

# Bounds on the slope and the curvature of the scalar $K\pi$ form factor at zero momentum transfer

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## Abstract

We derive and calculate unitarity bounds on the slope and curvature of the strangeness-changing scalar form factor at zero momentum transfer, using low-energy constraints and Watson final state interaction theorem. The results indicate that the curvature is important and should not be neglected in the representation of experimental data. The bounds can be converted also into an allowed region for the constants  $C_{12}^r$  and  $C_{34}^r$  of Chiral Perturbation Theory. Our results are consistent with, but weaker than the predictions made by Jamin, Oller and Pich in a coupled channel dispersion approach based on chiral resonance model. We comment on the differences between the two dispersive methods and argue that the unitarity bounds are useful as an independent check involving different sources of information.

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# 1 Introduction

The scalar form factor relevant for the decay  $K \rightarrow \pi \ell \nu$  is proportional to the matrix element  $\langle K | \bar{s} u | \pi \rangle$ . It is related to the vector form factors  $f_{\pm}(t)$  by the expression

$$f_0(t) = f_+(t) + \frac{t}{M_K^2 - M_\pi^2} f_-(t), \quad (1)$$

in the standard normalization where  $f_0(0)$  coincides with  $f_+(0)$ , a quantity that is of central importance for the determination of the CKM matrix element  $V_{us}$ . Writing the Taylor expansion of the form factor near  $t = 0$

$$f_0(t) = f_+(0) \left\{ 1 + \lambda_0 t + c t^2 + \dots \right\}, \quad (2)$$

where  $\lambda_0$  is the slope and  $c$  the curvature, it is customary to relate the slope with the scalar radius of the pion, through  $\lambda_0 \equiv \langle r^2 \rangle_s^{K\pi} M_\pi^2 / 6$ . This expansion is usually considered in the analysis of experimental data on  $K_{l3}$  decays of both charged and neutral kaons. The experimental situation on the  $K_{l3}$  decays improved very much recently. For the neutral kaons, the new result from KTeV collaboration is  $\langle r^2 \rangle_s^{K_L \pi} = 0.165 \pm 0.016 \text{ fm}^2$  [1]. This value now dominates the statistics and is consistent with the 1974 high statistics experiment [2]. For the charged kaons, the data collected at the ISTRA+ detector [3] give the radius  $\langle r^2 \rangle_s^{K^\pm \pi} = 0.235 \pm 0.014 \pm 0.007 \text{ fm}^2$ , which dominates the world average.

On the theoretical side there is also a considerable progress in the study of the scalar strangeness changing form factor  $f_0(t)$ . Low energy theorems impose strong constraints on the values of this function at some particular points. The value at the origin  $f_0(0) \approx 1$  is given by Ademollo-Gatto relation [4], while the theorem of Callan and Treiman, refined by Dashen and Weinstein [5], states that in the limit of vanishing quark masses the value of the form factor at the special point  $t = M_K^2 - M_\pi^2$  is

$$f_0(\Delta_{K\pi}) = F_K / F_\pi + \Delta_{CT}, \quad \Delta_{K\pi} = M_K^2 - M_\pi^2, \quad (3)$$

where the  $O(\hat{m})$  correction  $\Delta_{CT}$  was calculated to one loop in Chiral Perturbation Theory ( $\chi$ PT) [6] and found to be tiny:  $\Delta_{CT} \approx -3 \cdot 10^{-3}$ . Also, in Ref. [6] a first prediction to one loop for the radius was obtained,  $\langle r^2 \rangle_s^{K\pi} = 0.20 \pm 0.05 \text{ fm}^2$ . It is shown that the corrections of  $O(\hat{m})$  do not contain a chiral logarithm of the type  $M_\pi^2 \log M_\pi^2$  and therefore they are very small [6]. Recently, the  $K_{l3}$  form factors were calculated to two loops of  $\chi$ PT [7, 8].

Dispersion theory provides another useful tool for making predictions on the form factors that govern the  $K_{l3}$  decay. These functions are analytic in the  $t$ -plane cut from the unitarity threshold  $(M_K + M_\pi)^2$  to infinity. A first approach to derive bounds on the values of the form factors and their derivatives in the physical region was proposed in the years 70. It is based on analyticity, and exploited the Källén-Lehmann representation for a suitable two-point function  $\psi(q^2)$ , in combination with unitarity and low-energy theorems. However, these first studies [10]-[14] were based on an unsubtracted dispersion relation for  $\psi(q^2)$ , which turned out to be incorrect: this function, calculated in perturbative QCD [15], satisfies a dispersion relation which requires two subtractions. The correct dispersion relation was applied for the first time in Ref. [16]. Instead of using the low energy theorems on  $\psi(0)$  as input [10]-[14], the new approach involves the calculation of the derivative  $\psi''(Q^2)$  obtained by perturbative QCD in the deep euclidian region. The constraining power of this condition is weaker and therefore the strength of the bounds is reduced.

An alternative dispersive approach based on coupled channel unitarity equations for the  $K\pi$ ,  $K\eta$  and  $K\eta'$  form factors was considered recently in [17]. It is based on unitarized chiral perturbation plus a  $K$ -matrix fit of the S-wave scattering [18]. In a recent paper [19], the authors combine the values of the slope and curvature of the scalar form factor at zero momentum transfer, calculated in [17], with the two-loop calculation of  $\chi$ PT [7, 8], obtaining thereby predictions for the chiral constants  $C_{12}^r$  and  $C_{34}^r$ .

In the present work we revisit and extend the technique of unitarity bounds with the aim of deriving constraints on the slope and the curvature appearing in the Taylor expansion (2). The study is motivated by the recent interest in the determination of these parameters, both experimentally and in  $\chi$ PT. On the other hand, since the last work on unitarity bounds [16], the knowledge of the quantities entering as input improved considerably. For instance, the correlator  $\psi(q^2)$  is now calculated in perturbative QCD to order  $\alpha_s^3$  [21], [22], and recent lattice calculations of the light and strange quark masses [23], [24] and the ratio  $F_K/F_\pi$  [25] are available. We notice also that the implementation of Watson final state interaction theorem [26] was never done with the correct dispersion relation for  $\psi(q^2)$  (in Refs. [13]-[14], where this problem was considered, the old unsubtracted dispersion relation was used). In the present work we implement simultaneously both low energy constraints and the Watson theorem, exploiting in an optimal way the formalism, which can be compared with the alternative study made in Ref.

[19].

In Section 2, we give a short review of the mathematical technique we will use, and show how to incorporate additional constraints provided by  $\chi$ PT and Watson final state interaction theorem. In Section 3, we discuss our results and compare them with the recent predictions on the slope and curvature of the scalar  $K\pi$  form factor presented in [17]-[19]. As in these references we work in the isospin limit (the isospin breaking corrections calculated in [20] are important and should be included in a more precise analysis). We end the paper with some comments on the usefulness of the method and its possible generalizations.

## 2 Unitarity bounds

The starting point in the derivation of the bounds is the two-point function

$$\psi(q^2) = i \int d^4x e^{iq \cdot x} \langle 0 | T(\partial_\mu V^\mu(x) \partial_\mu V^\mu(0)^\dagger) | 0 \rangle, \quad (4)$$

where  $V_\mu = \bar{s} \gamma_\mu u$  is the strangeness changing vector current that governs  $K_{l3}$  decay. The function  $\psi(q^2)$  was calculated for euclidian arguments  $Q^2 = -q^2 > 0$  up to corrections of order  $\alpha_s^3$  [15], [21], [22]. It grows asymptotically as  $Q^2$  and satisfies a twice-subtracted Källén-Lehmann representation. One can use either the dispersion relation for the second derivative  $\psi''(Q^2)$

$$\psi''(Q^2) = \frac{2}{\pi} \int_{t_+}^{\infty} dt \frac{\text{Im} \psi(t)}{(t + Q^2)^3}, \quad (5)$$

where  $t_+ = (M_K + M_\pi)^2$  is the first unitarity threshold, or the relation satisfied by the derivative of the ratio  $\psi(Q^2)/Q^2$ :

$$\left( \frac{\psi(Q^2)}{Q^2} \right)' + \frac{\psi(0)}{Q^4} = \frac{1}{\pi} \int_{t_+}^{\infty} dt \frac{\text{Im} \psi(t)}{t(t + Q^2)^2}. \quad (6)$$

The advantage of the last relation is that it incorporates the value  $\psi(0)$ , which satisfies the inequality

$$\psi(0) < (M_K F_K - M_\pi F_\pi)^2, \quad (7)$$

derived by Mathur and Okubo [12] (we use the normalisation  $F_\pi = 92.4$  MeV). In our analysis we shall use the relations (5) and (6) successively.

Unitarity bounds [10] are obtained by combining the above dispersion relations with the inequality

$$\text{Im } \psi(t) \geq \frac{3(M_K^2 - M_\pi^2)^2}{64\pi} \frac{\sqrt{(t - t_+)(t - t_-)}}{t} |f_0(t)|^2, \quad t_\pm = (M_K \pm M_\pi)^2, \quad (8)$$

given by unitarity and the positivity of the spectral function. Then Eq. (5) gives [16]

$$\frac{1}{\pi} \int_{t_+}^{\infty} dt w(t) |f_0(t)|^2 \leq \psi''(Q^2), \quad (9)$$

where

$$w(t) = \frac{3(M_K^2 - M_\pi^2)^2}{32\pi} \frac{\sqrt{(t - t_+)(t - t_-)}}{t(t + Q^2)^3}. \quad (10)$$

Similarly Eq.(6) leads to

$$\frac{1}{\pi} \int_{t_+}^{\infty} dt \tilde{w}(t) |f_0(t)|^2 \leq \left( \frac{\psi(Q^2)}{Q^2} \right)' + \frac{\psi(0)}{Q^4}, \quad (11)$$

where

$$\tilde{w}(t) = w(t) \frac{t + Q^2}{2t}. \quad (12)$$

The r.h.s. of (9) is given by perturbative QCD [15],[22]

$$\begin{aligned} \psi''(Q^2) &= \frac{3}{8\pi^2} \frac{(m_s(Q^2) - m_u(Q^2))^2}{Q^2} \times \\ &\left[ 1 + \frac{11}{3} \frac{\alpha_s(Q^2)}{\pi} + 14.17 \left( \frac{\alpha_s(Q^2)}{\pi} \right)^2 + \dots + O\left(\frac{m^2}{Q^2}\right) + O\left(\frac{1}{Q^4}\right) \right] \end{aligned} \quad (13)$$

while the corresponding expansion for the r.h.s. of (11) reads [21]

$$\begin{aligned} \left( \frac{\psi(Q^2)}{Q^2} \right)' &= \frac{3}{8\pi^2} \frac{(m_s(Q^2) - m_u(Q^2))^2}{Q^2} \times \\ &\left[ 1 + \frac{17}{3} \frac{\alpha_s(Q^2)}{\pi} + 45.84 \left( \frac{\alpha_s(Q^2)}{\pi} \right)^2 + \dots + O\left(\frac{m^2}{Q^2}\right) + O\left(\frac{1}{Q^4}\right) \right] \end{aligned} \quad (14)$$

Using standard techniques [10]-[16], [27], the conditions (9) or (11) can be converted into bounds on the values of  $f_0(t)$  for points inside the analyticity

domain, in particular, on the slope and the curvature at the origin. We consider in detail the condition (9), indicating at the end of the section the modifications resulting for the condition (11).

The problem is brought into a canonical form by making the conformal mapping

$$z(t) = \frac{\sqrt{t_+} - \sqrt{t_+ - t}}{\sqrt{t_+} + \sqrt{t_+ - t}}, \quad (15)$$

which maps the  $t$ -plane cut for  $t > t_+$  onto the disk  $|z| < 1$ , such that  $z(0) = 0$ ,  $z(t_+) = 1$ ,  $z(\infty) = -1$ , and the upper edge of the cut  $t > t_+$  becomes the upper semicircle  $z = \exp(i\theta)$ ,  $0 < \theta < \pi$ . The inverse of (15) and its derivative are

$$t(z) = 4t_+ z/(1+z)^2, \quad dt/dz = 4t_+(1-z)/(1+z)^3, \quad (16)$$

while for points on the boundary,  $z = e^{i\theta}$ , Eqs. (15) and (16) write

$$\theta = 2 \arctan \sqrt{t/t_+ - 1}, \quad t(e^{i\theta}) = t_+ / \cos^2 \frac{1}{2}\theta. \quad (17)$$

In the new variable the relation (9) becomes

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \rho(\theta) |f(e^{i\theta})|^2 \leq \psi''(Q^2), \quad (18)$$

where  $f(z) = f_0(t(z))$  and

$$\rho(\theta) = w(t(e^{i\theta})) \left| \frac{dt(e^{i\theta})}{d\theta} \right|. \quad (19)$$

We define the outer function  $C(z)$ , analytic and without zeros inside  $|z| < 1$ , such that on the boundary of the unit disk:

$$|C(e^{i\theta})| = \sqrt{\rho(\theta)}. \quad (20)$$

A straightforward calculation [14], [27] gives

$$C(z) = \frac{1}{4} \sqrt{\frac{3}{2\pi}} \frac{M_K - M_\pi}{M_K + M_\pi} (1-z)(1+z)^{3/2} \frac{\sqrt{1-z+\beta(1+z)}}{(1-z+\beta_Q(1+z))^3}, \quad (21)$$

where  $\beta = \sqrt{1 - t_-/t_+}$  and  $\beta_Q = \sqrt{1 + Q^2/t_+}$ . Then the inequality (18) can be written in the form

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta |g(e^{i\theta})|^2 \leq \psi''(Q^2), \quad (22)$$

where the function  $g(z)$  defined by

$$g(z) = C(z) f(z) \quad (23)$$

is analytic inside the disk  $|z| < 1$ . Expanding this function as

$$g(z) = \sum_{n=0}^{\infty} g_n z^n, \quad (24)$$

we write the  $L^2$  norm condition (22) in terms of the Taylor coefficients  $g_n$  as

$$\sum_{n=0}^{\infty} g_n^2 \leq \psi''(Q^2). \quad (25)$$

We mention that the reality condition  $f_0(t^*) = f_0^*(t)$  means that the coefficients  $g_n$  are real.

From Eq. (23) it follows that each Taylor coefficient  $g_n$  is a linear expression of the first  $n$  derivatives of the form factor  $f_0(t)$  with respect to  $t$  at  $t = 0$  (we recall that  $t = 0$  is transformed by (15) into the origin  $z = 0$ ), in particular  $g_0, g_1$  and  $g_2$  are expressed in terms of the parameters  $f_0(0), \langle r^2 \rangle_s^{K\pi}$  and  $c$ . The coefficients depend on the conformal mapping  $z(t)$  given in (15) and the outer function  $C(z)$  given in (21), where  $Q^2$  enters as a parameter. For instance, for  $Q^2 = 4 \text{ GeV}^2$ , a straightforward calculation gives:

$$\begin{aligned} g_0 &= f_0(0) 0.00164, \\ g_1 &= f_0(0) [-0.00189 + 0.0113 \langle r^2 \rangle_s^{K\pi}], \\ g_2 &= f_0(0) [-0.000259 - 0.03579 \langle r^2 \rangle_s^{K\pi} + 0.00428 c]. \end{aligned} \quad (26)$$

In order to find the maximum domain allowed for the parameters  $f_0(0), \langle r^2 \rangle_s^{K\pi}$  and  $c$ , we fix the first three coefficients  $g_0, g_1$  and  $g_2$ , and take the minimum of the left hand side of (25) with respect to the remaining parameters  $g_n, n \geq 3$ , which are arbitrary. Clearly, the minimum is reached for  $g_n = 0, n \geq 3$ , so the allowed domain becomes

$$g_0^2 + g_1^2 + g_2^2 \leq \psi''(Q^2). \quad (27)$$

Using Eq. (26), this inequality defines a convex domain for the parameters  $f_0(0)$ ,  $\langle r^2 \rangle_s^{K\pi}$  and  $c$ . For a fixed  $f_0(0)$ , this domain is the interior of an ellipse in the plane  $(\langle r^2 \rangle_s^{K\pi}, c)$ .

The strength of the bounds, expressed by the size of the allowed domain, is increased by imposing additional constraints.

(1) *The Callan-Treiman condition*

Let us consider first the Callan-Treiman condition (3). We note by  $z_0$  the image of the point  $t_0 = \Delta_{K\pi}$  in the  $z$ -plane:  $z_0 = \frac{\sqrt{M_K/M_\pi+1}-\sqrt{2}}{\sqrt{M_K/M_\pi+1}+\sqrt{2}}$ . From Eqs. (23) and (24) we have the condition

$$\sum_{n=0}^{\infty} g_n z_0^n = C(z_0) f_0(\Delta_{K\pi}). \quad (28)$$

As explained in Ref. [27], in order to obtain the allowed domain, we have to evaluate the minimum of the left hand side of (25) with respect to the coefficients  $g_n$ , for  $n \geq 3$  subject to the condition (28). We impose this condition by the Lagrange multipliers method. The Lagrangian is

$$\mathcal{L} = \sum_{n=0}^{\infty} g_n^2 + \mu \left( \sum_{n=0}^{\infty} g_n z_0^n - C(z_0) f_0(\Delta_{K\pi}) \right) \quad (29)$$

where  $\mu$  is the Lagrange multiplier. The minimizing condition

$$\frac{\partial \mathcal{L}}{\partial g_n} = 0, \quad n \geq 3, \quad (30)$$

has the solution  $g_n = -\mu z_0^n / 2$  for  $n \geq 3$ . Inserting this solution in the condition (28), we find the Lagrange multiplier

$$39\mu = 2 \frac{1 - z_0^2}{z_0^6} \left( g_0 + g_1 z_0 + g_2 z_0^2 - C(z_0) f_0(\Delta_{K\pi}) \right). \quad (31)$$

By inserting the solution of Eqs. (30) with  $\mu$  from (31) into the left hand side of Eq. (25), we obtain the inequality

$$g_0^2 + g_1^2 + g_2^2 + \frac{1 - z_0^2}{z_0^6} \left( g_0 + g_1 z_0 + g_2 z_0^2 - C(z_0) f_0(\Delta_{K\pi}) \right)^2 \leq \psi''(Q^2), \quad (32)$$

which represents again, for each  $f_0(0)$ , the interior of an ellipse in the plane  $(\langle r^2 \rangle_s^{\pi K}, c)$ .



(2) *The Callan-Treiman condition and the Watson theorem*

We now include Watson theorem [26], which states that

$$\text{Arg}[f_0(t + i\epsilon)] = \delta_0^{1/2}(t), \quad (M_K + M_\pi)^2 < t < t_{in}, \quad (33)$$

where  $\delta_0^{1/2}(t)$  is the  $I = 1/2$  phase shift of the  $S$  wave  $K\pi$  elastic scattering and  $t_{in} = (M_K + 3M_\pi)^2$  is the inelastic threshold. The incorporation of Watson theorem in the unitarity bounds for the  $K_{l3}$  form factors was done in an approximate way in Ref. [13] and rigorously in Ref. [14]. However, as mentioned above, in these works it was assumed that  $\psi(q^2)$  satisfies an unsubtracted dispersion relation, which was not confirmed in perturbative QCD. Moreover, the authors derived bounds on the magnitude  $|f_0(t)|$  in the physical region, and did not consider the slope and the curvature in which we are now interested. In the present paper, we apply a mathematical method based on Lagrange multipliers, which is equivalent to the technique adopted in [14] but is conceptually simpler. The details are explained in Ref. [27], where the allowed domain for the coefficients  $g_n$  with the constraint (33) was obtained. Below, we consider the complete problem by imposing simultaneously the conditions (3) and (33).

Let us note by  $\theta_{in} = 2 \arctan \sqrt{t_{in}/t_+ - 1}$  the image of the point  $t_{in}$  on the boundary of the unit disk  $|z| < 1$  in the  $z$ -plane. We define the Omnès function [14], [28]

$$O(z) = \exp \frac{i}{\pi} \int_{-\pi}^{\pi} d\theta \frac{\delta(\theta)}{1 - ze^{-i\theta}}, \quad (34)$$

where  $\delta(\theta) = \delta_0^{1/2}(t(e^{i\theta}))$  for  $0 < \theta < \theta_{in}$ . When  $\theta > \theta_{in}$  the function  $\delta(\theta)$  is arbitrary, however these values will not enter the result. We recall again the reality condition  $f_0(t^*) = f_0^*(t)$ , which means that  $\delta(\theta)$  is an odd function of  $\theta$ :  $\delta(-\theta) = -\delta(\theta)$ . As in Ref. [14], we assume that  $\delta(\theta)$  is a Lipschitz continuous function of order larger than  $1/2$  on  $[0, \theta_{in}]$ . By construction,  $O(z)$  is analytic and without zeros inside  $|z| < 1$ , and  $\text{Arg} O(e^{i\theta}) = \delta(\theta)$ . Therefore, in the ratio  $f(z)/O(z)$  the phases of the numerator and the denominator compensate, leading to a real quantity along the elastic region. Expressed in terms of the function  $g(z)$  defined in (23) and its Taylor coefficients  $g_n$ , this condition reads

$$\lim_{r \rightarrow 1} \text{Im} \left[ \frac{g(re^{i\theta})}{W(re^{i\theta})} \right] = \lim_{r \rightarrow 1} \sum_{n=0}^{\infty} g_n \text{Im} \left[ \frac{r^n e^{in\theta}}{W(re^{i\theta})} \right] = 0, \quad -\theta_{in} < \theta < \theta_{in}, \quad (35)$$

where we introduced the function

$$W(z) = C(z) O(z). \quad (36)$$

We must find the minimum of the left hand side of (25) with the constraints (28) and (35). We consider the Lagrangian [27]

$$\begin{aligned} \mathcal{L} = & \sum_{n=0}^{\infty} g_n^2 + \mu \left( \sum_{n=0}^{\infty} g_n z_0^n - C(z_0) f_0(\Delta_{K\pi}) \right) \\ & + \lim_{r \rightarrow 1} \sum_{n=0}^{\infty} g_n \int_{-\theta_{in}}^{\theta_{in}} \frac{2d\theta'}{\pi} \lambda(\theta') |W(e^{i\theta'})| \operatorname{Im} \left[ \frac{r^n e^{in\theta'}}{W(re^{i\theta'})} \right], \end{aligned} \quad (37)$$

where  $\lambda(\theta)$  is a generalized Lagrange multiplier, which must be an odd function:  $\lambda(-\theta) = -\lambda(\theta)$ . As discussed in Ref. [27], the factor  $|W(e^{i\theta})|$  was inserted in the integrand in order to have a function  $\lambda(\theta)$  with integrable squared modulus (*i.e.* of class  $L^2$ ), while the coefficient  $2/\pi$  is introduced for convenience.

The equations (30) give the optimal coefficients  $g_n$ :

$$g_n = -\frac{\mu z_0}{2} - \lim_{r \rightarrow 1} \int_{-\theta_{in}}^{\theta_{in}} \frac{d\theta'}{\pi} \lambda(\theta') |W(e^{i\theta'})| \operatorname{Im} \left[ \frac{r^n e^{in\theta'}}{W(re^{i\theta'})} \right], \quad n \geq 3. \quad (38)$$

By imposing the constraint (28) we find the Lagrange multiplier  $\mu$

$$\mu = \frac{2(1 - z_0^2)}{z_0^6} \left[ g_0 + g_1 z_0 + g_2 z_0^2 - C(z_0) f_0(\Delta_{K\pi}) - z_0^3 \int_{-\theta_{in}}^{\theta_{in}} \frac{d\theta'}{\pi} \lambda(\theta') h(\theta') \right], \quad (39)$$

where we used the notation

$$h(\theta) = \operatorname{Im} \left[ \frac{e^{3i\theta - i\Phi}}{1 - z_0 e^{i\theta}} \right] = \frac{\sin[3\theta - \Phi(\theta)] - z_0 \sin[2\theta - \Phi(\theta)]}{1 + z_0^2 - 2z_0 \cos \theta}, \quad (40)$$

with

$$\Phi(\theta) = \operatorname{Arg} C(e^{i\theta}) + \delta(\theta). \quad (41)$$

This function is readily obtained using the expression (21) of the outer function  $C(z)$

$$\Phi(\theta) = \mp \frac{\pi}{2} + \frac{5}{4}\theta + \frac{1}{2} \operatorname{Arg}[1 - e^{i\theta} + \beta(1 + e^{i\theta})] - 3 \operatorname{Arg}[1 - e^{i\theta} + \beta_Q(1 + e^{i\theta})] + \delta(\theta), \quad (42)$$

the signs of the first term correspond to  $\theta > 0$  and  $\theta < 0$ , respectively. By including into the condition (35), the optimal coefficients (38), where  $\mu$  is given in (39), and using the Plemelj-Privalov relation [29]

$$\lim_{r \rightarrow 1} \frac{1}{\pi} \int_{-\pi}^{\pi} d\theta' \frac{F(\theta')}{1 - r e^{i(\theta - \theta')}} = F(\theta) + \frac{1}{\pi} \int_{-\pi}^{\pi} d\theta' \frac{F(\theta')}{1 - e^{i(\theta - \theta')}} , \quad (43)$$

where the last integral is a Principal Value, we obtain an integral equation for the function  $\lambda(\theta)$  given by

$$\begin{aligned} V(\theta) = \lambda(\theta) & - \int_{-\theta_{in}}^{\theta_{in}} \frac{d\theta'}{2\pi} \lambda(\theta') \frac{\sin[5/2(\theta - \theta') - \Phi(\theta) + \Phi(\theta')]}{\sin(\theta - \theta')/2} \\ & - (1 - z_0^2) \int_{-\theta_{in}}^{\theta_{in}} \frac{d\theta'}{\pi} \lambda(\theta') h(\theta) h(\theta') , \end{aligned} \quad (44)$$

where

$$V(\theta) = \sum_{n=0}^2 g_n \sin[n\theta - \Phi(\theta)] - \frac{1 - z_0^2}{z_0^3} \left( g_0 + g_1 z_0 + g_2 z_0^2 - C(z_0) f_0(\Delta_{K\pi}) \right) h(\theta). \quad (45)$$

As discussed in Refs. [14], [27], for  $\Phi(\theta)$  given in (42) and assuming that the physical phase  $\delta(\theta)$  is sufficiently smooth (more exactly Lipschitz continuous of order greater than 1/2 [14]), one can show that (44) is a Fredholm equation.

The minimum of the left hand side of (25) is obtained by inserting the solution (38) into this relation, taking into account the expression (39) of  $\mu$  and the integral equation (44). We thus obtain the inequality

$$\begin{aligned} g_0^2 + g_1^2 + g_2^2 + \frac{1 - z_0^2}{z_0^6} \left( g_0 + g_1 z_0 + g_2 z_0^2 - C(z_0) f_0(\Delta_{K\pi}) \right)^2 \\ + \int_{-\theta_{in}}^{\theta_{in}} \frac{d\theta}{\pi} \lambda(\theta) V(\theta) \leq \psi''(Q^2) , \end{aligned} \quad (46)$$

where  $\lambda(\theta)$  is the solution of the integral equation (44). The inequality (46) defines the allowed domain for the slope and curvature when both Callan-Treiman relation and Watson theorem are imposed. We notice that if the

Callan-Treiman relation (3) is removed, the domain allowed for the coefficients  $g_0$ ,  $g_1$  and  $g_2$  is given by the inequality [27]

$$g_0^2 + g_1^2 + g_2^2 + \int_{-\theta_{in}}^{\theta_{in}} \frac{d\theta}{\pi} \lambda(\theta) V(\theta) \leq \psi''(Q^2), \quad (47)$$

where  $\lambda(\theta)$  is the solution of the simpler equation [27]

$$\sum_{n=0}^2 g_n \sin[n\theta - \Phi(\theta)] = \lambda(\theta) - \int_{-\theta_{in}}^{\theta_{in}} \frac{d\theta'}{2\pi} \lambda(\theta') \frac{\sin[5/2(\theta - \theta') - \Phi(\theta) + \Phi(\theta')]}{\sin(\theta - \theta')/2}. \quad (48)$$

Before ending this section we mention that the alternative inequality (11) obtained from the dispersion relation (6) leads to similar bounds, which can be obtained by replacing in the r.h.s. of the inequalities (27), (32), (46) and (47) the quantity  $\psi''(Q^2)$  by  $(\psi(Q^2)/Q^2)' + \psi(0)/Q^4$ , and the outer function  $C(z)$  from (21) by

$$\tilde{C}(z) = C(z) \frac{1 - z + \beta_Q(1 + z)}{2\sqrt{2}}, \quad (49)$$

where  $\beta_Q$  was defined above (21). Then, instead of (26), we obtain the relations between  $g_n$  and the Taylor coefficients in (2) as:

$$\begin{aligned} g_0 &= f_0(0) 0.00249987, \\ g_1 &= f_0(0) [-0.00154365 + 0.0172731 \langle r^2 \rangle_s^{K\pi}], \\ g_2 &= f_0(0) [-0.00193761 - 0.0452122 \langle r^2 \rangle_s^{K\pi} + 0.00651435 c]. \end{aligned} \quad (50)$$

### 3 Results and discussion

The input required for the numerical evaluation of the bounds is represented by:

1. the function  $\psi''(Q^2)$  appearing in the right-hand side of the inequalities (27), (32), (46) and (47)
2. the ratio  $F_K/F_\pi$  appearing the Callan-Treiman relation (3),
3. the phase  $\delta(\theta) = \delta_0^{1/2}(t(e^{i\theta}))$  entering the expression (41) of the function  $\Phi$ .

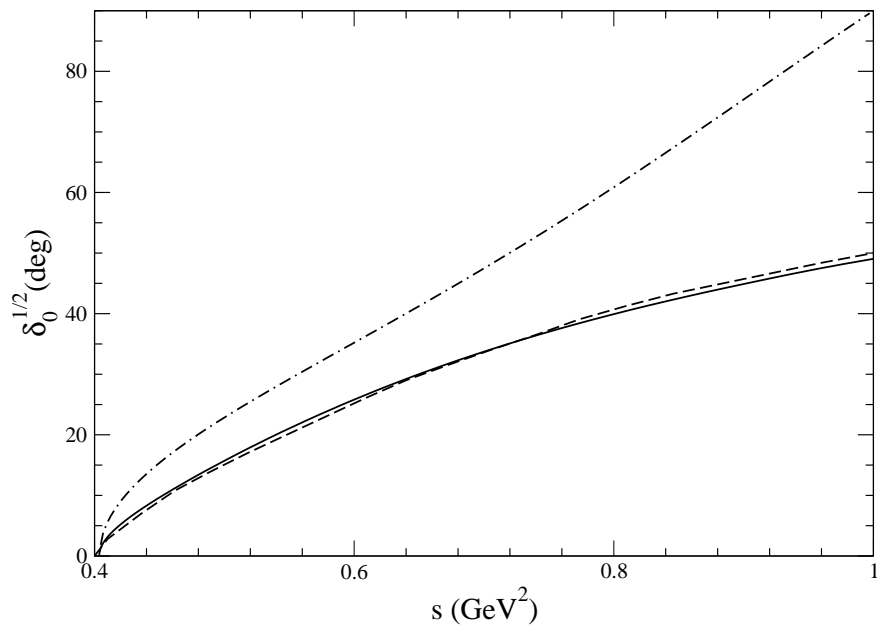


Figure 1: Phase shift of the  $I = 1/2$   $S$ -wave  $K\pi$  amplitude below 1 GeV. Solid: phase-shift  $\delta_0^{1/2}(t)$  calculated in [35], used in our analysis; dashed: one of the curves for  $\delta_0^{1/2}(t)$  given in Fig. 1 of [18]; dashed-dot: parametrization of  $\delta_0^{1/2}(t)$  used in [37].

We calculated  $\psi''(Q^2)$  given by the expression (13) taking  $Q^2 = 4 \text{ GeV}^2$  (see Ref.[16]), which is sufficiently large to ensure the validity of perturbative QCD. At this value of  $Q^2$  the  $\alpha_s$  correction in (13) is of about 30%, the  $\alpha_s^2$  correction is less than 9%, while the higher twist terms contribute with less than 1%. The choice  $Q^2 = 4 \text{ GeV}^2$  is also convenient since predictions for the light  $m_l$  and strange  $m_s$  quark masses at 2 GeV are now available from lattice calculations: QCDSF-UKQCD Collaboration [23] predicts  $m_s(2 \text{ GeV}) = 119(5)(8)\text{MeV}$  and  $m_l(2 \text{ GeV}) = 4.7(2)(3) \text{ MeV}$ , where the first quoted error is statistical and the second systematic, while MILC Collaboration [24] predicts  $m_s(2 \text{ GeV}) = 76(0)(3)(7)(0)\text{MeV}$  and  $m_l(2 \text{ GeV}) = 2.8(0)(1)(3)(0) \text{ MeV}$ , where the errors are from statistics, simulation, perturbation theory and electromagnetic effects, respectively. As concerns the strong coupling, the value  $\alpha_s(m_\tau^2)$  is now well known from hadronic  $\tau$  decays [31], [32]. The value given in PDG2004, which we used in our work, is  $\alpha_s(m_\tau^2) = 0.345 \pm 0.03$  [33], slightly larger than the renormalon-based estimates [34]. We notice that the strength of the bounds depend in a monotonous way on the magnitude of  $\psi''(Q^2)$ , a larger value of  $\psi''(Q^2)$  at a given  $Q^2$  leads to weaker bounds. Therefore, using instead of  $\alpha_s(Q^2)$  at  $Q^2 = 4 \text{ GeV}^2$  the value  $\alpha_s(m_\tau^2)$  is a conservative approximation. In this way, we calculate  $\psi''(4 \text{ GeV}^2)$  quite accurately using the running quark masses and the running coupling at 2 GeV, without going through the invariant masses and  $\Lambda_{QCD}$ . Introducing standard values of the quark condensates [30], we obtained  $\psi''(4 \text{ GeV}^2) = 0.00020$  with the quark masses from [23], and  $\psi''(4 \text{ GeV}^2) = 0.000079$  with the quark masses from [24].

The Particle Data Group [33] reports the value 1.22 for the ratio  $F_K/F_\pi$  appearing the Callan-Treiman relation (3). The pseudoscalar decay constants were recently calculated from the lattice [25], with the result  $F_K/F_\pi = 1.21(4)(13)$ , where the first error is statistical and the second systematic. Following Ref. [19], we shall use in our analysis  $F_K/F_\pi$  in the range 1.21 - 1.23. The last input is provided by the phase shift  $\delta_0^{1/2}(t)$  of the  $S$ -wave  $K\pi$  amplitude along the elastic region. We used in our work the phase shift derived from a new analysis of  $K\pi$  scattering from Roy-Steiner equation [35] and the experimental data [36]. We took the inelastic threshold at  $t_{in} = 1 \text{ GeV}^2$ , since the inelasticity is equal to 1 up to this energy. In Fig. 1 we show the phase  $\delta_0^{1/2}(t)$  used in our work, together with the phase shift derived in [18] and used in the subsequent analysis [17], [19] of the strangeness changing scalar form factors. For completeness we represent also in Fig. 1 a parametrization

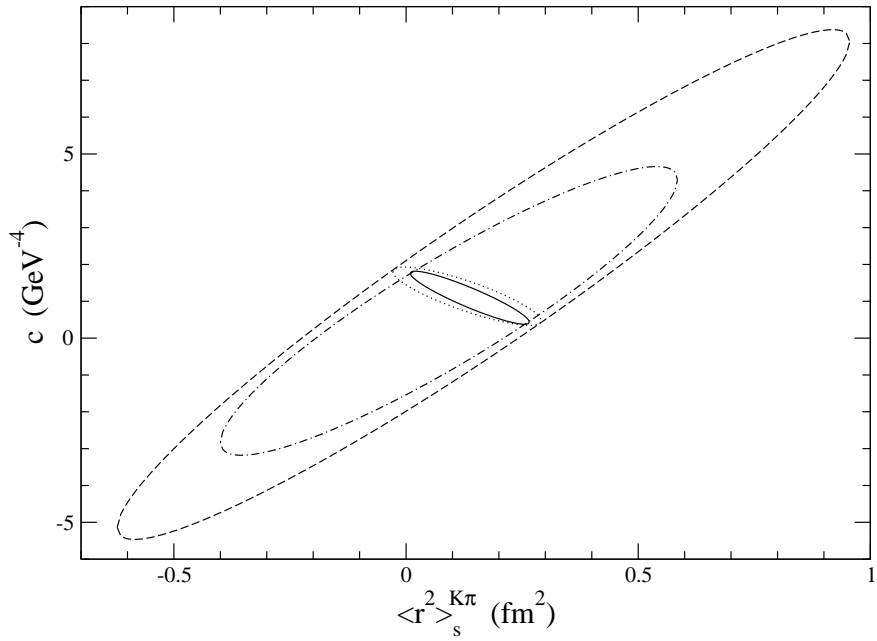


Figure 2: Allowed domains for the  $K\pi$  radius squared and the curvature for  $f_0(0) = 0.976$  and  $F_K/F_\pi = 1.21$ ; dashed: unitarity domain given by Eq. (27); dashed-dot: domain given by Eq. (47) imposing Watson theorem; dotted: domain given by Eq. (32) imposing Callan-Treiman condition; solid: domain given by Eq. (46) imposing Callan-Treiman and Watson theorem.

of  $\delta_0^{1/2}(t)$  used in [37] for a calculation of the radius  $\langle r^2 \rangle_s^{K\pi}$  based on the single-channel Omnès formalism.

With the input described above we evaluated the inequalities (27), (32), (46) and (47), derived in the previous section. The numerical solution of the integral equations (44) and (48) was found easily, since the coefficients  $g_0$ ,  $g_1$  and  $g_2$  do not appear in the kernel. For a fixed value of  $f_0(0)$ , the allowed domain for the slope and curvature is in all cases the interior of an ellipse in the plane  $(\langle r^2 \rangle_s^{K\pi}, c)$ . In order to show the effect produced by each additional constraint, we indicate in Fig. 2 the domains given by the inequalities (27), (32), (47) and (46), respectively. The curves shown were obtained with  $f_0(0) = 0.976$ , the quark masses from [24] and  $F_K/F_\pi = 1.21$ . One can see that the most important effect is produced by the Callan-Treiman condition (3). Imposing Watson theorem in addition to the Callan-Treiman relation leads only to a small reduction of the allowed domain.

In Fig. 3 we indicate the allowed domain given by the inequality (46), which takes into account both Callan-Treiman condition (3) and Watson theorem (33), using for the quark masses the values given in Refs. [24] and [23], respectively. As in Fig. 2 we took  $f_0(0) = 0.976$  and  $F_K/F_\pi = 1.21$ . Using the quark masses from [24] we obtain the upper bound  $\langle r^2 \rangle_s^{K\pi} \leq 0.26 \text{ fm}^2$ , while with the quark masses from [23] we obtain  $\langle r^2 \rangle_s^{K\pi} \leq 0.34 \text{ fm}^2$ . In Fig. 3 we indicate also the prediction of the radius and curvature made in [19] by means of an alternative dispersion theory, which gives  $\langle r^2 \rangle_s^{K\pi} = 0.192 \pm 0.012 \text{ fm}^2$ ,  $c = 0.855 \pm 0.051 \text{ GeV}^{-4}$ . These values are situated well inside the allowed domain derived with both values of the strange quark mass. Also, the  $\chi$ PT value  $\langle r^2 \rangle_s^{K\pi} = 0.20 \pm 0.05 \text{ fm}^2$  derived in [6] is consistent with the bounds, while the value  $\langle r^2 \rangle_s^{K\pi} = 0.31 \pm 0.06 \text{ fm}^2$  derived in [37] slightly violates the upper bound obtained with the  $s$  quark mass from [24].

Fig. 3 also predicts upper and lower limits on the curvature  $c$  for each value of  $\langle r^2 \rangle_s^{K\pi}$ . For instance, for the ISTRA value [3]:  $\langle r^2 \rangle_s^{K^\pm\pi} = 0.235 \text{ fm}^2$ , and the curvature  $c$  must be in the range from 0.41 to 0.78  $\text{GeV}^{-4}$ , while for the KTeV value [1]:  $\langle r^2 \rangle_s^{K_L\pi} = 0.165 \text{ fm}^2$ , and the curvature must be larger than 0.67 and smaller than 1.23  $\text{GeV}^{-4}$ . Thus, the second order term in the Taylor expansion (2) seems to be important and can not be neglected in experimental fits. The allowed range of  $c$  given in Fig. 3 for each value of the slope may be useful as an additional constraint in the representation of the experimental data. Up to now we indicated the allowed domains obtained with the form (5) of the dispersion relation. The relation (6) leads to similar



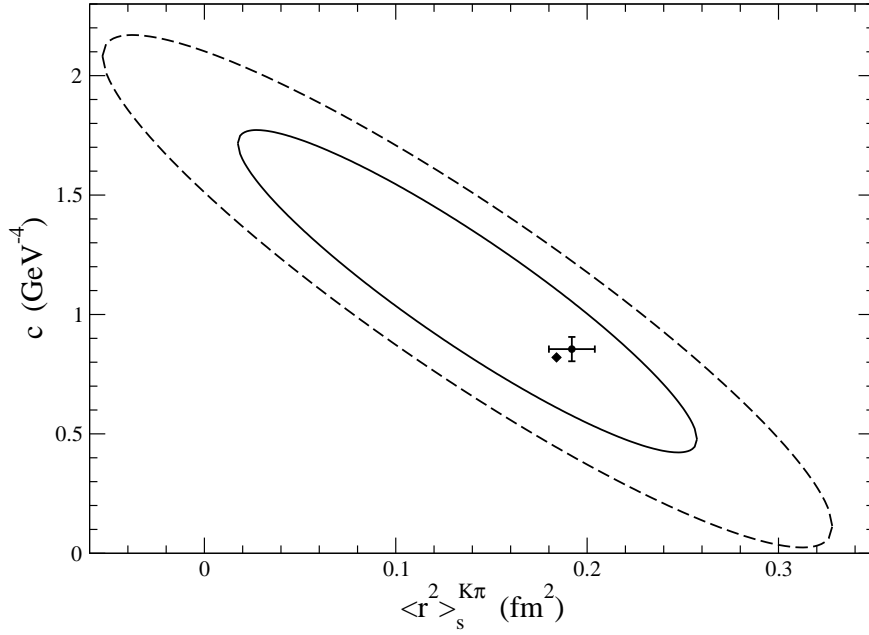


Figure 3: Allowed domain for the  $K\pi$  radius squared and curvature, given by Eq. (46) which incorporates Callan-Treiman relation and Watson theorem. Dashed: values obtained with  $m_s = 119$  MeV,  $m_u = 3$  MeV cf. [23]; solid: values obtained with  $m_s = 76$  MeV,  $m_u = 2.8$  MeV cf. [24]. The diamond is the determination made in [19] for the same input values  $f_0(0) = 0.976$  and  $F_K/F_\pi = 1.21$ , while the circle with error bars is the final determination in [19].

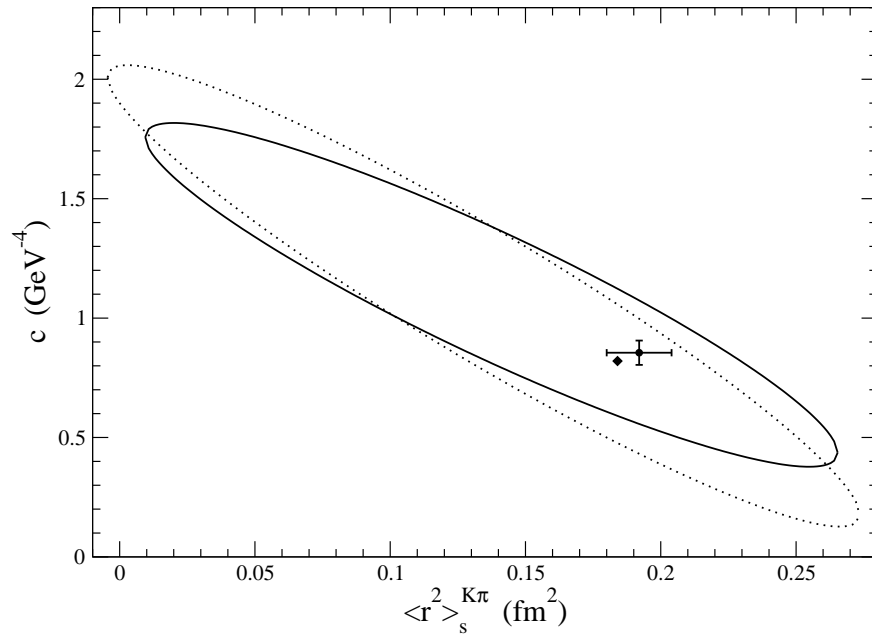


Figure 4: Allowed domain for the  $K\pi$  radius squared and curvature given by Eq.(46) using  $m_s = 76$  MeV,  $m_u = 2.8$  MeV: solid: domain obtained with the dispersion relation (5); dotted: domain obtained with the dispersion relation (6). The points are the determination made in [19].

results. In this case, in the r.h.s. of the inequalities (27), (32), (47) and (46) appears the quantity  $(\psi(Q^2)/Q^2)' + \psi(0)/Q^4$ . We evaluated this quantity at  $Q^2 = 4\text{GeV}^2$  using the QCD expression (14) and the inequality (7), which leads to the values 0.00022 with the quark masses from [24], and 0.00038 with the quark masses from [23]. In Fig. 4 we present for comparison the allowed domain for the radius and curvature obtained with the two forms of the dispersion relation discussed above.

It is of interest to combine the bounds on the Taylor coefficients of  $f_0(t)$  derived in the present work with the predictions of  $\chi$ PT. The expression of the scalar form factor at order  $p^6$  in  $\chi$ PT is [8]

$$f_0(t) = F_+(0) + \bar{\Delta}(t) + \frac{F_K/F_\pi - 1}{\Delta_{K\pi}} t + \frac{8}{F_\pi^4} (2C_{12}^r + C_{34}^r) \Sigma_{K\pi} t - \frac{8}{F_\pi^4} C_{12}^r t^2, \quad (51)$$

where

$$\begin{aligned} F_+(0) &= 1 - 0.008 - \frac{8}{F_\pi^4} (C_{12}^r + C_{34}^r) \Delta_{K\pi}^2, \\ \bar{\Delta}(t) &= -0.259t + 0.84t^2 + 1.291t^3, \end{aligned} \quad (52)$$

where the chiral constants are taken at the scale  $\mu = M_\rho$  [8], [19]. The inequalities derived in the previous section can be expressed as an allowed domain for the chiral constants  $C_{12}^r$  and  $C_{34}^r$ . As in Ref.[19] we use as independent parameters  $C_{12}^r$  and the sum  $C_{12}^r + C_{34}^r$ . For  $Q^2 = 4\text{GeV}^2$ , the first three Taylor coefficients  $g_n$  defined in (24) are expressed in terms of these parameters as

$$\begin{aligned} g_0 &= 0.00163 - 9.23 (C_{12}^r + C_{34}^r), \\ g_1 &= -0.000104 + 77.2 C_{12}^r + 87.84 (C_{12}^r + C_{34}^r). \\ g_2 &= -0.00226 - 713.3 C_{12}^r - 241.93 (C_{12}^r + C_{34}^r). \end{aligned} \quad (53)$$

Then inequality (46) gives the allowed domain shown in Fig.5, where we indicate also the result derived in [19] and the recent determination of the chiral low-energy constants in [38].

From Figs. 3-5 it is seen that the predictions made in [19] are much more precise than the unitarity bounds derived by us, even with the implementation of Callan-Treiman and Watson theorem. It is of interest to see what is the additional input used in [19], responsible for the very small uncertainties quoted for their results. In Ref. [17] the authors solve the coupled channel

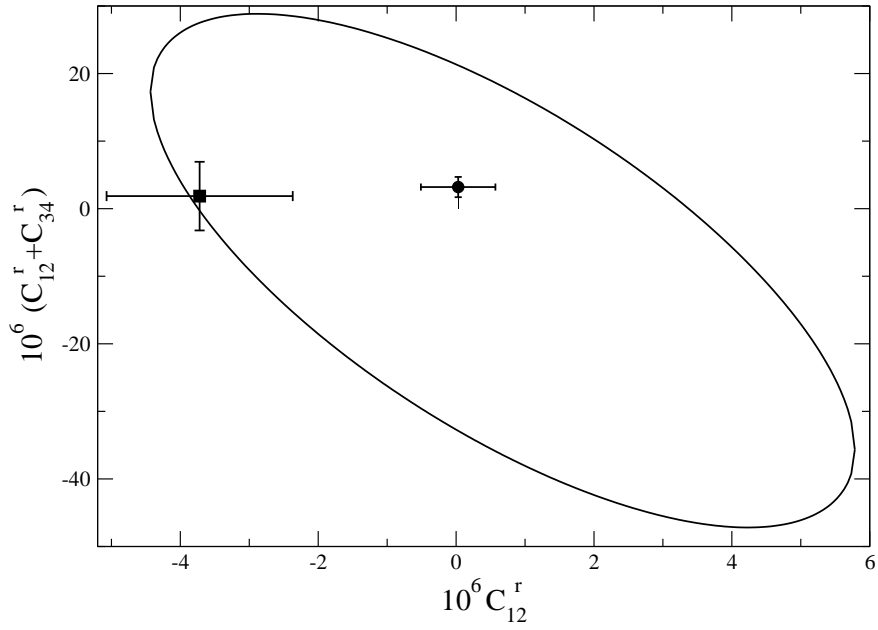


Figure 5: Allowed domains for the low energy constants  $C_{12}^r$  and  $C_{12}^r + C_{34}^r$  using Eq. (46), with the quark masses from [24] and  $F_K/F_\pi = 1.21$ . The circle with errors is the determination in [19], the square is the prediction in [38], where the quoted errors represent the variation with the scale  $\mu$  in the range from  $M_\eta$  to 1 GeV.

unitarity equations for the  $K\pi$ ,  $K\eta'$  and  $K\eta$  form factors, using the chiral resonance model and various  $K$ -matrix parametrizations of the scattering amplitudes. By construction, the Watson theorem is satisfied by the  $K\pi$  form factor  $f_0(t)$ , while the two integration constants entering the solution of the coupled channel equations are fixed by the values of  $f_0(t)$  at  $t = 0$  and  $t = \Delta_{K\pi}$  cf. Callan-Treiman relation (3). Therefore,  $f_0(t)$  calculated in [17] satisfies the same low energy constraints which we imposed. However, in our formalism the form factor is constrained above the inelastic threshold only by the  $L^2$  condition (9) (or (11)). This puts a weak restriction on the modulus  $|F(t)|$ , which is allowed to increase like  $t$  at asymptotic energies, while in [17] the  $1/t$  asymptotic decrease predicted by perturbative QCD is imposed. Analyticity is also implemented in a different way in the two approaches: in our formalism it is expressed by the relations (23) and (24), where the real coefficients  $g_n$  obey only the constraints that were explicitly incorporated. So, we work with the most general class of functions which satisfies these constraints. On the other hand, in [17] the form factor is expressed by a dispersion relation with the spectral function related to the meson - meson transition amplitudes, described by several specific parametrizations. Choosing a restricted class of functions leads to a smaller error, but may introduce a bias, so the errors given in [17] might be underestimated.

Of course, the information provided by the inelastic channels is important. The contribution of other two-particle states can be implemented in a rigorous way in the present formalism, as shown in [27] (see also [39]). The higher states in the unitarity relation (8) involve the modulus squared of the  $K\eta$  and  $K\eta'$  form factors above the corresponding unitarity thresholds. Their unphysical cuts can be eliminated by exploiting Watson theorem with a generalized Omnès formalism [27]. We shall investigate this problem in a future work. Other generalization, easily implementable, is the introduction of higher derivatives at zero momentum transfer, which are related to the chiral constants of  $\chi$ PT.

We mention finally that the inequalities derived above can be used also in the opposite way, for bounding the quark masses using the information on the low energy expansion parameters of the strangeness changing scalar form factors. This problem was investigated for instance in [40] and [41]. Our approach offers the possibility to include simultaneously various pieces of information, like Callan-Treiman constraint (3) and  $K\pi$  phase shift  $\delta_0^{1/2}$  through Watson theorem. For instance, using the predictions of the radius and the curvature made in [19] we obtain the lower bound  $m_s(2 \text{ GeV}) -$

$m_u(2 \text{ GeV}) > 38.6 \text{ MeV}$ . The prediction made in [41] based on a QCD sum rule is tighter since the authors use the spectral function of the two point function  $\psi$  calculated from the values of the  $K\pi$ ,  $K\eta$  and  $K\eta'$  form factors on the unitarity cut.

## 4 Conclusions

We have derived and evaluated unitarity bounds on the slope and curvature of the scalar strangeness changing form factor  $f_0(t)$ . The method incorporates in an exact mathematical way low energy theorems like Callan-Treiman condition and Watson final state interaction theorem. Our results indicate that the curvature can not be neglected in the representation of the experimental data on  $K_{l3}$  decay. Using the calculation of the form factor to two loops in  $\chi$ PT, we expressed our results as an allowed domain for the chiral constants  $C_{12}^r$  and  $C_{12}^r + C_{34}^r$ . Our bounds confirm the values obtained recently in an coupled channel dispersion approach, but are much weaker (however the uncertainties given in [19] might be underestimated). Generalizations of the method, including the  $K\eta$  and  $K\eta'$  form factors in the unitarity sum for the spectral function, and their higher derivatives at the origin, can increase the predictive power of the present formalism.

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