Bounded countable atomic compactness of ordered groups

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BOUNDED COUNTABLE ATOMIC COMPACTNESS OF ORDERED GROUPS.

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Abstract. We show that whenever A is a monotone σ-complete dimension group, then $A^+ \cup \{\infty\}$ is countably equationally compact, and we show how this property can supply the necessary amount of completeness in several kinds of problems. In particular, if A is a countable dimension group and E is a monotone σ-complete dimension group, then the ordered group of all relatively bounded homomorphisms from A to E is a monotone σ-complete dimension group.

§0. Introduction.

By definition, an ordered abelian group is *monotone σ-complete* whenever every bounded increasing sequence of elements admits a l.u.b. . We will be here concerned about those monotone σ-complete groups that are in addition *directed* and satisfy the *Riesz interpolation property*. One can then show that such groups are (strongly) *Archimedean*, thus they are *dimension groups* [**7**, **8**]. Monotone σ-complete dimension groups appear naturally as ordered Grothendieck groups K_0 of countably continuous regular rings and thus intervene also in the study of Rickart C*-algebras [**7**, **8**, **9**].

Intriguingly, there is another related large class of ordered structures, the class of Tarski's *cardinal algebras*, as well as other more general classes, as *e.g. refinement algebras* [**14**]. Basically, these objects also appear naturally in a lot of cases as algebras of isomorphism types of various structures such as sets under equipotence or σ-complete Boolean algebras. Many of their properties are already valid in the more general class of *weak cardinal algebras*, defined simultaneously and independently by K.P.S. Bhaskara Rao and R.M. Shortt on one side, the author on the other side in [**11**, **18**, **19**].

Then it is not difficult to verify that *if A is an abelian ordered group, then $A^+ \cup \{\infty\}$ is a weak cardinal algebra if and only if A is monotone σ-complete and satisfies the finite interpolation property*, and this gives a hint that methods used for one theory can often be used for the other. For example, norm-completeness of Archimedean abelian groups with unit with the countable interpolation property [**8**] is very closely related to metric completeness of weak cardinal algebras and more general objects [**19**].

There is another striking common point between the latter two results; it is that they are particular cases of *atomic compactness* [**1**, **10**, **15**, **16**, **17**]. Since most natural ordered groups fail to enjoy any *topological* compactness property under some intrinsic topology (like *e.g.* the order topology), atomic compactness could be the best one can hope in that direction. Furthermore, there are several other positive results of that flavor:

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— The characterization in [**18**] of injective positively ordered monoids.

— The result that *divisible* weak cardinal algebras (and more general objects) are countably injective, thus countably atomic compact [**20**]. In particular, if A is any divisible (not necessarily Archimedean) dimension group with the countable interpolation property, then $A^+ \cup \{\infty\}$ is countably injective.

— The result that if A is a *Dedekind complete* f-ring, then $(A, +, -, \cdot, \wedge, \vee, \leq)$ is "boundedly atomic compact" [**21**, theorem 5.5].

— The more recent result [**22**] that if A is a Dedekind σ-complete f-ring, then $(A, +, -, \cdot, \wedge, \vee, \leq)$ is "boundedly countably atomic compact".

In this paper, we generalize the latter result to monotone σ-complete dimension groups. The proof we present here does not use any forcing, but is valid only for the language $(+, \leq)$.

In section 1, we recall the classical definitions and terminology about atomic compactness [**1**, **10**, **15**, **16**], and we introduce *bounded* atomic compactness, which (roughly speaking) is to atomic compactness what local compactness is to compactness in general topology. Such a situation has already been encountered in [**21**, theorem 5.5] and [**22**].

In section 2, we present a proof of the fact that every monotone σ-complete dimension group is boundedly countably atomic compact in the language $(+, \leq)$ (theorem 2.9); unlike the proofs of atomic compactness in [**21**, **22**], it does not use any forcing, rather relying on the analysis of the structure of the set of solutions of a given finite linear system with parameters from the group in question. It is our belief that there are even stronger forms of compactness satisfied by monotone σ-complete dimension groups to be investigated yet. Furthermore, in theorem 2.11, we prove an algebraic analogue of the classical topological "closed projection theorem" (if K is compact, then the natural projection from $X \times K$ to X is closed).

In section 3, we show how bounded countable compactness can be applied to generalize some abstract measure extension problems (proposition 3.1) and a Hahn-Banach like theorem for monotone σ-complete dimension groups (theorem 3.4). Some of these results were known in the lattice case, but without the possibility to take countable suprema, atomic compactness properties are just what is needed to overcome this difficulty.

If X and Y are two sets, then ${}^X Y$ will denote the set of all maps from X to Y.

We will denote the set of all positive integers by ω or $\mathbb{N}$, according to the case where we consider it as an ordinal or a monoid; in the first case, recall that if n is a positive integer, then $n = \{x : \ x < n\}$.

Let $(P, \leq)$ be an ordered set. For all a, b in P, $[a, \ b]$ will denote the *interval* $\{x \in P : a \leq x \leq b\}$. If X, Y are two subsets of P, then $X \leq Y$ will stand for $(\forall x \in X)(\forall y \in Y)$ $(x \leq y)$; if $X = \{a\}$ (resp. $Y = \{a\}$), we will write $a \leq Y$ (resp. $X \leq a$); if $X = \{a_1, \ldots, a_m\}$ and $Y = \{b_1, \ldots, b_n\}$, we will also write $a_1, \ldots, a_m \leq b_1, \ldots, b_n$. We will say that P is *directed* (resp. *filtered*) when for all x, y in P, there exists z in P such that $x, y \leq z$ (resp. $z \leq x, y$). If x, x_n $(n \in \omega)$ are elements of P, say that $x = \bigvee_n \uparrow x_n$ (resp. $x = \bigwedge_n \downarrow x_n$) when $(x_n)_n$ is *increasing* (resp. *decreasing*) and x is the l.u.b. (resp. g.l.b.) of $(x_n)_n$. We will say that P is *monotone σ-complete* [**8**] when every increasing (resp. decreasing) bounded sequence of elements of P has a l.u.b. (resp. g.l.b.).

Formal variable symbols will be written in boldface roman letters **a**, **b**, **x**, **y**..., while elements of a given structure will be written in math italics a, b, x, y... . Lists of variables or of elements of a structure will often be denoted by *vectors* $\vec{\mathbf{x}}$, $\vec{x}$..., or $\underset{(n)}{\vec{\mathbf{x}}}$ if $\vec{\mathbf{x}} = (\mathbf{x}_i)_{i<n}$.

§1. Countable, bounded atomic compactness.

The goal of this section is to recall some classical definitions about (countable or arbitrary) atomic compactness, as well as the version we shall use throughout this paper, *bounded (countable or arbitrary) atomic compactness*. We refer to [**1**, **10**, **15**, **16**] for more precisions about atomic (and equational) compactness. We will be here concerned only about models of the language $(+, \leq)$, but what follows in this section could often be easily generalized to the case of an arbitrary first-order language; it could also be generalized to the case of any infinite cardinal instead of just $\aleph_0$.

Recall that an *atomic formula* is a formula of the form $\sum_{i<n} p_i \mathbf{x}_i \leq \sum_{i<n} q_i \mathbf{x}_i$ or $\sum_{i<n} p_i \mathbf{x}_i = \sum_{i<n} q_i \mathbf{x}_i$, where p_i and q_i are positive integers and $\mathbf{x}_i$ are variable symbols. A *formula* is a well-formed expression built up from atomic formulæ and the logical connectives and, or, $\neg$, $\forall$ and $\exists$.

A *structure* will be by definition a set A equipped with a binary operation $+$ and a binary relation $\leq$. One defines as usual formulæ with parameters from a given structure, and the satisfaction relation $A \models \varphi$ is to be read "A satisfies φ" [**4**]. The *reduced power* ${}^*A$ of A by the Fréchet filter over the integers is the quotient structure of ${}^\omega A$ (equipped with componentwise $+$ and $\leq$) by the equivalence modulo the Fréchet filter on ω [**4**]. We will as usual identify A with its image in ${}^*A$ under the natural embedding.

By analogy with equation systems, we will say that a set of formulæ is a *system*, then using "solvable" instead of "satisfiable". An *atomic system* is a system of atomic formulæ. If A is a structure, then we will say that A is *atomic compact* (resp. *countably atomic compact*) when for every system (resp. countable system) Σ of atomic formulæ of $(+, \leq)$, possibly with parameters from A, if Σ is finitely solvable in A (*i.e.* every finite subsystem of Σ is solvable in A), then Σ is solvable in A. When the formulæ that we consider are *equations*, we obtain the classical definition of *equational compactness* [**10**, **16**]. It is well-known that if A is a *compact* structure (*i.e.* there exists a compact Hausdorff topology on the underlying set for which $\leq$ is closed and $+$ is continuous), then A is atomic compact. The converse is false in a lot of [most] natural cases (*see* for example [**5**, **6**, **12**]).

It is well-known [**16**, theorem 2.3] that a structure is atomic compact if and only if it is a retract of all its reduced powers. The following proposition is a countable analogue of this result:

1.1. Proposition. *Let A be a structure. Then the following are equivalent:*

(i) *A is countably atomic compact;*

(ii) *Let A_0 and A_1 be countable structures such that $A_0 \subseteq A$ and $A_0 \subseteq A_1 \subseteq {}^*A_0$. Then there exists a* "partial retraction" *from A_1 to A,* i.e. *a $(+, \leq)$- homomorphism ρ from A_1 to A such that $\rho{\restriction}_{A_0} = \mathrm{id}$.*

Proof. (i)⇒(ii) Consider the following countable atomic system Σ, with parameters from A_0, with unknowns $\mathbf{x}_a$ ($a \in A_1$):

$$\Sigma : \begin{cases} \mathbf{x}_{a+b} = \mathbf{x}_a + \mathbf{x}_b & (\text{all } a, b \text{ in } A_1) \\ \mathbf{x}_a \leq \mathbf{x}_b & (\text{all } a, b \text{ in } A_1 \text{ such that } a \leq b) \\ \mathbf{x}_a = a & (\text{all } a \in A_0) \end{cases}$$

It is easy to see that Σ is finitely solvable in A (and even in A_0); by assumption, Σ is solvable in A. Any solution of Σ in ${}^{A_1}A$ defines a partial retraction from A_1 to A.

(ii)⇒(i) Let Σ be a countable atomic system with parameters from A, with unknowns $\mathbf{x}_n$ $(n \in \omega)$. There exists a countable substructure A_0 of A such that all parameters in Σ are in A_0 and Σ is finitely solvable in A_0. Hence, Σ is solvable in ${}^*A_0$ [**16**, lemma 2.2], thus in some countable substructure A_1 of ${}^*A_0$ containing A_0. By assumption, there exists a partial retraction ρ from A_1 to A; the image under ρ of a solution of Σ in A_1 is a solution of Σ in A. ∎

Note that the proof of (ii)⇒(i) above does not work when Σ consists on arbitrary *positive* formulæ, even by replacing ${}^*A_0$ by a ultrapower of A_0 and requiring $A_0 \prec A$ and $A_1 \prec {}^*A_0$. (and this is a difference with the case of full atomic compactness): indeed, $\rho[A_1]$ does not seem *a priori* to be a pure substructure of A.

A non zero directed ordered group A can never be atomic compact (consider the atomic system $\{\, a \leq \mathbf{x} \quad (\text{all } a \in A)\,)$; on the other hand, the author proved [**21**, theorem 5.5] that if A is a Dedekind complete *f*-ring, then A is "boundedly algebraically [or atomic] compact" in the language $(+, -, \cdot, \wedge, \vee, \leq)$, *i.e.* every finitely solvable system of the following form:

$$\begin{cases} \varphi_i\big(\underset{(J)}{\vec{\mathbf{x}}}\big) & (\text{all } i \in I; \text{ the } \varphi_i\text{'s are atomic formulæ of the language } (+, -, \cdot, \wedge, \vee, \leq)) \\ |\mathbf{x}_j| \leq a_j & (\text{all } j \in J; \text{ the } a_j\text{'s belong to } A) \end{cases}$$

is solvable.

1.2. Definition. Let A be a structure, let Σ be a system with parameters from A, with unknowns $\mathbf{x}_i$ $(i \in I)$. Then a family $(a_i, b_i)_{i \in I}$ of elements of $A \times A$ is *bounding* for Σ when the following system:

$$\begin{cases} \Sigma \\ a_i \leq \mathbf{x}_i \leq b_i \quad (\text{all } i \in I) \end{cases}$$

is finitely solvable. We will say that Σ is *bounding* when it admits a bounding family. We will say that A is *boundedly countably atomic compact* when every bounding countable system with parameters from A is solvable in A.

For every abelian ordered group A, equip $A^+ \cup \{\infty\}$ with the ordered monoid structure extending A^+ such that $\infty > 0$ and $(\forall x)(x + \infty = \infty + x = \infty)$. Then we have the following

1.3. Proposition. *Let A be a directed abelian ordered group. Then the following are equivalent:*

(i) *A is boundedly countably atomic compact.*

(ii) *$A^+ \cup \{\infty\}$ is countably atomic compact.*

(iii) *$A^+ \cup \{\infty\}$ is countably equationally compact.*

Proof. (ii)⇒(i) is easy. Now assume (i). Put $E = A^+ \cup \{\infty\}$. Let C be a countable submonoid of E, let C' be a countable monoid such that $C \subseteq C' \subseteq {}^*C$. Put $I = \{[x_n]_n \in C' : (\exists x \in A^+)(\forall n \in \omega)(x_n \leq x)\}$ (set of all "bounded" elements of C'). Then we see as in the proof of proposition 1.1, (i)⇒(ii) (the corresponding system Σ is bounding) that by bounded countable atomic compactness of A, there exists a partial retraction ρ from I to A^+. Thus, the extension ρ' of ρ to E defined by $\rho'(x) = \infty$ if $x \in C' \setminus I$ is a partial retraction from C' to E (see also [**21**, corollary 5.6] for a similar argument). Finally, (ii)⇔(iii) is easy, by observing that in $A^+ \cup \{\infty\}$, $x \leq y$ is equivalent to $(\exists z)(x + z = y)$.∎

Note that the analogue of proposition 1.3 fails for richer languages, as *e.g.* for the language $(+, \wedge, \leq)$ and $A = \mathbb{R} \times \mathbb{R}$, with $x \wedge y = \min(x, y)$.

§2. Special sentences. Case of monotone σ-complete groups.

Recall [**8**] that an abelian ordered group A satisfies the *interpolation property* (resp. the *countable interpolation property*) when for all finite (resp. countable) nonempty subsets X and Y of A such that $X \leq Y$, there exists z in A such that $X \leq z \leq Y$. Moreover, A is *unperforated* when for all $m \in \mathbb{N} \setminus \{0\}$, A satisfies $(\forall \mathbf{x})(m\mathbf{x} \geq 0 \Rightarrow \mathbf{x} \geq 0)$. A *dimension group* is a directed, abelian, unperforated ordered group satisfying the interpolation property. We will say that a *dimension cone* is the positive cone of a dimension group. Recall [**8**, theorem 16.10] that every abelian monotone σ-complete group with the (finite) interpolation property is Archimedean and satisfies the countable interpolation property.

The following definition of special sentences matches the one in [**3**, p. 181] in many aspects. The terminology "linear formula" is chosen by analogy with the already existing term "linear system".

2.1. Definition. A formula of the language $(+, \leq)$ is *linear* when it is a finite conjunction of atomic formulæ. A *special sentence* is a formula of the form

$$(\forall\, \vec{\mathbf{x}})\big[\varphi(\vec{\mathbf{x}}) \Rightarrow (\exists\, \vec{\mathbf{y}})\psi(\vec{\mathbf{x}}, \vec{\mathbf{y}})\big]$$

where φ and ψ are linear formulæ.

A key fact for this section is the following

2.2. Proposition. *There exist computable functions which to each linear formula* $\varphi(\underset{(n)}{\vec{\mathbf{x}}})$ *associate a strictly positive integer* N_φ *and a* N_φ*-uple* $(\vec{m}_j)_{j<N_\varphi}$ *of elements of* ${}^nN_\varphi$ *such that for every dimension group* A*, we have*

$$A^+ \models (\forall\, \vec{\mathbf{x}}) \left[\varphi(\vec{\mathbf{x}}) \Longleftrightarrow (\exists_{j<N_\varphi} \mathbf{y}_j) \left(\vec{\mathbf{x}} = \sum_{j<N_\varphi} \vec{m}_j \mathbf{y}_j \right) \right].$$

(Thus, taking $A = \mathbb{Z}$, we see that for all $j < N_\varphi$, $\mathbb{N} \models \varphi(\vec{m}_j)$).

Proof. When φ is $s = t$ where s and t are terms, this is [**8**, proposition 3.15] (the proof showed in [**8**] is obviously effective). When φ is $s(\vec{\mathbf{x}}) \leq t(\vec{\mathbf{x}})$ where s and t are terms, note

that in every dimension cone, the formulæ $\varphi(\vec{\mathbf{x}})$ and $(\exists \mathbf{y})\big(s(\vec{\mathbf{x}})+\mathbf{y}=t(\vec{\mathbf{x}})\big)$ are equivalent, and thus it suffices to apply the previous result to the formula $s(\vec{\mathbf{x}})+\mathbf{y}=t(\vec{\mathbf{x}})$. Now suppose that the theorem has been proved for conjunctions of n atomic formulæ ($n \in \omega\backslash\{0\}$) and let $\varphi(\underset{(n)}{\vec{\mathbf{x}}})$ be a conjunction of $n+1$ atomic formulæ. Thus φ is $\psi \wedge \theta$ where ψ is a conjunction of n atomic formulæ and θ is atomic. Let N_ψ and $\vec{m}_j$ ($j < N_\psi$) be associated with ψ (induction hypothesis). Then, in every dimension cone, $\varphi(\vec{\mathbf{x}})$ is equivalent to the formula

$$(\exists_{j<N_\psi} \mathbf{y}_j)\left(\vec{\mathbf{x}} = \sum_{j<N_\psi} \vec{m}_j \mathbf{y}_j \text{ and } \theta(\vec{\mathbf{x}})\right),$$

so that it suffices to apply the case $n=1$ to the atomic formula $\theta\left(\sum_{j<N_\psi} \vec{m}_j \mathbf{y}_j\right)$ and substitute back into $\vec{\mathbf{x}}$ the expressions found for the $\mathbf{y}_j$'s. ∎

In the context above, $\vec{\mathbf{x}} = \sum_{j<N_\varphi} \vec{m}_j \mathbf{y}_j$ will be called the *general solution* of $\varphi(\vec{\mathbf{x}})$ in dimension cones.

2.3. Corollary. *Consider a special sentence* θ

$$(\forall \underset{(m)}{\vec{\mathbf{x}}})\left(\varphi(\vec{\mathbf{x}}) \Rightarrow (\exists \underset{(n)}{\vec{\mathbf{y}}})\psi(\vec{\mathbf{x}},\vec{\mathbf{y}})\right),$$

where φ *and* ψ *are linear formulæ. Then the following are equivalent:*

(i) *Every dimension cone satisfies* θ*;*

(ii) $\mathbb{N}$ *satisfies* θ*;*

(iii) *For all* $\vec{x}$ *in* ${}^m N_\varphi$*, there exists* $\vec{y}$ *in* ${}^n\mathbb{N}$ *such that* $\mathbb{N} \models \psi(\vec{x},\vec{y})$.

Proof. (i)⇒(ii), (ii)⇒(iii) are trivial. Now assume (iii). Let A be a dimension group; we prove that $A^+ \models \theta$. So let $\vec{x}$ in ${}^m A^+$ such that $A^+ \models \varphi(\vec{x})$. Let N_φ, $\vec{m}_j$ ($j < N_\varphi$) be associated with φ as in proposition 2.2. By proposition 2.2, there are t_j ($j < N_\varphi$) in A^+ such that $\vec{x} = \sum_{j<N_\varphi} \vec{m}_j t_j$. Since $\vec{m}_j \in {}^m N_\varphi$ for all j and $\mathbb{N} \models \varphi(\vec{m}_j)$, there are $\vec{n}_j$ ($j < N_\varphi$) in ${}^n\mathbb{N}$ such that for all j, $\mathbb{N} \models \psi(\vec{m}_j,\vec{n}_j)$. Put $\vec{y} = \sum_{j<N_\varphi} \vec{n}_j t_j$. Then $\vec{y} \in {}^n A^+$ and $A^+ \models \psi(\vec{x},\vec{y})$. ∎

2.4. Corollary. *Let* $\varphi\Big(\mathbf{x}, \underset{(m)}{\vec{\mathbf{x}}}, \underset{(n)}{\vec{\mathbf{a}}}\Big)$ *be a linear formula. Then for every dimension group* A *and all* $\vec{a}$ *in* ${}^n A^+$*, the "projection"*

$$D = \big\{x \in A^+ :\ A^+ \models (\exists \vec{\mathbf{x}})\varphi(x,\vec{\mathbf{x}},\vec{a})\big\}$$

is both filtered and directed.

Proof. The statement that D is always directed means that the following sentence θ is satisfied by all dimension cones:

$$(\forall \mathbf{x},\mathbf{y},\vec{\mathbf{x}},\vec{\mathbf{y}},\vec{\mathbf{a}})\Big[\big(\varphi(\mathbf{x},\vec{\mathbf{x}},\vec{\mathbf{a}}) \text{ and } \varphi(\mathbf{y},\vec{\mathbf{y}},\vec{\mathbf{a}})\big) \Longrightarrow (\exists \mathbf{z},\vec{\mathbf{z}})\big(\mathbf{x}\le\mathbf{z} \text{ and } \mathbf{y}\le\mathbf{z} \text{ and } \varphi(\mathbf{z},\vec{\mathbf{z}},\vec{\mathbf{a}})\big)\Big]$$

However, θ is special, thus, by corollary 2.4, it suffices to show that θ holds in $\mathbb{N}$. However, since $\mathbb{N}$ is linearly ordered, this is obvious: if *e.g.* $x \leq y$, then take $z = y$ and $z_i = y_i$ for all i. The proof for "D is filtered" is similar. ∎

The idea underlying the use of corollary 2.4 for the proof of bounded countable atomic compactness of monotone σ-complete dimension groups is the following: if $S(\mathbf{x}_0, \mathbf{x}_1, \ldots)$ is a countable bounding atomic system with parameters from a given monotone σ-complete dimension group A, one constructs, using corollary 2.4, a bounded increasing sequence of x_0's such that larger and larger finite subsystems of $S(x_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ are finitely solvable. But now, we have to prove that the l.u.b. of such a sequence is in the first projection of S! Basically, it would be good if to an increasing sequence of x_0's, one could *e.g.* associate a decreasing, or increasing, sequence of x_1's; however, easy examples show that this is not always possible. Now, thinking "with bounded variation" instead of "increasing" or "decreasing" yields a small trick, summarized in the following proposition:

2.5. Corollary. *Let* $\varphi\Big(\mathbf{x}, \underset{(m)}{\vec{\mathbf{x}}}, \underset{(n)}{\vec{\mathbf{a}}}\Big)$ *be a linear formula; let* $N = N_\psi$ *where* ψ *is the following linear formula:*

$$\varphi(\mathbf{x}, \vec{\mathbf{x}}, \vec{\mathbf{a}}) \text{ and } \varphi(\mathbf{y}, \vec{\mathbf{y}}, \vec{\mathbf{a}}) \text{ and } \mathbf{x} \leq \mathbf{y}.$$

Let A *be a dimension group, let* $\vec{a}$ *in* ${}^nA^+$. *Put*

$$D = \big\{x \in A^+ :\ A^+ \models (\exists\, \vec{\mathbf{x}})\varphi(x, \vec{\mathbf{x}}, \vec{a})\big\},$$
$$S = \big\{(x, \vec{x}) \in A^+ \times {}^nA^+ :\ A^+ \models \varphi(x, \vec{x}, \vec{a})\big\}.$$

Then for all $(x, \vec{x}) \in S$ *and all* $y \in D$ *such that* $x \leq y$, *there exists* $\vec{y} \in {}^nA^+$ *such that* $(y, \vec{y}) \in S$ *and for all* $i < m$, $Nx + x_i \leq Ny + y_i$.

Proof. It suffices to prove that the following sentence θ holds in every dimension cone:

$$(\forall \mathbf{x}, \mathbf{y}, \vec{\mathbf{x}}, \vec{\mathbf{y}}, \vec{\mathbf{a}})\Bigg[\big(\varphi(\mathbf{x}, \vec{\mathbf{x}}, \vec{\mathbf{a}}) \text{ and } \varphi(\mathbf{y}, \vec{\mathbf{y}}, \vec{\mathbf{a}}) \text{ and } \mathbf{x} \leq \mathbf{y}\big) \Longrightarrow$$
$$(\exists\, \vec{\mathbf{z}})\Big(\bigwedge\!\!\!\!\bigwedge_{i<m} (N\mathbf{x} + \mathbf{x}_i \leq N\mathbf{y} + \mathbf{z}_i) \text{ and } \varphi(\mathbf{y}, \vec{\mathbf{z}}, \vec{\mathbf{a}})\Big)\Bigg]$$

However, θ is special, thus, by corollary 2.4, it suffices to show that it holds in $\mathbb{N}$ for x, y and components of $\vec{a}$, $\vec{x}$, $\vec{y}$ strictly less than N. If $x = y$ then just take $z_i = x_i$ (all $i < m$). So suppose $x < y$; take $z_i = y_i$ (all $i < m$). We verify that $Nx + x_i \leq Ny + y_i$, *i.e.* $x_i - y_i \leq N(y - x)$. However, this is the case, since $x_i - y_i \leq x_i < N \leq N(y - x)$. ∎

2.6. Lemma. *Let* $\varphi\big(\underset{(m)}{\vec{\mathbf{x}}}, \underset{(n)}{\vec{\mathbf{a}}}\big)$ *be a linear formula, let* A *be a monotone* σ*-complete dimension group, let* $\vec{a}$ *in* ${}^nA$, *let* $\vec{x}$, $\vec{x}_k$ $(k \in \omega)$ *be elements of* A *such that* $\vec{x} = \bigvee_k \uparrow \vec{x}_k$ *(resp.* $\vec{x} = \bigwedge_k \downarrow \vec{x}_k$*) and for all* k, $A \models \varphi(\vec{x}_k, \vec{a})$. *Then* $A \models \varphi(\vec{x}, \vec{a})$.

Proof. $\varphi(\vec{\mathbf{x}}, \vec{a})$ is equivalent to a conjunction of formulæ of the form $\sum_{i<m} p_i \mathbf{x}_i + a \leq \sum_{i<m} q_i \mathbf{x}_i + b$ ($p_i, q_i \in \mathbb{N}$ and $a, b \in A^+$). Since addition is distributive on countable $\bigvee \uparrow$ (resp. $\bigwedge \downarrow$), the conclusion follows immediately. ∎

2.7. Proposition. *Let $\varphi\big(\mathbf{x}, \underset{(m)}{\vec{\mathbf{x}}}, \underset{(n)}{\vec{\mathbf{a}}}\big)$ be a linear formula, let A be a monotone σ-complete dimension group, let $\vec{a}$ in ${}^n A$, let a in A^+. Then*

$$D_a = \big\{x \in A^+ : \ A \models (\exists_{i<m} \mathbf{x}_i \leq a)\varphi(x, \vec{\mathbf{x}}, \vec{a})\big\}$$

is closed under countable $\bigvee \uparrow$ and $\bigwedge \downarrow$.

Proof. We prove the result for countable $\bigvee \uparrow$; the proof for countable $\bigwedge \downarrow$ is similar. So let $(x^k)_{k\in\omega}$ be a countable increasing bounded sequence of elements of D_a, let $x = \bigvee_k \uparrow x^k$. We prove that $x \in D_a$. Let N be the integer associated as in corollary 2.5 with the formula $\big[\varphi(\mathbf{x}, \vec{\mathbf{x}}, \vec{\mathbf{a}})$ and $\mathbb{\Lambda}_{i<m}(\mathbf{x}_i \leq \mathbf{a})\big]$. For all $k \in \omega$, we construct inductively $\vec{x}^k = (x_i^k)_{i<m}$ ($0 \leq x_i^k \leq a$) the following way.

(•) Since $x^0 \in D_a$, there exists $\vec{x}^0$ in ${}^m[0,\ a]$ such that $A \models \varphi(x^0, \vec{x}^0, \vec{a})$.

(•) Suppose that $\vec{x}^k$ has been constructed (with $0 \leq x_i^k \leq a$) and $A \models \varphi(x^k, \vec{x}^k, \vec{a})$. Since $x^k \leq x^{k+1}$ and both belong to D_a, there exists, by corollary 2.5, $\vec{x}^{k+1}$ in ${}^m[0,\ a]$ such that $A \models \varphi(x^{k+1}, \vec{x}^{k+1}, \vec{a})$ and for all $i < m$, $Nx^k + x_i^k \leq Nx^{k+1} + x_i^{k+1}$.

Thus for all $i < m$, the sequence $(y_i^k)_k$ where $y_i^k = Nx^k + x_i^k$ is *increasing*. But it is bounded above (by $Nx + a$), thus it has a l.u.b., say y_i. Let $\psi\big(\mathbf{z}, \underset{(m)}{\vec{\mathbf{z}}}, \vec{\mathbf{a}}\big)$ be the linear formula $\varphi(\mathbf{z}, \vec{\mathbf{z}} - N\mathbf{z}, \vec{\mathbf{a}})$. Then for all k, we have $A \models \psi(x^k, \vec{y}^k, \vec{a})$, whence, by lemma 2.6, $A \models \psi(x, \vec{y}, \vec{a})$, whence $A \models \varphi(x, \vec{x}, \vec{a})$ where $\vec{x} = \vec{y} - Nx$. Finally, for all $i < m$, we have $0 \leq x_i^k \leq a$, which can be written $Nx^k \leq y_i^k \leq a + Nx^k$; taking l.u.b. in k of all members yields $Nx \leq y_i \leq a + Nx$, *i.e.* $0 \leq x_i \leq a$. Hence, $x \in D_a$. ∎

Now we can prove the decisive lemma:

2.8. Lemma. *Let A be a monotone σ-complete dimension group, let $S(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ be a bounding atomic system with parameters from A with a bounding sequence of the form $(0, a_n)_{n\in\omega}$. Then there exists x in $[0,\ a_0]$ such that $S(x, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ is a bounding system with bounding family $(0, a_n)_{n\in\omega\setminus\{0\}}$ (so that in particular, it is finitely solvable).*

Proof. Write S as a countable increasing sequence $S = \bigcup_{n\in\omega} S_n$ of finite subsystems. Note that each S_n can be naturally identified with a linear formula with parameters from A. Consider the strict well-ordering $\prec$ on $\omega \times \omega$ defined by $(i, j) \prec (i', j')$ if and only if $i + j < i' + j'$ or ($i + j = i' + j'$ and $j < j'$). Construct by $\prec$-induction x_i^j ($i, j \in \omega$) the following way. Suppose that all $x_{i'}^{j'}$ are constructed for $(i', j') \prec (i, j)$ in such a way that for all these (i', j'), $x_{i'}^{j'} \in [0,\ a_0]$ and $S_{i'}(x_{i'}^{j'}, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ is solvable in $\prod_{l\geq 1}[0,\ a_l]$. Put $n = i + j$. If $j = 0$, then $i = n$ and there is x_n^0 in $[0,\ a_0]$ such that $S_n(x_n^0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ is solvable in $\prod_{l\geq 1}[0,\ a_l]$. If $j > 0$, there exists by corollary 2.4 $x_i^j \leq x_{i+1}^{j-1}, x_i^{j-1}$ in $[0,\ a_0]$

such that $S_i(x_i^j, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ is solvable in $\prod_{l\geq 1}[0,\ a_l]$. The picture is as follows:

$$\begin{array}{ccccccc}
x_0^0 & & x_1^0 & & x_2^0 & & x_3^0 \ \ldots \\
\uparrow & \nearrow & \uparrow & \nearrow & \uparrow & \nearrow & \vdots \\
x_0^1 & & x_1^1 & & x_2^1 & \ldots & \\
\uparrow & \nearrow & \uparrow & \nearrow & \vdots & & \\
x_0^2 & & x_1^2 & \ldots & & & \\
\uparrow & \nearrow & \vdots & & & & \\
x_0^3 & \ldots & & & & & \\
\vdots & & & & & &
\end{array}$$

In this diagram, an arrow from x_i^j to $x_{i'}^{j'}$ means that $x_i^j \leq x_{i'}^{j'}$. Now, for all m in ω, put $x_m = \bigwedge_k \downarrow x_m^k$. By proposition 2.7, $S_m(x_m, \mathbf{x}_0, \mathbf{x}_1, \ldots)$ is solvable in $\prod_{l\geq 1}[0,\ a_l]$; thus $S_m(x_n, \mathbf{x}_0, \mathbf{x}_1, \ldots)$ is solvable in $\prod_{l\geq 1}[0,\ a_l]$ for all $n \geq m$. Furthermore, it results from the construction that $(x_n)_n$ is increasing; so put $x = \bigvee_n \uparrow x_n$. Thus $x \in [0,\ a_0]$, and by proposition 2.7 again, $S_m(x, \mathbf{x}_0, \mathbf{x}_1, \ldots)$ is solvable in $\prod_{l\geq 1}[0,\ a_l]$ for all $m \in \omega$. Thus x is as required. ∎

Now we can give the main theorem of this section.

2.9. Theorem. *Every monotone σ-complete dimension group is boundedly countably atomic compact. Equivalently, for every monotone σ-complete dimension group A, $(A^+ \cup \{\infty\}, +, \leq)$ is countably equationally (atomic) compact.*

Proof. Let A be a monotone σ-complete dimension group, let $S(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ be a countable bounding system with parameters from A. We show that S admits a solution in A. By appropriate translations of the variables, we may assume without loss of generality that S admits a bounding sequence of the form $(0, a_n)_n$. Now, using lemma 2.8, it is easy to construct inductively a sequence $(x_n)_n$ in $\prod_n[0,\ a_n]$ such that for all n, $S(x_0, \ldots, x_n, \mathbf{x}_{n+1}, \mathbf{x}_{n+2}, \ldots)$ admits $(0, a_l)_{l>n}$ as a bounding family (thus is finitely solvable). Since every subsystem of S involves only finitely many unknowns, $(x_n)_n$ is a solution of S in $\prod_n[0,\ a_n]$. ∎

Another fact is that in proposition 2.7, the restriction that $0 \leq \mathbf{x}_i \leq a$ may be removed as far as formulæ of the language $(+, \leq)$ are concerned (see the following counterexample 2.12 for a richer language).

2.10. Lemma. *Let $\varphi(\underset{(m)}{\vec{\mathbf{x}}}, \underset{(n)}{\vec{\mathbf{a}}})$ be a linear formula, let A be a monotone σ-complete dimension group, let $\vec{a} \in {}^n A^+$. Put $a = N_\varphi \sum_j a_j$. Then*

$$A^+ \models \left[(\exists \vec{\mathbf{x}})\varphi(\vec{\mathbf{x}}, \vec{a}) \Rightarrow (\exists \vec{\mathbf{x}}) \left(\bigwedge\!\!\!\!\bigwedge_{i<m} (0 \leq \mathbf{x}_i \leq a) \text{ and } \varphi(\vec{\mathbf{x}}, \vec{a}) \right)\right].$$

Proof. Put $N = N_\varphi$. Let θ be the following statement:

$$(\forall \vec{\mathbf{x}}, \vec{\mathbf{a}}) \left[\varphi(\vec{\mathbf{x}}, \vec{\mathbf{a}}) \Rightarrow (\exists \vec{\mathbf{y}}) \bigwedge\!\!\!\!\bigwedge_{i<m} \left(\left(0 \leq \mathbf{y}_i \leq N \sum_{j<n} \mathbf{a}_j \right) \text{ and } \varphi(\vec{\mathbf{y}}, \vec{\mathbf{a}}) \right) \right]$$

Since θ is a special sentence, it suffices to prove that θ is satisfied by $\mathbb{N}$, and furthermore, it suffices to verify θ for $x_i, a_j < N$ (corollary 2.3). So suppose that $x_i, a_j < N$ for all $i < m$, $j < n$ and that $\mathbb{N}$ satisfies $\varphi(\vec{x}, \vec{a})$. If $a_j = 0$ for all j, put $y_i = 0$ for all i; otherwise, take $y_i = x_i$ for all i. Then in both cases, $0 \leq y_i \leq N \sum_j a_j$ and $\varphi(\vec{y}, \vec{a})$. ∎

2.11. Theorem. *Let $\varphi\big(\mathbf{x}, \underset{(m)}{\vec{\mathbf{x}}}, \underset{(n)}{\vec{\mathbf{a}}}\big)$ be a linear formula, let A be a monotone σ-complete dimension group, let $\vec{a}$ in ${}^n A$. Then*

$$D = \big\{x \in A : \ A \models (\exists\, \vec{\mathbf{x}})\varphi(x, \vec{\mathbf{x}}, \vec{a})\big\}$$

is closed under countable $\bigvee \uparrow$ and $\bigwedge \downarrow$.

Proof. We prove for example that D is closed under countable $\bigvee \uparrow$; the proof for $\bigwedge \downarrow$ is similar. So let x in A, x_k $(k \in \omega)$ in D such that $x = \bigvee_k \uparrow x_k$. We prove that $x \in D$. Since A is directed, one can assume without loss of generality that $x_0 \geq 0$ and $a_j \geq 0$ for all $j < n$, whence $x \geq 0$ and $x_k \geq 0$ for all k. Furthermore, for all y in A^+, we have

$$y \in D \Longleftrightarrow A \models (\exists\, \vec{\mathbf{z}}, \vec{\mathbf{t}} \geq 0)\varphi(y, \vec{\mathbf{z}} - \vec{\mathbf{t}}, \vec{a}).$$

Thus there exists a linear formula $\psi(\mathbf{x}, \underset{(2m)}{\vec{\mathbf{x}}}, \underset{(n)}{\vec{\mathbf{a}}})$ such that $D \cap A^+ = \{y \in A^+ : \ A^+ \models \psi(y, \vec{\mathbf{x}}, \vec{a})\}$. By lemma 2.10, if $N = N_\psi$, we also have

$$D = \left\{y \in A^+ : \ A^+ \models \left(\exists_{i<2m} \mathbf{x}_i \leq N\Big(y + \sum_{j<n} a_j\Big)\right)\psi(y, \vec{\mathbf{x}}, \vec{a})\right\}.$$

Thus for all k in ω, $x_k \in D_a$ where $a = N\Big(x + \sum_{j<n} a_j\Big)$ and

$$D_a = \big\{y \in A^+ : \ A^+ \models (\exists_{i<2m} \mathbf{x}_i \leq a)\psi(y, \vec{\mathbf{x}}, \vec{a})\big\}$$

By proposition 2.7, $x \in D_a$, whence $x \in D$. ∎

2.12. Example. The analogue of theorem 2.11 may fail for richer languages. Let $\mathcal{L} = (+, \cdot, \leq)$ where $+$ and $\cdot$ are binary operation symbols and $\leq$ is a binary relation symbol. Consider the natural realization of $\mathbb{R}$ as a model of L. Then

$$D = \{x \in \mathbb{R} : \ \mathbb{R} \models (\exists \mathbf{y})(x \cdot \mathbf{y} = 1)\}$$

is *not* closed under countable increasing supremum or infimum.

2.13. Question. Let A be a monotone σ-complete dimension group. Is the structure $(A \cup \{\infty\}, +, \leq)$ countably *positive* compact? Positive formulæ are as usual those which are built up only from the connectors and, or, $\exists$, $\forall$ (no negation). The answer to this question is affirmative for *Dedekind complete ℓ-groups* by [**21**, corollary 5.6] (atomic compactness implies positive compactness, [**16**], but observe that the proof does not apply to the countable case).

2.14. Question. Is there a "compactness-like" statement which characterizes monotone σ-complete dimension groups? *See* [**21**, proposition 5.8] for a related statement.

2.15. Question. It is easy to verify that any boundedly countably atomic compact dimension group satisfies the countable interpolation property. Is the converse true (at least for Archimedean groups)?

§3. Consequences of bounded countable atomic compactness.

We choose in this chapter two kinds of problems that can be formulated in terms of equations-and-inequalities systems. The first kind has been studied in [**13**] and it deals with extensions of semigroup-valued charges. By definition, if A is a Boolean algebra and E is a preordered commutative monoid, $\mu : \mathcal{A} \to E$ is a *charge* when $\mu(0) = 0$ and whenever x, y are disjoint elements of A, then $\mu(x \vee y) = \mu(x) + \mu(y)$. If A and B are Boolean subalgebras of some Boolean algebra C and $\mu : \mathcal{A} \to E$ and $\nu : \mathcal{B} \to E$ are charges, then μ and ν are said to be *consistent* when for all $x \in \mathcal{A}$ and $y \in \mathcal{B}$, $x \leq y$ (resp. $x \geq y$, $x = y$) implies that $\mu(x) \leq \nu(y)$ (resp. $\mu(x) \geq \nu(y)$, $\mu(x) = \nu(y)$). This does not imply in general that μ and ν admit a common extension to a E-valued charge on C, but there are important cases where it does. In [**13**, theorem 3.2], the class of those positively preordered commutative monoids E such that any two consistent E-valued charges with finite domains admit a common extension is characterized by a very simple finite set of conditions. Among these conditions is "minimality" of the preordering, *i.e.* for all x, y in E, $x \leq y$ if and only if there exists z such that $x + z = y$. Structures with this "grid property" are for example all abelian groups (with the coarse preordering), but also $A^+ \cup \{\infty\}$ where A is any interpolation group. Now, in the case where $(A^+ \cup \{\infty\}, +, \leq)$ is atomic compact, the extension property is satisfied in the general case, *i.e.* any two consistent $A^+ \cup \{\infty\}$-valued charges admit a common extension; this includes the case where A is a Dedekind complete ℓ-group [**21**, corollaries 5.6, 5.7]. Now, when A is just monotone σ-complete, this may fail, but the argument above is easily seen to work in the case where only *countable* systems are involved. Thus we get the following

3.1. Proposition. *Let A be a boundedly countably atomic compact interpolation group. Then $A^+ \cup \{\infty\}$ has the* 2*-charge extension property for* countable *charges. This holds in particular when A is a monotone σ-complete dimension group.* ∎

Another field of application of the results of previous chapter lies in the generalization of the results of [**8**, pp. 37-43] about relatively bounded homomorphisms. First recall the

3.2. Definition. Let X and Y be ordered sets. A map $f : X \to Y$ is *relatively bounded* when for every bounded subset W of X, $f[W]$ is bounded in Y.

Then we can obtain the following analogue of [**8**, proposition 2.25].

3.3. Proposition. *Let A be a countable dimension group, let E be a boundedly countably atomic compact dimension group, let f in* $\mathrm{Hom}_{\mathbb{Z}}(A, E)$*. Then f is relatively bounded if and only if there are positive homomorphisms g and h such that $f = g - h$. This holds in particular when A is a countable dimension group and E is a monotone σ-complete dimension group.*

Proof. If $f = g - h$ with positive g, h, then it is immediate that f is relatively bounded. So suppose now that f is relatively bounded. For all a in A^+, there exists $\varphi(a)$ in E^+ such that $(\forall x \in [0,\ a])\big(0 \leq f(x) \leq \varphi(a)\big)$. We need a

Claim. *Let G be a dimension group, let m, n in $\omega \setminus \{0\}$, a_i $(i < m)$ in G, $\vec{p}_j = (p_{ij})_{i<m}$ in ${}^m\mathbb{N}$ (all $j < n$), b_j $(j < n)$ in G^+ such that for all $I \subseteq m$ and all $j < n$, $\sum_{i\in I} p_{ij}a_i \leq b_j$. Then the following system with unknowns $\mathbf{x}_i$ $(i < m)$ is solvable:*

$$\begin{cases} 0, a_i \leq \mathbf{x}_i & (\text{all } i < m) \\ \sum_{i<m} p_{ij}\mathbf{x}_i \leq b_j & (\text{all } j < n) \end{cases}$$

Proof of claim. By corollary 2.3, it is sufficient to prove the claim in the case where $G = \mathbb{Z}$. But in this case, put $x_i = \max(a_i, 0)$. Then for all $i < m$, we have $0, a_i \leq x_i$. Furthermore, let $j < n$. Put $I = \{i < m :\ a_i \geq 0\}$. Then $\sum_{i<m} p_{ij}x_i = \sum_{i\in I} p_{ij}a_i \leq b_j$. Hence, $(x_i)_{i<m}$ is a solution of the system. ∎ Claim

Now, consider the following atomic system S, with unknowns $\mathbf{x}_a$, $a \in A^+$:

$$S : \begin{cases} 0, f(a) \leq \mathbf{x}_a \leq \varphi(a) & (\text{all } a \in A^+) \\ \mathbf{x}_{a+b} = \mathbf{x}_a + \mathbf{x}_b & (\text{all } a, b \text{ in } A^+) \end{cases}$$

We prove that S is finitely solvable. So let P be a finite subset of A^+, and consider the following finite system S_P:

$$S_P : \begin{cases} 0, f(a) \leq \mathbf{x}_a \leq \varphi(a) & (\text{all } a \in P) \\ \mathbf{x}_{a+b} = \mathbf{x}_a + \mathbf{x}_b & (\text{all } a, b \text{ in } P) \end{cases}$$

Let S'_P be the system $\{\, \mathbf{x}_{a+b} = \mathbf{x}_a + \mathbf{x}_b \quad (\text{all } a, b \text{ in } P)$. It is a finite equation system with unknowns $\mathbf{x}_a$ $(a \in Q)$ where $Q = P \cup (P + P)$. By proposition 2.2, S'_P admits a general solution $\mathbf{x}_a = \sum_{k<N} n_{ak}\mathbf{t}_k$ in all dimension cones. In particular, for all a, b in P and all $k < N$, we have $n_{a+b,k} = n_{ak} + n_{bk}$. Furthermore, since $(\mathbf{x}_a \mapsto a)$ is a solution of S'_P in A^+ and A is a dimension group, there are b_k $(k < N)$ in A^+ such that $(\forall a \in Q)\big(a = \sum_{k<N} n_{ak}b_k\big)$. Now consider the following system R with unknowns $\mathbf{y}_k$ $(k < N)$:

$$R : \begin{cases} 0, f(b_k) \leq \mathbf{y}_k & (\text{all } k < N) \\ \sum_{k<N} n_{ak}\mathbf{y}_k \leq \varphi(a) & (\text{all } a \in Q) \end{cases}$$

Let $a \in Q$, $I \subseteq N$. We have $\sum_{k\in I} n_{ak}f(b_k) = f\Big(\underbrace{\sum_{k\in I} n_{ak}b_k}_{\in[0,\ a]}\Big) \leq \varphi(a)$, so that the condition of the claim is satisfied. Therefore, R admits a solution, say $(y_k)_{k<N}$. For all $a \in Q$, put $x_a = \sum_{k<N} n_{ak}y_k$. Then for all $a \in P$, $x_a \geq 0$, $x_a \geq \sum_{k<N} n_{ak}f(b_k) = f(a)$, and $x_a \leq \varphi(a)$. Furthermore, for all a, b in P, we have $(\forall k < N)(n_{a+b,k} = n_{ak} + n_{bk})$, whence $x_{a+b} = x_a + x_b$. Thus $(x_a)_{a\in Q}$ is a solution of S_P. Therefore, S is finitely solvable, hence bounding with bounding family $(0, \varphi(a))_{a\in A^+}$. By assumption, S admits a solution, say $(x_a)_{a\in A^+}$. The map $a \mapsto x_a$ extends naturally to a positive homomorphism g from A

to E, and by assumption, $f \leq g$. Then g and $h = g - f$ satisfy the required conditions. The final conclusion comes from theorem 2.9. ∎

When A and E are directed abelian ordered groups, equip as in [**8**] $\mathrm{Hom}_{\mathbb{Z}}(A, E)$ with the partial ordering $\leq^+$ whose positive cone consists on the positive homomorphisms. The set $\mathrm{Hom}^*_{\mathbb{Z}}(A, E)$ of all relatively bounded homomorphisms from A to E is an *ideal* of $\mathrm{Hom}_{\mathbb{Z}}(A, E)$. The first statement of the following theorem is a Hahn-Banach like theorem for boundedly countably atomic compact dimension groups.

3.4. Theorem. *Let A be a countable dimension group, let E be a boundedly countably atomic compact dimension group. Then the following holds:*

(a) *Let X and Y be nonempty, countable sets of maps from A^+ to E such that $X \leq Y$ and the following holds:*

(i) $(\forall f \in X)(\forall a, b \in A^+)\big(f(a+b) \leq f(a) + f(b)\big)$;

(ii) $(\forall g \in Y)(\forall a, b \in A^+)\big(g(a+b) \geq g(a) + g(b)\big)$.

Then there exists h in $\mathrm{Hom}_{\mathbb{Z}}(A, E)$ such that

$$(\forall f \in X)(\forall g \in Y)\big(f \leq h\restriction_{A^+} \leq g\big).$$

(b) *Suppose that E is monotone σ-complete. Then $\mathrm{Hom}_{\mathbb{Z}}(A, E)$ is a monotone σ-complete interpolation group and $\mathrm{Hom}^*_{\mathbb{Z}}(A, E)$ is a monotone σ-complete dimension group.*

Proof. (a) Consider the following atomic system S, with unknowns $\mathbf{x}_a$ ($a \in A^+$):

$$S : \begin{cases} f(a) \leq \mathbf{x}_a \leq g(a) & (\text{all } f \in X,\, g \in Y,\, a \in A^+) \\ \mathbf{x}_{a+b} = \mathbf{x}_a + \mathbf{x}_b & (\text{all } a,\, b \text{ in } A^+) \end{cases}$$

We show that S is finitely solvable. So let $X_0 \subseteq X$, $Y_0 \subseteq Y$, $P \subseteq A^+$ be finite and nonempty, and consider the following system S_0:

$$S_0 : \begin{cases} f(a) \leq \mathbf{x}_a \leq g(a) & (\text{all } f \in X_0,\, g \in Y_0,\, a \in P) \\ \mathbf{x}_{a+b} = \mathbf{x}_a + \mathbf{x}_b & (\text{all } a,\, b \text{ in } P) \end{cases}$$

We prove that S_0 admits a solution. Consider first the following system S_1:

$$S_1 : \{\, \mathbf{x}_{a+b} = \mathbf{x}_a + \mathbf{x}_b \quad (\text{all } a,\, b \text{ in } P).$$

Let $Q = P \cup (P + P)$, and let $\mathbf{x}_a = \sum_{k<N} n_{ak}\mathbf{t}_k$ be the general solution of S_1 in dimension cones; thus for all a, b in P and $k < N$, we have $n_{a+b,k} = n_{ak} + n_{bk}$. Furthermore, $(\mathbf{x}_a \mapsto a)$ is a solution of S_1 in A^+; therefore, there are b_k ($k < N$) in A^+ such that for all $a \in Q$, we have $a = \sum_{k<N} n_{ak} b_k$. Now, since E satisfies the interpolation property, there are c_k ($k < N$) in E such that for all $f \in X_0$, $g \in Y_0$ and $k < N$, we have $f(b_k) \leq c_k \leq g(b_k)$. For all $a \in Q$, let $x_a = \sum_{k<N} n_{ak} c_k$. We check that $(x_a)_{a \in Q}$ is a solution of S_0.

(•) Let $f \in X_0$, $g \in Y_0$, $a \in P$. Then we have

$$\begin{aligned} f(a) = f\Big(\sum_{k<N} n_{ak} b_k\Big) &\leq \sum_{k<N} n_{ak} f(b_k) \\ &\leq \sum_{k<N} n_{ak} c_k \\ &= x_a \\ &\leq \sum_{k<N} n_{ak} g(b_k) \\ &\leq g\Big(\sum_{k<N} n_{ak} b_k\Big) \\ &= g(a). \end{aligned}$$

(•) Let a, b in P. Since for all $k < N$, we have $n_{a+b,k} = n_{ak} + n_{bk}$, we have $x_{a+b} = x_a + x_b$.

Thus $(x_a)_{a\in Q}$ is a solution of S_0. But X and Y are nonempty, thus S is bounding. By assumption, S admits a solution $(x_a)_{a\in A^+}$. The map $a \mapsto x_a$ extends naturally to a homomorphism h from A to E. Then h satisfies the required condition.

(b) Let f, f', g, g' in $\mathrm{Hom}_{\mathbb{Z}}(A, E)$ such that $f, f' \leq^+ g, g'$. By (a), there exists h in $\mathrm{Hom}_{\mathbb{Z}}(A, E)$ such that $f, f' \leq^+ h \leq^+ g, g'$. Therefore, $\mathrm{Hom}_{\mathbb{Z}}(A, E)$ satisfies the interpolation property. Furthermore, it is immediate (taking componentwise suprema and infima) that $\mathrm{Hom}_{\mathbb{Z}}(A, E)$ is monotone σ-complete. The conclusion follows from the fact that $\mathrm{Hom}^*_{\mathbb{Z}}(A, E)$ is an ideal of $\mathrm{Hom}_{\mathbb{Z}}(A, E)$, directed by proposition 3.3. ∎

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