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A COMPACTNESS PROPERTY OF DEDEKIND σ-COMPLETE *f*-RINGS.

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Abstract. We prove that Dedekind σ-complete *f*-rings are boundedly countably atomic compact in the language $(+,-,\cdot,\wedge,\vee,\leq)$. This means that whenever Γ is a countable set of atomic formulæ with parameters from some Dedekind σ-complete *f*-ring A every finite subsystem of which admits a solution in some fixed product K of bounded closed intervals of A, then Γ admits a solution in K.

In [**16**, Theorem 5.5], we prove the following theorem:

$(*)$ *Let A be a Dedekind complete f-ring. Then every system of the form*

$$\begin{cases} \varphi_i(\vec{\mathbf{x}}) & (\text{all } i \in I;\ \text{the } \varphi_i\text{'s are atomic formulæ of } (+,-,\cdot,\wedge,\vee,\leq)) \\ |\mathbf{x}_j| \leq a_j & (\text{all } j \in J;\ \text{the } a_j\text{'s are elements of } A^+) \end{cases}$$

every finite subsystem of which admits a solution in A, admits a solution in A.

The method of proof consists roughly speaking of forcing with the Boolean algebra of polars of subsets of A and observing that in the Boolean universe, A becomes a *linearly ordered* complete *f*-ring, where the usual argument of topological compactness can be performed. Then the question of generalizing this result to the *Dedekind σ-complete case* (for I and J countable) is natural, and it seems also natural to think that this relativization is easily obtained by just writing in more detail the proof of the result above.

But rather surprisingly, in trying to do so, one encounters an important difficulty: namely, one needs to be able to go from "bounded" Boolean-valued names of elements of A to elements of A [**16**, Lemma 5.2], and for this, (full) conditional lateral completeness is needed. However, this is not always the case in Dedekind σ-complete *f*-rings. Note that in a similar way, Stone's general representation theorem for Dedekind complete ℓ-groups [**12**, Theorem 3] fails to generalize to Dedekind σ-complete ℓ-groups. Thus the step from Dedekind completeness to Dedekind σ-completeness is not trivial.

This we do here, proving that Dedekind σ-complete *f*-rings satisfy the version of $(*)$ above with I and J *countable*. There is nothing special about the cardinal $\aleph_0$ in this proof, and it could be generalized to any uncountable cardinal. The main idea is to combine the proof in the complete case with the countable version of Sikorski's extension theorem for Boolean algebras, constructing this way something close to a Boolean-valued name of a non-trivial ultrafilter on ω and allowing a more effective construction of a global

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solution of the atomic system under consideration (see also remark 13). The formalism involving forcing used here could be (in a somewhat laborious way) "translated" into a proof without forcing, at the expense of introducing very cumbersome and unnatural notations and statements.

Note that there is a close similarity between the forcing description of an ordered structure used here (and in [**16**]) and the theory of *general comparability* [**4**, **5**], but the former offers the very important additional convenience of the generic model theorem [**6**, **8**, **16**], which expresses that arguments in usual logic can be carried over to Boolean-valued models.

We will use the notations and basic results of [**16**] throughout this paper, and we refer to [**3**] for the basic terminology and facts about model theory, reduced powers, *etc.*. Furthermore, some of the ideas underlying [**16**] originate in [**8**], which the reader may want to consult for further reference about forcing and Boolean-linear spaces; *see* also [**6**, **7**].

Let $(P, \leq)$ be an ordered set. For any two subsets X and Y of P, $X \leq Y$ will stand for $(\forall x \in X)(\forall y \in Y)(x \leq y)$. We will say that P has the *countable interpolation property* when for any two nonempty countable subsets X and Y of P such that $X \leq Y$, there exists z in P ("interpolant") such that $X \leq \{z\} \leq Y$.

We will denote by fin the Fréchet filter on ω, and for every set X, we will denote by ${}^*X = {}^\omega X/fin$ the *reduced power* of X by fin. For all $(x_n)_n$ in ${}^\omega X$, we will denote by $[x_n]_n$ its image in ${}^*X$. Furthermore, we will identify as usual X with its image by the natural embedding from X into ${}^*X$. It will also sometimes be convenient to identify X with its image in ${}^\omega X$ under the diagonal map.

Formal variable symbols will be written in boldface roman letters $\mathbf{x}$, $\mathbf{y}$..., while elements of a given structure will be written in math italics a, b, x, y... .

Let $\mathcal{L}$ be a first-order language, let M be a model of $\mathcal{L}$. Recall [**13**] that an *atomic system* with parameters from M is a set of atomic formulæ with parameters from M. Here, instead of being concerned with atomic compactness, we will be interested in *bounded atomic compactness*. This notion will be defined in the case where $\mathcal{L}$ has a distinguished *binary relation symbol*, $\leq$.

1. Definition. Let Σ be an atomic system with parameters from M, with unknowns $\mathbf{x}_i$ $(i \in I)$. Then a family $(a_i, b_i)_{i \in I}$ of elements of $M \times M$ is *bounding* for Σ when the following system:

$$\begin{cases} \Sigma \\ a_i \leq \mathbf{x}_i \leq b_i \quad (\text{all } i \in I) \end{cases}$$

is finitely solvable in M (thus Σ is itself finitely solvable in M). The system Σ is *bounded* when it admits a bounding family. The model M is *[boundedly] countably atomic compact* when every [bounded] countable atomic system with parameters from M is solvable in M.

Now we have the following analogue of [**14**, Theorem 2.3].

2. Proposition. *Let $\mathcal{L}$ be a first-order language containing a distinguished relation symbol $\leq$, let M be a model of $\mathcal{L}$ such that $\leq^M$ is a partial preordering and the operations*

in M are bounded (i.e. *they send bounded sets on bounded sets). Then the following are equivalent:*

(i) *M is boundedly countably atomic compact;*

(ii) *Let M_0 and M_1 be countable models such that $M_0 \subseteq M$, $M_0 \subseteq M_1 \subseteq {}^{\omega}M_0$ and M_1 consists only of eventually bounded sequences. Then there exists a homomorphism $r : M_1/fin \to M$ such that $r|_{M_0} = id$.*

Proof. (i)⇒(ii) Let $N = M_1/fin$; thus up to natural isomorphisms, we have $M_0 \subseteq N \subseteq {}^*M_0$. For all a in ${}^{\omega}M_0$, denote by $[a]$ the fin-equivalence class of a. Consider the following countable atomic system Σ, with parameters from M_0, with unknowns $\mathbf{x}_\alpha$ ($\alpha \in N$):

$$\Sigma : \begin{cases} \mathbf{x}_\alpha = f(\mathbf{x}_{\alpha_1}, \ldots, \mathbf{x}_{\alpha_n}) & \text{(all } n \in \omega \setminus \{0\}\text{, all } n\text{-ary function symbols } f \text{ and all} \\ & \alpha,\ \alpha_i\ (1 \le i \le n) \text{ in } N \text{ such that } \alpha = f^N(\alpha_1, \ldots, \alpha_n)) \\ R(\mathbf{x}_{\alpha_1}, \ldots, \mathbf{x}_{\alpha_n}) & \text{(all } n \in \omega \setminus \{0\}\text{, all } n\text{-ary relation symbols } R \text{ and all} \\ & \alpha_i\ (1 \le i \le n) \text{ in } N \text{ such that } R^N(\alpha_1, \ldots, \alpha_n)) \\ \mathbf{x}_{[a]} = a & \text{(all } a \in M_0) \end{cases}$$

For all α in N, let u_α and v_α in M such that for some representative sequence $(a_n)_n$ of α in M_1 and all but finitely many n, $u_\alpha \le a_n \le v_\alpha$. Then it is easy to see that $(u_\alpha, v_\alpha)_{\alpha \in N}$ is a bounded family for Σ. Thus, by assumption, Σ admits a solution in M, say $(x_\alpha)_{\alpha \in N}$. Define r by $r(\alpha) = x_\alpha$.

(ii)⇒(i) Let Σ be a countable atomic system with parameters from M, with unknowns $\mathbf{x}_n$ ($n \in \omega$) and with bounding sequence $(a_n, b_n)_{n \in \omega}$. There exists a countable substructure M_0 of M such that all parameters in Σ and all a_n, b_n's are in M_0 and Σ is finitely solvable in M_0. Hence, Σ is solvable in ${}^*M_0$ [**14**, Lemma 2.2], with solutions x_n ($n \in \omega$) satisfying $a_n \le x_n \le b_n$. Let N be the submodel of ${}^*M_0$ generated by M_0 and $\{x_n : n \in \omega\}$; let M_1 be the inverse image of N by the natural projection ${}^{\omega}M_0 \to {}^*M_0$. Then M_1 is countable, and, by the assumption on the operations, all elements of M_1 are eventually bounded. By assumption, there exists a homomorphism r from $M_1/fin = N$ to M such that $r|_{M_0} = id$. Then $(r(x_n))_n$ is a solution of Σ in M. ∎

Now we will specialize to *f*-rings — *see* [**2**] for reference about these objects.

Let $\mathcal{F}$ be a *filterbasis* on a set I, *i.e.* a nonempty set of nonempty subsets of I that is closed under finite intersection. We will say that a subset X of I is *$\mathcal{F}$-measurable* when X either contains or is disjoint from an element of $\mathcal{F}$. Note that $\mathcal{F}$ generates a ultrafilter of the Boolean algebra of $\mathcal{F}$-measurable subsets of I.

3. Lemma. *Let $(a_n)_n$ be a bounded real sequence, let D be a subgroup of $(\mathbb{R}, +)$ containing $\{a_n : n \in \omega\}$, let $\mathcal{F}$ be a filterbasis on ω. Suppose that for all a in D, both sets $\{n \in \omega : a_n \le a\}$ and $\{n \in \omega : a_n \ge a\}$ are $\mathcal{F}$-measurable. Then $(a_n)_n$ is convergent relative to $\mathcal{F}$.*

Proof. Since $(a_n)_{n \in \omega}$ is bounded, $a = \underline{\lim}_{\mathcal{F}}(a_n)_n$ and $b = \overline{\lim}_{\mathcal{F}}(a_n)_n$ belong to $\mathbb{R}$, and the convergence of $(a_n)_n$ is equivalent to the fact that $b \le a$ (we always have $a \le b$). We distinguish two cases.

Case 1. D is discrete.

Then both a and b belong to D. Let $d \in D \cap \mathbb{R}^+$ be a generator of D. Since $(a_n)_n$ is bounded and the a_n's belong to D, there exists $N \in \mathbb{N} \setminus \{0\}$ such that for all n, we have $-Nd \leq a_n \leq Nd$. Thus $\omega = \bigcup_{k=-N}^{N} X_k$ where we put $X_k = \{n \in \omega : \ a_n = kd\}$. But by assumption, all the X_k's are $\mathcal{F}$-measurable (and $\omega \in \mathcal{F}$), whence there exists k such that $X_k \in \mathcal{F}$. Thus for $\mathcal{F}$-almost every n, we have $a_n = kd$, whence $a = b = kd$.

Case 2. D is dense.

Towards a contradiction, assume that $a < b$. Since D is dense, there exists d in D such that $0 < d < b - a$. Since $(a_n)_n$ is bounded, there exists N in $\mathbb{N} \setminus \{0\}$ such that for all n, we have $-Nd \leq a_n \leq Nd$. Thus $\omega = \bigcup_{k=-N}^{N-1} X_k$ where we put $X_k = \{n \in \omega : \ kd \leq a_n \leq (k+1)d\}$. Since the X_k's are by assumption $\mathcal{F}$-measurable, there exists k such that $X_k \in \mathcal{F}$. It follows that $kd \leq a < b \leq (k+1)d$, which contradicts the fact that $d < b - a$.

Thus in both cases, $a = b$, which completes the proof. ∎

Note that in the context of Lemma 3, one cannot necessarily conclude that $\lim_{\mathcal{F}} a_n$ belongs to D; thus if one is restrained to state Lemma 3 "inside D", as it will be the case in Lemma 9, the conclusion has to be restated as "$(a_n)_n$ is a $\mathcal{F}$-Cauchy sequence".

4. Lemma. *Let E be a Boolean algebra satisfying the countable interpolation property. Then E is countably atomic compact in the language* $(\wedge, \vee, \neg, \leq)$.

Note: the converse is trivial, thus we have equivalence.

Proof. It suffices to prove that E is *countably injective, i.e.* for every subalgebra A of a countable Boolean algebra B, every homomorphism from A to E extends to a homomorphism from B to E (this is in fact equivalent to countable atomic compactness, since every extension of Boolean algebras is *pure*). Furthermore, an easy induction argument shows that it suffices to consider the case where B is *monogenic over A, i.e.* B is generated by $A \cup \{b\}$ for some $b \in B$. Then, for an extension as desired to exist, it is necessary and sufficient [**9**, Corollary 5.8] that there exists β in E such that for all x in A, $x \leq b$ (resp. $x \geq b$) implies $f(x) \leq \beta$ (resp. $f(x) \geq \beta$). But by the countable interpolation property, such a β always exists. ∎

Recall [**9**, 5.26 *and* 5.27] that an example of a non σ-complete Boolean algebra with the countable interpolation property is the quotient Boolean algebra $\mathcal{P}(\omega)/fin$ of the powerset algebra $\mathcal{P}(\omega)$ by the equivalence modulo finite sets.

Now, in the sequel, let A be a Dedekind σ-complete *f*-ring. Thus A is Archimedean, thus commutative [**2**]. For every subset X of A, we put as usual $\perp X = \{y \in A : (\forall x \in X)(|x| \wedge |y| = 0)\}$, the *polar* of X. The polars of subsets of A form a complete Boolean algebra under inclusion, say $\mathcal{B}$. For all $p \in A^+$, denote by $[p]$ the natural image of p in $\mathcal{B}$, *viz.* $[p] = \perp\perp\{p\}$. Let B be the σ-complete subalgebra of $\mathcal{B}$ generated by $\{[p] : \ p \in A^+\}$. As in [**16**, sections 4 *and* 5], A is equipped with a structure of $\mathcal{B}$-valued model of the theory of linearly ordered commutative rings, defined by $\|a \leq b\| = \perp\{(a-b)_+\}$ for all a, b in A; note that $\|a \leq b\| = \perp[(a-b)_+] \in B$. Furthermore, there is no loss of generality in assuming that elements of A are atoms of the universe and that the $\mathcal{B}$-valued universe extends the

$\mathcal{B}$-valued A [**16**, Lemmas 2.1 *and* 2.2]; then, such statements as $\|\check{A}$ is Archimedean$\| = 1$ [**16**, Lemma 4.13] ($\check{A}$ is the canonical name for A) become meaningful.

5. Lemma. *For every u in B, we have $u + \perp u = A$.*

Proof. Let $B' = \{u \in \mathcal{B} : \ u + \perp u = A\}$. We prove that B' contains B.

First, for all p in A^+, $[p] \in B'$ since A, being a Dedekind σ-complete ℓ-group, is *projectable* [**2**, Proposition 11.2.3]. It is also clear that B' is closed under 'complementation' $u \mapsto \perp u$. Finally, let $(u_n)_{n\in\omega}$ be a sequence of elements of B', let $u = \bigvee_n u_n$. We prove that $u \in B'$. It suffices to prove that for all a in A^+, there exists x in u such that $a - x \in \perp u$. For all n, let x_n be the unique element of u_n such that $a - x_n \in \perp u_n$. For all n, $0 \le x_n \le a$, so that $x = \bigvee_n x_n$ exists and belongs to $[0,\ a]$. For all n, $0 \le a - x \le a - x_n \in \perp u_n$, so that $a - x \in \bigcap_n \perp u_n = \perp u$. On the other hand, $x_n \in u_n \subseteq u$ for all n, whence $x \in u$.

Thus we have proved that B' is a σ-subalgebra of $\mathcal{B}$ containing $\{[p] : \ p \in A^+\}$; thus it contains B. ∎

For every u in B and every a in A, denote by $a|u$ the unique element x of u such that $a - x \in \perp u$.

Now, let us study $\mathcal{B}$-valued names of subsets of ω. In general, for every $\mathcal{B}$-valued name $\boldsymbol{X}$, we have $\|\boldsymbol{X} \subseteq \check{\omega}\| = \|\boldsymbol{X} = \boldsymbol{X}'\|$ where $\boldsymbol{X}' = \{(\check{n}, \|\check{n} \in \boldsymbol{X}\|) : \ n \in \omega\}$. In that sense, $\mathcal{B}$-valued names of subsets of ω can be identified with sequences of elements of $\mathcal{B}$ (just as subsets of ω can be identified with sequences of elements of $\{0, 1\}$). For every sequence $\vec{u} = (u_n)_{n\in\omega}$ of elements of $\mathcal{B}$, let $\boldsymbol{X}_{\vec{u}} = \{(\check{n}, u_n) : \ n \in \omega\}$ be the $\mathcal{B}$-valued name associated with $\vec{u}$.

Now, suppose that $\vec{u} \in {}^{\omega}B$. Let $(a_n)_{n\in\omega}$ be a bounded sequence of elements of A; we construct a name for $\bigwedge_{n\in\boldsymbol{X}_{\vec{u}}} a_n$. Put $a = \bigvee_n a_n$ (any upper bound of the a_n's would do). For all n, let $x_n = a_n|u_n + a|\perp u_n$ (use Lemma 5). Since $(a_n)_n$ is bounded, $x = \bigwedge_n x_n$ exists. In the sequel, we will denote this x by $\bigwedge^{\vec{u}}(a_n)_n$.

6. Lemma. $\|\boldsymbol{X}_{\vec{u}} \ne \emptyset\| \le \|x = \bigwedge_{n\in\boldsymbol{X}_{\vec{u}}} a_n\|$. *Furthermore, if $\bigvee_n u_n = 1$, then x is the largest element y of A such that for all n, $u_n \le \|y \le a_n\|$.*

Proof. Let $u = \|\boldsymbol{X}_{\vec{u}} \ne \emptyset\|$, $v = \|x = \bigwedge_{n\in\boldsymbol{X}_{\vec{u}}} a_n\|$. Note that $u = \bigvee_n u_n$. For all n, we have $x|u_n \le x_n|u_n = a_n|u_n$, whence $u_n \le \|x \le a_n\|$, whence

$$\|(\forall n \in \boldsymbol{X}_{\vec{u}})(x \le a_n)\| = \bigwedge_{n\in\omega} (u_n \to \|x \le a_n\|) = 1.$$

On the other hand, let y in A. Let $w = \|(\forall n \in \boldsymbol{X}_{\vec{u}})(y \le a_n)\|$. We prove that $u \wedge w \le \|y \le x\|$. We have $w = \bigwedge_{n\in\omega}(u_n \to \|y \le a_n\|) \in B$. Let b be a lower bound of $\{a_n : \ n \in \omega\}$, let $z = y|w + b|\perp w$. Since $w \le \|z = y\|$ and $\perp w \le \|z = b\|$, we have $\|(\forall n \in \boldsymbol{X}_{\vec{u}})(z \le a_n)\| = 1$, *i.e.* for all n, $u_n \le \|z \le a_n\|$. Thus for all n, $u_n \le \|z \le a\|$, thus $u \le \|z \le a\|$, thus $z' \le a$ where $z' = z|u + a|\perp u$. For all n, we have

$$z'|u_n = z|u_n \le a_n|u_n = x_n|u_n,$$

and

$$z'|\perp u_n \le a|\perp u_n = x_n|\perp u_n,$$

whence $z' \le x_n$. Thus, $z' \le x$, whence $u \le \|z \le x\|$, whence $u \wedge w \le \|y \le x\|$. So we have proved that

$$u \le \big\|(\forall y \in \check{A})\big((\forall n \in \boldsymbol{X}_{\vec{u}})(y \le a_n) \Rightarrow y \le x\big)\big\|,$$

which completes the first part of the proof.

As to the second part of the proof (in the case $\bigvee_n u_n = 1$), note first that for all n, $u_n \le \|x \le a_n\|$. If y satisfies this property, then we have $\|(\forall n \in \boldsymbol{X}_{\vec{u}})(y \le a_n)\| = 1$, whence, by the previous conclusion, $\|y \le x\| = 1$, whence $y \le x$. ∎

Similarly, if $(a_n)_n$ is a bounded sequence of elements of A and $\vec{u} \in {}^{\omega}B$, one can define canonically $x = \bigvee^{\vec{u}}(a_n)_n$, in such a way that $\|\boldsymbol{X}_{\vec{u}} \ne \emptyset\| \le \|x = \bigvee_{n \in \boldsymbol{X}_{\vec{u}}} a_n\|$.

Now, let A_0 be a countable *f*-subring of A; identify A_0 with its image in ${}^{\omega}A_0$ under the diagonal map, and let A_1 be a countable *f*-subring of ${}^{\omega}A_0$ containing A_0 and consisting only of *bounded* sequences. We shall prove that there is a *f*-ring homomorphism $\rho : A_1/fin \to A$ such that $\rho|_{A_0} = id$; this will suffice (by Proposition 2) to prove the countable bounded atomic compactness of $(A, +, -, \cdot, \wedge, \vee, \le)$.

Let B_0 be the subalgebra of B generated by $\{[p] : \ p \in A_0\}$, let B_1 be the subalgebra of ${}^{\omega}B_0$ generated by all sequences $(\|a_n \le a\|)_n$ and $(\|a_n \ge a\|)_n$ for $a \in A_0$, $(a_n)_n \in A_1$ and let B_2 be the subalgebra of ${}^{\omega}B_0$ generated by B_1 and all $\bigvee_{n \ge m} u_n$ where $m \in \omega$ and $(u_n)_n \in B_1$. Thus B_0, B_1 and B_2 are countable, and $B_0 \subseteq B_1 \subseteq B_2 \subseteq {}^{\omega}B_0$. Since B is σ-complete, there exists by Lemma 4 a homomorphism $r' : B_2 \to B$ such that r' factors through fin, yielding a homomorphism from B_2/fin to B, and $r'|_{B_2 \cap B} = id$. Let $r = r'|_{B_1}$. Note that for all $\vec{u} = (u_n)_n$ in B_1 and all m in ω, we have, putting $\bar{u} = \bigvee_{n \ge m} u_n$, that $u_n \le \bar{u}$ for all $n \ge m$, thus $\vec{u} \le \bar{u}$ (mod fin) and $\vec{u} \in B_1 \subseteq B_2$ and $\bar{u} \in B_2$, thus $r(\vec{u}) = r'(\vec{u}) \le r'(\bar{u}) = \bar{u} = \bigvee_{n \ge m} u_n$.

Now, we shall define something as close as possible to the $\mathcal{B}$-valued name of a ultrafilter on ω: it will be

$$\boldsymbol{\mathcal{F}} = \big\{\big(\boldsymbol{X}_{\vec{u}}, r(\vec{u})\big) : \ \vec{u} \in B_1\big\}.$$

7. Lemma. $\|\boldsymbol{\mathcal{F}}$ is a filterbasis on $\check{\omega}\| = 1$.

Proof. Let $\vec{1}$ be the constant sequence with value 1. Then $\boldsymbol{\mathcal{F}}(\boldsymbol{X}_{\vec{1}}) = 1$, whence $\|\check{\omega} \in \boldsymbol{\mathcal{F}}\| = \|\boldsymbol{X}_{\vec{1}} \in \boldsymbol{\mathcal{F}}\| = 1$. Furthermore,

$$\|\emptyset \notin \boldsymbol{\mathcal{F}}\| = \bigwedge_{\vec{u} \in B_1} \big(\boldsymbol{\mathcal{F}}(\boldsymbol{X}_{\vec{u}}) \to \|\boldsymbol{X}_{\vec{u}} \ne \emptyset\|\big) = \bigwedge_{\vec{u} \in B_1} \left(r(\vec{u}) \to \bigvee_n u_n\right) = 1.$$

Finally, let $\vec{u}$ and $\vec{v}$ in B_1. Put $\vec{w} = \vec{u} \wedge \vec{v}$ (*i.e.* for all n, $w_n = u_n \wedge v_n$). It is easy to see that $\|\boldsymbol{X}_{\vec{w}} = \boldsymbol{X}_{\vec{u}} \cap \boldsymbol{X}_{\vec{v}}\| = 1$. Thus, $r(\vec{u}) \wedge r(\vec{v}) = r(\vec{w}) \le \|\boldsymbol{X}_{\vec{u}} \cap \boldsymbol{X}_{\vec{v}} \in \boldsymbol{\mathcal{F}}\|$. This proves that $\|(\forall \boldsymbol{X}, \boldsymbol{Y} \in \boldsymbol{\mathcal{F}})(\boldsymbol{X} \cap \boldsymbol{Y} \in \boldsymbol{\mathcal{F}})\| = 1$. ∎

Now, let $B_1 = \{\vec{u}^k : \ k \in \omega\}$ be an enumeration of B_1. Fix in Lemmas 8 and 9 an element $(a_n)_n$ of A_1. For all k, put $v_k = r(\vec{u}^k)$ and $y_k = \bigwedge^{\vec{u}^k}(a_n)_n$. The sequence $(y_k)_k$ is bounded (it lies in $[\bigwedge_n a_n, \ \bigvee_n a_n]$), so that one can define $a = \bigvee^{\vec{v}}(y_k)_k$.

8. Lemma. $\|a = \underline{\lim}_{\boldsymbol{\mathcal{F}}}(a_n)_n\| = 1$.

Proof. We prove that $\|a = \bigvee_{\boldsymbol{X}\in\boldsymbol{\mathcal{F}}} \bigwedge_{n\in\boldsymbol{X}} a_n\| = 1$. Let us first prove that

$$\left\|(\forall \boldsymbol{X} \in \boldsymbol{\mathcal{F}})\left(a \geq \bigwedge_{n\in\boldsymbol{X}} a_n\right)\right\| = 1.$$

For this, it suffices to prove that for all $\vec{u}$ in B_1, we have $r(\vec{u}) \leq \left\|a \geq \bigwedge_{n\in\boldsymbol{X}_{\vec{u}}} a_n\right\|$, *i.e.* for all k in ω, $v_k \leq \|a \geq y_k\|$. But this comes from Lemma 6.

Conversely, let b in A. We prove that

$$\left\|(\forall \boldsymbol{X} \in \boldsymbol{\mathcal{F}})\left(\bigwedge_{n\in\boldsymbol{X}} a_n \leq b\right)\right\| \leq \|a \leq b\|.$$

Let u be the left-hand side. Then we have

$$u = \bigwedge_{k\in\omega} \left(r(\vec{u}^k) \to \|y_k \leq b\|\right) \in B.$$

Thus (Lemma 5) we can define $c = b|u + (\bigvee_n a_n)\,|\perp u$. Then we have

$$\left\|(\forall \boldsymbol{X} \in \boldsymbol{\mathcal{F}})\left(\bigwedge_{n\in\boldsymbol{X}} a_n \leq c\right)\right\| = 1,$$

whence for all k in ω, $v_k \leq \|y_k \leq c\|$. Since $\bigvee_k v_k = 1$, we obtain by Lemma 6 (analogue for $\geq$) that $a \leq c$, so that $u \leq \|a \leq b\|$. ∎

9. Lemma. $\|(a_n)_n$ is a $\boldsymbol{\mathcal{F}}-$ Cauchy sequence$\| = 1$.

The reason why we state this lemma with Cauchy sequences rather than with convergent sequences is that we do not know at that point whether $\check{A}$ is complete in the $\mathcal{B}$-valued universe (*see* also the comments following the proof of Lemma 3).

Proof. Since the following holds

$$\begin{cases} \|(a_n)_n \text{ is bounded}\| = 1, \\ \|\check{A} \text{ is Archimedean linearly ordered}\| = 1, \\ \|\check{A}_0 \text{ is a subgroup of } \check{A}\| = 1, \\ \|(\forall n \in \check{\omega})(a_n \in \check{A}_0)\| = 1, \end{cases}$$

it suffices by Lemma 3 to prove that the following holds:

$$\begin{cases} \|(\forall x \in \check{A}_0)(\{n : a_n \leq x\} \text{ is } \boldsymbol{\mathcal{F}}\text{−measurable})\| = 1, \\ \|(\forall x \in \check{A}_0)(\{n : a_n \geq x\} \text{ is } \boldsymbol{\mathcal{F}}\text{−measurable})\| = 1. \end{cases}$$

Let us prove *e.g.* the first statement; the proof of the second statement is similar. So let x in A_0, we prove that

$$\|\{n : a_n \leq x\} \text{ is } \boldsymbol{\mathcal{F}}\text{−measurable}\| = 1.$$

For all n, put $u_n = \|a_n \leq x\|$; note that $\vec{u} = (u_n)_n$ belongs to B_1. Then we have $\|\boldsymbol{X}_{\vec{u}} = \{n : a_n \leq x\}\| = 1$. Furthermore, we have

$$\begin{cases} r(\vec{u}) \leq \|\boldsymbol{X}_{\vec{u}} \in \boldsymbol{\mathcal{F}}\|, \\ r(\neg\vec{u}) \leq \|\boldsymbol{X}_{\neg\vec{u}} \in \boldsymbol{\mathcal{F}}\|, \end{cases}$$

whence, since $\|\boldsymbol{X}_{\neg\vec{u}} = \breve{\omega} \setminus \boldsymbol{X}_{\vec{u}}\| = 1$,

$$\begin{cases} r(\vec{u}) \leq \|\{n : a_n \leq x\} \in \boldsymbol{\mathcal{F}}\|, \\ r(\neg\vec{u}) \leq \|\breve{\omega} \setminus \{n : a_n \leq x\} \in \boldsymbol{\mathcal{F}}\|. \end{cases}$$

Since $r(\vec{u}) \vee r(\neg\vec{u}) = 1$, we can conclude. ∎

Now, from Lemmas 8 and 9, we deduce immediately the following

10. Lemma. *For all $(a_n)_n$ in A_1, there exists a [unique] a in A such that $\|a = \lim_{\boldsymbol{\mathcal{F}}}(a_n)_n\| = 1$.* ∎

(The uniqueness is trivial, *see* [**16**, Proposition 4.3]).

Now, Lemma 10 allows us to define a map ρ from A_1/fin to A the following way: if $(a_n)_n \in A_1$, then $\rho\big([a_n]_n\big) =$ the unique a in A such that $\|a = \lim_{\boldsymbol{\mathcal{F}}}(a_n)_n\| = 1$; it is clear that ρ is well-defined.

11. Lemma. *ρ is a f-ring homomorphism from A_1/fin to A and $\rho|_{A_0} = id$.*

Proof. Let us prove *e.g.* that ρ is a $\wedge$-homomorphism; the proofs for $+$, $-$, $\cdot$, $\vee$, $\leq$ are similar. So let $(a_n)_n$, $(b_n)_n$, $(c_n)_n$ in A_1 such that for all but finitely many n, $c_n = a_n \wedge b_n$. Let $a = \rho([a_n]_n)$, $b = \rho([b_n]_n)$, $c = \rho([c_n]_n)$. Since $\| \wedge$ is continuous$\| = 1$, we have $\|c = \lim_{\boldsymbol{\mathcal{F}}}(a_n)_n \wedge \lim_{\boldsymbol{\mathcal{F}}}(b_n)_n\| = 1$, thus $\|c = a \wedge b\| = 1$, thus $c = a \wedge b$ [**16**, Proposition 4.5]. ∎

Now, we can deduce immediately the

12. Theorem. *Every Dedekind σ-complete f-ring is boundedly countably atomic compact in the language $(+, -, \cdot, \wedge, \vee, \leq)$.* ∎

13. Remark. Theorem 12 has the following immediate consequence:

Let A be an Archimedean f-ring, let $\bar{A}$ be the free Dedekind σ-complete f-ring above A. Then every countable bounded atomic system in A admits a solution in $\bar{A}$.

If Σ is the atomic system under consideration, then the solutions of Σ in $\bar{A}$ will be generalized terms of countable length (of the language $(+, -, \cdot, \wedge, \vee)$ and countable meet and join) in the parameters and partial solutions of Σ; these expressions will also yield solutions of Σ in *any* Dedekind σ-complete *f*-ring containing A. This conclusion provides additional information that [**16**, Theorem 5.5] (Dedekind complete case) does not provide, since there are *e.g.* no free countably generated Dedekind-complete *f*-rings.

14. Example. Let S be the positive cone of a dense linearly ordered abelian group satisfying the countable strict interpolation property (*i.e.* if X and Y are two nonempty countable subsets of S such that $X < Y$, then there exists z in S such that $X < \{z\} < Y$);

one can for example take S to be any ω_1-saturated elementary extension of $(\mathbb{R}^+, +, \leq)$, although this is not the "cheapest" method from the axiomatic point of view. Let A be the set of all formal linear combinations $x = \sum_{\xi<\alpha} x_\xi X^{s_\xi}$ where X is an indeterminate, the x_ξ's are reals, α is a countable ordinal and $(s_\xi)_{\xi<\alpha}$ is a strictly decreasing α-sequence of elements of S; we identify X^0 to 1. We leave as an exercise the proof of the fact that with componentwise addition and "polynomial-like" (convolution) product (so that $X^a \cdot X^b = X^{a+b}$), A has a natural structure of commutative linearly ordered ring with the countable interpolation property, whose positive cone consists of those x as above such that $x = 0$ or $x_0 > 0$ (thus for every $\varepsilon > 0$ in S, $\mathbb{R} < X^\varepsilon$). However, the following system with unknown $\mathbf{x}$,

$$\begin{cases} X\mathbf{x} \geq 1 \\ n\mathbf{x} \leq 1 \quad (\text{all } n \in \mathbb{N}) \end{cases}$$

is finitely solvable in A, though not solvable. Note that replacing S by $\mathbb{R}^+$ in the previous construction yields a linearly ordered ring without the countable interpolation property (the sets $\mathbb{N}$ and $\{X^{1/n} : n \in \mathbb{N} \setminus \{0\}\}$ cannot be interpolated in this structure), although the "set of exponents" $\mathbb{R}^+$ has the countable interpolation property.

15. Question. Does the conclusion of Theorem 12 still hold for abelian (at least Archimedean) ℓ-groups satisfying the countable interpolation property? (By the previous example, this fails for f-rings).

16. Question. Using the method of [**16**, Corollary 5.6], it is not difficult to prove that if A is a Dedekind σ-complete f-ring, then $A^+ \cup \{\infty\}$ is countably atomic compact in the language $(+, \vee, \leq)$. Does this still hold for *positive* compactness? This is true for (full) Dedekind completeness by [**16**, Corollary 5.6] and [**14**, Theorem 2.3]. The reader can find some more results (similar to the ones in this paper) in [**17**].

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