $I_r(\alpha) := \{1, \ldots, n\}$ otherwise.
(a) Let T be a compact metric space and let μ be a continuous finite Borel measure on T. If $p = 1$, if X is pathwise continuous with

$$supp(\mathbb{P}_X) = \{f \in C_{\mathbb{R}^d}(T) : f(t) = x, t \in T_0\}$$

(in case X is viewed as a $(C_{\mathbb{R}^d}(T), \|\cdot\|_{sup})$-random vector) for some $x \in \mathbb{R}^d$ and some closed subset T_0 of T with $\mu(T_0) = 0$ and if the distribution $\mathbb{P}_{X_{j,t}}$ is continuous on $\mathbb{R}$ for every $t \in T \setminus T_0, j \in \{1, \ldots, d\}$, then the components of α have μ-versions consisting of continuous functions such that $a_i(t) = x, t \in T_0, i = 1, \ldots, n$.
(b) If $p \in (1, \infty)$, then the components $a_i, i \in I_r(\alpha)$ of α have μ-versions consisting of ρ^{r-1}_X-continuous functions. Furthermore, if $X_t = x \in \mathbb{R}^d, t \in T_0 \subset T$, then there are such versions with $a_i(t) = x, t \in T_0$.
(c) If $p = 2$, then the components $a_i, i \in I_r(\alpha)$ of α have μ-versions consisting of ρ^{r-1}_X-Lipschitz continuous functions.

Remarks. • If $\mathbb{P}_X$ is continuous then $I_r(\alpha) = \{1, \ldots, n\}$.
• If $r \ge p = 2, \mathbb{E}X = 0$ and $\mathbb{E}\|X\|_2^{2r-4} < \infty$, then $\{a_i : i \in I_r(\alpha)\}$ even lies in the reproducing kernel Hilbert space of X. This is a consequence of (3.5).
• Let (T, ρ) be a separable metric space and μ a finite Borel measure on (T, ρ). If $p > 1$ and $t \mapsto X_t$ from (T, ρ) into $L^{r-1}_{\mathbb{R}^d}(\mathbb{P})$ is continuous that is ρ^{r-1}_X is majorized by the initial metric ρ on T, then the $I_r(\alpha)$-components of α have versions consisting of ρ-continuous functions. The $L^{r-1}_{\mathbb{R}^d}(\mathbb{P})$-continuity assumption is fulfilled e.g. if X is pathwise ρ-continuous and $\|X\|_{\sup} \in L^{r-1}(\mathbb{P})$.

Proof of Theorem 5. For every $i \in I_r(\alpha)$, set $\mathbb{Q}_{i,r} = \mathbf{1}_{C_i(\alpha)}(X)\|X - a_i\|_p^{r-p}.\mathbb{P}$. The measure $\mathbb{Q}_{i,r}$ is finite: if $r = p$, this is obvious, otherwise,

$$\mathbb{Q}_{i,r}(\Omega) \le \mathbb{E}\,\|X - a_i\|_p^{r-p} \le \left(\mathbb{E}\|X - a_i\|_p^r\right)^{1-\frac{p}{r}} < +\infty.$$

On the other hand, $\mathbb{Q}_{i,r}$ is a nonzero measure equivalent to $1_{C_i(\alpha)}(X).\ \mathbb{P}$ since $\mathbb{P}(X \in C_i(\alpha)) > 0$ and for $r > p$, $\mathbb{P}(X = a_i) = 0$. Now, define on $\mathbb{R} \times T$ the function Φ_{ij} by

$$\Phi_{ij}(y,t) := \int_\Omega \varphi_{p-1}(y - X_{j,t})d\mathbb{Q}_{i,r} \qquad \text{where} \qquad \varphi_q(x) = \text{sign}(x)|x|^q.$$

First note that the function Φ_{ij} is real valued. If $r > p > 1$, the Young inequality with $p' = \frac{r-1}{p-1}$ and $q' = \frac{r-1}{r-p}$ implies

$$\begin{aligned} |y - X_{j,t}|^{p-1}\|a_i - X\|_p^{r-p} &\leq C(|y - X_{j,t}|^{r-1} + \|X - a_i\|_p^{r-1}) \\ &\leq C(|y|^{r-1} + \|a_i\|_p^{r-1} + |X_{j,t}|^{r-1} + \|X\|_p^{r-1}) \end{aligned}$$

so that $|y - X_{j,t}|^{p-1}\|a_i - X\|_p^{r-p} \in L1(\mathbb{P})$. When $r = p$ (or $p = 1$), the result is obvious.
(b) For every fixed $t \in T$ and $p > 1$, $y \mapsto \varphi_{p-1}(y - X_{j,t})$ is (strictly) increasing, hence $y \mapsto \Phi_{ij}(y,t)$ is strictly increasing too. The continuity of $y \mapsto \Phi_{ij}(y,t)$ on $\mathbb{R}$ for every $t \in T$ follows from the Lebesgue dominated convergence Theorem. Furthermore, for every $t \in T, y \geq 0$,

$$\Phi_{ij}(y,t) \geq \int_{\{X_{j,t} \leq y\}} \varphi_{p-1}(y - X_{j,t}) d\mathbb{Q}_{i,r} - \int |X_{j,t}|^{p-1} d\mathbb{Q}_{i,r}$$

so that $\lim_{y \to +\infty} \Phi_{ij}(y,t) = +\infty$ by Fatou's Lemma. Similarly, $\lim_{y \to -\infty} \Phi_{ij}(y,t) = -\infty$.

The proof reduces to providing an argument for the ρ_X^{r-1}-continuity of $t \mapsto \Phi_{ij}(y,t)$ for every $y \in \mathbb{R}$.

If $1 < p \leq 2$, one starts from the inequality

$$|\varphi_{p-1}(u) - \varphi_{p-1}(v)| \leq 2^{2-p}|u - v|^{p-1} \quad u,\, v \in \mathbb{R}.$$

When $r > p$, the Hölder inequality applied with the conjugate exponents $\frac{r-1}{p-1}$ and $\frac{r-1}{r-p}$ yields

$$\begin{aligned} |\Phi_{ij}(y,t) - \Phi_{ij}(y,s)| &\leq 2^{2-p}\|X_{j,t} - X_{j,s}\|_{L^{r-1}(\mathbb{P})}^{p-1} \|\, \|X - a_i\|_p\|_{L^{r-1}(\mathbb{P})}^{r-p} \\ &\leq 2^{2-p}(\rho_X^{r-1}(s,t))^{\frac{p-1}{(r\wedge 2)-1}} \|\, \|X - a_i\|_p\|_{L^{r-1}(\mathbb{P})}^{r-p}. \end{aligned}$$

This still holds if $r = p$.

If $p > 2$, one starts from

$$|\varphi_{p-1}(u) - \varphi_{p-1}(v)| \leq (p-1)(|u| \vee |v|)^{p-2}|u - v|, \quad u,\, v \in \mathbb{R}.$$

Since $r > 2$ the Holder Inequality applied with $r - 1$ and $\frac{r-1}{r-2}$ yields

$$\begin{aligned} |\Phi_{ij}(y,t) - \Phi_{ij}(y,s)| &\leq (p-1)\mathbb{E}\left(|X_{j,t} - X_{j,s}|\,(|y - X_{j,t}| \vee |y - X_{j,s}|)^{p-2}\,\|X - a_i\|_p^{r-p}\mathbf{1}_{C_i(\alpha)}(X)\right) \\ &\leq (p-1)\|X_{j,t} - X_{j,s}\|_{L^{r-1}(\mathbb{P})}\left[\mathbb{E}\left((|y - X_{j,t}| \vee |y - X_{j,s}|)^{\frac{(p-2)(r-1)}{r-2}}\|X - a_i\|_p^{\frac{(r-p)(r-1)}{r-2}}\right)\right]^{\frac{r-2}{r-1}}. \end{aligned}$$

A new application of the Holder Inequality to the expectation in the right hand side of the above inequality yields

$$\begin{aligned} |\Phi_{ij}(y,t) - \Phi_{ij}(y,s)| &\leq (p-1)\|X_{j,t} - X_{j,s}\|_{L^{r-1}(\mathbb{P})}\||y - X_{j,t}| \vee |y - X_{j,s}|\|_{L^{r-1}(\mathbb{P})}^{p-2} \|\, \|X - a_i\|_p\|_{L^{r-1}(\mathbb{P})}^{r-p} \\ &\leq C_{p,a_i}\|X_{j,t} - X_{j,s}\|_{L^{r-1}(\mathbb{P})}\left(|y|^{p-2} + \|X_{j,s}\|_{L^{r-1}(\mathbb{P})}^{p-2} + \|X_{j,t}\|_{L^{r-1}(\mathbb{P})}^{p-2}\right) \\ &\leq C_{p,a_i}\rho_X^{r-1}(s,t)(|y|^{p-2} + \|X_s\|_{L_{\mathbb{R}^d}^{r-1}(\mathbb{P})}^{p-2} + \|X_t\|_{L_{\mathbb{R}^d}^{r-1}(\mathbb{P})}^{p-2}). \end{aligned}$$

Owing to these properties, one easily checks that for every $t \in T$, the equation $\Phi_{ij}(y,t) = 0$ admits a unique solution $y_{ij}(t)$ and that the implicitly defined function $t \mapsto y_{ij}(t)$ is ρ_X^{r-1}-continuous. On the other hand the function a_i satisfies $\mu(dt)$-a.e. $\Phi_{ij}(a_{ij}(t),t) = 0$ so that $y_{ij}(t) = a_{ij}(t)$ $\mu(dt)$-a.e..

If $X_t = x \in \mathbb{R}^d, t \in T_0$ then $\Phi_{ij}(y,t) = \varphi_{p-1}(y - x_j)\mathbb{Q}_{i,r}(\Omega), t \in T_0$ so that $y_{ij}(t) = x_j$.
(a) Now let T be a compact metric space. When $p = 1$,

$$\Phi_{ij}(y,t) = \int_\Omega \operatorname{sign}(y - X_{j,t})\, d\mathbb{Q}_{i,r}.$$

The continuity of $y \mapsto \Phi_{ij}(y,t)$ on $\mathbb{R}$ for every $t \in T \setminus T_0$ and the continuity of $t \mapsto \Phi_{ij}(y,t)$ at every point $t \in T \setminus T_0$ for every $y \in \mathbb{R}$ follows from the pathwise continuity of X and from the continuity of $\mathbb{P}_{X_{j,t}}, t \in T \setminus T_0$, by the Lebesgue dominated convergence Theorem: the sign function is bounded and $\mathbb{Q}_{i,r} \ll \mathbb{P}$. Similarly one shows that $\lim_{y \to \pm\infty} \Phi_{ij}(y,t) = \pm\mathbb{Q}_{i,r}(\Omega)\ \forall t \in T$. It is also obvious that $y \mapsto \Phi_{ij}(y,t)$ is nondecreasing $\forall t \in T$. To establish strict monotonicity $\forall t \in T \setminus T_0$, one proceeds as follows: let us consider the subset of $C_i(\alpha)$ defined by

$$U_i(\alpha) := \{f \in C_{\mathbb{R}^d}(T, T_0) : \|f - a_i\|_1 < \min_{j \neq i} \|f - a_j\|_1\}$$

where

$$C_{\mathbb{R}^d}(T, T_0) := \{f \in C_{\mathbb{R}^d}(T) : f(t) = x, t \in T_0\}.$$

It is a nonempty open subset of $(C_{\mathbb{R}^d}(T,T_0), \|\cdot\|_1)$ since $C_{\mathbb{R}^d}(T,T_0)$ is everywhere $\|\cdot\|_1$-dense in $L^1_{\mathbb{R}^d}(\mu)$ in view of $\mu(T_0) = 0$. Now, for $t \in T \setminus T_0$ and every nonempty open interval I the set $\{f \in C_{\mathbb{R}^d}(T,T_0) : f_j(t) \in I\}$ is clearly everywhere dense in $(C_{\mathbb{R}^d}(T,T_0), \|\cdot\|_1)$ since $\mu(\{t\}) = 0$ so that

$$U_i(\alpha) \cap \{f \in C_{\mathbb{R}^d}(T,T_0) : f_j(t) \in I\}$$

is a nonempty set. On the other hand, $f \mapsto \|f\|_1$ and $f \mapsto f_j(t)$ are both continuous as functionals on $(C_{\mathbb{R}^d}(T,T_0), \|\cdot\|_{\sup})$ so that $U_i(\alpha) \cap \{f \in C_{\mathbb{R}^d}(T,T_0) : f_j(t) \in I\}$ is a (nonempty) open subset of $(C_{\mathbb{R}^d}(T,T_0), \|\cdot\|_{\sup})$. Now, if $\Phi_{ij}(y,t) = \Phi_{ij}(y',t)$ for some $y < y'$, then $\mathbb{Q}_{i,r}(X_{j,t} \in (y,y')) = 0$. $\mathbb{Q}_{i,r}$ is equivalent to $\mathbf{1}_{C_i(\alpha)}(X).\mathbb{P}$. Consequently

$$\mathbb{P}(\{X \in U_i(\alpha)\} \cap \{X_{j,t} \in (y,y')\}) = 0.$$

This is impossible owing to the assumption on the support of $\mathbb{P}_X$. Consequently $y \mapsto \Phi_{ij}(y,t)$ is strictly increasing for every $t \in T \setminus T_0$ and one concludes like in the case $p > 1$ to the existence of a continuous version of α in $C_{\mathbb{R}^d}(T,T_0)$.

To be a bit more precise, the equation $\Phi_{ij}(y,t) = 0$ has for $t \in T \setminus T_0$ a unique solution $y_{ij}(t) \in \mathbb{R}$ and for $t \in T_0$, since $X_{j,t} = x_j$ $\mathbb{P}$-a.s., $y_{ij}(t) = x_j$ is the unique solution. The function $y_{ij} : T \to \mathbb{R}$ is continuous at every $t \in T \setminus T_0$ since $\Phi_{ij}(\cdot,t)$ is strictly increasing on $\mathbb{R}$ and $\Phi_{ij}(y,\cdot)$ is continuous at t for every $y \in \mathbb{R}$. One must consider the behaviour of y_{ij} at $t \in T_0$ more carefully. First note that $\Phi_{ij}(y,\cdot)$ is continuous at $t \in T_0$ for every $y \neq x_j$ since X_j is pathwise continous. Now let (s_n) be a sequence in T going to t such that $y_{ij}(s_n) \geq x_j + \eta$ for some $\eta > 0$. Then, $\Phi_{ij}(x_j + \eta, s_n) \leq \Phi_{ij}(y_{ij}(s_n), s_n) = 0$ for every $n \geq 1$ so that $\operatorname{sign}(\eta)\mathbb{Q}_{i,r}(\Omega) = \Phi_{ij}(x_j + \eta, t) = \lim_{n\to\infty} \Phi_{ij}(x_j + \eta, s_n) \leq 0$ which is impossible. Hence $\limsup_{s\to t} y_{ij}(s) \leq x_j$. One shows similarly that $\liminf_{s\to t} y_{ij}(s) \geq x_j$ i.e. $\lim_{s\to t} y_{ij}(s) = x_j = y_{ij}(t)$.
(c) It is a consequence of Equation (3.5):

$$|a_{ij}(t) - a_{ij}(s)| = \frac{\mathbb{E}(|X_{j,t} - X_{j,s}|L_i)}{\mathbb{E}(L_i)} \quad \text{with} \quad L_i = \mathbf{1}_{C_i(\alpha)}(X)\|X - a_i\|_2^{r-2}.$$

When $r > 2$, The Holder Inequality yields the announced result

$$\max_{i \in I_r(\alpha)} |a_i(t) - a_i(s)|_1 \leq C_{X,\alpha}\rho_X^{r-1}(s,t)$$

with

$$C_{X,\alpha} := d \max_{i\in I_r(\alpha)} (\mathbb{E}(\|X-a_i\|_2^{r-1}))^{(r-2)/(r-1)}/(\mathbb{E}(\mathbf{1}_{C_i(\alpha)}(X)\|X-a_i\|_2^{r-2})).$$

When $r=2$, one sets accordingly $C_{X,\alpha} := 1/\min_{1\le i\le n}\mathbb{P}(X\in C_i(\alpha))$. □

Examples First consider real or $\mathbb{R}^d$-valued processes with $T=[0,t_0]$ and $\mu(dt)=dt$.

• The $(L^r, \|\,.\,\|_p)$-stationary n-quantizers, $1\le p\le r<+\infty$, of the *standard Brownian motion* and are made up with continuous functions which are null at 0, 1/2-Hölder if $p=2$. The same result holds for the *Brownian bridge* over $[0,t_0]$ where any of its stationary quantizers are 0 at t_0 and for the *standard d-dimensional Brownian motion.*

• One considers a $\mathbb{R}^d$-valued *Brownian diffusion process*

$$\begin{array}{rcl} dX_t &=& b(t,X_t)dt+\sigma(t,X_t)dW_t, t\in[0,t_0] \\ X_0 &=& x, x\in\mathbb{R}^d, \end{array}$$

where W is a m-dimensional standard Brownian motion and $b:[0,t_0]\times\mathbb{R}^d\to\mathbb{R}^d$, $\sigma:[0,t_0]\times\mathbb{R}^d\to\mathbb{R}^{d\times m}$ are Borel functions with linear growth such that the above SDE admits at least one (weak) solution over $[0,t_0]$. This solution is pathwise continuous and it is classical background (see [17]) that $\|X\|_{\sup}\in L^r(\mathbb{P})$ for every $r\in(0,\infty)$ and

$$\mathbb{E}|X_s-X_t|_q^q \le C_q|s-t|^{q/2}$$

for every $q\in(0,\infty)$. Thus the $(L^r,\|\cdot\|_p)$-stationary n-quantizers, $1<p=r<\infty$, are made up with continuous functions which are x at $t=0$, 1/2-Holder if $p=2$. The same holds if $1\le p\le r<\infty$ for the homogeneous SDE with b and σ independent of t and $d=m$ provided b_i and σ_{ij} are bounded with bounded derivatives up to order 3 and $\sigma\sigma^T$ is uniformly elliptic. In fact, the assumptions imply that $\mathbb{P}_{X_{j,t}}$ has a Lebesgue density for every $j\in\{1,\dots,d\}, t\in(0,t_0]$ and by the support theorem, in $C_{\mathbb{R}^d}([0,t_0])$,

$$\mathrm{supp}(\mathbb{P}_X)=\{f\in C_{\mathbb{R}^d}([0,t_0]) : f(0)=x\}$$

(see [3], p. 11 and [1], p. 25).

• The *fractional Brownian motion* W^H on $[0,t_0]$ with Hurst exponent $H\in(0,1)$ is a centered continuous Gaussian process having the covariance function

$$EW_s^HW_t^H=\frac{1}{2}(|s|^{2H}+|t|^{2H}-|s-t|^{2H})$$

and thus satisfies for every $q\in(0,\infty)$

$$\mathbb{E}|W_s^H-W_t^H|^q=C_{H,q}|s-t|^{qH}.$$

Consequently, $(L^r,\|\cdot\|_p)$-stationary n-quantizers, $1\le p\le r<\infty$ are made up with continuous functions which are null at $t=0$, H-Hölder if $p=2$.

• We consider some examples of (cadlag) real Lévy processes $X=(X_t)_{t\in\mathbb{R}_+}$ restricted to $[0,t_0]$ (without Brownian component). Since the increments of X are stationary and $X_0=0$,

$$\mathbb{E}|X_s-X_t|^q=\mathbb{E}|X_{|s-t|}|^q$$

so that the behaviour of the semimetric ρ_X^q reduces to the behaviour of $t\mapsto\mathbb{E}|X_t|^q$.

- The *ρ-stable Lévy motions* indexed by $\rho\in(0,2)$ satisfy a self-similarity property, namely

$$X_t\overset{d}{=}t^{1/\rho}X_1.$$

Furthermore,

$$\sup\{q > 0 : \mathbb{E}|X_1|^q < \infty\} = \rho \text{ and } \mathbb{E}|X_1|^\rho = \infty.$$

For this background see [27]. It follows that for every $q \in (0, \rho)$

$$\mathbb{E}|X_t|^q = t^{q/\rho}\mathbb{E}|X_1|^q < \infty.$$

Consequently, since the ρ-stable distributions $\mathbb{P}_{X_t}, t > 0$ have a Lebesgue density, the $(L^r, \|\cdot\|_p)$-stationary n-quantizers, $1 < p \leq r < \rho$, are made up with continuous functions which are null at 0.

- The Γ-*processes* are Lévy processes whose distribution $\mathbb{P}_{X_t}$ at $t > 0$ is a $\Gamma(a, t)$-distribution

$$\mathbb{P}_{X_t}(dx) = \frac{a^t}{\Gamma(t)} 1_{(0,\infty)}(x) x^{t-1} e^{-ax} dx,$$

$a > 0$. So, for every $q > 0$

$$\mathbb{E}|X_t|^q = \frac{\Gamma(t+q)}{a^q \Gamma(t+1)} t.$$

Consequently, $(L^r, \|\cdot\|_p)$-stationary n-quantizers, $1 < p \leq r < \infty$, are made up with continuous functions, $1/(r-1)$-Hölder if $p = 2$.

- The *compound Poisson process* is given by $X_t = \sum_{j=1}^{N_t} U_j$, where $U_1, U_2, \ldots$ are i.i.d. real random variables with $\mathbb{P}(U_1 = 0) = 0$ and $N = (N_t)_{t \geq 0}$ is a standard Poisson process (with intensity λ) independent of $(U_j)_{j \geq 1}$. If $q \in (0, \infty)$ and $\mathbb{E} \mid U_1 \mid^q < \infty$, easy computations show that

$$\mathbb{E} \mid X_t \mid^q \leq \mathbb{E} \mid U_1 \mid^q \mathbb{E} N_t^{1 \vee q} \leq C_{q,\lambda,U} t < \infty$$

Assume $\mathbb{E} \mid U_1 \mid^r < \infty$. Then the $(L^r, \|\cdot\|_p)$-stationary n-quantizers, $1 < p \leq r < \infty$, are made up with continuous functions, $1/(r-1)$-Hölder if $p = 2$. Here it has to be noticed that the function $f = 0$ is the only atom of $\mathbb{P}_X$ in $L^p([0, t_0], dt)$.

Theorem 5(a) does not apply to the above examples because of the pathwise continuity assumption so that the case $p = 1$ remains open.

As for a real multiparameter process on $T = [0, t_0]^k$ with $\mu(dt) = dt$:

• The $(L^r, \|\cdot\|_p)$-stationary n-quantizers, $1 \leq p \leq r < \infty$, of the *standard Brownian sheet* are made up with continous functions which are null on $\bigcup_{i=1}^{k} \{t \in T : t_i = 0\}$ and 1/2-Hölder if $p = 2$.

As for an example with noncompact T consider $T = \mathbb{R}_+$ and $\mu(dt) = e^{-bt}dt, b > 0$.

• The stationary *Ornstein-Uhlenbeck process* $X = (X_t)_{t \geq 0}$ on $\mathbb{R}_+$ is a centered continuous Gaussian process having the covariance function

$$\mathbb{E} X_s X_t = e^{-c|s-t|}, c > 0.$$

Clearly, X can be seen as $L^p(\mathbb{R}_+, \mu)$-valued random vector for every $p \in [1, \infty)$. The process satisfies for every $q \in (0, \infty)$

$$\mathbb{E} \mid X_s - X_t \mid^q = C_q (1 - e^{-c|s-t|})^{q/2}.$$

Consequently, $(L^r, \|\cdot\|_p)$-stationary n-quantizers, $1 < p \leq r < \infty$ have components consisting of continuous functions, 1/2-Hölder if $p = 2$.

A COUNTEREXAMPLE WHEN $p = +\infty$ We will exhibit a bounded pathwise continuous process

X on $T = [0,1]$ having a discontinuous $(L^r, \|\,.\,\|_\infty)$-optimal 1-quantizer. Consider functions $f_n \in C([0,1]), n \in \mathbb{N}$ and $\mathbb{P}_X$ from the $C([0,1])$-counterexample following Theorem 4. Then set $h := \frac{1}{2}(\mathbf{1}_{[0,1/2]} - \mathbf{1}_{(1/2,1]})$. One checks that in $L^\infty([0,1], dt)$, for every $n \geq 1$,

$$\|f_n - h\|_\infty = 1/2$$

so that

$$\forall\, r \in [1, +\infty], \qquad \|\,\|X - h\|_\infty\|_{L^r(\mathbb{P})} = 1/2.$$

On the other hand,

$$e_{1,1}(X, L^\infty) = e_{1,1}(X, C([0,1]) = 1/2 = \|\,\|X - h\|_\infty\|_{L^1(\mathbb{P})}$$

by the $C([0,1])$-counterexample and Theorem 4. Consequently, the $\|\,.\,\|_{L^r(\mathbb{P})}$-norm being nondecreasing as a function of r,

$$\forall\, r \in [1, +\infty], \qquad e_{1,r}(X, L^\infty) = \|\,\|X - h\|_\infty\|_{L^r(\mathbb{P})} = 1/2$$

with obvious definition of $e_{1,\infty}$. The function h is an $(L^r, \|\,.\,\|_\infty)$-optimal 1-quantizer without continuous dt-version of the pathwise continuous process X, $1 \leq r \leq +\infty$.

Note that $t \mapsto X_t$ from $[0,1]$ into $L^p(\mathbb{P})$ is continuous for any $p \in [1, +\infty)$ since X is pathwise continuous and uniformly bounded by 1. Consequently it follows from Theorem 5 that, as soon as $1 < p \leq r < +\infty$, any $(L^r, \|\,.\,\|_p)$ optimal n-quantizer of X (has a dt-version which) consists of continuous functions. However, $t \mapsto X_t$ from $[0,1]$ into $L^\infty(\mathbb{P})$ is not continuous (at $t = 1/2$), so the pathwise regularity of an optimal $(L^r, \|\,.\,\|_\infty)$-optimal n-quantizer of an $L^\infty(\mathbb{P})$-continuous process remains open. But the $\|\,.\,\|_\infty$-norm being nowhere Gateaux-differentiable, the very notion of $(L^r, \|\cdot\|_\infty)$-stationary quantizer no longer exists. So this would require to develop a new approach.

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