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THREE OSCULATING WALKERS

MIREILLE BOUSQUET-MÉLOU

To Tony Guttmann, on the occasion of his 60th birthday

Abstract. We consider three directed walkers on the square lattice, which move simultaneously at each tick of a clock and never cross. Their trajectories form a *non-crossing configuration of walks*. This configuration is said to be *osculating* if the walkers never share an edge, and *vicious* (or: non-intersecting) if they never meet.

We give a closed form expression for the generating function of osculating configurations starting from prescribed points. This generating function turns out to be algebraic. We also relate the enumeration of osculating configurations with prescribed starting and ending points to the (better understood) enumeration of non-intersecting configurations.

Our method is based on a step by step decomposition of osculating configurations, and on the solution of the functional equation provided by this decomposition.

1. Introduction

Consider p directed walkers on the (rotated) square lattice, labelled from 1 to p (Figure 1). At time 0, all of them are located at abscissa 0, at respective (even) ordinates $j_{0,1}, \dots, j_{0,p}$, with $j_{0,1} \leq j_{0,2} \leq \cdots \leq j_{0,p}$. Then, at each tick of a clock, each of them moves to the right. More precisely, at each $m \in [\![1, n]\!]$, each walker takes either a North-East step $(1, 1)$ or a South-East step $(1, -1)$. The set of trajectories of these walkers, stopped at time n, is called a *configuration of paths* of length n. This configuration is *non-crossing* if, at each time m, the ordinates $j_{m,1}, \dots, j_{m,p}$ of the p walkers remain ordered as they were at time 0, that is, if $j_{m,1} \leq j_{m,2} \leq \cdots \leq j_{m,p}$. The configuration is *non-intersecting* (or *vicious*) if $j_{m,1} < j_{m,2} < \cdots < j_{m,p}$ for all $m \in [\![0, n]\!]$. The configuration is *osculating* if, as soon as $j_{m,k} = j_{m,k+1}$ for some $m \in [\![0, n-1]\!]$ and $k \in [\![1, p-1]\!]$, then $j_{m+1,k} < j_{m+1,k+1}$. That is, the walkers are allowed to meet, but cannot share an edge nor cross. In an osculating configuration, every pair (m, k) such that $j_{m,k} = j_{m,k+1}$ and $m < n$ is called an *osculation*. For instance, the second configuration of Figure 1 has 3 osculations (the final contact of the endpoints is not counted as an osculation). Observe that in an osculating configuration of positive length, three walkers never occupy the same site.

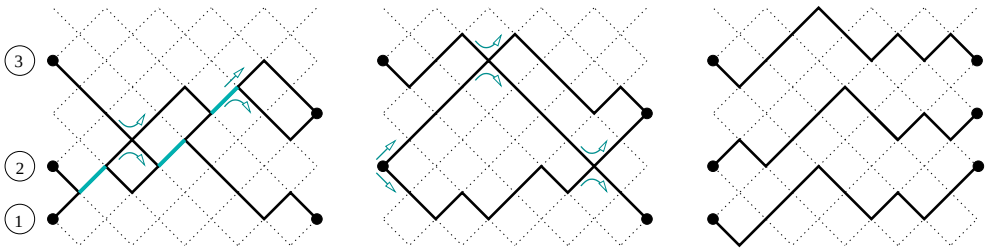

Figure 1. A non-crossing configuration, an osculating configuration, and a non-intersecting (vicious) configuration.

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Configurations of vicious and osculating walkers have attracted a lot of attention in the past 20 years, both in combinatorics and in statistical physics[1]. Vicious walkers are known to be related to many important combinatorial objects, like plane partitions, Young tableaux, symmetric functions, perfect matchings... to name just a few. See [27, Ch. 2] and [28, Ch. 7], as well as [8, 15, 30], for instance. In physics, they were introduced by Fisher as a model of "wetting and melting", and they can be considered as networks of polymers [11, 13, 17, 21]. The enumeration of non-intersecting configurations of walks is well understood: in many cases, it reduces to the evaluation of a determinant [16], or a Pfaffian [29], for which efficient tools are now available [20].

Configurations of *osculating* walkers naturally arise in physics, in the ice model (or 6-vertex model) [2]. More recently, it was realized that they are also connected to some famous matrices, called *alternating sign matrices* [5]. These matrices are renowned for having defeated the combinatorial community for more than a decade: it was conjectured in 1983 that their number is given by a remarkably simple product formula [25], but this formula was only proved, with tremendous difficulty, in 1996 [31]. A bit later, a second proof was found, based on some former work on the 6-vertex model [22, 19]. Let us finally mention that there exists a conjectural formula for the number of osculating configurations with fixed endpoints [7]. In this formula, the determinant that usually appears for non-intersecting configurations is replaced by a more complicated sum on permutations.

In this note, we consider configurations of *three* walkers. Following a terminology inspired by Duplantier [10], and now commonly used in many physics papers, we say that a non-crossing configuration of three walkers starting respectively at ordinates $0, 2i$ and $2(i+j)$, with $i, j \geq 0$, is an (i, j)-*star*. We give explicitly the length generating function of *osculating* (i, j)-stars, which turns out to be a simple algebraic (quadratic) series (Proposition 1). The case $i = j = 1$ of our expression proves a conjecture of Guttmann and Vöge [18]. We refine our result by taking into account, in the enumeration, the number of osculations, thus proving a refined conjecture of Essam [11]. We also obtain the length generating function of *vicious* (i, j)-stars.

Finally, we consider the enumeration of osculating stars in which the distances between the three endpoints are also fixed. More precisely, we study the generating function

$$\mathcal{O}_{i,j}(t; x, y) = \sum_{k,\ell,n\geq 0} o_{i,j}^{(k,\ell)}(n) x^k y^\ell t^n,$$

where $o_{i,j}^{(k,\ell)}(n)$ is the number of osculating (i, j)-stars in which the three walkers end at time n at ordinates j_1, j_2, j_3, with $j_2 - j_1 = 2k$ and $j_3 - j_2 = 2\ell$. We call $\mathcal{O}_{i,j}$ the *complete generating function* of (i, j)-stars. We find an intriguing relation between this series and the complete generating function $\mathcal{V}_{i,j}$ defined similarly for the (better understood) *vicious* walkers (Proposition 2). This relation proves that $\mathcal{O}(t; x, y)$ is *D-finite*, and allows us to compute $o_{i,j}^{(k,\ell)}(n)$ explicitly for given values of i, j, k, ℓ. In particular, we prove a second conjecture of Guttmann and Vöge on the number of osculating *watermelons*. Note that $\mathcal{O}_{i,j}(t; 1, 1)$ is simply the length generating function of osculating (i, j)-stars.

Let us conclude this introduction by recalling some definitions and notation on formal power series. Given a ring $\mathbb{L}$ and k indeterminates $x_1, \ldots, x_k$, we denote by $\mathbb{L}[x_1, \ldots, x_k]$ the ring of polynomials in $x_1, \ldots, x_k$ with coefficients in $\mathbb{L}$. We denote by $\mathbb{L}[[x_1, \ldots, x_k]]$ the ring of formal power series in the x_i with coefficients in $\mathbb{L}$. A *Laurent polynomial* in the x_i is a polynomial in both the x_i and the $\bar{x}_i = 1/x_i$. For $F \in \mathbb{L}[[t]]$, we denote by $[t^n]F$ the coefficient of t^n in $F(t)$. If F is a formal series in t whose coefficients are Laurent polynomials

[1]Moreover, Tony Guttmann has personally observed non-crossing configurations of vacillating and oscillating runners in the final portion of the Marathon du Médoc.

in x, we denote by F^+ the *positive part of F in x*, that is,

$$F = \sum_{n\geq 0} t^n \sum_{i\in\mathbb{Z}} f_i(n)x^i \;\Rightarrow\; F^+ = \sum_{n\geq 0} t^n \sum_{i>0} f_i(n)x^i. \tag{1}$$

We define similarly the negative part of F.

Assume, from now on, that $\mathbb{L}$ is a field. We denote by $\mathbb{L}(x_1,\ldots,x_k)$ the field of rational functions of $x_1,\ldots,x_k$ with coefficients in $\mathbb{L}$. A series F in $\mathbb{L}[[x_1,\ldots,x_k]]$ is *algebraic* if there exists a non-trivial polynomial P with coefficients in $\mathbb{L}$ such that $P(F,x_1,\ldots,x_k)=0$. The sum and product of algebraic series is algebraic. The series F is *D-finite* if the partial derivatives of F span a finite dimensional vector space over the field $\mathbb{L}(x_1,\ldots,x_k)$; see [26] for the one-variable case, and [23, 24] otherwise. In other words, for $1\leq i\leq k$, the series F satisfies a non-trivial partial differential equation of the form

$$\sum_{\ell=0}^{d_i} P_{\ell,i}\ \frac{\partial^\ell F}{\partial x_i^\ell} = 0,$$

where $P_{\ell,i}$ is a polynomial in the x_j. Any algebraic series is D-finite. The sum and product of D-finite series are D-finite. Finally, if F is D-finite, then any *diagonal* of F is also D-finite [23] (the diagonal of F in x_1 and x_2 is obtained by keeping only those monomials for which the exponents of x_1 and x_2 are equal). We shall use the following consequence of this result: if $F(t,x)\in\mathbb{L}[x,\bar{x}][[t]]$ is algebraic, then the positive part of F in x is D-finite, as well as its negative part.

2. The complete generating function of osculating stars

The results stated in this section will be proved in the next section. Our first proposition deals with the *length* generating function of (i,j)-stars.

Proposition 1. *For $i,j\geq 0$ and $(i,j)\neq(0,0)$, the length generating function of osculating (i,j)-stars is algebraic and belongs to $\mathbb{Q}(t,\sqrt{1-8t})$. For instance,*

$$\mathcal{O}_{1,1}(1,1) = \frac{3-15t-4t^2-3(1-t)\sqrt{1-8t}}{8t^2(1+t)}.$$

More generally, let $T\equiv T(t)$ be the unique power series in t satisfying $T=2t(1+T)^2$:

$$T = \frac{1-4t-\sqrt{1-8t}}{4t}.$$

Then

$$\begin{aligned}(1-8t)\mathcal{O}_{i,j}(1,1) &= 1-3\frac{T^{j+1}}{1+2T}+3\frac{T^{i+j+1}}{2+T}-3\frac{T^{i+1}}{1+2T}\\ &= 1-3\,\frac{t}{1+t}\left(T^j(2+T)-T^{i+j}(1+2T)+T^i(2+T)\right).\end{aligned}$$

For $i,j\geq 0$, the length generating function of vicious *(i,j)-stars is algebraic and belongs to $\mathbb{Q}(t,\sqrt{1-8t})$:*

$$(1-8t)\mathcal{V}_{i,j}(1,1) = (1-T^i)(1-T^j).$$

These results have also been obtained, independently and via a different approach, by Gessel [14]. We compare both approaches after the proof of Proposition 1. The expression of $\mathcal{O}_{1,1}(1,1)$ was conjectured in [18]. In Section 4, we refine the above result by taking into account the number of osculations: we prove that the refined generating function belongs to $\mathbb{Q}(t,u,\sqrt{1-8t})$ (where the variable u counts the osculations). This interpolates between osculating stars and vicious stars.

Note that

$$1-8t=\frac{(1-T)^2}{(1+T)^2}=\frac{2t}{T}(1-T)^2.$$

Hence the above result for vicious (i,j)-stars specializes, when $i=j=1$, to

$$\mathcal{V}_{1,1}(1,1)=\frac{T}{2t}=\sum_{n\geq 0}\frac{2^n}{n+2}\binom{2n+2}{n+1}t^n,$$

as was already proved in [17]. As explained there, counting vicious $(1,1)$ stars is equivalent to counting semi-standard Young tableaux having at most 3 columns.

For the *complete* generating function of stars, we obtain the following result.

Proposition 2. *For $i,j\geq 0$, the complete generating function of osculating (i,j)-stars is D-finite, and can be expressed in terms of the complete generating functions of vicious stars:*

$$(1+t)\mathcal{O}_{i,j}(x,y)=x^iy^j+t\,\frac{x+y+xy}{xy}\Big(\mathcal{V}_{i,j}(x,y)+\mathcal{V}_{i+1,j}(x,y)+\mathcal{V}_{i,j+1}(x,y)\Big).$$

Let us make two comments on this result.

1. D-finite series. The number of vicious (i,j)-stars of length n such that the endpoints of the three paths are respectively $-n+2r, -n+2r+2k$ and $-n+2r+2k+2\ell$ can be expressed, using the Gessel-Viennot method [16], as the following determinant:

$$v_{i,j}^{(k,\ell)}(r,n)=\begin{vmatrix} \binom{n}{r} & \binom{n}{r+k} & \binom{n}{r+k+\ell} \\ \binom{n}{r-i} & \binom{n}{r+k-i} & \binom{n}{r+k+\ell-i} \\ \binom{n}{r-i-j} & \binom{n}{r+k-i-j} & \binom{n}{r+k+\ell-i-j} \end{vmatrix}.$$

Hence the complete generating function of vicious (i,j)-stars reads

$$\mathcal{V}_{i,j}(t;x,y)=\sum_{k,\ell,n\geq 0}\sum_{r=0}^{n}v_{i,j}^{(k,\ell)}(r,n)x^ky^\ell t^n,$$

and the closure properties of D-finite series [24] imply that $\mathcal{V}_{i,j}(t;x,y)$ is D-finite. The expression of Proposition 2 shows that $\mathcal{O}_{i,j}(t;x,y)$ is also D-finite.

2. Watermelons of all sorts. In particular, when $i=j=1$, we obtain

$$\begin{aligned}[t^nxy]\mathcal{V}_{1,1}(t;x,y) &= \sum_{r=0}^{n}v_{1,1}^{(1,1)}(r,n)\\ &= \frac{2}{(n+1)(n+2)^2}\sum_{r=0}^{n}\binom{n+2}{r}\binom{n+2}{r+1}\binom{n+2}{r+2}:=b_{n+1}.\end{aligned}$$

The configurations counted by the series $[xy]\mathcal{V}_{1,1}$ are sometimes called (vicious) *watermelons*. The number of watermelons of length n, given above, is also the number of Baxter permutations of length $n+1$ (see [9] and references therein). Let us now set $i=0$ and $j=1$ in Proposition 2. Since $\mathcal{V}_{0,j}=0$ for all j, this gives

$$(1+t)\mathcal{O}_{0,1}(x,y)=y+t\,\frac{x+y+xy}{xy}\,\mathcal{V}_{1,1}(x,y).$$

Similarly,

$$(1+t)\mathcal{O}_{1,0}(x,y)=x+t\,\frac{x+y+xy}{xy}\,\mathcal{V}_{1,1}(x,y).$$

Recall that $\mathcal{V}_{1,1}(x,y)$ is a multiple of xy, and extract from these two identities the coefficient of x^1y^0. This gives

$$[x^1y^0]\mathcal{O}_{0,1}(x,y) = [x^1y^0]\mathcal{O}_{1,0}(x,y) - \frac{1}{1+t} = \frac{t}{1+t}[xy]\,\mathcal{V}_{1,1}(t;x,y) = \frac{B(t)}{1+t}, \qquad (2)$$

where $B(t) = \sum_{n\geq 1} b_n t^n$ is the generating function of Baxter permutations. Hence

$$o_{0,1}^{(1,0)}(n) = \sum_{k=1}^{n} (-1)^{n-k} b_k,$$

where b_k is the number of Baxter permutations of length k. Note also that

$$o_{1,0}^{(1,0)}(n) = o_{0,1}^{(1,0)}(n) + (-1)^n,$$

which does not seem to be combinatorially obvious.

Now the first 3-tuple of steps in an osculating $(0,1)$-star is very constrained: only two possibilities are allowed for these first steps (Figure 2). This observation implies that

$$[x^1y^0]\mathcal{O}_{0,1}(x,y) = t[x^1y^0]\mathcal{O}_{1,0}(x,y) + t[x^1y^0]\mathcal{O}_{1,1}(x,y).$$

From (2), we obtain

$$[x^1y^0]\mathcal{O}_{1,1}(x,y) = \frac{(1-t)B(t)-t}{t(1+t)}.$$

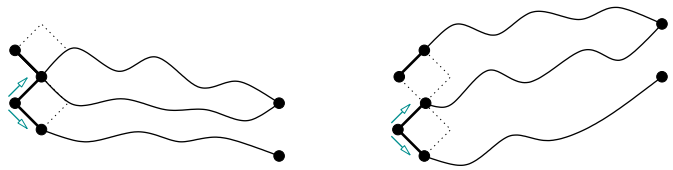

FIGURE 2. The first three steps in an osculating $(0,1)$-star.

Similarly, there are only two possibilities for the last 3-tuple of steps in a configuration counted by $[x^1y^0]\mathcal{O}_{1,1}(x,y)$. This gives:

$$\begin{aligned} [x^1y^0]\mathcal{O}_{1,1}(x,y) &= t[x^0y^1]\mathcal{O}_{1,1}(x,y) + t[x^1y^1]\mathcal{O}_{1,1}(x,y) \\ &= t[x^1y^0]\mathcal{O}_{1,1}(x,y) + t[x^1y^1]\mathcal{O}_{1,1}(x,y) \quad \text{by symmetry.} \end{aligned}$$

Hence the generating function of "osculating watermelons" is finally

$$[x^1y^1]\mathcal{O}_{1,1}(x,y) = \frac{1-t}{t^2(1+t)}\Big((1-t)B(t)-t\Big). \qquad (3)$$

Using the MAPLE packages `EKHAD` and `GFUN`, one can prove that the series $B(t)$ satisfies the following linear differential equation:

$$12\,t - 6\,(1-2\,t)\,B(t) - 2\,t\left(3-14\,t-8\,t^2\right)B'(t) - t^2\,(t+1)\,(1-8\,t)\,B''(t) = 0.$$

By combining the last two equations, we obtain a differential equation satisfied by the generating function $[x^1y^1]\mathcal{O}_{1,1}(t;x,y)$ of osculating watermelons. This equation was conjectured in [18, Eq. (4.38)].

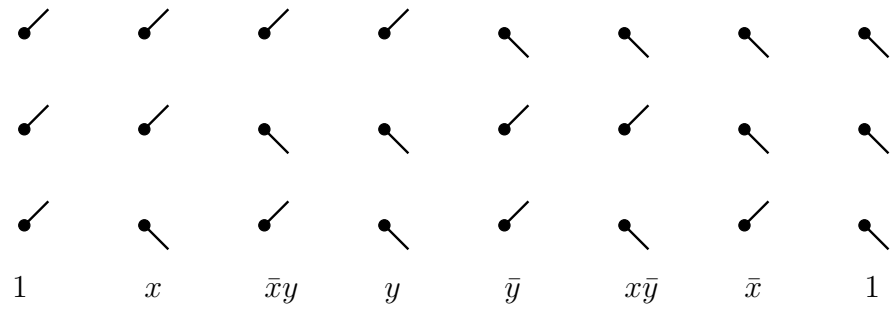

FIGURE 3. The eight possible moves.

3. **Proofs**

3.1. **Proof of Proposition 1**

For the sake of simplicity, let us denote by $\mathcal{O}(x,y)$ the complete generating function of osculating (i,j)-stars (instead of $\mathcal{O}_{i,j}(x,y)$). Imagine we construct these stars step by step as follows: we start from the star reduced to three points, and add, at each tick of the clock, one step to each of the three walks. In general, there are $2^3 = 8$ ways of adding these steps. The way in which they modify the distances between the endpoints of the walks is summarized in Figure 3. However, if two walks end at the same place, exactly 6 of these 8 moves are illegal. This simple construction translates into the following equation:

$$\begin{aligned}\mathcal{O}(x,y) = x^i y^j + t\,(1 + x + \bar{x}y + y + \bar{y} + x\bar{y} + \bar{x} + 1)\,\mathcal{O}(x,y)\\ -t\,(1 + x + \bar{y} + x\bar{y} + \bar{x} + 1)\,\mathcal{O}(x,0) - t\,(1 + \bar{x}y + y + \bar{y} + \bar{x} + 1)\,\mathcal{O}(0,y),\end{aligned}$$

which can be rewritten as

$$\begin{aligned}&(xy - t(1+x)(1+y)(x+y))\,\mathcal{O}(x,y)\\ &\quad = \quad x^{i+1}y^{j+1} - t(x+y+xy)(1+x)\mathcal{O}(x,0) - t(x+y+xy)(1+y)\mathcal{O}(0,y),\\ &\quad = \quad x^{i+1}y^{j+1} - (x+y+xy)P(x) - (x+y+xy)Q(y), \qquad (4)\end{aligned}$$

where $P(x) = t(1+x)\mathcal{O}(x,0)$ and $Q(y) = t(1+y)\mathcal{O}(0,y)$. We call the coefficient of $\mathcal{O}(x,y)$ the *kernel* $K(x,y)$ of the equation:

$$K(x,y) = xy - t(1+x)(1+y)(x+y). \qquad (5)$$

We are going to apply to (4) the *obstinate kernel method* that has already been used in [3, 4]. The classical kernel method consists in coupling the variables x and y so as to cancel the kernel $K(x,y)$. This gives some "missing" information about the series $P(x)$ and $Q(y)$ (see for instance [6, 1]). In its obstinate version, the kernel method is combined with a procedure that constructs and exploits several (related) couplings (x,y). This procedure is essentially borrowed from [12], where similar functional equations occur in a probabilistic context.

Let us first fix x, and consider the kernel as a quadratic polynomial in y. Its two roots are:

$$\begin{aligned}Y_0(x) &= \frac{1 - t(1+x)(1+\bar{x}) - \sqrt{1 - 2t(1+x)(1+\bar{x}) - t^2(1-x^2)(1-\bar{x}^2)}}{2t(1+\bar{x})}\\ &= (1+x)t + (1+x)^2(1+\bar{x})t^2 + O(t^3),\end{aligned}$$

$$\begin{aligned}Y_1(x) &= \frac{1 - t(1+x)(1+\bar{x}) + \sqrt{1 - 2t(1+x)(1+\bar{x}) - t^2(1-x^2)(1-\bar{x}^2)}}{2t(1+\bar{x})}\\ &= \frac{x}{1+x}\frac{1}{t} - (1+x) - (1+x)t + O(t^2).\end{aligned}$$

Observe that $Y_0Y_1 = x$. The first root Y_0 is a formal power series in t, and can thus be substituted for y in (4). This gives a functional equation relating P and Q:

$$P(x) + Q(Y_0) = \frac{x^{i+1}Y_0^{j+1}}{x + Y_0 + xY_0}. \tag{6}$$

Replacing y by Y_1 in $\mathcal{O}(x, y)$ would not give a well-defined power series in t, so that we must resist the temptation of this substitution. However, the following procedure will produce other interesting pairs (x, y) that cancel the kernel.

Let $(X, Y) \neq (0, 0)$ be a pair of Laurent series in t with coefficients in a field $\mathbb{K}$ such that $K(X, Y) = 0$. Recall that K is quadratic in x and y. In particular, the equation $K(x, Y) = 0$ admits a second solution X'. Define $\Phi(X, Y) = (X', Y)$. Similarly, define $\Psi(X, Y) = (X, Y')$, where Y' is the second solution of $K(X, y) = 0$. Note that Φ and Ψ are involutions. Moreover, with the kernel given by (5), one has $Y' = X/Y$ and $X' = Y/X$. Let us examine the action of Φ and Ψ on the pair (x, Y_0): we obtain an orbit of cardinality 6 (Figure 4).

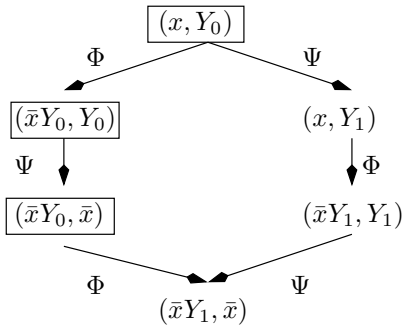

FIGURE 4. The orbit of (x, Y_0) under the action of Φ and Ψ.

The 6 pairs of power series given in Figure 4 cancel the kernel, and we have framed the ones that can be legally substituted for (x, y) in the main functional equation (4). We thus obtain *three* equations relating the unknown series $P(x)$ and $Q(x)$:

$$\begin{cases} P(x) + Q(Y_0) & = & \dfrac{x^{i+1}Y_0^{j+1}}{x + Y_0 + xY_0}, \\ P(\bar{x}Y_0) + Q(Y_0) & = & \dfrac{\bar{x}^i Y_0^{i+j+1}}{1 + x + Y_0}, \\ P(\bar{x}Y_0) + Q(\bar{x}) & = & \dfrac{\bar{x}^{i+j}Y_0^{i+1}}{x + Y_0 + xY_0}. \end{cases}$$

By combining these three equations, we obtain a relation between $P(x)$ and $Q(\bar{x})$:

$$P(x) + Q(\bar{x}) = \frac{x^{i+1}Y_0^{j+1}}{x + Y_0 + xY_0} - \frac{\bar{x}^i Y_0^{i+j+1}}{1 + x + Y_0} + \frac{\bar{x}^{i+j}Y_0^{i+1}}{x + Y_0 + xY_0}. \tag{7}$$

Setting $x = 1$ in the above equation gives

$$P(1) + Q(1) = \frac{T^{j+1}}{1 + 2T} - \frac{T^{i+j+1}}{2 + T} + \frac{T^{i+1}}{1 + 2T},$$

where $T = Y_0(1)$ is the series defined in Proposition 1. Setting $x = y = 1$ in (4) gives

$$(1 - 8t)\,\mathcal{O}_{i,j}(1, 1) = 1 - 3P(1) - 3Q(1).$$

The first part of Proposition 1 follows.

We now apply the same approach to the enumeration of vicious (i,j)-stars, or, more precisely, to the enumeration of *quasi-vicious* stars: these are the osculating (i,j)-stars that are only allowed to meet at their (rightmost) endpoint. Let $\mathcal{W}_{i,j} \equiv \mathcal{W}$ denote their complete generating function. Note that the complete generating function of vicious (i,j)-stars is, for $i,j \geq 1$,

$$\mathcal{V}(x,y) \equiv \mathcal{V}_{i,j}(x,y) = \mathcal{W}_{i,j}(x,y) - \mathcal{W}_{i,j}(x,0) - \mathcal{W}_{i,j}(0,y).$$

We can construct quasi-vicious stars step by step, as we did for general osculating stars. The difference is that now, no further move is possible when two walkers are the same place. This gives

$$\mathcal{W}(x,y) = x^i y^j + t(1+\bar{x})(x+y)(1+\bar{y})\left(\mathcal{W}(x,y) - \mathcal{W}(x,0) - \mathcal{W}(0,y)\right),$$

that is,

$$(1 - t(1+\bar{x})(x+y)(1+\bar{y}))\mathcal{V}(x,y) = x^i y^j - \mathcal{W}(x,0) - \mathcal{W}(0,y). \tag{8}$$

The rest of the argument copies what we did for osculating stars. In particular,

$$\mathcal{W}(x,0) + \mathcal{W}(0,\bar{x}) = x^i Y_0^j - \bar{x}^i Y_0^{i+j} + \bar{x}^{i+j} Y_0^i, \tag{9}$$

hence

$$\mathcal{W}(1,0) + \mathcal{W}(0,1) = T^j - T^{i+j} + T^i,$$

and the expected expression of $\mathcal{V}(1,1)$ follows using (8).

■

Note. Proposition 1 has also been obtained by Gessel [14]. Here, we sketch his approach and compare it to ours. Gessel considers the generating function

$$G(t;u,v) \equiv G(u,v) = \sum_{i,j\geq 0} u^i v^j \mathcal{O}_{i,j}(t;1,1).$$

This series counts all stars, by their length and by the position of their starting points. He then writes a recurrence relation for the coefficients of G, which is equivalent to the following functional equation:

$$G(u,v) = \frac{1}{(1-u)(1-v)} - 1 + t(1+\bar{u})(1+\bar{v})(u+v)G(u,v) \\ -t(1+u)(1+\bar{u}+\bar{v})G(u,0) - t(1+v)(1+\bar{u}+\bar{v})G(0,v).$$

This equation reflects a recursive description of stars based on the deletion of the *first* step of each path. Then, he conjectures that $G(u,v)$ is a rational function of u, v and the series T, *guesses* this rational function with the help of MAPLE, and finally checks that it satisfies the functional equation (or the corresponding recurrence relation on the coefficients of G).

The main difference between his approach and ours is that we derive the solution of the functional equation without having to guess anything. This allows us to generalize easily Proposition 1 in various ways, as shown by Propositions 2 and 3.

3.2. **Proof of Proposition 2**

We now wish to evaluate the complete generating function of (i,j)-stars, not only their length generating function. Let us go back to (7). The series $P(x) = t(1+x)\mathcal{O}(x,0)$ is a formal power series in t with coefficients in $x\mathbb{Q}[x]$, while $Q(\bar{x})$ is a formal power series in t with coefficients in $\bar{x}\mathbb{Q}[\bar{x}]$. Hence $P(x)$ and $Q(\bar{x})$ are respectively the positive part and the negative part of the right-hand side of (7), as defined by (1). But this right-hand side is an algebraic series, and this implies $P(x)$ and $Q(x)$ are D-finite. Going back to the main equation (4), we conclude that the complete generating function $\mathcal{O}(x,y)$ is D-finite too.

A similar treatment may be applied to quasi-vicious stars: since $\mathcal{W}(x,0)$ and $\mathcal{W}(0,\bar{x})$ are power series in t with coefficients in $x\mathbb{Q}[x]$ and $\bar{x}\mathbb{Q}[\bar{x}]$ respectively, it follows from (9) that they are, respectively, the positive and the negative part of $x^i Y_0^j - \bar{x}^i Y_0^{i+j} + \bar{x}^{i+j} Y_0^i$.

The only information that we have used to determine $\mathcal{W}(x,0)$ and $\mathcal{W}(0,x)$ is the fact that, for each pair (X,Y) framed in the diagram of Figure 4,

$$\mathcal{W}_{i,j}(X,0)+\mathcal{W}_{i,j}(0,Y)=X^iY^j. \tag{10}$$

Similarly, our determination of $P(x)$ and $Q(y)$ is based on the fact that, for each such pair (X,Y),

$$P(X)+Q(Y)=\frac{X^{i+1}Y^{j+1}}{X+Y+XY}. \tag{11}$$

Now observe that, for any pair (X,Y) such that $K(X,Y)=0$,

$$\frac{1}{X+Y+XY}=\frac{t}{1+t}\frac{1+X+Y}{XY}.$$

In particular, the identity (11) can be rewritten

$$P(X)+Q(Y)=\frac{tX^iY^j}{1+t}(1+X+Y).$$

Comparing with (10) gives, by linearity,

$$\begin{aligned}(1+t)P(x)/t &= \mathcal{W}_{i,j}(x,0)+\mathcal{W}_{i+1,j}(x,0)+\mathcal{W}_{i,j+1}(x,0),\\ (1+t)Q(y)/t &= \mathcal{W}_{i,j}(0,y)+\mathcal{W}_{i+1,j}(0,y)+\mathcal{W}_{i,j+1}(0,y).\end{aligned}$$

We now plug these expressions of $P(x)$ and $Q(y)$ into (4), use (8), and obtain

$$(1+t)\mathcal{O}_{i,j}(x,y)=x^iy^j+t\,\frac{x+y+xy}{xy}\Big(\mathcal{V}_{i,j}(x,y)+\mathcal{V}_{i+1,j}(x,y)+\mathcal{V}_{i,j+1}(x,y)\Big)$$

as stated in Proposition 2.

4. **The number of osculations**

In this section, we refine the generating function of osculating (i,j)-stars by adding a new indeterminate u, which keeps track of the number of osculations. We denote by $\mathcal{O}_{i,j}(t;u,x,y)\equiv\mathcal{O}_{i,j}(u,x,y)$ the refined generating function. For instance, the $(0,1)$-star of Figure 1 has a contribution $t^{10}x^2y^0u^3$ is this generating function.

Proposition 3. *For $i,j\geq 0$ and $(i,j)\neq(0,0)$, the generating function that counts of (i,j)-stars by their length and number of osculations is algebraic and belongs to $\mathbb{Q}(t,u,\sqrt{1-8t})$. More precisely, let $T\equiv T(t)$ be the unique power series in t satisfying $T=2t(1+T)^2$. Then*

$$(1-8t)\mathcal{O}_{i,j}(1,1)=1-\frac{4-u}{(1+T)^2-uT^2}\left(T^{j+1}-\frac{T^{i+j+1}(2(1+T)-u)}{2(1+T)-uT}+T^{i+1}\right).$$

Proof. As in the proof of Proposition 1, we first write a functional equation defining $\mathcal{O}_{i,j}(u,x,y)\equiv\mathcal{O}(x,y)$. We have to weight each osculation by u, which gives:

$$\begin{aligned}\mathcal{O}(x,y) &= x^iy^j+t(1+\bar{x})(1+\bar{y})(x+y)\mathcal{O}(x,y)\\ &\quad -t(\bar{x}+\bar{y}+1)(1+x)\mathcal{O}(x,0)-t(\bar{x}+\bar{y}+1)(1+y)\mathcal{O}(0,y)\\ &\quad +ty(u-1)(1+\bar{x})\mathcal{O}(x,0)+tx(u-1)(1+\bar{y})\mathcal{O}(0,y).\end{aligned}$$

This can be rewritten as

$$K(x,y)\mathcal{O}(x,y)=x^{i+1}y^{j+1}-(x+y+xy+y^2(1-u))P(x)-(x+y+xy+x^2(1-u))Q(y), \tag{12}$$

where $P(x)=t(1+x)\mathcal{O}(x,0)$ and $Q(y)=t(1+y)\mathcal{O}(0,y)$, and the kernel $K(x,y)$ is still given by (5). The rest of the proof follows the same principles as the proof of Proposition 1. We successively replace the pair (x,y) by the three framed pairs of Figure 4. This gives three linear equations that relate $P(x),P(\bar{x}Y_0),Q(Y_0)$ and $Q(\bar{x})$. We eliminate $P(\bar{x}Y_0)$ and

$Q(Y_0)$ between these three equations to obtain a linear relationship between $P(x)$ and $Q(\bar{x})$. Setting $x = 1$ and $Y_0 = T$ in this equation gives:

$$\Big(P(1) + Q(1)\Big)\left((1+T)^2 - uT^2\right) = T^{j+1} - \frac{(2-u+2T)\,T^{i+j+1}}{2(1+T)-uT} + T^{i+1}.$$

But according to (12),

$$(1-8t)\mathcal{O}(1,1) = 1 - (4-u)\,(P(1)+Q(1))\,.$$

Proposition 3 follows. The case $i = j = 1$ of our result was conjectured in [11].

5. **Discussion**

The main question raised by this paper is whether this approach can be extended to more than three walkers. Of course, the step by step construction can still be exploited: in general, it gives a functional equation that defines the generating function $\mathcal{O}_{i_1,\dots,i_{p-1}}(t; x_1, \dots, x_{p-1})$ counting osculating $(i_1, \dots, i_{p-1})$-stars according to their length and the distances between their endpoints. But the problem is how to *solve* this equation...

The connection between osculating and vicious walkers stated in Proposition 2 is intriguing. Since we are at a loss to extend it to more walkers, let us, very modestly, state the corresponding results for two walkers. We use notations that are similar to those used for three walkers, and should be self-explanatory. We take into account the number of osculations using an indeterminate u. The functional equation that defines the complete generating function $\mathcal{O}_i(t; x) \equiv \mathcal{O}(x)$ of osculating i-stars is

$$\mathcal{O}_i(x) = x^i + t(2 + x + \bar{x})\mathcal{O}_i(x) - t(2 + \bar{x} + x(1-u))\mathcal{O}_i(0),$$

that is,

$$(1 - t(1+x)(1+\bar{x}))\mathcal{O}_i(x) = x^i - t(2 + \bar{x} + x(1-u))\mathcal{O}_i(0). \tag{13}$$

The equation satisfied by the generating function $\mathcal{W}_i(t; x)$ of quasi-vicious i-stars reads

$$(1 - t(1+x)(1+\bar{x}))\mathcal{V}_i(x) = x^i - \mathcal{W}_i(0), \tag{14}$$

where $\mathcal{V}_i(x) = \mathcal{W}_i(x) - \mathcal{W}_i(0)$ is the length generating function of vicious i-stars. The standard kernel method gives

$$\mathcal{W}_i(0) = X^i, \quad \mathcal{O}_i(0) = \frac{X^{i+1}}{t((1+X)^2 - uX^2))} = \frac{X^i}{1-tuX},$$

where $X \equiv X(t)$ is the only power series in t that cancels the kernel $K(x) = 1 - t(1+x)(1+\bar{x})$:

$$X = \frac{1 - 2t - \sqrt{1-4t}}{2t}.$$

Setting $x = 1$ in (13) and (14) gives the counterpart of Propositions 1 and 3:

$$(1-4t)\mathcal{O}_i(t;1) = 1 - \frac{(4-u)X^{i+1}}{(1+X)^2 - uX^2} = 1 - \frac{(4-u)tX^i}{1-tuX} \quad \text{and} \quad (1-4t)\mathcal{V}_i(t;1) = 1 - X^i.$$

To obtain a relation between *complete* generating functions of osculating and vicious stars, we observe that

$$(1-tuX)(1-u+2tu+tuX) = (1+ut)^2 - u.$$

Hence the above expression of $\mathcal{O}_i(0)$ can be rewritten as

$$\mathcal{O}_i(0) = \frac{X^i(1-u+2tu+tuX)}{(1+ut)^2-u} = \frac{1}{(1+ut)^2-u}\Big((1-u+2tu)\mathcal{W}_i(0) + tu\mathcal{W}_{i+1}(0)\Big).$$

We now plug this expression back in (13) and use (14) to obtain the following counterpart of Proposition 2:

$$((1+ut)^2-u)\mathcal{O}_i(x) =$$
$$x^i(1-u+2tu+xtu(1-u)) + t\,\frac{1+2x+x^2(1-u)}{x}\Big((1-u+2tu)\mathcal{V}_i(x)+tu\mathcal{V}_{i+1}(x)\Big).$$

When $u=1$, this specializes to

$$(2+t)\mathcal{O}_i(x) = 2x^i + t\,\frac{1+2x}{x}\Big(2\mathcal{V}_i(x)+\mathcal{V}_{i+1}(x)\Big).$$

Finally, it would be interesting to find purely combinatorial proofs of Propositions 1 and 2.

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