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POLYNOMIAL EQUATIONS WITH ONE CATALYTIC VARIABLE, ALGEBRAIC SERIES, AND MAP ENUMERATION

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Abstract. Let $F(t,u) \equiv F(u)$ be a formal power series in t with polynomial coefficients in u. Let $F_1, \dots, F_k$ be k formal power series in t, independent of u. Assume all these series are characterized by a polynomial equation

$$P(F(u), F_1, \dots, F_k, t, u) = 0.$$

We prove that, under a mild hypothesis on the form of this equation, these $(k+1)$ series are algebraic, and we give a strategy to compute a polynomial equation for each of them. This strategy generalizes the so-called *kernel method* and *quadratic method*, which apply respectively to equations that are linear and quadratic in $F(u)$. Applications include the solution of numerous map enumeration problems, among which the hard-particle model on general planar maps.

1. Introduction

Let us begin with a classical enumeration problem. We consider walks on the half-line $\mathbb{N}$, that start from 0 and consist of unit steps ± 1. Let $F(t,u) \equiv F(u)$ be their generating function, where t counts the length (the number of steps) and u the position of the endpoint. That is to say, $F(t,u) = \sum_{n,k} a_{n,k} t^n u^k$, where $a_{n,k}$ is the number of n-step walks that end at level k. Note that $F(t,0) \equiv F(0)$ is the length generating function of the celebrated *Dyck paths*, which are the walks ending at 0 [41, p. 173]. A step-by-step construction of these walks gives either a recurrence relation of the numbers $a_{n,k}$ or, equivalently, the following functional equation:

$$F(u) = 1 + tuF(u) + \frac{t}{u}\Big(F(u) - F(0)\Big). \tag{1}$$

The second (resp. third) term on the right-hand side counts walks ending with a step $+1$ (resp. -1). Clearly, this equation defines $F(u)$ uniquely as a formal power series in t (with rational coefficients in u). Observe that the equation

$$F(u) = 1 + tuF(u) + \frac{t}{u}\Big(F(u) - F_1\Big) \tag{2}$$

defines uniquely *both* $F(u)$ and F_1 as formal power series in t, if we impose that $F(u)$ has *polynomial* coefficients in u and that F_1 is independent of u. Indeed, after multiplying the equation by u and setting $u = 0$, we find $F_1 = F(0)$, and we are thus back to (1). Finally, we recall that $F(0)$ is well-known to be *algebraic* of degree 2,

$$F(0) = \frac{1 - \sqrt{1 - 4t^2}}{2t^2}.$$

Consequently, $F(u)$ is algebraic too (meaning that it satisfies a non-trivial polynomial equation, $Q(t,u,F(u)) = 0$, with rational coefficients).

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The above example is an instance of the general situation we study in this paper. We assume that a $(k+1)$-tuple $(F(u), F_1, F_2, \dots, F_k)$ of formal power series in t is completely determined by a polynomial equation

$$P(F(u), F_1, F_2, \dots, F_k, t, u) = 0. \tag{3}$$

Typically, $F(u)$ has polynomial coefficients in u, and F_i is the coefficient of u^{i-1} in $F(u)$. Following Zeilberger's terminology [49, p. 457], we say that (3) is a *polynomial equation with one catalytic variable* u. The aim of this paper is twofold: we prove that the solution of a (well-founded) equation of the form (3) is *always algebraic* and we present a strategy to obtain a polynomial equation it satisfies.

There are several reasons why we like to know that the generating function of some class of objects is algebraic. Firstly, the set of algebraic series is closed under natural operations (sum, product, derivatives, composition...). Secondly, these series are reasonably easy to handle (via resultants or Gröbner bases). In particular, several computer algebra packages are now able to make the above closure properties effective. Thirdly, algebraic series are also *D-finite* and this implies that their coefficients can be computed in a linear number of operations [41, Ch. 6]. The asymptotic behaviour of these coefficients has a generic form, the details of which are usually not to hard to obtain. Finally – and, to many combinatorialists, most importantly – the fact that a class of objects is counted by an algebraic series suggests that it should be possible to construct these objects recursively by *concatenation* of objects of the same type. For many objects, such a construction is easily found, but for others, among which planar maps [25, 38], the algebraic structure of the objects is far from clear, and the algebraicity of the generating function gives rise to challenging combinatorial problems. See [41, Ch. 6] or [31] for a presentation of algebraic series in enumeration.

But let us return to polynomial equations with one catalytic variable. Many combinatorialists have fought them before us, and we want to recall some milestones in this history.

1.1. **A partial historical account**

In 1956 already, Temperley writes, for the perimeter enumeration of *column-convex polyominoes*, a set of recurrence relations [42, Eq. (7)] that is equivalent, after summation, to

$$F(u) = \frac{ut^2}{1-ut} + \frac{t^3u^2F(u)}{(1-ut)^2} + 2\,\frac{t^2u^2}{1-ut}\,\frac{F(u)-F(1)}{u-1} + ut\,\frac{uF(u)-uF(1)-(u-1)F'_u(1)}{(u-1)^2}.$$

He proves that $F(1)$ is algebraic, without being able to compute it explicitly (see [29] for a simple expression of $F(1)$). Like (1), the above equation is linear in $F(u)$, but it contains *two* additional unknown functions, $F(1)$ and $F'_u(1)$.

The first non-linear equations appear in the early sixties, in the work of Tutte and Brown on planar maps. For instance, Tutte publishes in 1962 the following equation [43, Eq. (3.7)], which rules the enumeration of certain triangulations:

$$F(u) = 1 + \frac{t}{u}\left(\frac{F(u)}{1-uF(u)} - F(0)\right). \tag{4}$$

In the following years, more equations of this type are published for various families of planar maps (non-separable [18, 23], general [44], other triangulations [19], quadrangulations [20]). All of them involve only one unknown function F_1 (and thus read $P(F(u), F_1, t, u) = 0$), and are quadratic in $F(u)$, apart from the equation on quadrangulations which is cubic. In the first papers, Tutte and Brown solve these equations by *guessing and checking*: either they guess the expansions of F_1 and $F(u)$, and then check that their guesses satisfy the functional equation, or they only guess the expansion of F_1, and then prove that the polynomial equation $P(F(u), F_1, t, u)$, taken with the conjectured value of F_1, admits one root $F(u)$ that is a formal power series in t with polynomial coefficients in u. Of course, any equation for which the value of F_1 cannot be guessed remains hopeless with this strategy.

In 1965, Brown publishes a theorem that deals, at first sight, with a different topic: with the conditions satisfied by a series in t and u that admits a square root (which is itself assumed to be a formal power series) [21]. He shows that this theorem allows to solve, in a systematic way, all equations of the type (3) that are quadratic in $F(u)$ and only involve one unknown function F_1. The *quadratic method* is born (see [33, Section 2.9] for a modern account). Brown even manages to solve, with some contorsions, the above-mentioned cubic equation for quadrangulations [21, Section 4]. At the end of [22], he writes *"It is possible that the method may be effective when more than one unknown series is present"*. This hope was confirmed many years later, in 1994, when Bender and Canfield applied the method to a quadratic equation with arbitrarily many unknown functions [5].

But let us go back to the sixties. In 1968, in the first volume of *The art of computer programming*, Knuth gives for the classical *ballot problem* an equation that is equivalent to (1), and presents a "trick" that solves it [34, Section 2.2.1, Ex. 4]. This may have been the unnoticed birth of the *kernel method*, which allows to solve systematically equations of the form (3) that are linear in $F(u)$. This trick may have been better known at that time in probability theory. At least, the same idea definitely appears in a 1979 paper [28], in a more difficult, analytic context. The kernel method is currently the subject of a certain revival in combinatorics [1, 2, 11, 27, 37].

In 1972, Cori and Richard solve again certain linear equations, and also some polynomial equations with one unknown series F_1 [24]. Their technique is very interesting, but the fact that they deal with equations in non-commuting variables makes it both deeper and more obscure. Still, the strategy we present here to attack (3) owes a lot to [24].

Since then, equations of the form (3) have continued to appear in various enumeration problems, mostly involving maps [5, 32], but also polyominoes [7, 8, 30], stack-sortable permutations [9, 48] and their generalizations [47], lattice walks [2, 10], etc. Examples can be found were both the degree of the equation and the number of unknown functions is arbitrarily large. For instance, such equations are hiding in Tutte's work on the chromatic polynomial of triangulations [45, Section 5]. Another such set of equations, unbounded in degree and number of unknowns, is presented in Section 5.3. It deals with the enumeration of certain Eulerian maps called *constellations*.

1.2. **Contents**

The general strategy. The method presented in this paper to solve equations of the form (3) encapsulates and simplifies all previous approaches, in particular the kernel method and the quadratic method. It works without any restriction on the degree of the equation or on the number of unknowns F_i. The general strategy is described in Section 2. Its justification only takes a few lines. It yields a system of 3ℓ polynomial equations that relate $k + 2\ell$ series in t: the unknowns $F_1, \dots, F_k$, first, then ℓ series named $U_1, \dots, U_\ell$, which are defined as the roots of a certain equation (simply related to the original functional equation), and finally the values of $F(u)$ at $u = U_i$, for $i = 1, \dots, \ell$. The strategy "works" if, first, $\ell = k$ (so that we have as many equations as unknown series), and if the $3k$ polynomial equations thus obtained imply the algebraicity of the F_i.

First examples. In Section 3, we apply this strategy to several examples. For each of them, we observe that the strategy works: we find as many series U_i as we have unknowns F_i, and we can derive from the system of $3k$ polynomial equations an algebraic equation for each F_i. We also relate our approach to the earlier *kernel method* and *quadratic method*.

A generic algebraicity theorem. Will this strategy *always* work? Section 4 answers this question positively, at least for a well-founded equation of the form

$$F(u) = F_0(u) + t\, Q\Big(F(u), \Delta F(u), \Delta^{(2)} F(u), \dots, \Delta^{(k)} F(u), t, u\Big), \tag{5}$$

where $F_0(u)$ is a given polynomial in u (with coefficients in a field $\mathbb{K}$ of characteristic 0), $Q(x_0, \dots, x_k, t, v)$ is another polynomial, and

$$\Delta^{(i)} F(u) = \frac{F(u) - F_1 - uF_2 - \cdots - u^{i-1}F_i}{u^i}.$$

If we require $F(u)$ to be a formal power series in t with polynomial coefficients in u, and F_i to be the coefficient of u^{i-1} in $F(u)$, then this equation defines uniquely $F(u)$. We prove that $F(u)$, and hence all the F_i, are algebraic.

Algebraicity results for planar maps. Thus the solution of every (well-founded) equation with one catalytic variable is algebraic. This result urges a combinatorial interlude, in which we establish for several families of planar maps an equation of this type. The generic algebraicity theorem tells us, without going further, that their generating functions are algebraic. Our examples include some already studied problems (like the face-distribution of Eulerian maps, for which we answer positively a question left open in [13]), and some new ones, like the hard-particle model on general planar maps.

From $3k$ to $2k$, and then k equations. The next question that we address is both theoretical and practical: it deals with the size of our polynomial system. Assume the general strategy works and provides a system of $3k$ equations. Even when $k = 2$, even for a computer algebra system, this can be hard to handle. In Section 6 we reduce the system to $2k$ equations which involve only the series F_i and U_i. This new system can be described simply in terms of the *discriminant* of the polynomial P occurring in (3), taken with respect to its first variable (we assume that P is at least quadratic in this variable). Our $2k$ equations say that this discriminant, evaluated at $F_1, \dots, F_k, t, u$ and considered as a polynomial in u, *has k multiple roots* $U_1, \dots, U_k$. This extends a result that was known to hold in the quadratic case and is one of the possible formulations of the quadratic method [33, Section 2.9].

Hence the discriminant and its derivative with respect to u have k roots in common. It is well-known that two polynomials have *one* root in common if their resultant is zero. In Section 7, we recall how to express, by a set of k determinants, the fact that two polynomials have k roots in common. Applying this to the discriminant and its derivative, we obtain a set of k polynomial equations that relate $F_1, \dots, F_k$.

A new proof of Brown's theorem. Before turning our attention to specific examples, we give in Section 8 a "modern", and maybe clearer proof of Brown's theorem on square roots of bivariate power series[1]. Recall that this theorem is the basis of the quadratic method.

Practical examples. We discuss in Section 9 how to derive in practise an algebraic equation for, say, the unknown series F_1. We suggest various approaches, which we exemplify on certain maps called 3-constellations. The associated equation is cubic and involves two unknown series F_i. In Section 10, we walk in the steps of Bender and Canfield [5] to find the face-distribution of planar maps. This problem was already solved in two other ways [14], and we prove that our results are equivalent to the former ones. Finally, we solve in Section 11 the hard-particle model on general planar maps. For other recent applications of our method, see [6].

Finally, Section 12 discusses a number of open questions.

1.3. **Formal power series and their relatives**

Let us conclude this introduction with some notation. Let $\mathbb{K}$ be a commutative ring. We denote by $\mathbb{K}[t]$ the set of polynomials in t with coefficients in $\mathbb{K}$. If $\mathbb{K}$ is a field, then $\mathbb{K}(t)$ denotes the field of fractions in t with coefficients in $\mathbb{K}$. We denote by $\overline{\mathbb{K}}$ the algebraic closure

[1] As mentioned in Section 8, it seems that there may be a mistake in Brown's original proof.

of $\mathbb{K}$. We also consider several sets of series of the form

$$A(t) = \sum_{n \geq n_0} a_n t^{n/d},$$

where $n_0 \in \mathbb{Z}$, $a_{n_0} \neq 0$ and $d \in \mathbb{N} \setminus \{0\}$. The number n_0/d is called the *valuation* of $A(t)$. We use the standard notation for the coefficients of a series:

$$[t^{n/d}]A(t) := a_n.$$

In particular,

- $\mathbb{K}[[t]]$ is the set of formal power series in t with coefficients in $\mathbb{K}$ ($n_0 \geq 0$ and $d = 1$),
- $\mathbb{K}((t))$ is the set of Laurent series in t with coefficients in $\mathbb{K}$ ($d = 1$),
- $\mathbb{K}^{\mathrm{fr}}[[t]]$ is the set of fractional power series in t with coefficients in $\mathbb{K}$ ($n_0 \geq 0$),
- $\mathbb{K}^{\mathrm{fr}}((t))$ is the set of fractional Laurent series in t (a.k.a. Puiseux series) with coefficients in $\mathbb{K}$ (no condition).

Each of these sets is a commutative ring, and the second and fourth are fields if $\mathbb{K}$ is a field. More precisely, $\mathbb{K}((t))$ is the fraction field of $\mathbb{K}[[t]]$, and $\mathbb{K}^{\mathrm{fr}}((t))$ is the fraction field of $\mathbb{K}^{\mathrm{fr}}[[t]]$. If, moreover, $\mathbb{K}$ is algebraically closed and has characteristic 0, then so is $\mathbb{K}^{\mathrm{fr}}((t))$ [41, Thm. 6.1.5].

These notations generalize to series in several indeterminates. In this paper, we will mostly use series in t and u. Note the following inclusions:

$$\mathbb{K}[t,u] \subset \mathbb{K}[[t]][u] \subset \mathbb{K}[u][[t]] \subset \mathbb{K}[[t,u]] = \mathbb{K}[[t]][[u]].$$

The second set above is the set of polynomials in u whose coefficients are formal power series in t. The third set is the set of formal power series in t whose coefficients are polynomials in u. The notation $\mathbb{K}[[u]]^{\mathrm{fr}}[[t]]$ stands for the set of power series in u and t that are fractional in t.

All the fields considered in this paper have implicitly characteristic 0.,

2. **The general strategy**

Let $\mathbb{K}$ be a field. In our examples, $\mathbb{K}$ will be $\mathbb{C}$, or a field of fractions like $\mathbb{C}(s_1, \ldots, s_m)$. Let $F(t,u) \equiv F(u)$ be a series of $\mathbb{K}[u][[t]]$, and let $F_1(t) \equiv F_1, \ldots, F_k(t) \equiv F_k$ be k series of $\mathbb{K}[[t]]$. In our framework, these $k+1$ series are the generating functions of certain families of objects, counted according to one or two parameters. Assume these series are related by an equation of the form

$$P(F(u), F_1, F_2, \ldots, F_k, t, u) = 0, \tag{6}$$

where $P(x_0, x_1, \ldots, x_k, t, v)$ is a non-trivial polynomial in $k+3$ variables, with coefficients in $\mathbb{K}$. Assume, moreover, that the above equation defines the $(k+1)$-tuple $(F(u), F_1, \ldots, F_k)$ *uniquely* in the set $\mathbb{K}[u][[t]] \times \mathbb{K}[[t]]^k$. Some examples were given in the introduction, and numerous examples will be given below.

Let us differentiate (6) with respect to u:

$$F'(u)\frac{\partial P}{\partial x_0}(F(u), F_1, \ldots, F_k, t, u) + \frac{\partial P}{\partial v}(F(u), F_1, \ldots, F_k, t, u) = 0.$$

Let $U(t) \equiv U$ be a series of $\mathbb{K}^{\mathrm{fr}}[[t]]$. The series $F(U) \equiv F(t,U)$ is a well-defined fractional power series in t. The same holds for $F'(U)$. If, moreover,

$$\frac{\partial P}{\partial x_0}(F(U), F_1, \ldots, F_k, t, U) = 0, \tag{7}$$

then the above identity implies that

$$\frac{\partial P}{\partial v}(F(U), F_1, \ldots, F_k, t, U) = 0.$$

This simple observation is the key of our solution of equations of the form (6). *If we can prove the existence of k distinct series $U_1, \dots, U_k$, belonging to $\mathbb{K}^{\mathrm{fr}}[[t]]$, that satisfy* (7)*, then the following system of $3k$ polynomial equations holds*: for $1 \leq i \leq k$,

$$P\Big(F(U_i), F_1, \dots, F_k, t, U_i\Big) = 0, \tag{8}$$

$$\frac{\partial P}{\partial x_0}\Big(F(U_i), F_1, \dots, F_k, t, U_i\Big) = 0, \tag{9}$$

$$\frac{\partial P}{\partial v}\Big(F(U_i), F_1, \dots, F_k, t, U_i\Big) = 0. \tag{10}$$

A bit of optimism allows us to hope that this system characterizes completely the $3k$ unknown series it involves, namely $F_1, \dots, F_k$, $U_1, \dots, U_k$ and $F(U_1), \dots, F(U_k)$, so that each unknown series (in particular each F_i) is algebraic. More precisely, we would like this system to have only a finite number of solutions *under the assumption that the series U_i are distinct*. This assumption can be encoded by adding a new unknown X and a new polynomial equation:

$$X \prod_{1 \leq i < j \leq k} (U_i - U_j) = 1. \tag{11}$$

We prove in Section 4 that this optimism is justified: the solution of a well-founded equation of the form (5) is indeed shown to be algebraic. However, we do not need this general theorem to examine and solve *specific* examples, like (2) or (4). What we *do* need is a way to determine how many series U satisfy (7), without knowing the value of $F(u)$ or $F_1, \dots, F_k$. This turns out to be easy. Let us first clarify what we mean by a *root* U of a series $\Phi(t,u)$.

Lemma 1. *Let $\Phi(t,u) \in \mathbb{K}[u]^{\mathrm{fr}}[[t]]$, and $U \in \mathbb{K}^{\mathrm{fr}}[[t]]$. Then $\Phi(t,U)$ is a well-defined series of $\mathbb{K}^{\mathrm{fr}}[[t]]$. If this series is zero, we say that U is a root of $\Phi(t,u)$. In this case, there exists $\Psi(t,u) \in \mathbb{K}[u]^{\mathrm{fr}}[[t]]$ such that*

$$\Phi(t,u) = (u-U)\Psi(t,u).$$

More generally, if $\Phi(t,u)$ factors as

$$\Phi(t,u) = (u-U)^m \Psi(t,u),$$

where $\Psi(t,u) \in \mathbb{K}[u]^{\mathrm{fr}}[[t]]$, the series U belongs to $\mathbb{K}^{\mathrm{fr}}[[t]]$ and $\Psi(t,U) \neq 0$, we say that U is a root of $\Phi(t,u)$ of multiplicity m.

This extends to the case where $\Phi(t,u)$ belongs to $\mathbb{K}[[u]]^{\mathrm{fr}}[[t]]$, if we require that U has no constant term (that is, vanishes at $t=0$). In this case, $\Psi(t,u)$ also belongs to $\mathbb{K}[[u]]^{\mathrm{fr}}[[t]]$.

Proof. The fact that $\Phi(t,U)$ is well-defined is obvious, by definition of the substitution of series: If

$$\Phi(t,u) = \sum_{n \geq 0} t^{n/d} \phi_n(u),$$

where $\phi_n(u)$ is a polynomial in u, then

$$\Phi(t,U) = \sum_{n \geq 0} t^{n/d} \phi_n(U),$$

and the coefficient of $t^{p/q}$ in $\Phi(t,U)$, for $p/q \leq k/d$, depends only on the polynomials $\phi_0(u), \dots, \phi_k(u)$. Now for any indeterminate v,

$$\Phi(t,u) - \Phi(t,v) = (u-v) \sum_{n \geq 0} t^{n/d} \phi'_n(u,v),$$

where

$$\phi'_n(u,v) = \frac{\phi_n(u) - \phi_n(v)}{u-v}$$

is a polynomial in u and v. The case $v = U$ proves the second statement of the lemma.

The argument can be adapted without any difficulty to the case where $\Phi(t, u)$ belongs to $\mathbb{K}[[u]]^{\mathrm{fr}}[[t]]$ and U has no constant term, upon writing

$$\Phi(t, u) = \sum_{n,m\geq 0} \phi_{m,n} u^m t^{n/d}$$

with $\phi_{m,n} \in \mathbb{K}$.

■

The next theorem tells how many roots a series $\Phi(t, u)$ has.

Theorem 2. *Let $\Phi(t, u) \in \mathbb{K}[u]^{\mathrm{fr}}[[t]]$, where $\mathbb{K}$ is an algebraically closed field. Assume that the coefficient of t^0 in Φ, that is to say, the polynomial $\Phi(0, u)$, is non-zero and has degree k. Then $\Phi(t, u)$ has exactly k roots in $\mathbb{K}^{\mathrm{fr}}[[t]]$, counted with multiplicities. Let $U_1, \ldots, U_k$ denote these roots. Then*

$$\Phi(t, u) = (u - U_1) \cdots (u - U_k)\Psi(t, u)$$

where $\Psi(t, u) \in \mathbb{K}[u]^{\mathrm{fr}}[[t]]$.

Proof. The proof is a harmless extension of the proof of the Puiseux theorem, which establishes the above result (and more) in the case where $\Phi(t, u) \in \mathbb{K}^{\mathrm{fr}}((t))[u]$. We refer the reader to [46, Ch. 4]. The coefficients of the U_i can be computed inductively using Newton's polygon.

■

3. First examples

We now apply our general strategy to a few examples.

3.1. Walks on a half-line and the kernel method

We consider here some equations of the form (6) that are linear in $F(u)$. The reader familiar with the *kernel method* will not find our calculations very original, and this is normal: beyond solving these equations, our objective here is to show that our general strategy reduces to the kernel method when the equation is linear. We refer to [2] for a systematic treatment of walks on the half-line, based on the kernel method.

Let us first go back to the simplest equation we have met so far, Eq. (2). It can be rewritten under the form (6):

$$P(F(u), F_1, t, u) = 0,$$

where

$$P(x_0, x_1, t, v) = \left(v - t\left(1 + v^2\right)\right) x_0 - v + t x_1.$$

Condition (7) reads in this case:

$$U - t(1 + U^2) = 0.$$

In accordance with Theorem 2, we find that there exists a unique fractional power series in t that satisfies this equation, namely

$$U = \frac{1 - \sqrt{1 - 4t^2}}{2t}.$$

The system (8–10) now reads

$$\begin{aligned} \left(U - t\left(1 + U^2\right)\right) F(U) &= U - tF_1, \\ U - t(1 + U^2) &= 0, \\ (1 - 2tU)F(U) &= 1. \end{aligned}$$

The first and second equations together imply that

$$F_1 = \frac{U}{t} = \frac{1-\sqrt{1-4t^2}}{2t^2}.$$

We have recovered the classical expression of the generating function of Dyck paths. An expression for $F(u)$ now follows from the original equation $P(F(u), F_1, t, u) = 0$.

Let us now study a problem with more unknown functions. We still consider walks on the half-line $\mathbb{N}$ that start from 0, but they now consist of steps $+3$ and -2. A step-by-step construction of these walks gives, for their bivariate generating function $F(t,u) \equiv F(u)$, the equation

$$F(u) = 1 + tu^3F(u) + \frac{t}{u^2}\left(F(u) - F_1 - uF_2\right) \tag{12}$$

where F_1 (resp. F_2) is the length generating function of walks ending at 0 (resp. 1). This equation can be rewritten as $P(F(u), F_1, F_2, t, u) = 0$, with

$$P(x_0, x_1, x_2, t, v) = \left(v^2 - t(1+v^5)\right) x_0 - v^2 + tx_1 + tvx_2.$$

Condition (7) now reads

$$U^2 - t(1+U^5) = 0.$$

By Theorem 2, exactly two fractional power series U_1 and U_2 satisfy this equation, and we happily observe that two is also the number of unknown series F_i. One may compute the first terms of the U_i's using Newton's polygon:

$$U_{1,2} = \pm t^{1/2} + \frac{1}{2}\,t^3 \pm \frac{9}{8}\,t^{11/2} + \frac{7}{2}\,t^8 + O(t^{21/2}).$$

In particular, these two series are distinct. The system (8–10) now reads, for $i = 1, 2$,

$$\begin{aligned}
\left({U_i}^2 - t(1+{U_i}^5)\right) F(U_i) &= {U_i}^2 - tF_1 - tU_iF_2, &(13)\\
U_i^2 - t(1+U_i^5) &= 0, &(14)\\
U_i\left(2 - 5\,t{U_i}^3\right) F(U_i) &= 2\,U_i - tF_2.
\end{aligned}$$

We have thus obtained six equations that relate $F_1, F_2, U_1, U_2, F(U_1)$ and $F(U_2)$. At this point, there are several ways to conclude. The fastest one is probably to observe that, by (13) and (14), the series U_1 and U_2 are the two roots of the following polynomial in u:

$$R(u) = u^2 - tuF_2 - tF_1.$$

Thus this polynomial factors as $(u-U_1)(u-U_2)$, which implies

$$-tF_1 = U_1U_2 \quad \text{and} \quad tF_2 = U_1 + U_2.$$

One can then eliminate U_1 and U_2 using (14), and obtain polynomial equations for F_1 and F_2. In particular, the generating function F_1 of walks ending at 0 satisfies:

$$F_1 = 1 + 2\,t^5{F_1}^5 - t^5{F_1}^6 + t^5{F_1}^7 + t^{10}{F_1}^{10}.$$

Consider, more generally, the case where the functional equation (6) has degree 1 in $F(u)$ and can be written as

$$K(t,u)F(u) = P(F_1, \ldots, F_k, t, u)$$

where $K(t,u) \in \mathbb{K}[t,u]$ is the *kernel* of the equation, and $P(x_1, \ldots, x_k, t, u)$ is a polynomial in $k+2$ indeterminates. The system (8–10) reads

$$\begin{aligned}
K(t,U)F(U) &= P(F_1, \ldots, F_k, t, U),\\
K(t,U) &= 0,\\
K'_u(t,U)F(U) &= P'_u(F_1, \ldots, F_k, t, U).
\end{aligned}$$

By combining the first and second equations, we see that every root of the kernel that is finite at $t = 0$ gives a polynomial equation relating the k unknown series $F_1, \ldots, F_k$. This is

exactly the principle of the *kernel method*, which has been around since the late 60's, and is currently the subject of a certain revival (see [1, 2, 11, 27, 37] and references therein).

3.2. **Planar maps and the quadratic method**

We consider here *rooted planar maps* (see Section 5 or [33] for definitions). Let $F(t,u) \equiv F(u)$ be their generating function, where t counts the number of edges, and u the degree of the root-face. Deleting the root-edge gives [44, Eq. (4)]:

$$F(u) = 1 + tu^2F(u)^2 + tu\,\frac{uF(u) - F(1)}{u-1}. \tag{15}$$

Multiplying this equation by $(u-1)$ gives a polynomial equation of the form (6), with one unknown function $F_1 := F(1)$. Condition (7) reads in this case:

$$U - 1 = 2tU^2(U-1)F(U) + tU^2.$$

By Theorem 2, this equation has a (unique) solution U in the set of fractional power series in t. It is actually clear on the equation that such a series exists, and is a formal power series in t (think of extracting the coefficient of t^n). Moreover, $U \neq 0, 1$. From (8–10), we obtain

$$\begin{aligned}(U-1)F(U) &= U - 1 + tU^2(U-1)F(U)^2 + tU^2F(U) - tUF_1,\\ U-1 &= 2tU^2(U-1)F(U) + tU^2,\\ F(U) &= 1 + tU(3U-2)F(U)^2 + 2tUF(U) - tF_1.\end{aligned}$$

One can eliminate $F(U)$ between the first and second equation, and then between the second and the third. This gives two equations relating U and F_1. We ignore the irrelevant factors U and $U-1$, and eliminate U. This gives an algebraic equation satisfied by F_1, containing three distinct factors. The right one is easily identified, given that $F_1 = 1 + O(t)$, and one concludes that the generating function of planar maps, counted by edges, satisfies

$$F_1 = 1 - 16\,t + 18tF_1 - 27\,t^2{F_1}^2.$$

More generally, an equation of the form (6) having degree 2 in $F(u)$ can be written as

$$\Big(2aF(u) + b\Big)^2 = b^2 - 4ac = \Delta(u),$$

where a, b, c and Δ lie in $\mathbb{K}[t, u, F_1, \ldots, F_k]$. The system (8–10) reads

$$\begin{aligned}\Big(2aF(U) + b\Big)^2 &= \Delta(U),\\ 2aF(U) + b &= 0,\\ 2\Big(2a'_uF(U) + b'_u\Big)\Big(2aF(U) + b\Big) &= \Delta'_u(U).\end{aligned}$$

By combining the first and second equations, we see that every fractional power series U that cancels $2aF(u) + b$ cancels the discriminant Δ. By combining the second and third equations, we see that U is actually a *multiple root of the discriminant*.

When there is only one unknown function F_1, we recover exactly the *quadratic method*, as described in [33]: if there exists a series U such that $2aF(U) + b = 0$, then $\Delta(u)$ admits a multiple root. Hence the discriminant of $\Delta(u)$ with respect to u is zero: this gives an algebraic equation satisfied by F_1.

This will be generalized in this paper to functional equations of the form (6) and of degree at least two in $F(u)$: we will prove that the discriminant Δ of P, taken with respect to its first variable and evaluated at $F_1, \ldots, F_k, t, u$, admits each U_i as a multiple root (Section 6).

3.3. **Quadrangular dissections of the disk**

Let us now consider a *cubic* example with one unknown function. This example was solved by Brown, with some difficulties [21]. Our strategy works without any restriction on the degree of the equation, and the solution of this cubic example will be just as easy as the solution of, say, the quadratic equation (15).

The *quadrangular dissections of the disk* studied by Brown in [20] can be described as the rooted, non-separable planar maps, with no multiple edges, in which each non-root face has degree 4 (see Section 5 for definitions). It is easy to see that the root-face of such maps has an even degree, at least equal to 4. Let $a_{n,k}$ be the number of such maps with $n+4$ vertices in which the root-face has degree $2k$, and let

$$F(t,u) \equiv F(u) = \sum_{n\geq 0, k\geq 2} a_{n,k} t^n u^{k-2}.$$

Eq. (5.1) of [20] can be rewritten as

$$F(u) = \frac{F(u)-F_1}{u} - t^2 F_1 F(u) + 2tF(u)(1+ut^2F(u)) + (1+ut^2F(u))^3,$$

where $F_1 \equiv F(0)$ is the generating function of dissections of squares. Condition (7) reads:

$$U = 1 - Ut^2F_1 + 2\,Ut\left(1+2Ut^2F(U)\right) + 3\,U^2t^2\left(1+Ut^2F(U)\right)^2.$$

By Theorem 2, this equation has a (unique) solution U in the set of fractional power series in t. (It is again clear on the equation itself that such a series exists, and is a formal power series in t.) Moreover, $U \neq 0$. From (8–10), we obtain

$$\begin{aligned} UF(U) &= F(U) - F_1 - Ut^2F_1F(U) + 2UtF(U)(1+Ut^2F(U)) + U(1+Ut^2F(U))^3, \\ U &= 1 - Ut^2F_1 + 2\,Ut\left(1+2Ut^2F(U)\right) + 3\,U^2t^2\left(1+Ut^2F(U)\right)^2, \\ F(U) &= -t^2F_1\,F(U) + 2\,tF(U)\left(1+2Ut^2F(U)\right) + \left(1+Ut^2F(U)\right)^2(1+4Ut^2F(U)). \end{aligned}$$

One can eliminate $F(U)$ between the first and second equation, and then between the second and the third. This gives two equations relating U and F_1. Ignoring the irrelevant factors U, we then eliminate U. This gives an algebraic equation satisfied by F_1, containing three distinct factors. The right one is easily identified, given that $F_1 = 1+O(t)$, and one concludes that the generating function of quadrangular dissections of a square, counted by the number of vertices, satisfies

$$F_1 = 1 - 8t + 2t\,(5-6\,t)\,F_1 - 2\,t^2\,(1+3\,t)\,{F_1}^2 - t^4{F_1}^3.$$

4. **A generic algebraicity theorem**

Let $Q(y_0, y_1, \ldots, y_k, t, v)$ be a polynomial in $k+3$ indeterminates, with coefficients in a field $\mathbb{K}$. We consider the functional equation

$$F(u) \equiv F(t,u) = F_0(u) + t\,Q\Big(F(u), \Delta F(u), \Delta^{(2)}F(u), \ldots, \Delta^{(k)}F(u), t, u\Big), \qquad (16)$$

where $F_0(u) \in \mathbb{K}[u]$ is given explicitly and the operator Δ is the divided difference (or discrete derivative):

$$\Delta F(u) = \frac{F(u)-F(0)}{u}.$$

Note that

$$\lim_{u\to 0} \Delta F(u) = F'(0),$$

where the derivative is taken with respect to u. The operator $\Delta^{(i)}$ is obtained by applying i times Δ, so that:

$$\Delta^{(i)}F(u) = \frac{F(u) - F(0) - uF'(0) - \cdots - u^{i-1}/(i-1)!\,F^{(i-1)}(0)}{u^i}.$$

Observe that all the equations met in Sections 1 to 3 are of the form (16), or can be easily transformed into an equation of this form. Clearly, (16) has a unique solution $F(t,u)$ in $\mathbb{K}[u][[t]]$ (think of extracting from (16) the coefficient of t^n, for $n = 0, 1, 2\ldots$). Upon multiplying (16) by a large power of u, one obtains a polynomial equation of the form

$$P\Big(F(u), F_1, \ldots, F_k, t, u\Big) = 0,$$

where $F_i = F^{(i-1)}(0)/(i-1)!$ is the coefficient of u^{i-1} in $F(u)$, for $1 \leq i \leq k$. Here is the main result of this section.

Theorem 3. *The formal power series $F(t,u)$ defined by* (16) *is algebraic over $\mathbb{K}(t,u)$.*

The proof requires the following result [35, Prop. X.8].

Theorem 4. *Let $\mathbb{K} \subset \mathbb{L}$ be a field extension. For $1 \leq i \leq n$, let $P_i(x_1, \ldots, x_n)$ be a polynomial in n indeterminates $x_1, \ldots, x_n$, with coefficients in the (small) field $\mathbb{K}$. Assume $F_1, \ldots, F_n$ are n elements of the (big) field $\mathbb{L}$ that satisfy $P_i(F_1, \ldots, F_n) = 0$ for all $i \leq n$. Let J be the Jacobian matrix*

$$J = \left(\frac{\partial P_i}{\partial x_j}(F_1, \ldots, F_n)\right)_{1 \leq i,j \leq n}.$$

If $\det(J) \neq 0$, then each F_j is algebraic over $\mathbb{K}$.

Proof of Theorem 3. The idea is of course to apply the general strategy of Section 2. However, in order to avoid multiplicities in the roots U_i, we first introduce a small perturbation of (16). Let ϵ be a new indeterminate, and consider the equation

$$G(u) \equiv G(z,u,\epsilon) = F_0(u) + \epsilon^k z \Delta^{(k)} G(u) + z^2 Q\Big(G(u), \Delta G(u), \Delta^{(2)} G(u), \ldots, \Delta^{(k)} G(u), z^2, u\Big) \tag{17}$$

where F_0 and Q are the same polynomials as above. Again, this equation admits a unique solution in the ring of formal power series in z with coefficients in $\mathbb{K}[u,\epsilon]$. Moreover, $G(z,u,0) = F(z^2,u)$, so that it suffices to prove that $G(z,u,\epsilon)$ is algebraic over $\mathbb{K}(z,u,\epsilon)$.

We now apply to (17) our general strategy. Our first task will be to convert (17) into a polynomial equation of the form (6). Let $x_0, x_1, \ldots, x_k$ and v be some indeterminates. For $0 \leq i \leq k$, let

$$Y_i = \frac{x_0 - x_1 - v x_2 - \cdots - v^{i-1} x_i}{v^i}$$

and let

$$R(x_0, x_1, \ldots, x_k, z, v) = x_0 - F_0(v) - \epsilon^k z Y_k - z^2 Q\Big(Y_0, Y_1, \ldots, Y_k, z^2, v\Big). \tag{18}$$

Then

$$R\Big(G(u), G_1, \ldots, G_k, z, u\Big) = 0,$$

with $G_i = G^{(i-1)}(0)/(i-1)!$. Moreover, R is a polynomial in z and the x_i, but a rational function in v. So let m be the smallest integer such that

$$P(x_0, x_1, \ldots, x_k, z, v) := v^m R(x_0, x_1, \ldots, x_k, z, v) \tag{19}$$

is a polynomial in z, v and the x_i (with coefficients in $\mathbb{K}(\epsilon)$). Then $m \geq k$ (because of the term $\epsilon^k z Y_k$ occurring in R) and Eq. (17) now reads

$$P\Big(G(u), G_1, \ldots, G_k, z, u\Big) = 0. \tag{20}$$

Let us apply to (20) the general strategy of Section 2. We need to find sufficiently many fractional power series U in z, with coefficients in some algebraic closure of $\mathbb{K}(\epsilon)$, satisfying

$$\frac{\partial P}{\partial x_0}\Big(G(U), G_1, \ldots, G_k, z, U\Big) = 0.$$

Let us focus on the non-zero solutions U. The above condition is then equivalent to

$$U^k = \epsilon^k z + z^2 \sum_{i=0}^{k} U^{k-i} \frac{\partial Q}{\partial y_i}\Big(F(U), \dots, \Delta^{(k)}F(U), z^2, U\Big).$$

By Theorem 2, this equation has exactly k solutions $U_1, \dots, U_k$, which are fractional power series in z with coefficients in an algebraic closure of $\mathbb{K}(\epsilon)$. More precisely, the Newton-Puiseux algorithm shows that these series can be written as

$$U_i = \epsilon\, \xi^i s\, (1 + V(\xi^i s)) \tag{21}$$

where $s = z^{1/k}$, ξ is a primitive kth root of unity and $V(s)$ is a formal power series in s with coefficients in $\mathbb{K}(\epsilon)$, having constant term 0. In particular, the k series U_i are distinct.

The following system of $3k$ polynomial equations thus holds:

$$\forall i \in [1,k], \qquad \left\{ \begin{array}{rcl} P\Big(G(U_i), G_1, \dots, G_k, z, U_i\Big) & = & 0, \\ P_0'\Big(G(U_i), G_1, \dots, G_k, z, U_i\Big) & = & 0, \\ P_v'\Big(G(U_i), G_1, \dots, G_k, z, U_i\Big) & = & 0, \end{array} \right.$$

where P_0' and P_v' respectively denote the derivatives of P with respect to x_0 and v. The above system relates $3k$ unknowns, namely the U_i, the $G(U_i)$, and the series $G_1, \dots, G_k$, and has coefficients in $\mathbb{K}(\epsilon, z)$. Let us now apply Theorem 4. The Jacobian matrix is represented below for $k = 3$. The rows are indexed by the $3k$ equations, and the columns by the $3k$ unknowns, taken in the following order: $G(U_1), U_1, \dots, G(U_k), U_k$ and finally $G_1, \dots, G_k$. We denote any series of the form $S\,(G(U_i), G_1, \dots, G_k, z, U_i)$ by $S(U_i)$ for short. The notation P_i' means that the derivative of P is taken with respect to the variable x_i.

$$\left(\begin{array}{cc|cc|cc|ccc} P_0'(U_1) & P_v'(U_1) & 0 & 0 & 0 & 0 & P_1'(U_1) & \cdots & P_k'(U_1) \\ P_{0,0}''(U_1) & P_{0,v}''(U_1) & 0 & 0 & 0 & 0 & \star & \cdots & \star \\ P_{0,v}''(U_1) & P_{v,v}''(U_1) & 0 & 0 & 0 & 0 & \star & \cdots & \star \\ \hline 0 & 0 & P_0'(U_2) & P_v'(U_2) & 0 & 0 & P_1'(U_2) & \cdots & P_k'(U_2) \\ 0 & 0 & P_{0,0}''(U_2) & P_{0,v}''(U_2) & 0 & 0 & \star & \cdots & \star \\ 0 & 0 & P_{0,v}''(U_2) & P_{v,v}''(U_2) & 0 & 0 & \star & \cdots & \star \\ \hline 0 & 0 & 0 & 0 & P_0'(U_k) & P_v'(U_k) & P_1'(U_k) & \cdots & P_k'(U_k) \\ 0 & 0 & 0 & 0 & P_{0,0}''(U_k) & P_{0,v}''(U_k) & \star & \cdots & \star \\ 0 & 0 & 0 & 0 & P_{0,v}''(U_k) & P_{v,v}''(U_k) & \star & \cdots & \star \end{array} \right).$$

Recall that

$$P_0'(U_j) = P_v'(U_j) = 0 \tag{22}$$

for all j, so that the top line in each 3×2 rectangle is actually zero. Consequently, the determinant factors into k blocks of size 2 and one block of size k:

$$\det(J) = \pm \prod_{j=1}^{k} \left(P_{0,0}''(U_j) P_{v,v}''(U_j) - P_{0,v}''(U_j)^2\right) \det\left(P_i'(U_j)\right)_{1 \le i,j \le k}. \tag{23}$$

Our aim is to prove that this Jacobian is not zero.

1. Assume

$$P_{0,0}''(U_j) P_{v,v}''(U_j) - P_{0,v}''(U_j)^2 = 0. \tag{24}$$

Let us differentiate twice the functional equation (20) with respect to u. We first obtain

$$G'(u) P_0'(G(u), \dots, u) + P_v'(G(u), \dots, u) = 0$$

and then

$$G''(u)P_0'(u) + G'(u)^2 P_{0,0}''(u) + 2G'(u)P_{0,v}''(u) + P_{v,v}''(u) = 0,$$

where, as above, the notation $S(u)$ actually stands for $S(G(u), G_1, \ldots, G_k, t, u)$. For $u = U_j$, in view of (22), the latter equation becomes

$$G'(U_j)^2 P_{0,0}''(U_j) + 2G'(U_j)P_{0,v}''(U_j) + P_{v,v}''(U_j) = 0.$$

The assumption (24) implies that the quadratic equation in x

$$x^2 P_{0,0}''(U_j) + 2xP_{0,v}''(U_j) + P_{v,v}''(U_j) = 0$$

has a double root. The previous equation shows that this root is $G'(U_j)$, so that

$$G'(U_j)P_{0,0}''(U_j) + P_{0,v}''(U_j) = 0.$$

Given that $P_0'(U_j) = 0$, this is equivalent to saying that the series

$$P_0'(G(u), G_1, \ldots, G_k, t, u)$$

admits $u = U_j$ as a *multiple* root, whereas we have seen that the k non-zero roots of this equation are distinct. We have thus obtained a contradiction, and so (24) cannot hold.

2. Let us now focus on the second part of the expression (23) of the Jacobian. From (19) and (18), we derive that for $j \geq 1$, and indeterminates $x_0, x_1, \ldots, x_k, z$ and v:

$$P_j'(x_0, \ldots, x_k, z, v) = v^m R_j'(x_0, \ldots, x_k, z, v) = -v^m \left(\epsilon^k z \frac{\partial Y_k}{\partial x_j} + z^2 \sum_{\ell=j}^{k} \frac{\partial Y_\ell}{\partial x_j} Q_\ell'(Y_0, \ldots, Y_k, z^2, v) \right)$$

where Q_ℓ' denotes the derivative of $Q(y_0, \ldots, y_k, t, v)$ with respect to y_ℓ. Given that

$$\frac{\partial Y_\ell}{\partial x_j} = -v^{j-\ell-1},$$

the above derivative can be rewritten

$$P_j'(x_0, \ldots, x_k, z, v) = v^{m-k} \left(\epsilon^k z v^{j-1} + z^2 \sum_{\ell=j}^{k} v^{k-\ell+j-1} Q_\ell'(Y_0, \ldots, Y_k, z^2, v) \right). \tag{25}$$

Let us specialize this to $P_j'(U_i) \equiv P_j'(G(U_i), G_1, \ldots, G_k, z, U_i)$. By (21), this is a formal power series in $s = z^{1/k}$, with coefficients in $\mathbb{K}(\epsilon, \xi)$. Moreover, $z = s^k = o(U_i^{j-1})$ for $1 \leq j \leq k$, so that, in view of (25), the first term in the expansion of $P_j'(U_i)$ in s is

$$(\xi^i \epsilon s)^{m+j-1}.$$

(Recall that $\xi^{ik} = 1$.) The last factor in the determinant (23) of the Jacobian matrix J reads

$$\det\left(P_i'(U_j)\right)_{1 \leq i,j \leq k} = \det\left((\xi^i \epsilon s)^{m+j-1}\right)_{1 \leq i,j \leq k} + \text{higher powers of } s.$$

But

$$\det\left((\xi^i \epsilon s)^{m+j-1}\right)_{i,j} = \prod_{j=1}^{k} (\epsilon s)^{m+j-1} \prod_{i=1}^{k} (\xi^i)^m \det\left((\xi^i)^{j-1}\right)_{i,j}.$$

The last term is the VanderMonde of the ξ^i. It equals

$$\pm \prod_{1 \leq i < j \leq k} \left(\xi^i - \xi^j\right)$$

and it is not zero, since ξ is a kth primitive root of unity.

We have at last proved that the determinant of the Jacobian matrix associated with our system of $3k$ polynomial equations is not zero. By Theorem 4, the series G_i are algebraic over $\mathbb{K}(z, \epsilon)$. Recall that G_i is, up to a multiplicative constant, the derivative $G^{(i-1)}(0)$ of

$G(u)$. In view of (17), the series $G(z,u,\epsilon)$ is algebraic over $\mathbb{K}(z,u,\epsilon)$. By specializing ϵ to 0, we conclude that $F(t,u) = G(\sqrt{t},u,0)$ is algebraic over $\mathbb{K}(t,u)$.

■

5. **Algebraicity results for planar maps**

A *planar map* is a 2-cell decomposition of the oriented sphere into vertices (0-cells), edges (1-cells), and faces (2-cells). Loops and multiple edges are allowed (Figure 1(a)). The degree of a vertex (or a face) is the number of incidences of edges to this vertex (or face). Two maps are isomorphic if there exists an orientation preserving homeomorphism of the sphere that sends cells of one of the maps onto cells of the same type of the other map and preserves incidences. We shall consider maps up to isomorphisms.

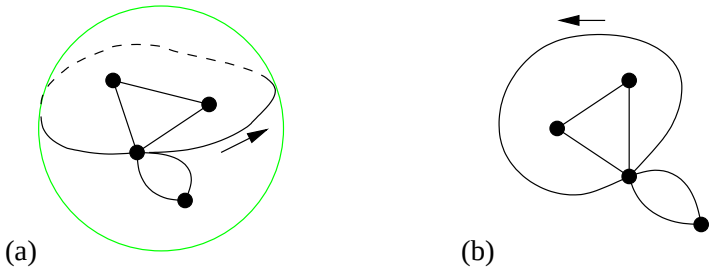


FIGURE 1. (a) A rooted planar map on the sphere – (b) Canonical representation on the plane.

A map is *rooted* if one of its edges, called the root edge, is distinguished and oriented. In this case, the map can be drawn in a canonical way in the plane, by deciding that the infinite face lies to the right of the root-edge. This face is sometimes called the root-face. Its degree is called the outer-degree. The starting point of the root-edge is the root-vertex. A *corner* of a face F is a 3-tuple (e_1, v, e_2), where e_1 and e_2 are edges, v is a vertex, and e_1, v and e_2 are met consecutively when walking around the face F in counterclockwise order. The number of corners of F is thus its degree. In the map of Figure 1, the root-face has three corners. In what follows, we consider only rooted maps, and the word "rooted" is often omitted.

A map M is separable if it contains a vertex whose deletion disconnects M. For instance, the map of Figure 1 is separable, since deleting the root-vertex disconnects it.

The dual map M^* of a map M describes the incidence relation between the faces of M (Figure 2). To construct M^*, create a vertex in every face of M: this gives the vertices of M^*. The edges of M^* are in bijection with the edges of M: for each edge e of M, incident to the faces f_1 and f_2, create an edge of M^* that crosses e and joins the vertices of M^* corresponding to f_1 and f_2. The root-edge of M^* is chosen canonically.

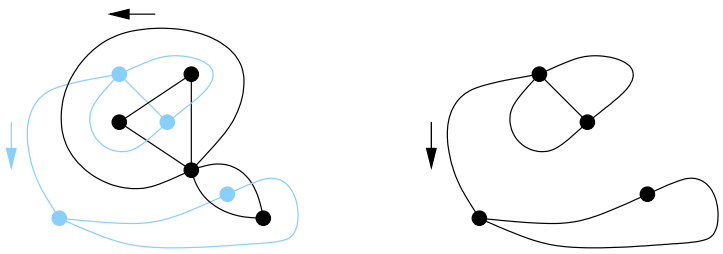


FIGURE 2. Construction of the dual map.

5.1. **The face-distribution of planar maps**

Many functional equations for planar maps are based on the deletion of the root-edge. Here, we write an equation for the series $F(t,u;z_1,\dots,z_m,\dots) = F(t,u;\boldsymbol{z})$ that counts rooted planar maps by the number of edges (variable t), the outer-degree (variable u) and the number of *finite* faces of degree i (variable z_i) for all $i \geq 1$. This equation essentially appears in an old paper of Tutte [44, Eq. (1)].

Lemma 5. *The generating function $F(t,u;\boldsymbol{z}) \equiv F(u)$ satisfies*

$$F(u) = 1 + tu^2F(u)^2 + t\sum_{i\geq 1} z_i \,\frac{F(u) - \sum_{j=0}^{i-2} u^j F_j}{u^{i-2}},$$

where F_j is the coefficient of u^j in $F(u)$.

Proof. Take a planar map M. If it is not reduced to a single vertex, delete the root-edge (but not its endpoints). Then

- either two connected components are left, which we can root in a canonical way (Figure 3). The generating function of such maps is $tu^2F(u)^2$,
- or only one connected component is left, which we can root in a canonical way. Let j be its outer-degree, and let i be the degree of the finite face that has been deleted with the root-edge of M. Then $i \in [1, j+1]$. The generating function of maps of this second type is

$$t\sum_{j\geq 0}\left(F_j \sum_{i=1}^{j+1} z_i u^{j-i+2}\right).$$

Adding the two contributions gives a functional equation for $F(u)$ which

- specializes to (15) when $z_i = 1$ for all i,
- gives the equation of Lemma 5 upon exchanging the order of the summations on i and j.

■

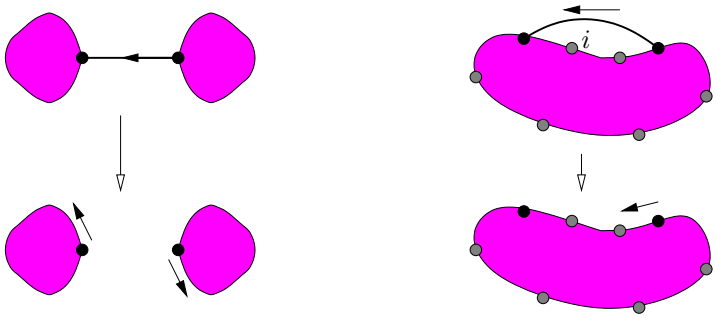


FIGURE 3. The decomposition of planar maps.

One may think that there is in $F(t,u;\boldsymbol{z})$ an unpleasant lack of symmetry: why should one count only the *finite* faces of a given degree? Let $G(t;z_1,\dots,z_m,\dots) = G(t;\boldsymbol{z})$ count rooted planar maps by the number of edges (variable t) and the number of faces (finite or not) of degree i (variable z_i). Observe that, by duality, $G(t;\boldsymbol{z})$ also counts planar maps by the number of edges and the number of *vertices* of degree i. We call G the *face-distribution generating function* of planar maps (equivalently, the vertex-distribution generating function of planar maps).

Lemma 6. *The face-distribution generating function of planar maps, $G(t;\boldsymbol{z})$, is related to the series $F(t,u;\boldsymbol{z})$ of Lemma 5 by*

$$G(t;\boldsymbol{z}) = \frac{1}{t}[u^2]F(t,u;\boldsymbol{z}).$$

Proof. Take a map M with outer-degree 2. The root-face is incident to two edges: delete the non-root one to obtain a planar map M'. This transformation is bijective and the degree distribution of *finite* faces in M coincides with the degree distribution of all faces in M'. ∎

The equation of Lemma 5 was solved in [5] in the case where $z_i = 1$ if $i \in D$ and $z_i = 0$ otherwise, for a given set D. More recently, the vertex-distribution generating function of planar maps was characterized in [14] via two methods: first, by a matrix integral calculation, and then using a purely bijective approach. In Section 10, we provide an alternative solution to this problem, and prove that it is equivalent to [14]. For the moment, observe that the generic algebraicity theorem of Section 4 (Theorem 3) implies the following:

Corollary 7. *Let $m \geq 1$, and let $F(t,u;z_1,\dots,z_m)$ be the generating function of rooted planar maps in which no* finite *face has a degree larger than m (as above, t counts edges, u the outer-degree, and z_i the number of finite faces of degree i). Similarly, let $G(t;z_1,\dots,z_m)$ be the face-distribution generating function of rooted planar maps in which no face has a degree larger than m. Then both series are algebraic.*

Proof. These series F and G are obtained by setting $z_i = 0$ for all $i > m$ in the series F and G of Lemmas 5 and 6. The equation of Lemma 5 has then the generic form (16). By Theorem 3, its solution $F(t,u;z_1,\dots,z_m)$ is algebraic over $\mathbb{Q}(t,u,z_1,\dots,z_m)$. Since the extraction of coefficients preserves algebraicity, Lemma 6 implies that $G(t;z_1,\dots,z_m)$ is algebraic too. ∎

5.2. **The face-distribution of Eulerian planar maps**

The question we address here is similar to that of Section 5.1, but is made harder by the fact that we now deal with *Eulerian* maps, that is, with maps in which all vertices have an even degree. The faces of an Eulerian map can be uniquely coloured in black and white in such a way

- the infinite face is white,
- every black face is only adjacent to white faces, and vice-versa.

Let $F(t,u;x_1,x_2,\dots;y_1,y_2,\dots) = F(t,u;\boldsymbol{x},\boldsymbol{y})$ be the generating function of these maps, where t counts edges, u the outer-degree, x_i the number of (finite) white faces of degree i, and y_i the number of black faces of degree i (all black faces are finite).

If we set $y_i = 0$ for $i \neq 2$, the series $F(t,u;\boldsymbol{x},\boldsymbol{y})$ only count those Eulerian maps in which every black face has degree 2. Contracting every black face into a single edge gives a planar map whose face-distribution coincides with the white face-distribution of the original Eulerian map. Consequently, $F(t,u;z_1,z_2,\dots;0,1,0,\dots)$ is the series studied in Lemma 5, and the problem addressed here generalizes the previous one.

In order to obtain a functional equation for $F(t,u;\boldsymbol{x},\boldsymbol{y})$, we will delete all the edges of the black face incident to the root-edge. We call this face the *black root-face*. A face is called a *polygon* if the number of vertices it contains coincides with its degree.

Definition 8. *An Eulerian map M is a* skeleton *if the following conditions hold:*

(*i*) *each of the connected components that remain after deleting the edges of the black root-face R is either a single vertex or a polygon,*
(*ii*) *every edge that is incident to the white root-face is also incident to the black root-face.*

A connected component of $M \setminus R$ is called an internal component of M *if none of its vertices belong to the infinite face. Otherwise, it is said to be an* external component of M.

The fourth map of Figure 4 is a skeleton. Among its non-root black faces, two are external, and two are internal. The following observation will be useful to prove that the face-distribution generating function of Eulerian maps with faces of bounded degree is algebraic.

Lemma 9. *Let $m \geq 1$. There exists only a finite number of skeletons in which the black root-face and all the finite white faces have degree at most m.*

Proof. Let us first bound the number of white faces. Condition (i) implies that each white face of a skeleton shares at least one edge with the black root-face. Conversely, each edge of the black root-face belongs to exactly one white face. Since there are, by assumption, at most m such edges, the number of white faces is at most m. By assumption, the finite white faces have degree at most m. Condition (ii) implies that this is also true for the infinite white face. Consequently, the total number of edges that are incident to a white face — that is, the total number of edges — is at most m^2. Since there only exists a finite number of maps having a given number of edges, the result follows.

■

Proposition 10. *Let $\mathcal{S}$ denote the set of skeletons. The generating function $F(t, u; \boldsymbol{x}; \boldsymbol{y}) \equiv F(u)$ counting Eulerian maps according to the above-defined parameters satisfies*

$$F(u) = 1 + \sum_{S \in \mathcal{S}} \left(u^{d(S)} t^{i(S)} y_{i(S)} \prod_{k \geq 1} x_k^{w_k(S)} \prod_{k \geq 1} F_k^{I_k(S)} \prod_{k \geq 0} \left(\Delta^{(k)} F(u) \right)^{E_k(S)} \right),$$

where for any skeleton S, $d(S)$ is the outer-degree, $i(S)$ is the degree of the black root-face, $w_k(S)$ is the number of finite white faces of degree k, and $I_k(S)$ (resp. $E_k(S)$) is the number of internal (resp. external) components of degree k. As above, F_j denotes the coefficient of u^j in $F(u)$, and for $k \geq 0$,

$$\Delta^{(k)} F(u) = \frac{F(u) - \sum_{j=0}^{k-1} u^j F_j}{u^k}.$$

Proof. Take an Eulerian map M, not reduced to a single vertex. We first describe how to associate a skeleton to M. This construction is illustrated in Figure 4. Let R denote the black root-face of M. Consider the set of connected components that are left after the deletion of the edges of R (since we do not delete the vertices of R, some of these components may be reduced to a single vertex). The corresponding sub-maps of M are called, for short, the *components* of M. Each component is itself an Eulerian map. In order to obtain a skeleton, we are going to modify the components of M, while keeping the black root-face R unchanged. In each component, delete every edge that is not in the infinite face of $M \setminus R$: in the resulting map M_1, every component has only black (finite) faces (Figure 4(b)). Then "inflate" each component into a black polygon having the same outer-degree (Figure 4(c)). This gives an Eulerian map M_2. Finally, contract every edge of M_2 that is incident to the white root-face but not to the black root-face. This gives a skeleton S (Figure 4(d)). The finite white faces of S are in one-to-one correspondence with the finite white faces of M that are adjacent to R, and this correspondence preserves the degree.

Conversely, take a skeleton S with black root-face of degree i. We wish to find the generating function of Eulerian maps M associated with S. To obtain these maps, one must:

– Replace every internal component of degree k by an Eulerian map of outer-degree k; this gives the factors F_k in the functional equation of Proposition 10.

– Replace every external component of degree k by an Eulerian map of outer-degree $j \geq k$. Then $j-k$ edges of this map contribute to the outer-degree of the final map M. This gives the factors $\Delta^{(k)}F(u)$ in the equation.

The remaining factors take care of R and its edges, and of the contribution of the white faces of S. The result follows.

■

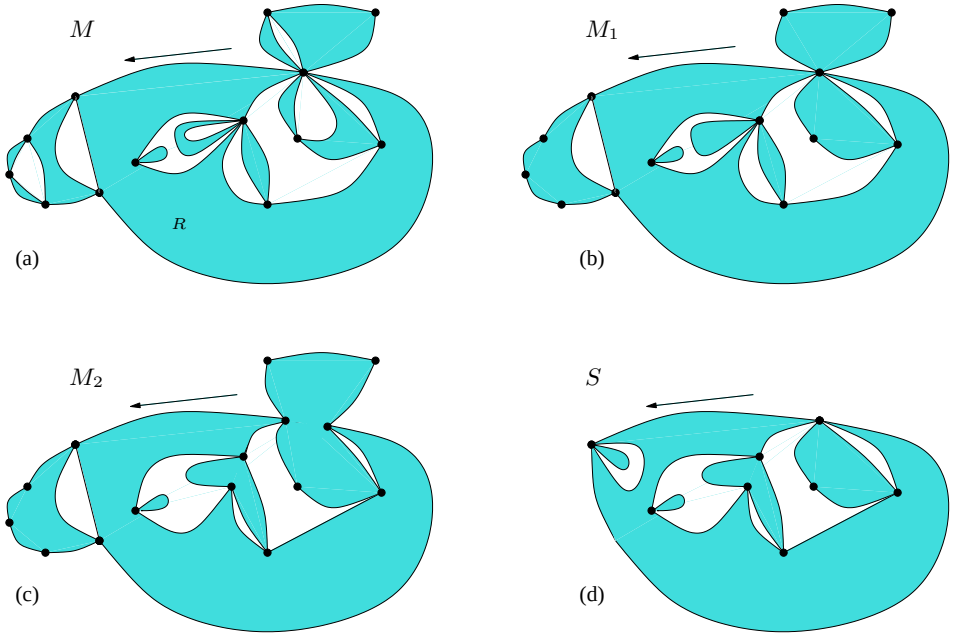


FIGURE 4. From an Eulerian map M to a skeleton S. In step (b), all white faces that are not adjacent to the black root-face R disappear. In step (c), all connected component that are left after deleting the black root-face are inflated to polygons. Finally, in step (d), all edges that are incident to the white root-face but not to the black root-face are contracted.

Corollary 11. *Let $m \geq 2$. Let $F(t,u;x_1,\dots,x_m;y_1,\dots,y_m)$ be the generating function of Eulerian planar maps whose* finite *faces have degree at most m, counted, as above, by the number of edges, the outer-degree, and the degree-distribution of black and white* finite *faces.*

Similarly, let $G(t,u;x_1,\dots,x_m;y_1,\dots,y_m)$ be the generating function of Eulerian planar maps in which all faces have degree at most m, counted by the number of edges, the outer-degree, and the degree-distribution of white and black faces.

Then F and G are algebraic.

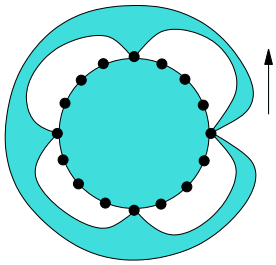
Proof. The series F is obtained by setting $x_i = y_i = 0$ for all $i > m$ in the series of Proposition 10. In the equation given in this proposition, it is clear that the skeletons in which either the black root-face, or one of the finite white faces, has degree more than m, have a zero contribution. By Lemma 9, the right-hand side of the functional equation contains only finitely many terms, so that one can apply Theorem 3, and conclude that $F(t,u;x_1,\dots,x_m;y_1,\dots,y_m)$ is algebraic.

In particular, the coefficient of u^i in this series is algebraic. Given that

$$G(t,u;x_1,\dots,x_m;y_1,\dots,y_m) = \sum_{i=0}^{m} x_i u^i\, [u^i] F(t,u;x_1,\dots,x_m;y_1,\dots,y_m),$$

the algebraicity of G follows.

■



Note. It was already proved in [13] that F_2, the coefficient of u^2 in the series $F(t,u;\boldsymbol{x},\boldsymbol{y})$, is algebraic. The above corollary thus extends this earlier result, and actually seems difficult to obtain via the combinatorial approach of [13]. However, as far as F_2 is concerned, the result of [13] is more precise than a simple algebraicity statement, since a system of $2m+3$ polynomial equations defining F_2 is given explicitly, together with its combinatorial interpretation. Let us compare the size of this system with the number of unknown series in our functional equation. The skeleton of an Eulerian map in which all finite faces have degree at most m may contain a component of degree $(m-1)^2$ (see the figure for an example with $m=5$), but no more, so that the functional equation contains approximately m^2 unknown functions. Consequently, the size of the polynomial system given by our general strategy is *quadratic* in m.

Example. Let us illustrate Proposition 10 by writing a functional equation for the generating function of Eulerian maps in which all finite faces have degree 2 or 3. The corresponding skeletons are shown in Figure 5. Proposition 10 gives the contribution of each skeleton in the functional equation (for the sake of simplicity, the variable t is omitted: it is easily recovered upon replacing y_i by $t^i y_i$).

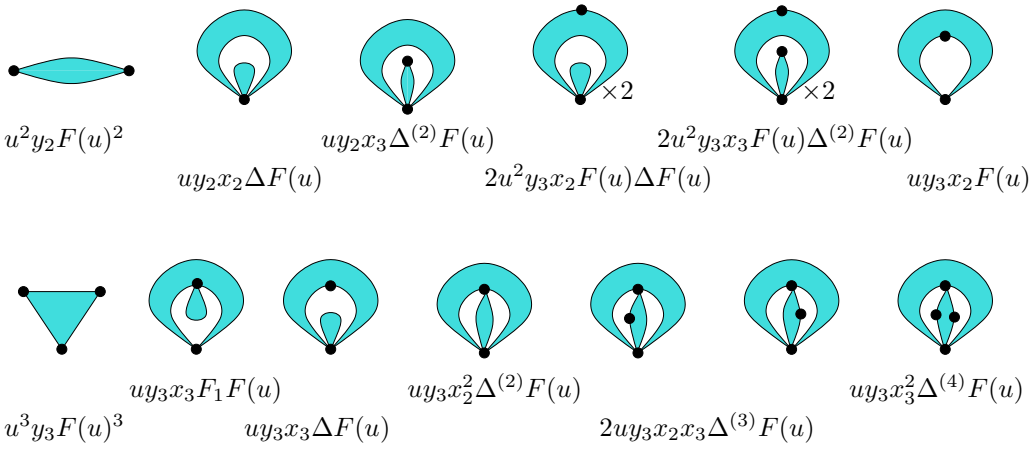


FIGURE 5. The skeletons that are involved in the enumeration of Eulerian maps with finite faces of degree 2 and 3. The multiplicities account for the number of possible rootings.

The functional equation reads:

$$F(u) = 1+u^2y_2F(u)^2+u^3y_3F(u)^3+uy_3(x_2+x_3F_1)F(u)+u\left(x_2y_2+2ux_2y_3F(u)+x_3y_3\right)\Delta F(u)$$
$$+u\left(x_3y_2+2ux_3y_3F(u)+x_2^2y_3\right)\Delta^{(2)}F(u)+2ux_2x_3y_3\Delta^{(3)}F(u)+ux_3^2y_3\Delta^{(4)}F(u).$$

We may check the validity of this equation as follows. Replacing y_i by t^iy_i, we derive from this equation the first terms of the expansion in t of $F(u)$. Retaining only the coefficient of u^2, we obtain the expansion of the series F_2 that counts maps of outer-degree 2, and we check that this expansion is (fortunately!) in adequation with the algebraic equations of [13].

5.3. **Constellations**

We focus in this section on the enumeration of certain Eulerian planar maps defined by constraints on their face degrees. Let $m \geq 2$. An Eulerian planar map M, having its faces bicolored in such a way the infinite face is white, is an *m-constellation* if

– the degree of every black face is m,
– the degree of every white face is a multiple of m.

An example of a 3-constellation is given in Figure 6. As explained in [12], these maps are closely connected to *minimal transitive factorizations* of permutations.

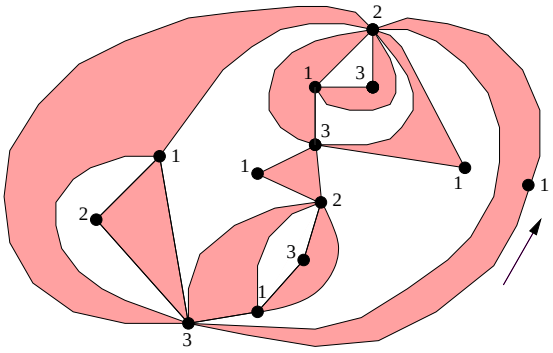


FIGURE 6. A 3-constellation with its canonical labelling of root 3.

The above conditions guarantee that it is possible to label all vertices, with labels taken from the set $\{1, 2, \ldots, m\}$, in such a way that in every black face, the vertices are labelled $1, 2, \ldots, m$ in counterclockwise order. Moreover, if we fix the label of the root-vertex to be i, then there is a unique labeling satisfying the above property, which we call the *canonical labeling of root i*.

Let

$$F(t,u) \equiv F(u) = \sum_{n,d} a_{n,d} t^n u^d = \sum_d F_d u^d, \tag{26}$$

where $a_{n,d}$ is the number of m-constellations having n black faces and outer-degree md. This series is a specialization of the face-distribution generating function of Eulerian planar maps studied in Section 5.2. More precisely, if, in the series $F(t,u;\boldsymbol{x},\boldsymbol{y})$ of Proposition 10, we set

$$\begin{cases} x_i &= 1 \quad \text{if } m \text{ divides } i, \\ x_i &= 0 \quad \text{otherwise,} \end{cases} \qquad \text{and} \qquad \begin{cases} y_m &= 1, \\ y_i &= 0 \quad \text{if } i \neq m, \end{cases}$$

we obtain the series $F(t^m, u^m)$, with $F(t,u)$ defined by (26). However, the functional equation of Proposition 10, specialized to the above values of x_i and y_i, contains infinitely many terms. We give in Proposition 12 an equation with *finitely* many terms defining $F(t,u)$. Before we do so, let us examine cases $m = 2$ and $m = 3$.

2-Constellations. Take a 2-constellation not reduced to a single vertex, label the root-vertex with 2 and the other vertices canonically. Each black face has degree 2 and contains a vertex labelled 1 and a vertex labelled 2. Contract each black face to a single edge: this gives a *bipartite* map, that is, a map in which every face has an even degree. The series $F(t,u)$ thus counts bipartite maps by the number of edges (t) and half the outer-degree (in other words, the number of corners labelled 1 in the infinite face). Deleting the root-edge as we did in Section 5.1 for general maps now gives

$$\begin{aligned} F(u) &= 1 + tuF(u)^2 + t \sum_{d \geq 0} F_d \left(u^d + \cdots + u\right) \\ &= 1 + tuF(u)^2 + tu\, \frac{F(u) - F(1)}{u - 1}. \end{aligned} \tag{27}$$

Observe that the deletion of the root-edge in a bipartite map corresponds to the deletion of the black root-face in the associated 2-constellation. The study of 2-constellations will be

useful in Section 5.4, where we count certain maps with bicolored vertices. However, it is a bit too simple to foresee what happens for general m-constellations. This is why we also treat below the case of 3-constellations.

3-Constellations. Take a 3-constellation C not reduced to a single vertex, label the root-vertex with 3 and the other vertices canonically. Let R denote the black root-face. Erase all the edges of R (but not its vertices). This leaves a set of connected components, which are constellations, and which we root in a canonical way (Figure 7).

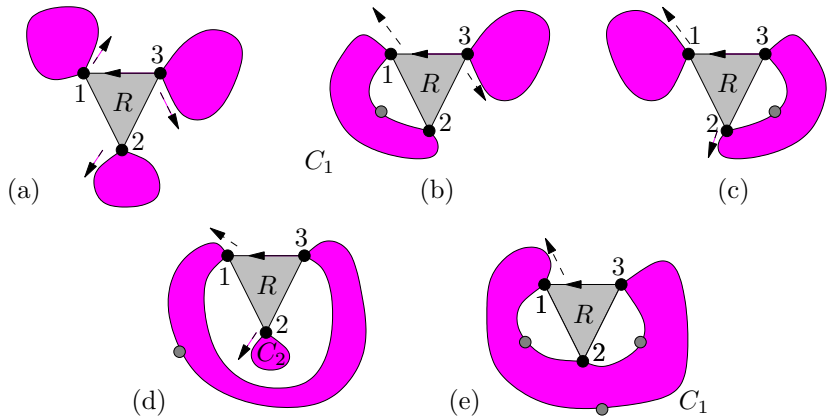


FIGURE 7. The decomposition of 3-constellations. The dashed arrows indicate how to root the components after the deletion of the black root-face.

Five cases occur, depending on which vertices of R end up in the same component. For the first case, the generating function is clearly $tuF(u)^3$. The second and third cases are symmetric and thus give the same generating function. Note that the component C_1 in Figure 7(b) must have outer-degree 3 at least, and that the number of ways to glue a (rooted) 3-constellation C_1 of outer-degree $3d$ to the face R is d. If the jth corner labelled 2 of the infinite face of C_1 is glued to R, then $1+3(j-1)$ edges of C_1 contribute to the outer-degree of C. Thus the generating function in the second case is

$$tu^{2/3}F(u)\sum_{d\geq 1}\left(F_d\sum_{j=1}^{d}u^{1/3+j-1}\right)=tuF(u)\,\frac{F(u)-F(1)}{u-1}.$$

In the fourth case, the component C_2 does not contribute to the outer-degree of C, but this case is otherwise similar to the previous one. The generating function is now

$$tuF(1)\,\frac{F(u)-F(1)}{u-1}.$$

Finally, in the fifth case, the component C_1 has degree $3d$ with $d\geq 2$. Assume the jth corner labelled 3 of the infinite face of C_1 is glued to R, as well as the kth corner labelled 2. Then $1\leq j<k\leq d$ and the generating function of this last case is

$$tu^{1/3}\sum_{d\geq 2}\left(F_d u^{2/3}\sum_{j=1}^{d-1}\left(u^{j-1}\sum_{k=j+1}^{d}1\right)\right),$$

which, after two summations, reduces to

$$tu\frac{F(u)-F(1)-(u-1)F'(1)}{(u-1)^2}.$$

Finally, the generating function of 3-constellations satisfies

$$F(u) = 1 + tuF(u)^3 + tu(2F(u) + F(1))\,\frac{F(u) - F(1)}{u-1} + tu\frac{F(u) - F(1) - (u-1)F'(1)}{(u-1)^2}. \quad (28)$$

m-Constellations. In order to write a functional equation for general m-constellations, we need the notion of *non-crossing partitions* [40]. A partition P of the set $\{1, 2, \ldots, m\}$ is *non-crossing* if one cannot find $i < j < k < \ell$ such that i and k are in the same block, j and ℓ are in the same block, but i and j are *not* in the same block. A block B of a non-crossing partition P is *internal* if there exists another block B' such that $\min B' < \min B \leq \max B < \max B'$. Otherwise, it is external. Let $\mathcal{P}_m$ denote the set of non-crossing partitions of $\{1, 2, \ldots, m\}$.

Proposition 12. *Let $m \geq 2$. The generating function $F(t, u) \equiv F(u)$ of m-constellations, defined by* (26)*, satisfies:*

$$F(u) = 1 + tu \sum_{P \in \mathcal{P}_m} \prod_{k=1}^{m-1} (G_{k-1})^{I_k(P)} \prod_{k=1}^{m} \left(\frac{F(u) - \sum_{i=0}^{k-2} (u-1)^i G_i}{(u-1)^{k-1}} \right)^{E_k(P)},$$

where

$$G_i = \frac{1}{i!} \frac{\partial^i F}{\partial u^i}(1)$$

and $I_k(P)$ (resp. $E_k(P)$) denotes the number of internal (resp. external) blocks of cardinality k in the partition P.

Note that

$$G_{k-1} = \lim_{u \to 1} \frac{F(u) - \sum_{i=0}^{k-2} (u-1)^i G_i}{(u-1)^{k-1}}.$$

The above equation defining $F(u)$ has degree m in $F(u)$ and involves $m-1$ additional unknowns series G_i, for $0 \leq i \leq m-2$.

Proof. The proof is based again on the deletion of the black root-face. We call this face the *root m-gon* and denote it by R.

1. Decomposition of constellations. Take a constellation C that is not reduced to a single vertex. Label the root-vertex by m, and all the other vertices in a canonical way. Erase all the edges of the root m-gon R (but not its vertices). This leaves a number of constellations, which we root in a canonical way (Figure 8). For each of them, the label of the root-vertex is minimal among the labels of the vertices that it shares with R.

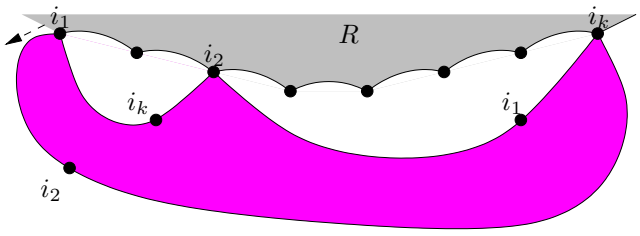


FIGURE 8. The decomposition of m-constellations. The dashed arrow indicates how to root the component after the deletion of the black root-face. One has $i_1 < i_2 < \cdots < i_k$.

Associate with C the partition P of $\{1, 2, \ldots, m\}$ defined as follows: i and j belong to the same block if and only if the vertices labeled i and j in R end up in the same connected component after deleting the edges of R. By planarity of C, the partition P is non-crossing. To each block B of P, there corresponds a constellation C_B (the associated connected component).

The outer-degree of C is

$$\mathrm{ext}(P) + \sum_{B \text{ external}} \delta(C_B, C), \tag{29}$$

where $\mathrm{ext}(P)$ is the number of external blocks of P and $\delta(C_B, C)$ is the number of edges of C_B that contribute to the outer-degree of C.

2. Construction of constellations. Conversely, let P be a non-crossing partition of $\{1, 2, \ldots, m\}$. We wish to find the generating function of the m-constellations associated with P.

Take first a root m-gon R, and label it canonically, the root-vertex being labeled m. Then, for each block B of P, take a constellation C_B of outer-degree md, for some $d \geq 0$. Label its root-vertex by $\min B$, and the other vertices in a canonical way.

For each block B, we need to glue the component C_B to the m-gon R, and to keep track of the number of edges of C_B that will contribute to the outer-degree of the final constellation C. Let $B = \{i_1, i_2, \ldots, i_k\}$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq m$. If one walks around the root-face of C_B, starting from the root-edge, the labels read at the corners of the root-face form the word $u = (i_1 \cdots i_2 \cdots i_k \cdots)^d$. From now on, we identify the corners of this face with the letters of u. For $r = 1, \ldots, k$, glue the j_rth corner labeled i_r to the (unique) vertex labeled i_r in R. To be consistent with the way we have chosen to root the components in the decomposition of a constellation, j_1 must be 1. The condition for the final map to be planar is

$$1 \leq j_k < \cdots < j_2 \leq d$$

(see Figure 8). Hence the component C_B must have outer-degree at least $m(k-1)$ and there are $\binom{d}{k-1}$ ways of gluing C_B to R.

If B is an internal block, none of its edges contribute to the outer-degree of C. Otherwise,

$$\delta(C_B, C) = \begin{cases} md & \text{if } k = 1, \\ i_k - i_1 + m(j_k - 1) & \text{if } k \geq 2. \end{cases}$$

For j_k fixed, the number of ways of choosing $j_{k-1}, \ldots, j_2$ is $\binom{d-j_k}{k-2}$.

By (29), the outer-degree of the final constellation C is thus

$$\mathrm{ext}(P) + \sum_{B \text{ external}} \big(\max B - \min B + m\,(j(B,C) - 1)\big) = m + m \sum_{B \text{ external}} \big(j(B,C) - 1\big),$$

where $j(B,C) = d+1$ if B is a singleton and C_B has outer-degree md, and $j(B,C)$ is the number j_k defined above if B has at least two elements.

Putting together the above results, one can write the generating function of m-constellations associated with the partition P as

$$tu \prod_{k=1}^{m-1} \left(\sum_{d \geq k-1} \binom{d}{k-1} F_d \right)^{I_k(P)} \left(\sum_{d \geq 0} F_d u^d \right)^{E_1(P)} \prod_{k=2}^{m} \left(\sum_{d \geq k-1} \left(F_d \sum_{j=1}^{d} u^{j-1} \binom{d-j}{k-2} \right) \right)^{E_k(P)},$$

where F_d is the coefficient of u^d in $F(u)$, that is, the generating function of constellations having outer-degree md, and $I_k(P)$ (resp. $E_k(P)$) denotes the number of internal (resp. external) blocks of cardinality k in the partition P.

Clearly, with the notation defined in the proposition,

$$\sum_{d \geq k-1} \binom{d}{k-1} F_d = G_{k-1} \quad \text{and} \quad \sum_{d \geq 0} F_d u^d = F(u).$$

Now

$$\begin{aligned}
\sum_{j=1}^{d} u^{j-1}\binom{d-j}{k-2} &= \sum_{i=0}^{d-1} u^{d-i-1}\binom{i}{k-2} \\
&= \frac{u^{d-k+1}}{(k-2)!}\,\frac{d^{k-2}}{dv^{k-2}}\left(\frac{1-v^d}{1-v}\right)\Big|_{v=1/u} \qquad \text{(use Leibnitz' formula)} \\
&= \frac{u^{d-k+1}}{(k-2)!}\left(\frac{1-v^d}{(1-v)^{k-1}}(k-2)! - \sum_{i=1}^{k-2}\binom{k-2}{i}\binom{d}{i}\frac{i!(k-2-i)!v^{d-i}}{(1-v)^{k-i-1}}\right)\Big|_{v=1/u} \\
&= \frac{u^{d-k+1}}{(k-2)!}\left((k-2)!\frac{u^d-1}{(u-1)^{k-1}}u^{k-1-d} - \sum_{i=1}^{k-2}(k-2)!\binom{d}{i}\frac{u^{k-d-1}}{(u-1)^{k-i-1}}\right) \\
&= \frac{1}{(u-1)^{k-1}}\left(u^d - 1 - \sum_{i=1}^{k-2}\binom{d}{i}(u-1)^i\right) \\
&= \frac{1}{(u-1)^{k-1}}\left(u^d - \sum_{i=0}^{k-2}\binom{d}{i}(u-1)^i\right).
\end{aligned}$$

Consequently,

$$\sum_{d\geq k-1}\left(F_d\sum_{j=1}^{d} u^{j-1}\binom{d-j}{k-2}\right) = \frac{1}{(u-1)^{k-1}}\left(F(u) - \sum_{i=0}^{k-2}\frac{(u-1)^i}{i!}F^{(i)}(1)\right),$$

and the proposition follows.

■

Note. The above functional equations for constellations were obtained a few years ago by the first author of this paper. They were used to conjecture that the number of m-constellations having n black faces is

$$C_m(n) = \frac{(m+1)m^{n-1}}{[(m-1)n+2][(m-1)n+1]}\binom{mn}{n}.$$

This conjecture was then proved in a bijective way [12].

5.4. **Hard particles on planar maps**

We consider here rooted planar maps in which the vertices are either vacant, or occupied by a particle, with the constraint that two adjacent vertices cannot be both occupied. In [13], it was shown that the generating function of such decorated maps (rooted at an edge with vacant endpoints) is a specialization of the vertex-distribution generating function of bipartite planar maps, and it was proved to be algebraic as soon as the degree of the vertices is bounded.

Here, we provide an independent approach for the case of unbounded degrees. We say that an edge is *frustrated* if it has an occupied endpoint (so that the other endpoint is vacant)[2]. Let $F(t,s,x,y,u) \equiv F(u)$ be the generating function of maps with hard particles rooted at a *vacant* vertex, counted by the number of edges (t), frustrated edges (s), vacant vertices (x), occupied vertices (y), and number of white corners in the infinite face (u). Let $G(t,s,x,y,u) \equiv G(u)$ be defined similarly for maps with hard particles rooted at an *occupied* vertex. As observed by Gilles Schaeffer [39], it is not hard to adapt the equation written for bipartite maps (27) so as to obtain equations for $F(u)$ and $G(u)$.

[2]The terminology is standard in magnetism models like the Ising model.

Lemma 13. *The series $F(u)$ and $G(u)$ defined above are related by*

$$\begin{aligned} F(u) &= x - y + G(u) + tu^2F(u)^2 + tu\,\frac{uF(u) - F(1)}{u-1}, \\ G(u) &= y + tsuF(u)G(u) + tsu\,\frac{G(u) - G(1)}{u-1}. \end{aligned}$$

Proof. As in Section 5.1, these equations follow from the deletion of the root-edge. From Figure 9 one derives

$$\begin{array}{rclcrclcl} F(u) & = & x & + & G(u) - y & + & tu^2F(u)^2 & + & t\sum_{j\geq 0} F_j\left(u + \cdots + u^{j+1}\right), \\ G(u) & = & y & + & & + & tsuF(u)G(u) & + & ts\sum_{j\geq 0} G_j\left(u + \cdots + u^{j}\right), \end{array}$$

where F_j (resp. G_j) is the coefficient of u^j in $F(u)$ (resp. $G(u)$). The result follows. ∎

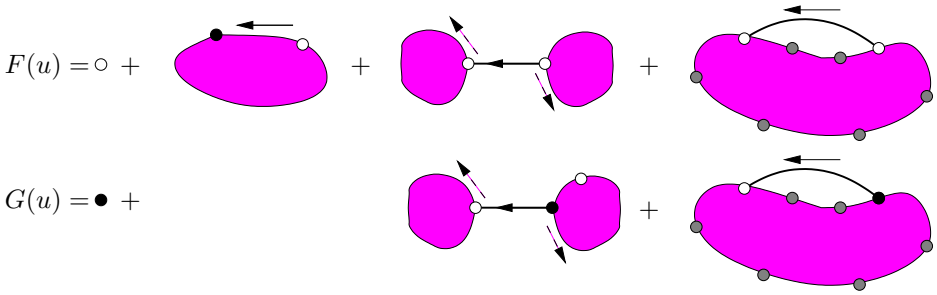


FIGURE 9. The decomposition of planar maps carrying hard particles.

Since the second equation is linear in $G(u)$, it is easy to eliminate $G(u)$. This gives a polynomial equation involving $F(u), F(1)$ and $G(1)$, and we can foresee that its solution will be algebraic. We solve this equation in Section 11 (in the case $x = y = 1$).

6. From $3k$ to $2k$ equations: the role of the discriminant

We assume again that $k+1$ power series in t, denoted $F(u), F_1, \ldots, F_k$, are related by a functional equation of the form

$$P(F(u), F_1, \ldots, F_k, t, u) = 0. \tag{30}$$

Here, $P(x_0, x_1, \ldots, x_k, t, v)$ is a polynomial with coefficients in a field $\mathbb{K}$, the F_i belong to $\mathbb{K}[[t]]$ and $F(u)$ belongs to $\mathbb{K}[u][[t]]$. As discussed in Section 2, for every fractional power series $U \equiv U(t)$ such that

$$P_0'(F(U), F_1, \ldots, F_k, t, U) = 0, \tag{31}$$

a system of three polynomial equations relating $U, F(U)$ and the unknown functions F_i holds:

$$\left\{ \begin{array}{rcl} P\Big(F(U), F_1, \ldots, F_k, t, U\Big) & = & 0, \\ P_0'\Big(F(U), F_1, \ldots, F_k, t, U\Big) & = & 0, \\ P_v'\Big(F(U), F_1, \ldots, F_k, t, U\Big) & = & 0. \end{array} \right.$$

We say that the functional equation is *generic* if there exist k distinct series U_i in $\overline{\mathbb{K}}^{\mathrm{fr}}[[t]]$ satisfying (31). In this case, the strategy of Section 2 provides a system of $3k$ polynomial equations relating the series $U_i, F(U_i)$ and F_i for $1 \leq i \leq k$ (more precisely, a system of $3k+1$ equations, since one has to take into account the fact that the U_i are distinct, thanks to an equation of the form (11)).

The aim of this section is to eliminate the series $F(U_i)$, and to reduce the system to $2k\,(+1)$ equations involving only the series U_i and F_i. The key of this reduction is the following theorem, which also considers the case of multiple roots U_i.

Theorem 14. *Assume that the functional equation* (30) *holds, and that the series* $U \in \overline{\mathbb{K}}^{\mathrm{fr}}[[t]]$ *is a root of multiplicity* ℓ *of* $P_0'(F(u), F_1, \dots, F_k, t, u)$*, with* $\ell \geq 1$*. Assume also that the degree of* $P(x_0, \dots, x_k, t, v)$ *in* x_0 *is at least* 2*, and let* $\Delta(x_1, \dots, x_k, t, v)$ *be the discriminant of* $P(x_0, \dots, x_k, t, v)$ *with respect to* x_0*. Then, as a polynomial in* v*,* $\Delta(F_1, \dots, F_k, t, v)$ *admits the series* U *as a root of multiplicity at least* 2ℓ*. In other words, for* $0 \leq i \leq 2\ell - 1$*,*

$$\frac{\partial^i \Delta}{\partial v^i}(F_1, \dots, F_k, t, U) = 0.$$

Recall that the discriminant of a polynomial $P(x) = a_n x^n + \cdots + a_0$ such that $a_n \neq 0$ can be expressed as

$$\Delta = (-1)^{n(n-1)/2} \begin{vmatrix} 1 & a_{n-1} & \cdots & a_2 & a_1 & a_0 & & & \\ & a_n & \cdots & & a_2 & a_1 & a_0 & & \\ & & \ddots & & & \ddots & & \ddots & \\ & & & a_n & & \cdots & & & a_0 \\ n & (n-1)a_{n-1} & \cdots & 2a_2 & a_1 & 0 & & & \\ & na_n & \cdots & & & a_1 & & & \\ & & \ddots & & & & \ddots & & \\ & & & & na_n & & \cdots & & a_1 \end{vmatrix}. \tag{32}$$

The above square matrix has size $2n-1$, and the coefficients that are not indicated equal 0. In the generic case, Theorem 14 provides a system of $2k$ equations:

$$\forall i \in [1, k], \qquad \begin{cases} \Delta(F_1, \dots, F_k, t, U_i) &= 0, \\ \Delta_v'(F_1, \dots, F_k, t, U_i) &= 0, \end{cases} \tag{33}$$

which we complete with the distinctness condition (11). But it may also happen that the series $P_0'(F(u), F_1, \dots, F_k, t, u)$ has a multiple root. In this case, the system derived from Theorem 14 contains more equations than unknowns. An example is provided in Section 9.2.2.

In order to simplify the proof of Theorem 14, we first reduce it to the case $U = 0$. Define

$$S(x, v) := P(x, F_1, \dots, F_k, U + v) \quad \text{and} \quad G(u) := F(u + U).$$

Then $S(x, v)$ is a polynomial in x and v with coefficients in $\mathbb{L} = \overline{\mathbb{K}}^{\mathrm{fr}}((t))$, and $G(u)$ is a series of $\overline{\mathbb{K}}[u]^{\mathrm{fr}}[[t]]$, and hence of $\mathbb{L}[[u]]$. The functional equation (30) and the assumption of Theorem 14 respectively imply

$$S(G(u), u) = 0 \qquad \text{and} \qquad \frac{\partial S}{\partial x}(G(u), u) = u^\ell \Phi(u) \tag{34}$$

with $\Phi(u) \in \overline{\mathbb{K}}[u]^{\mathrm{fr}}[[t]] \subset \mathbb{L}[[u]]$ (the second identity follows from Lemma 1). Thanks to this reduction, we will derive Theorem 14 from the following proposition.

Proposition 15. *Let* $\mathbb{L}$ *be an algebraically closed field, and let* $S(x, v)$ *be a polynomial in* x *with coefficients in* $\mathbb{L}^{\mathrm{fr}}[[v]]$*, of degree* $n \geq 2$ *in* x*. Suppose that there exist two elements* $G(u)$ *and* $\Phi(u)$ *in* $\mathbb{L}^{\mathrm{fr}}[[u]]$ *such that*

$$S(G(u), u) = 0 \qquad \textit{and} \qquad \frac{\partial S}{\partial x}(G(u), u) = u^\ell \Phi(u).$$

Then the discriminant of $S(x, v)$ *with respect to* x*, denoted* $\Delta(v)$*, is divisible by* $v^{2\ell}$ *in* $\mathbb{L}^{\mathrm{fr}}[[v]]$*.*

The first step in the proof of Proposition 15 is the following "exchange" lemma.

Lemma 16. *Under the assumptions of Proposition* 15*, suppose, moreover, that* $\frac{\partial^2 S}{\partial x^2}(G(0),0) \neq 0$*. Then there exists* $H(u)$ *and* $\Psi(u)$ *in* $\mathbb{L}^{\mathrm{fr}}[[u]]$ *such that*

$$S(H(u),u) = u^{2\ell}\Psi(u) \qquad \text{and} \qquad \frac{\partial S}{\partial x}(H(u),u) = 0.$$

Proof of Lemma 16. We look for a solution of the equation $S'_x(H(u),u) = 0$ in the form $H(u) = G(u) + u^\ell Y(u)$, with $Y(u) \in \mathbb{L}^{\mathrm{fr}}[[u]]$. Using Taylor's formula, we write

$$S(G(u)+z,u) = \sum_{i=1}^{n} \frac{z^i}{i!}\frac{\partial^i S}{\partial x^i}(G(u),u), \tag{35}$$

and

$$\frac{\partial S}{\partial x}(G(u)+z,u) = \sum_{j=0}^{n-1} \frac{z^j}{j!}\frac{\partial^{j+1} S}{\partial x^{j+1}}(G(u),u).$$

We thus want to find out whether there exists $Y \equiv Y(u)$ in $\mathbb{L}^{\mathrm{fr}}[[u]]$ satisfying

$$\sum_{j=0}^{n-1} \frac{u^{\ell j}Y^j}{j!}\frac{\partial^{j+1} S}{\partial x^{j+1}}(G(u),u) = 0,$$

that is,

$$\frac{\partial S}{\partial x}(G(u),u) + \sum_{j=1}^{n-1} \frac{u^{\ell j}Y^j}{j!}\frac{\partial^{j+1} S}{\partial x^{j+1}}(G(u),u) = 0,$$

which, after dividing by u^ℓ, reduces to

$$\Phi(u) + \sum_{j=1}^{n-1} \frac{u^{\ell(j-1)}Y^j}{j!}\frac{\partial^{j+1} S}{\partial x^{j+1}}(G(u),u) = 0.$$

This is a polynomial equation in Y with coefficients in $\mathbb{L}^{\mathrm{fr}}[[u]]$. By Theorem 2, the number of roots lying in $\mathbb{L}^{\mathrm{fr}}[[u]]$ is

$$\deg_Y\left(\Phi(0) + Y\frac{\partial^2 S}{\partial x^2}(G(0),0)\right).$$

The assumption of Lemma 16 implies that this degree is 1, so that the equation $S'_x(H(u),u) = 0$ admits a solution of the form $H(u) = G(u)+u^\ell Y$, with $Y \equiv Y(u)$ in $\mathbb{L}^{\mathrm{fr}}[[u]]$. Then, by (35),

$$\begin{aligned} S(H(u),u) &= \sum_{i=1}^{n} \frac{u^{\ell i}Y^i}{i!}\frac{\partial^i S}{\partial x^i}(G(u),u) \\ &= u^{2\ell}Y\Phi(u) + \sum_{i=2}^{n} \frac{u^{\ell i}Y^i}{i!}\frac{\partial^i S}{\partial x^i}(G(u),u), \end{aligned}$$

which is divisible by $u^{2\ell}$ in $\mathbb{L}^{\mathrm{fr}}[[u]]$.

■

Proof of Proposition 15. Let $\tilde{S}(x,v) = S(x,v) + \epsilon(x-G(v))^2 + \epsilon(x-G(v))^n$, where ϵ is a new indeterminate. Then $\tilde{S}(x,v)$ belongs to $\mathbb{M}^{\mathrm{fr}}[[v]][x]$, where $\mathbb{M}$ is the algebraic closure of $\mathbb{L}(\epsilon)$. Moreover,

$$\tilde{S}(G(u),u) = 0 \qquad \text{and} \qquad \frac{\partial \tilde{S}}{\partial x}(G(u),u) = \frac{\partial S}{\partial x}(G(u),u) = u^\ell\Phi(u).$$

Also,

$$\frac{\partial^2 \tilde{S}}{\partial x^2}(G(0),0) = \frac{\partial^2 S}{\partial x^2}(G(0),0) + 2\epsilon(1+\delta_{n,2}) \neq 0 \quad \text{and} \quad \frac{\partial^n \tilde{S}}{\partial x^n}(x,0) = \frac{\partial^n \tilde{S}}{\partial x^n}(x,0) + \epsilon(2\delta_{n,2}+n!) \neq 0.$$

The discriminant of $\tilde{S}(x,v)$ with respect to x can be written as follows [35, Ch. V, § 10]:

$$\tilde{\Delta}(v) = \pm n^n a_n(v)^{n-1} \prod_{X(v)\in\mathcal{R}} \tilde{S}(X(v),v) \tag{36}$$

where $a_n(v) \in \mathbb{M}^{\mathrm{fr}}[[v]]$ is the coefficient of x^n in $\tilde{S}(x,v)$ and $\mathcal{R} = \{X(v) \in \mathbb{M}^{\mathrm{fr}}((v)) \ : \ \tilde{S}'_x(X(v),v) = 0\}$.

The condition $\frac{\partial^n \tilde{S}}{\partial x^n}(x,0) \neq 0$, combined with Theorem 2, implies that all the elements of $\mathcal{R}$ are actually in $\mathbb{M}^{\mathrm{fr}}[[v]]$. Hence all the series $\tilde{S}(X(v),v)$, for $v \in \mathcal{R}$, lie in $\mathbb{M}^{\mathrm{fr}}[[v]]$. The condition $\frac{\partial^2 \tilde{S}}{\partial x^2}(G(0),0) \neq 0$, combined with Lemma 16, implies that one of the elements of $\mathcal{R}$, say $H(v)$, is such that $\tilde{S}(H(v),v)$ is divisible by $v^{2\ell}$. By (36), $v^{2\ell}$ divides $\tilde{\Delta}(v)$ in $\mathbb{M}^{\mathrm{fr}}[[v]]$.

Since $\tilde{\Delta}(v)$ is a polynomial in ϵ, this implies that each of its coefficients is divisible by $v^{2\ell}$. Since its constant coefficient is equal to $\Delta(v)$, we conclude that $v^{2\ell}$ divides $\Delta(v)$ in $\mathbb{L}^{\mathrm{fr}}[[v]]$. ∎

Proof of Theorem 14. Let us return to (34). By Proposition 15, *if the degree of $S(x,v)$ in x is at least 2*, the discriminant of $S(x,v)$ with respect to x, denoted here $\delta(v)$, has a root of multiplicity at least 2ℓ at $v = 0$. We want to prove that the same holds for $\Delta(F_1,\dots,F_k,t,U+v)$. How is this polynomial (in v) related to $\delta(v)$?

- If $S(x,v)$ has degree n in x, then $\Delta(F_1,\dots,F_k,t,U+v) = \delta(v)$, and the theorem follows from Proposition 15.
- If $S(x,v)$ has degree at most $n-2$, then (32) shows that $\Delta(F_1,\dots,F_k,t,v) = 0$, and the result is trivial.
- If $S(x,v)$ has degree $n-1$, then (32) gives $\Delta(F_1,\dots,F_k,t,U+v) = a_{n-1}(v)^2\delta(v)$, where $a_{n-1}(v)$ is the coefficient of x^{n-1} in $P(x,F_1,\dots,F_k,t,U+v)$.
  - If $n = 2$, then $P(x,F_1,\dots,F_k,t,U+v) = a_1(v)x + a_0(v)$, where $a_0(v)$ and $a_1(v)$ belong to $\overline{\mathbb{K}}^{\mathrm{fr}}[[t]][v]$. Then $P'_0(x,F_1,\dots,F_k,t,U+v) = a_1(v)$ and the assumption of Theorem 14 tells us that 0 is a root of $a_1(v)$ of multiplicity ℓ, and hence a root of multiplicity 2ℓ of $\Delta(F_1,\dots,F_k,t,U+v) = a_{n-1}(v)^2$.
  - If $n \geq 3$, then $S(x,v)$ has degree at least 2, and the theorem follows again from Proposition 15.

∎

7. From $2k$ to k equations: resultants and their generalization

In the previous section, we have shown how to reduce our polynomial system to $2k+1$ equations, at least in a generic situation. This system says that the polynomials (in v) $\Delta(F_1,\dots,F_k,t,v)$ and $\Delta'_v(F_1,\dots,F_k,t,v)$ have k distinct roots in common. It is well-known that two polynomials have one root in common if and only if their resultant vanishes. This is the result we generalize in this section: we give a criterion that tells when two polynomials P and Q have k roots in common. If the respective degrees of P and Q are m and n, this criterion involves k determinants of respective order $m+n$, $m+n-2$,..., $m+n-2k+2$. In a generic situation, these determinants directly provide k equations between the series $F_1,\dots,F_k$, with no mention of the series U_i. Whether these equations are as small as they can be is another story...

Let $P(X) = \sum_{i=0}^m a_i X^i$ and $Q(X) = \sum_{i=0}^n b_i X^i$, where the coefficients a_i and b_i belong to a field $\mathbb{L}$. For $0 \leq k < \min(m,n)$, we define a matrix $\mathcal{S}_k(P,Q)$ having $m+n-2k$ rows

and columns by:

$$\mathcal{S}_k(P,Q) = \begin{pmatrix} a_m & & \cdots & & a_0 & & \\ & \ddots & & & & \ddots & \\ & & a_m & & \cdots & & a_0 \\ & & & \ddots & & & \vdots \\ & & & & a_m & \cdots & a_k \\ b_n & & \cdots & & & b_0 & \\ & \ddots & & & & & \\ & & b_n & & \cdots & & b_k \end{pmatrix}$$

where the first $n-k$ rows are filled up with the coefficients a_i of P and the $m-k$ last ones by the coefficients b_i of Q. The other entries are zero. In particular, $\mathcal{S}_0(P,Q)$ is the *Sylvester matrix* of P and Q (and its determinant is the resultant of P and Q). In general, $\mathcal{S}_k(P,Q)$ is obtained by deleting the k last rows of a's, the k last rows of b's, and the $2k$ rightmost columns in the Sylvester matrix of P and Q.

The following theorem is a simple adaptation of [4, Prop. 4.33].

Theorem 17. *Let $k \leq \min(m,n)$. If the polynomials P and Q have k common roots, counted with multiplicities, then for $0 \leq i \leq k-1$,*

$$\det \mathcal{S}_i(P,Q) = 0.$$

Conversely, if the above determinants vanish, then either P and Q have k common roots, or $a_m = b_n = 0$.

8. **A new proof of Brown's theorem**

Theorem 18 below is essentially due to Brown [21]. It has been used several times in the past to solve functional equations of the form (3). Its application is straightforward for quadratic equations (see [5, 32] and the discussion at the end of this section), but more elusive when the degree in $F(u)$ is larger (see [21, Section 4] for a solution of the cubic equation of Section 3.3 based on this theorem).

In this section, we give a new proof of, and a new point of view on Brown's theorem[3]. Here is our formulation of this theorem.

Theorem 18. *Let $\Delta(t,u) \in \mathbb{K}[[t]][u]$, where $\mathbb{K}$ is a field. If Δ has a square root in $\mathbb{K}[[t,u]]$, then it can be factored as*

$$\Delta(t,u) = c^2 t^{2p}(1+tS(t))\Big(1+tuR_1(t,u)\Big)\Big(u^d + tR_2(t,u)\Big)^2 \prod_{i=1}^{k} \left(\left(1-\frac{u}{\alpha_i}\right)^{d_i} + tuQ_i(t,u) \right),$$

where

- *p, d, and the d_i's are nonnegative integers,*
- *c belongs to $\mathbb{K}$ and the α_i's belong to $\overline{\mathbb{K}}$, the algebraic closure of $\mathbb{K}$,*
- *$S(t) \in \mathbb{K}[[t]]$,*
- *$R_1(t,u)$ and $R_2(t,u)$ belong to $\mathbb{K}[[t]][u]$, with $\deg_u(R_2) < d$,*
- *$Q_i(t,u)$ belongs to $\overline{\mathbb{K}}[[t]][u]$, and $\deg_u(Q_i) < d_i$.*

Moreover, if Δ has a square root in $\mathbb{K}[u][[t]]$, then it can be factored as

$$\Delta(t,u) = c^2 t^{2p}(1+tS(t))\Big(1+tuR_1(t,u)\Big)\Big(u^d + tR_2(t,u)\Big)^2 \prod_{i=1}^{k} \left(\left(1-\frac{u}{\alpha_i}\right)^{d_i} + tuQ_i(t,u) \right)^2,$$

[3]Or maybe we should write that *we give a proof of* this theorem, since there seems to be a mistake in Brown's proof [21]: in the equation that follows (2.12), why aren't there any terms $U_{r-1}V_1, \ldots, U_0V_r$?

with the same conditions as above.

What is remarkable in the above factorizations is the fact that some factors are squared. We will derive this theorem from the combination of two results. The first one is a factorization theorem which has an independent interest and will be used in Sections 9 and 10.

Theorem 19 (Factorization Theorem). *Let $\Delta(t,u)$ be a non-zero polynomial of $\mathbb{K}[[t]][u]$, where $\mathbb{K}$ is a field. Then Δ admits a unique factorization as*

$$\Delta(t,u) = ct^p(1+tS(t))\Big(1+tuR_1(t,u)\Big)\Big(u^d+tR_2(t,u)\Big)\prod_{i=1}^{k}\left(\left(1-\frac{u}{\alpha_i}\right)^{d_i}+tuQ_i(t,u)\right),$$

with the same conditions as in Theorem 18. *The roots of $\Delta(t,u)$ that are infinite (resp. zero, finite and non-zero) at $t=0$ are the roots of the first (resp. second, third) factor above.*

Proof. Let us first recall that the units of $\mathbb{K}[[t]][u]$ and $\mathbb{K}[[t]]$ coincide, and are the series $c(1+tS(t))$, where $c \in \mathbb{K}\setminus\{0\}$ and $S(t) \in \mathbb{K}[[t]]$.

Now consider an irreducible polynomial of $\mathbb{K}[[t]][u]$, denoted $P(t,u)$, of degree d in u. By definition, P is not a unit. If $d=0$, then

$$P(t,u) = tI(t),$$

where $I(t)$ is unit of $\mathbb{K}[[t]]$. If $d>0$, then $P(0,u)\neq 0$ (otherwise P would be divisible by t). Moreover, $P(t,u)$ is also irreducible in $\mathbb{K}((t))[u]$. The roots $U_1,\dots,U_d$ of P are of the form [41, Prop. 6.1.6]

$$U_i = \sum_{n\geq n_0} a_n\left(\xi^i t^{1/d}\right)^n$$

where $n_0 \in \mathbb{Z}\cup\{+\infty\}$, $a_{n_0}\neq 0$, the coefficients a_n lie in $\overline{\mathbb{K}}$, and ξ is a primitive dth root of unity in $\overline{\mathbb{K}}$. We consider three cases, depending on whether n_0 is negative, positive or zero.

Case 1: $n_0<0$**.** Then all the roots of P are infinite at $t=0$. By Theorem 2, $\deg_u P(0,u) = 0$. Thus P can be written $P(t,u)=P(t,0)+tuR(t,u)$ with $\deg_u R(t,u) = d-1$. Since by assumption $P(0,u)\neq 0$, we have $P(0,0)\neq 0$, so that $P(t,0)$ is a unit in $\mathbb{K}[[t]]$. Denoting $P(t,0)=P_0(t)$, we have

$$P(t,u) = P_0(t)\left(1+tuR_1(t,u)\right)$$

with $R_1(t,u)\in\mathbb{K}[[t]][u]$.

Case 2: $n_0>0$**.** All the roots of P are zero at $t=0$. By Theorem 2, $\deg_u P(0,u)=d$. More precisely, $P(0,u)=cu^d$ for some $c\in\mathbb{K}\setminus\{0\}$. Denoting by $P_d(t)$ the coefficient of u^d in P, we thus have $P(t,u)=P_d(t)u^d+tR(t,u)$ where $\deg_u R(t,u)<d$, and $P_d(0)=c\neq 0$. Thus $P_d(t)$ is a unit of $\mathbb{K}[[t]]$, and we can write

$$P(t,u) = P_d(t)\Big(u^d+tR_2(t,u)\Big),$$

where $R_2(t,u)\in\mathbb{K}[[t]][u]$ and $\deg_u R_2(t,u)<d$.

Case 3: $n_0=0$**.** All the roots of P are equal to some $\alpha\neq 0$ when $t=0$, with $\alpha\in\overline{\mathbb{K}}$. As in Case 2, $\deg_u P(0,u)=d$, and more precisely

$$P(0,u) = c\left(1-\frac{u}{\alpha}\right)^d$$

where $c\in\mathbb{K}\setminus\{0\}$. In particular, $P(0,0)=c\neq 0$, so that $P(t,0)\equiv P_0(t)$ is a unit in $\mathbb{K}[[t]]$. Thus we can write $P(t,u)=P_0(t)\left(1+uR(t,u)\right)$ where $R(t,u)\in\mathbb{K}[[t]][u]$ and $\deg_u R(t,u)=d-1$. Setting $t=0$ gives $P(0,u)=P_0(0)\left(1+uR(0,u)\right)=P_0(0)\left(1-\frac{u}{\alpha}\right)^d$. Finally,

$$P(t,u) = P_0(t)\left(\left(1-\frac{u}{\alpha}\right)^d+tuQ(t,u)\right)$$

where

$$Q(t,u) = \frac{R(t,u) - R(0,u)}{t}$$

belongs to $\mathbb{K}[[t]][u]$ and has degree at most $d-1$ in u.

Now take $\Delta \in \mathbb{K}[[t]][u]$, as in the statement of the theorem. Factor Δ into irreducible polynomials of $\mathbb{K}[[t]][u]$. Write each irreducible factor in the above canonical form. Then, group together the irreducible factors whose roots are infinite (resp. zero, equal to $\alpha_i \neq 0$) at $t = 0$. This gives for Δ a factorization of the prescribed form.

The uniqueness of this factorization is a consequence of the two following facts:

– the roots of the first (resp. second, third) factor are exactly the roots of $\Delta(t,u)$ that are infinite (resp. zero, equal to $\alpha_i \neq 0$) at $t = 0$,

– these factors are normalized (they have either constant term 1, or leading coefficient 1). This concludes the proof.

■

In order to prove Brown's theorem (Theorem 18), we only need to combine the above factorization theorem with the following proposition.

Proposition 20. *Let $\Delta(t,u) \in \mathbb{K}^{\mathrm{fr}}[[t]][u]$, where $\mathbb{K}$ is a field. The roots of $\Delta(t,\cdot)$ belong to $\overline{\mathbb{K}}^{\mathrm{fr}}((t))$.*

If Δ has a square root in $\mathbb{K}[[u]]^{\mathrm{fr}}[[t]]$, then every root U of Δ that vanishes at $t = 0$ has an even multiplicity in Δ.

If Δ has a square root in $\mathbb{K}[u]^{\mathrm{fr}}[[t]]$, then every root U of Δ that is finite at $t = 0$ has an even multiplicity.

Proof. Assume $\Delta(t,u) = \delta(t,u)^2$, with $\delta \in \mathbb{K}[[u]]^{\mathrm{fr}}[[t]]$. Let $U \equiv U(t)$ be a root of Δ that vanishes at $t = 0$. Then $\delta(t,U)$ is a well-defined series in $\mathbb{K}^{\mathrm{fr}}[[t]]$, which must be 0. Thus U is a root of $\delta(t,u)$, and by Lemma 1,

$$\delta(t,u) = (u-U)\Psi(t,u) \tag{37}$$

where $\Psi(t,u) \in \overline{\mathbb{K}}[[u]]^{\mathrm{fr}}[[t]]$, so that

$$\Delta(t,u) = (u-U)^2\Psi(t,u)^2.$$

Thus U is a root of Δ of multiplicity at least 2.

More generally, let us prove by induction on $m \geq 0$ that, if U has multiplicity at least $2m+1$ in Δ, then it actually has multiplicity at least $2m+2$. The case $m = 0$ has just been proved. Now take $m \geq 1$, and assume

$$\Delta(t,u) = (u-U)^{2m+1}\Delta_1(t,u) = \delta(t,u)^2$$

with $\Delta_1(t,u) \in \overline{\mathbb{K}}^{\mathrm{fr}}[[t]][u]$. As argued above, U is a root of $\delta(t,u)$, and the factorization (37) gives

$$\tilde{\Delta}(t,u) := (u-U)^{2m-1}\Delta_1(t,u) = \Psi(t,u)^2.$$

The induction hypothesis implies that U is a root of $\tilde{\Delta}(t,u)$ of multiplicity at least $2m$, and thus a root of $\Delta(t,u)$ of multiplicity at least $2m+2$. This completes the proof of the first statement of the proposition.

The proof of second statement is very similar. It relies on the fact that if $\delta(t,u)$ lies in $\mathbb{K}[u]^{\mathrm{fr}}[[t]]$, then all roots U of Δ that are finite at $t = 0$ can be substituted for u in $\delta(t,u)$.

■

We are finally ready for a

Proof of Theorem 18. Take $\Delta(t,u) \in \mathbb{K}[[t]][u]$ and consider its canonical factorization, given by Theorem 19. Assume $\Delta(t,u) = \delta(t,u)^2$, with $\delta \in \mathbb{K}[[u,t]]$. If q is the valuation in t of δ, then the valuation in t of Δ is $p = 2q$. Thus p is even. Now

$$t^{-2q}\Delta(t,u) = \left(t^{-q}\delta(t,u)\right)^2.$$

Setting $t = 0$ in this identity shows that the constant c occurring in the canonical factorization of Δ is a square of $\mathbb{K}$.

By Proposition 20, each root of Δ that vanishes at $t = 0$ has an even multiplicity in Δ. This means that every irreducible factor of Δ occurring in the term $(u^d + tR_2(t,u))$ actually occurs an even number of times. This implies that d is even, and that this term can be factored as $(u^{d/2} + t\tilde{R}_2(t,u))^2$. This completes the proof of the first statement.

The proof of the second statement is very similar: now, each root of Δ that is finite at $t = 0$ must have an even multiplicity.

■

Let us finally discuss how Brown's theorem may be used to solve a *quadratic* equation with one catalytic variable [5, 32]. We start from a $(k+1)$-tuple of series, denoted $F(u), F_1, \dots, F_k$, such that $F(u) \in \mathbb{K}[u][[t]]$ and $F_i \in \mathbb{K}[[t]]$ for all i. We assume they satisfy

$$\Big(2aF(u) + b\Big)^2 = \Delta(u), \tag{38}$$

where a, b and Δ are polynomials in $F_1, \dots, F_k, t$ and u, with coefficients $\mathbb{K}$. Obviously, Δ has a square root in $\mathbb{K}[u][[t]]$ (namely, the series $2aF(u) + b$). Hence the second part of Theorem 18, applies: the canonical factorisation of Δ contains several squared factors.

Let us now adopt the notation of Theorem 18. In order to determine the degrees in u of R_1, R_2 and the Q_i, one has to decide how many roots of Δ are infinite (resp. equal to zero, equal to α_i) when $t = 0$. This can be done routinely using Newton's polygon method. (Curiously, these degrees are only *guessed* in [5] and [32]. This forces the authors to check afterwards the validity of their assumption.) One then introduces a new set of indeterminates (the coefficients of the polynomials R_1, R_2 and Q_i) and obtains a system of polynomial equations by comparing the coefficient of u^j in Δ and in its factorisation, for all j. This is illustrated in Sections 9.3 and 10 (even though we do not use Brown's theorem, but rather a combination of our general strategy with the factorization theorem, Theorem 19).

To conclude, let us underline one important difference between the quadratic case and the general case. As shown by (38), in the quadratic case, *every* root of Δ that is finite at $t = 0$ has an even multiplicity. For a general (i.e., non-quadratic) equation, Theorem 14 exhibits a certain number of multiple roots of Δ, which are finite at $t = 0$. But Δ may also have simple roots (or roots of odd multiplicity) that are finite at $t = 0$. In the example of Section 11 below, Δ has two simple roots that are finite at $t = 0$.

9. **Practical strategies**

The general strategy presented in Section 2 to solve functional equations of the form $P(F(u), F_1, \dots, F_k, t, u) = 0$ yields a system of polynomial equations ($3k + 1$ equations in a generic case) relating the unknown series F_i, some auxilliary series U_i, and the values of $F(U_i)$. Section 6 performs the elimination of the $F(U_i)$, yielding a system of $2k+1$ equations. Section 7 even goes further by eliminating the U_i, but the k equations it provides are often, in practise, unnecessarily big. At any rate, it is always easy to write a system of $2k + 1$ equations relating the series F_i and U_i.

In most combinatorial problems, one is interested in finding the minimal equation satisfied by F_1, or at least a "nice" system involving all the F_i, if such a system exists. As discussed in [31, Section 4], three main methods can be used to reduce further the size of our system:

the paper-and-pencil approach, the resultant approach, and the Grbner basis approach. Note that our system of $2k+1$ equations contains k times the "same" pair of equations (see (33)), which means that the elimination of the U_i must be performed with care not to loose any information.

The paper and-pencil approach has been amply illustrated in Section 3. In almost all examples presented there, there was actually a *single* unknown function F_1. In this case, as soon as one finds a series U that cancels $P_0'(F(u),\dots,t,u)$, the discriminant Δ has a double root (Theorem 14), and one obtains immediately an equation for F_1 by writing that

the discriminant of the discriminant vanishes.

We also studied in Section 3 one equation involving two unknown series F_i, but it was linear in $F(u)$ and of an especially simple form.

In this section, we gather a number of practical strategies that permit to solve bigger examples. We advise the reader who would be interested in the practical aspects of our method to read what follows with a computer algebra system at hand. All the strategies we suggest have been tested on the same example (except the Grbner one, which seems to be too brutal to work). Two more examples are provided in Sections 10 and 11.

9.1. **Brute force on $3k+1$ equations**

The laziest approach naturally consists in feeding a Grbner basis package with the $3k(+1)$ equations obtained in the generic case, and let it work. The aim is to obtain either a polynomial system defining the series F_i, or a single algebraic equation for, say, F_1. One has to choose carefully a monomial order. See [26] for generalities on Grbner bases, and [3] for a recent study of the complexity of Grbner computations.

Unfortunately, this lazy approach often fails, because the computation tends to take forever. This is why we only give here a very simple — and somewhat degenerate — example.

Return to the second example of Section 3.1, Eq. (12). Form a set S of 5 equations consisting of (14) for $i=1,2$, the right-hand side of (13) for $i=1,2$ again, and the distinctness condition $X(U_1-U_2)=1$. The Maple command
`Groebner[univpoly]`$(F_1, S, \{X, U_1, U_2, F_1, F_2\})$ directly gives

$$F_1 = 1 + 2\,t^5 {F_1}^5 - t^5 {F_1}^6 + t^5 {F_1}^7 + t^{10} {F_1}^{10}.$$

9.2. **Bare hands elimination on $2k+1$ equations**

9.2.1. ***The number of* 3*-constellations.*** Let us consider now the equation (28) that defines the generating function of 3-constellations. It has degree 3 in $F(u)$ and contains two unknown series $F_1 = F(1)$ and $F_2 = F'(1)$. Multiplying by $(u-1)^2$ gives an equation of the form $P(F(u), F_1, F_2, t, u) = 0$. Theorem 2, applied to $P_0'(F(u), F_1, F_2, t, u)$, shows that this series has two roots, U_1 and U_2. Indeed, $P_0'(F(u), F_1, F_2, t, u)$ reduces to $(u-1)^2$ when $t=0$. The original functional equation gives the first terms of $F(u)$:

$$F(u) = 1 + tu + 3\,(u+1)\,ut^2 + 2\left(6\,u^2 + 10\,u + 11\right)ut^3 + O(t^4),$$

and the equation $P_0'(F(U_i), F_1, F_2, t, U_i) = 0$ provides the first terms of U_1 and U_2:

$$U_{1,2} = 1 \pm t^{1/2} + 2\,t \pm 5\,t^{3/2} + 15\,t^2 \pm 48\,t^{5/2} + O(t^3).$$

In particular, the series U_i are distinct. Let $\Delta(F_1, F_2, t, v) \equiv \Delta(v)$ be the discriminant of $P(x, F_1, F_2, t, v)$, taken with respect to x. By Theorem 14, $\Delta(v)$ admits U_1 and U_2 as multiple roots. But $\Delta(v)$ factors as $tv(v-1)^4 R_1(v)$, where R_1 is a polynomial in F_1, F_2, t and v of degree 5 in v. Since $U_i \neq 0$ and $U_i \neq 1$, we conclude that U_1 and U_2 are double roots of R_1. Let $R_2(v)$ be the derivative of R_1 with respect to v. Then R_1 and R_2 have the roots U_1 and U_2 in common.

The rest of the elimination procedure is schematized in Figure 10. The labels on the arrows indicate which variable is eliminated (using a resultant) at each stage.

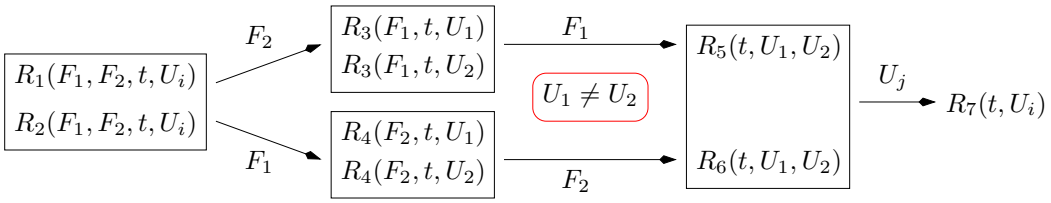


FIGURE 10. The bare-hand elimination procedure for $k = 2$.

We first eliminate F_2 between $R_1(F_1, F_2, t, U_i)$ and $R_2(F_1, F_2, t, U_i)$. The polynomial thus obtained factors in much smaller terms than R_1 or R_2. Knowing the first terms of $F(u), U_1$ and U_2 allows us to decide which factor vanishes. We thus obtain $R_3(F_1, t, U_i) = 0$, with

$$R_3(F_1, t, v) = t^2 v^4 {F_1}^2 + 2\,tv^2 F_1 (v-1)(v-3) - 4\,tv^2 + v^4 - 8\,v^3 + 22\,v^2 - 24\,v + 9.$$

Similarly, if we eliminate F_1 between $R_1(F_1, F_2, t, U_i)$ and $R_2(F_1, F_2, t, U_i)$, and then choose the right factor, we obtain an equation of the form $R_4(F_2, t, U_i) = 0$, of degree 8 in U_i.

We now eliminate F_1 between $R_3(F_1, t, U_1)$ and $R_3(F_1, t, U_2)$. The resultant naturally contains a factor $(U_1 - U_2)$, *which we know to be non-zero*. Choosing the right factor among the remaining ones provides a first equation between U_1 and U_2, of the form $R_5(t, U_1, U_2) = 0$. Similarly, eliminating F_2 between $R_4(F_2, t, U_1)$ and $R_4(F_2, t, U_2)$ provides another such equation, say $R_6(t, U_1, U_2) = 0$. Finally, eliminating one of the U_i's between R_5 and R_6 gives $R_7(t, U_i) = 0$, with

$$R_7(t, v) = (t-4)^3\, v^6 - 4\,(21\,t + 44)\,(t-4)\,v^5 - \left(180\,t + 2944 + 27\,t^2\right) v^4$$

$$-18\,(-332 + 15\,t)\,v^3 + 27\,(-235 + 9\,t)\,v^2 + 3402\,v - 729.$$

We have finally obtained the algebraic equation (on $\mathbb{Q}(t)$, of degree 6) satisfied by each of the series U_i. It remains to eliminate U_1 between $R_3(F_1, t, U_1)$ and $R_7(t, U_1)$ to obtain, by extraction of the relevant factor, the (cubic) algebraic equation satisfied by F_1:

$$F_1 = 1 - 47t + 3t^2 + 3t(22 - 9t)F_1 + 9t(9t - 2)F_1^2 - 81t^2 F_1^3. \tag{39}$$

Recall that F_1 counts 3-constellations by their number of black triangles.

9.2.2. ***An example with multiple roots*** U_i**.** Consider the functional equation

$$F(u) = u + t\left(F(u)^3 - 3 + 2\,\frac{F(u) - F(0)}{u} - t\,\frac{F(u) - F(0) - uF'(0)}{u^2}\right). \tag{40}$$

Clearly, it has a unique power series solution. The first terms of the expansion of F are:

$$F(u) = u + (u^3 - 1)t + u^2(3u^3 - 1)t^2 + 3u^4(4u^3 - 1)t^3 + u^6(55u^3 - 12)t^4 + \cdots$$

After multiplying by u^2, our functional equation reads $P(F(u), F(0), F'(0), t, u) = 0$, for some polynomial $P(x_0, x_1, x_2, t, u)$. We are looking for fractional series U that satisfy $P_0'(F(U), F(0), F'(0), t, U) = 0$, that is

$$(U - t)^2 = 3tU^2 F(U)^2. \tag{41}$$

By Theorem 2, this equation has two solutions, counted with multiplicities. Let us denote them U_1 and U_2. Using the first terms of $F(u)$, one derives from (41) the first terms of the series U_i. Remarkably, one finds $U_i = t + O(t^9)$ for $i = 1, 2$.

This observation leads us to *conjecture* that the series U_i are the same, so that (41) has a only one solution, of multiplicity 2. Let $\Delta(x_1, x_2, t, v)$ be the discriminant of $P(x_0, x_1, x_2, t, v)$ taken with respect to x_0. If $U_1 = U_2 \equiv U$, then, by Theorem 14, the series U is a root of Δ

of multiplicity at least 4. As Δ factors as tv^2D, for some polynomial $D \equiv D(v)$ of degree 8 in v, our assumption implies that for $0 \leq i \leq 3$,

$$\frac{\partial^i D}{\partial v^i}(F(0), F'(0), t, U) = 0.$$

This gives 4 equations involving 3 unknowns, namely $F(0)$, $F'(0)$ and U.

Let us first eliminate $F'(0)$ between $D(U)$ and $D'(U)$. The resultant thus obtained reads $t^{10}U^6(U-t)^6R_1$, where R_1 is a polynomial in t, U and $F(0)$, of degree 8 in U. The first few terms of $F(0)$ and U, which we have computed, rule out the possibility that $U = 0$, but are not sufficient to decide which of the factors $(U-t)$ and R_1 are zero.

So let us first assume that $U = t$. Taking the resultant in $F'(0)$ of $D(U)$ and $D''(U)$ gives $F(0) = -t$. Returning to Δ provides $F'(0) = 1$. Set now $F(0) = -t$ and $F'(0) = 1$ in the original functional equation (40). This gives the following cubic equation in $F(u)$:

$$tu^2F(u)^3 - (u-t)^2F(u) + (u-t)^3 = 0.$$

By Theorem 2, this equation has only one solution that is a formal power series in t. The form of the equation suggests to write $F = (u-t)G$, so that G satisfies $G = 1 + tu^2G^3$.

Hence, our assumption that (41) has a double root has led us to the conjecture that the solution of (40) is $F = (u-t)G$, where G is the unique series in t satisfying $G = 1 + tu^2G^3$. It is now straightforward to check that this series F satisfies $F(0) = -t$, $F'(0) = 1$, and that the original functional equation (40) holds. Given that this equation has a unique power series solution, we have solved it.

9.3. **Applying the factorization theorem to the discriminant**

We exploit here the factorization theorem, Theorem 19, in combination with Theorem 14, which implies that the discriminant $\Delta(F_1, \dots, F_k, t, v)$ admits k multiple roots.

Our example is again the equation for 3-constellations studied in Section 9.2.1. There, $k = 2$, and the discriminant reads $\Delta(F_1, F_2, t, v) = tv(v-1)^4R_1$, where R_1 is a polynomial in F_1, F_2, t and v, of degree 5 in v. By Theorem 14, Δ admits two double roots U_1 and U_2, and we have seen that they are actually double roots of R_1. What about the fifth root of R_1? Setting $t = 0$ in R_1 gives a polynomial in v of degree 4, so the fifth root of R_1 is infinite at $t = 0$.

Theorem 19, combined with the form of the series U_i, implies that R_1 factors as

$$R_1 = ct^p(1+tS)(1+tvR)\Big((1-v)^2 + tvQ_0 + tv^2Q_1\Big)^2 \tag{42}$$

where S, R, Q_0 and Q_1 belong to $\mathbb{C}[[t]]$. Setting $t = 0$ in this identity immediately gives $c = -4$ and $p = 0$. Setting $v = 0$ gives $S = 0$. Extracting the coefficient of v gives an expression of Q_0 in terms of R and F_1. Extracting the coefficient of v^2 gives an expression of Q_1 in terms of R, F_1 and F_2. We now replace Q_0 and Q_1 by their expressions in (42). The extraction of the coefficients of v^i, for $i = 3, 4, 5$ gives a system of 3 polynomial equations relating R, F_1 and F_2. The elimination of R and F_2 yields back (after some heavy intermediate steps) the algebraic equation (39) satisfied by F_1.

9.4. **Writing directly k equations**

We exploit here the results of Section 7 (Theorem 17), which provide directly a system of k equations between the series F_i. Our guinea-pig is again the equation for 3-constellations studied in Section 9.2.1. In order to apply Theorem 17, we need two polynomials P and Q having 2 roots in common. With the notation of Section 9.2.1, these polynomials can be either Δ and Δ'_v, or R_1 and R_2, or R_3 and R_4: the latter pair being the simplest, we decide to start from it. Then P has degree $m = 4$ and Q has degree $n = 8$. The Sylvester matrix of P and Q, denoted $\mathcal{S}_0(P, Q)$ in Section 7, has size 12. By Theorem 17, its determinant D_0 — the resultant of P and Q — is zero, as well as the determinant D_1 of the matrix $\mathcal{S}_1(P, Q)$,

obtained by deleting the last two columns, as well as the last row of a's (the 8th row) and the last row of b's (the 12th row).

The determinant D_0 is found to factor into two terms. The relevant one has degree 8 in F_1 and degree 4 in F_2. The second determinant, D_1, does not factor, and has degrees 14 and 6 in F_1 and F_2 respectively. Still, Maple agrees to eliminate F_2 between D_1 and the relevant factor of D_0. The corresponding resultant contains four different factors, and the one that vanishes yields (39) again.

We observe that two of the above factors are squared. The occurrence of repeated factors in iterated resultants is a systematic phenomenon, which we discuss in Section 12.

10. **The degree distribution of planar maps**

Let us return to the equations of Lemmas 5 and 6, which characterize the face-distribution of rooted planar maps. In this section, we solve these equations by generalizing the approach of [5]. Then we compare our solution to the result obtained in [14] for the same problem.

Theorem 21. *There exists a unique pair (R_1, R_2) of formal power series in t with coefficients in $\mathbb{Q}[z_1, z_2, \ldots]$ such that*

$$R_1 = \frac{t}{2}\sum_{i\geq 1} z_i [u^{i-1}] R^{-1/2} \quad \text{and} \quad R_2 = t - 3R_1^2 + \frac{t}{2}\sum_{i\geq 1} z_i [u^i] R^{-1/2}, \tag{43}$$

where

$$R = 1 - 4uR_1 - 4u^2 R_2.$$

Let $G(t;\boldsymbol{z}) = G(t; z_1, z_2, \ldots)$ be the generating function of rooted planar maps, counted by the number of edges (variable t) and the number of faces of degree i (variable z_i). Then

$$t^2 (tG(t;\boldsymbol{z}))' = (R_2 + R_1^2)(R_2 + 9R_1^2),$$

where the derivative is taken with respect to t, and

$$tG(t;\boldsymbol{z}) = \frac{1}{t}\left(R_2 + R_1^2\right)\left(3R_2 + 15R_1^2 - 2t\right) + R_1[u]\frac{\beta}{\sqrt{R}} - \frac{1}{2}[u^2]\frac{\beta}{\sqrt{R}}$$

where

$$\beta = \sum_{i\geq 1} z_i u^{-i}.$$

Comments

1. The equations defining R_1 and R_2 can also be written in terms of β:

$$R_1 = \frac{t}{2}[u^{-1}]\frac{\beta}{\sqrt{R}} \quad \text{and} \quad R_2 = t - 3R_1^2 + \frac{t}{2}[u^0]\frac{\beta}{\sqrt{R}}. \tag{44}$$

2. Let m be a positive integer, and set $z_i = 0$ for $i > m$. Then $G(t;\boldsymbol{z})$ is the face-distribution generating function of planar maps in which all faces have degree at most m. The right-hand sides of the equations defining R_1 and R_2 now involve only finitely many terms, so that these two series are actually algebraic. The same holds for $G(t;\boldsymbol{z})$ (as stated in Corollary 7), and the above theorem makes this algebraicity explicit by providing a system of three polynomial equations defining R_1, R_2 and G. For instance, the generating function of planar maps in which all faces have degree 3, counted by edges and faces, satisfies

$$t^2 G(t;z) = \left(R_2 + {R_1}^2\right)\left(3R_2 + 15R_1^2 - 2t - 24\,tzR_1R_2 - 56\,tz{R_1}^3\right)$$

with

$$R_1 = zt\left(R_2 + 3{R_1}^2\right) \quad \text{and} \quad R_2 = t - 3{R_1}^2 + zt\left(6R_1R_2 + 10{R_1}^3\right).$$

Proof. The existence and uniqueness of the series R_j is clear: think of extracting inductively from the equations (43) the coefficient of t^n. The fact that R_1 and R_2 are multiples of t implies that only finitely many values of i are involved in this extraction, so that the coefficient of t^n in R_1 and R_2 is a *polynomial* in the z_i.

We now want to relate R_1 and R_2 to the face-distribution of planar maps. As noted in [5, p. 13], it suffices to prove our results when $z_i = 0$ for $i > m$, for any $m \geq 3$. Then the equation of Lemma 5 may be written the form $P(F(u), F_1, \dots, F_{m-2}, t, u) = 0$:

$$u^{m-2}F(u) = u^{m-2} + tu^m F(u)^2 + t\theta_1(u)F(u) - t\sum_{j=0}^{m-2} u^j F_j\, \theta_{j+2}(u), \tag{45}$$

where $\theta_k(u)$ is the following polynomial in u, of degree $m-k$:

$$\theta_k(u) = \sum_{i=k}^{m} z_i u^{m-i}.$$

Note that $F_0 = 1$. This equation coincides with Eq. (2.2) of [5], apart for the value of θ_k. We now apply the general strategy of Section 2. The condition $P_0'(F(U), F_1, \dots, t, U) = 0$ reads:

$$U^{m-2} = 2tU^m F(U) + t\theta_1(U).$$

By Theorem 2, this equation has $m-2$ solutions, $U_1, \dots, U_{m-2}$, which are fractional series in t (with coefficients in an algebraic closure of $\mathbb{Q}(z_1, \dots, z_m)$). All of them vanish at $t = 0$. By Theorem 14, these series are multiple roots of the discriminant $\Delta(u) \equiv \Delta(F_1, \dots, F_{m-2}, t, u)$. This discriminant is found to be

$$\Delta(u) = \big(t\theta_1(u) - u^{m-2}\big)^2 - 4tu^m \left(u^{m-2} - t\sum_{j=0}^{m-2} u^j F_j \theta_{j+2}(u) \right). \tag{46}$$

It has degree (at most) $2m-2$ in u, and it reduces to $u^{2(m-2)}$ when $t = 0$. By the Newton-Puiseux theorem, this implies that the series U_i, for $1 \leq i \leq m-2$, are the only roots of $\Delta(u)$ that are finite at $t = 0$, and that they have multiplicity 2 exactly. The remaining roots are infinite. The canonical factorization of $\Delta(u)$ (Theorem 19) thus reads:

$$\Delta(u) = ct^p(1 + tS(t))\Big(1 + tuS_1(t,u)\Big)\Big(u^{m-2} + tS_2(t,u)\Big)^2$$

where S_1 has degree (at most) 1 in u and S_2 has degree at most $m-3$. Setting $t = 0$ in (46) shows that $p = 0$ and $c = 1$. Setting $u = 0$ then gives $t^2(1 + tS(t))S_2(t,0)^2 = t^2 z_m^2$, and we finally choose to write the canonical factorization of $\Delta(u)$ with the notation of [5]:

$$\Delta = RQ^2$$

with

$$R = 1 - 4uR_1 - 4u^2R_2 \quad \text{and} \quad Q = tz_m + \sum_{i=1}^{m-2} Q_i u^i.$$

The R_i and Q_i are power series in t with coefficients in $\mathbb{Q}(z_1, \dots, z_m)$.

The derivations of the equations defining the R_i, and of the expression of $(tG)'$, now faithfully follow [5]. Let us simply recall where (43) comes from. Solving (45) gives

$$2tu^m F(u) = u^{m-2} - t\theta_1(u) \pm \sqrt{\Delta(u)} = u^{m-2} - t\theta_1(u) + Q\sqrt{R}, \tag{47}$$

so that

$$Q = \frac{2tu^m + t\theta_1(u) - u^{m-2}}{\sqrt{R}} + O(u^{m+1}). \tag{48}$$

Recall that $Q(u)$ has degree $m-2$ in u. Extracting the coefficients of u^{m-1} and u^m in the above identity gives (43).

Let us finally derive an expression for $G(t; \boldsymbol{z})$. By (48),

$$Q_i = [u^i]\frac{t\theta_1(u) - u^{m-2}}{\sqrt{R}} \quad \text{for } 0 \leq i \leq m-2. \tag{49}$$

Now by Lemma 6 and (47),

$$\begin{aligned}
2t^2G = 2tF_2 &= [u^{m+2}]Q\sqrt{R} = \sum_{i=0}^{m-2} Q_i[u^{m+2-i}]\sqrt{R} \\
&= \sum_{i=0}^{m-2}[u^i]\frac{t\theta_1(u)-u^{m-2}}{\sqrt{R}}[u^{m+2-i}]\sqrt{R} \qquad \text{by (49)} \\
&= \sum_{i=0}^{m+2}[u^i]\frac{t\theta_1(u)-u^{m-2}}{\sqrt{R}}[u^{m+2-i}]\sqrt{R} - \sum_{i=m-1}^{m+2}[u^i]\frac{t\theta_1(u)-u^{m-2}}{\sqrt{R}}[u^{m+2-i}]\sqrt{R} \\
&= [u^{m+2}]\left(t\theta_1(u)-u^{m-2}\right) - \sum_{i=m-1}^{m+2}[u^i]\frac{t\theta_1(u)-u^{m-2}}{\sqrt{R}}[u^{m+2-i}]\sqrt{R} \\
&= -\sum_{i=-1}^{2}[u^i]\frac{t\beta-u^{-2}}{\sqrt{R}}[u^{2-i}]\sqrt{R},
\end{aligned}$$

where $\beta = u^{-m}\theta_1(u)$. The expected expression of $G(t;\boldsymbol{z})$ follows, upon using (44).

■

In [14], another characterization of the face-distribution of planar maps was obtained, using two different methods: first, using matrix integrals, and then by a purely combinatorial approach. Both methods yield the same expression for the series $G(t;\boldsymbol{z})$, but this expression differs from that of Theorem 21. Our aim is now to relate these two different expressions. Let us first recall the expression of [14].

Theorem 22 ([14])**.** *There exists a unique pair (S_1, S_2) of formal power series in t with coefficients in $\mathbb{Q}[z_1, z_2, \ldots]$ such that*

$$S_1 = t[v^0]W \quad \text{and} \quad S_2 = t + t[v^{-1}]W \tag{50}$$

where

$$W = \sum_{i\geq 1} z_i P^{i-1} \quad \text{and} \quad P = v + S_1 + S_2/v.$$

The face-distribution generating function of rooted planar maps, denoted above $G(t;\boldsymbol{z})$, satisfies

$$tG(t;\boldsymbol{z}) = S_1^2 + S_2 - 2S_1[v^{-2}]W - [v^{-3}]W.$$

Proposition 23. *The solutions to the face-distribution problem given by Theorems* 21 *and* 22 *are related as follows.*

(*i*) *The auxilliary series R_i and S_i satisfy*

$$S_1 = 2R_1 \quad \text{and} \quad S_2 = R_2 + R_1^2.$$

Moreover, for all $\ell \geq 0$,

$$[u^\ell]\frac{\beta}{\sqrt{R}} = [v^0]P^{\ell+1}W.$$

(*ii*) *The following identities, valid for all $k \geq 0$ and $j \in \mathbb{Z}$,*

$$\begin{aligned}
[v^j]P^{k+1}W &= [v^{j-1}]P^kW + S_1[v^j]P^kW + S_2[v^{j+1}]P^kW, \\
[v^j]\,P^kW &= S_2^{-j}[v^{-j}]P^kW,
\end{aligned}$$

allow one to express any term $[v^j]P^kW$ as a linear combination of terms $[v^{-i}]W$, for $i \geq 0$, with coefficients in $\mathbb{Q}(S_1, S_2)$.

(*iii*) *Rewrite the expression of* $G(t;\boldsymbol{z})$ *given in Theorem* 21 *in terms of* S_1, S_2 *and* P *using* (*i*). *Then, use* (*ii*) *to rewrite this in terms of* S_1, S_2 *and the* $[v^{-i}]W$, *for* $i \geq 0$. *Use finally* (50) *to express* $[v^0]W$ *and* $[v^{-1}]W$ *in terms of* S_1 *and* S_2*: The resulting expression of* $G(t;\boldsymbol{z})$ *is that of Theorem* 22.

Proof. (*i*) Let us introduce the series

$$\bar{R}_1 := \frac{S_1}{2}, \quad \bar{R}_2 := S_2 - \frac{S_1^2}{4} \quad \text{and} \quad \bar{R} := 1 - 4u\bar{R}_1 - 4u^2\bar{R}_2 = (1-uS_1)^2 - 4u^2S_2.$$

We want to prove that $\bar{R}, \bar{R}_1$ and $\bar{R}_2$ satisfy (43) (with bars over all unknowns). In view of (50), the first equation in (43) holds if and only if

$$[v^0]W = \sum_{j\geq 0} z_{j+1}[u^j]\bar{R}^{-1/2}.$$

Given that $W = \sum_j z_{j+1}P^j$, it suffices to prove that for all $j \geq 0$,

$$[v^0]P^j = [u^j]\bar{R}^{-1/2}, \tag{51}$$

or, upon taking generating functions, that

$$\sum_{j\geq 0} u^j[v^0]P^j = \bar{R}^{-1/2}.$$

But

$$\begin{aligned}
\sum_{j\geq 0} u^j[v^0]P^j &= \sum_{j\geq 0} u^j \sum_{k\geq 0} \binom{j}{2k}\binom{2k}{k} S_1^{j-2k}S_2^k \\
&= \sum_{k\geq 0} \binom{2k}{k} S_2^k \frac{u^{2k}}{(1-uS_1)^{2k+1}} \\
&= \frac{1}{1-uS_1}\left(1 - \frac{4u^2S_2}{(1-uS_1)^2}\right)^{-1/2} = \bar{R}^{-1/2}.
\end{aligned}$$

(By convention, $\binom{b}{a} = 0$ unless $0 \leq a \leq b$.) The first equation of (43) follows. The second one reads, in view of (50),

$$2[v^{-1}]W + S_1[v^0]W = \sum_{i\geq 1} z_i[u^i]\bar{R}^{-1/2}.$$

In order to prove it, it suffices to check that for all $j \geq 0$,

$$\begin{aligned}
2[v^{-1}]P^j + S_1[v^0]P^j &= [u^{j+1}]\bar{R}^{-1/2}, \\
&= [v^0]P^{j+1} \qquad \text{by (51)}.
\end{aligned}$$

This is easily proved by first extracting the coefficient of v^0 in $P^{j+1} = (v+S_1+S_2/v)P^j$, and then noticing that $S_2[v]P^j = [v^{-1}]P^j$ (this comes from the fact that P^j is left unchanged when replacing v by S_2/v). Since $\bar{R}_1$ and $\bar{R}_2$ satisfy (43), they coincide respectively with the series R_1 and R_2. The second result of (*i*) follows from (51).

The first identity of (*ii*) is simply obtained by writing

$$P^{k+1}W = (v + S_1 + S_2/v)P^kW,$$

and extracting the coefficient of v^j. The second one follows from the fact that P, and hence W, is left invariant upon replacing v by S_2/v.

Finally, (*iii*) is a straightforward verification.

■

11. **Hard particles on planar maps**

Let us return to the equations established in Lemma 13 for planar maps carrying hard particles. We will solve this system when $x = y = 1$. That is, the series $F(u) \equiv F(t, s, u)$ counts maps rooted at a vacant vertex by the total number of edges (variable t), the number of frustrated edges (variable s) and the number of white corners in the root-face (variable u). The series $G(u) \equiv G(t, s, u)$ counts maps rooted at an occupied vertex, according to the same statistics.

The first step consists in eliminating $G(u)$. This gives an equation of the form $P(F(u), F(1), G(1), t, u) = 0$, which is cubic in $F(u)$.

The next steps require a computer, but otherwise copy faithfully the bare-hands strategy of Section 9.2.1. We do not give the details. Let us simply mention that, when $s = 1$, the two series U_i that cancel $P_0'(F(u), F(1), G(1), t, u)$ are formal power series in $\sqrt{t}$:

$$U_{1,2} = 1 + t \pm t^{3/2} + 4t^2 \pm 17/2t^{5/2} + O(t^3)$$

while, when $s \neq 1$, they are formal series in t with coefficients in $\mathbb{Q}(s)$:

$$U_1 = 1 + t + \frac{(3\,s-4)\,t^2}{s-1} + \frac{\left(-25+64\,s+s^4+12\,s^3-51\,s^2\right)t^3}{(s-1)^3} + O(t^4),$$

$$U_2 = 1 + st + \frac{s^2\,(-2+3\,s)\,t^2}{s-1} + \frac{s^2\left(-28\,s^3+13\,s^4+16\,s^2-2\right)t^3}{(s-1)^3} + O(t^4).$$

Another interesting observation is that, in all cases, some of the roots of the discriminant $\Delta(u) \equiv \Delta(F(1), G(1), t, u)$ that are finite at $t = 0$ are *simple*. For instance, when $s = 1$, this discriminant has two simple roots, U_3 and U_4, of the following form:

$$U_{3,4} = 1 \pm 2\,it^{1/2} - 5\,t \mp 13\,it^{3/2} + O(t^2).$$

In other words, $\Delta(u)$ does not have a square root in $\mathbb{K}[u][[t]]$ (by Proposition 20). The other roots of $\Delta(u)$ that are finite at $t = 0$ are U_1, U_2, 0 and 1, and have an even multiplicity.

At the end of the elimination procedure, one obtains a pair of quartic polynomial equations for the series $F(1)$ and $G(1)$. The corresponding curves have genus 0 (as is often the case for hard-particle models [15]), and we have found, with the help of the `algcurves` package of Maple, a simple parametrization of them. In this form, our results are begging for a purely combinatorial derivation, in the vein of [13, 16, 17].

Proposition 24. *Let $T \equiv T(t, s)$ be the unique formal power series in t with constant term 0 satisfying*

$$T(1-2T)(s-3T+3T^2) = s^2t.$$

Then T has actually coefficients in $\mathbb{N}[s]$. Morever, the generating functions $F(t, s, 1) \equiv F(1)$ and $G(t, s, 1) \equiv G(1)$ that count planar maps carrying hard particles (rooted, respectively, at an empty and an occupied vertex) satisfy

$$\begin{aligned} s^3t^2F(1) &= T^2\left(s - 4\,T - 3\,sT + 15\,T^2 + sT^2 - 15\,T^3 + 4\,T^4\right), \\ s^4t^2G(1) &= T^3\left(s - 3\,T + 3\,T^2\right)\left(s - 4\,T - 3\,sT + 14\,T^2 - 9\,T^3\right). \end{aligned}$$

Proof. The only result that does not follow from the elimination of U_1 and U_2 is the fact that T has coefficients in $\mathbb{N}[s]$. By writing

$$T = sS, \quad S = \frac{t}{(1-2sS)(1-3S(1-sS))}, \quad S(1-sS) = t\,\frac{1-sS}{1-2sS}\,\frac{1}{1-3S(1-sS)},$$

it is easy to prove by induction on n that the coefficient of t^n in S and in $S(1-sS)$ belongs to $\mathbb{N}[s]$. Since $T = sS$, the same holds for the coefficients of T.

■

By studying the singular behaviour of the series $F(1)$ and $G(1)$ when $s = 1$, one obtains the following corollary.

Corollary 25. *The number of n-edge planar maps carrying hard particles is equivalent to*

$$\alpha \left(\frac{\sqrt[3]{39509 + 23436\sqrt{62}} - \sqrt[3]{-39509 + 23436\sqrt{62}} + 38}{3} \right)^n n^{-5/2} \simeq \alpha(15.4...)^n n^{-5/2},$$

for some positive constant α.

Proof. We apply the general principles that relate the singularities of an algebraic series to the asymptotic behaviour of its coefficients [31]. Since T, $F(1)$ and $G(1)$ have non-negative coefficients, their radius of convergence is one of their singularities. The expressions of $F(1)$ and $G(1)$ show that their singularities are also singularities of T. As the leading coefficient of the equation defining T does not vanish, the singularities of T are among the roots of the discriminant of its minimal polynomial, that is, among the roots of

$$\delta = 18432t^3 - 1545t^2 + 38t - 1.$$

The only real root of δ is $\rho \simeq .065$. Hence ρ is the radius of convergence of T, $F(1)$ and $G(1)$. The modulus of the other two roots of δ is less than ρ. So ρ is actually the only singularity of T, $F(1)$ and $G(1)$. A local expansion of these three series in the neighborhood of $t = \rho$ shows that T has a square root type singularity, while $F(1)$ and $G(1)$ have a singularity in $(1 - t/\rho)^{3/2}$. This implies that the nth coefficient of $F(1)$ and $G(1)$ is asymptotic to $\alpha\rho^{-n}n^{-5/2}$ for some positive constant α (which is not the same for $F(1)$ and $G(1)$). But ρ^{-1} is exactly the constant occurring above in the corollary.

■

Note. A similar study can be conducted for a generic value of $s \in (0, +\infty)$. We have not worked out all the details, but it seems that the above pattern persists for all $s \in \mathbb{R}_+$. In other words, there is no (physical) phase transition in this model. At any rate, it is not very hard to prove that the radius of convergence $\rho(s)$ is a smooth function of s, equal to the branch of

$$18432\, s^4\rho^3 - 3\, s^2\rho^2 \left(963 - 2496\, s + 2048\, s^2\right) + 2\, \rho \left(16\, s^2 + 21\, s - 18\right) (4\, s - 3)^2 - (4\, s - 3)^3$$

that equals $1/12$ at $s = 0$.

12. **Concluding remarks and questions**

Let us begin by bragging a bit about some positive points of this paper. We have proved that the series $F(t, u)$ given by a functional equation of a certain type (see (16)) are algebraic. As illustrated in Section 5, this tells us that a number of enumerative problems have an algebraic generating function, without having to solve them in detail. Our general strategy gives a system of $3k$ polynomial equations. Its reduction to $2k$ equations (Section 6) has a theoretical interest, and tells us what is left of the "quadratic method" for equations that are no longer quadratic.

However, the practical aspects of our approach probably require more work. Generally speaking, we are lacking an efficient elimination theory for polynomial systems which, as (8–10) or (33), are highly symmetric. The case of 3-constellations, solved by different approaches in Section 9, shows that even when the final result is relatively simple (here, F_1 satisfies a cubic equation), the intermediate steps may involve big polynomials. Does this mean that 3-constellations are somehow pathological, or that we have not conducted the calculations in the best possible way (or both...)? We have no definite answer to this question, but the following observations may be of some interest:

1. The degree of $F(u)$ may be very big compared to the degrees of the original functional equation. Consider, for instance, the enumeration of walks on the half-line $\mathbb{N}$, that start

from 0 and take steps $+k$ and $-\ell$, where k and ℓ are coprime. The case $(k, \ell) = (3, 2)$ was solved in Section 3.1. In general, the equation reads

$$(u^\ell - t(1 + u^{k+\ell}))F(u) = u^\ell - t\sum_{i=0}^{\ell-1} u^i F_i.$$

Its solution satisfies [11, Ex. 4]:

$$tF_0 = tF(0) = (-1)^{\ell+1} \prod_{i=1}^{\ell} U_i,$$

where $U_1, \ldots, U_\ell$ are the ℓ roots of the kernel that are finite at $t = 0$. It can be proved that F_0 has degree exactly $\binom{k+\ell}{k}$. When $\ell = k - 1$, this degree is exponential in k, even though the original equation is linear in $F(u)$ and all the F_i (and has degree $2k - 1$ in u).

2. Certain resultant calculations yield systematically repeated factors. Imagine we are trying to find a polynomial equation for F_1, starting from $P(F(u), F_1, F_2, t, u) = 0$. At some point, we end up with two polynomials $R(u)$ and $Q(u)$, with coefficients in $\mathbb{K}[F_1, F_2, t]$, which have two roots in common. We thus apply Theorem 17: the determinants D_0 and D_1 of $\mathcal{S}_0$ and $\mathcal{S}_1$ vanish. We take the resultant of D_0 and D_1 in F_2 to obtain a polynomial equation for F_1. Then every factor in this resultant has multiplicity at least 2 [36, Thm. 3.4].

Moreover, a similar reduction might apply to equations with a single unknown function F_1. For such equations, we obtain a polynomial equation for F_1 by writing that the iterated discriminant $discrim_v(discrim_x P(x, F_1, t, v))$ vanishes. Again, if P is a generic polynomial in x, F_1 and v of total degree $d \geq 3$, then it is conjectured that this iterated discriminant has repeated factors [36, p. 384]. Note that this does not mean that we will always meet such a factor in our examples, since they do not have generic coefficients. This is illustrated by the following example, which is a generalization of the equation for planar maps (Section 3.2):

$$F(u) = 1 + atu^2 F(u)^2 + tu\,\frac{uF(u) - F(1)}{u - 1}.$$

There, the iterated discriminant is an irreducible polynomial of degree 4 in F_1 – but the equation we start from is not a *generic* equation of total degree 5.

Finally, let us underline that it is still an unsolved problem to enumerate m-constellations starting from the equations of Proposition 12 (even though the result is known to be remarkably simple [12]).

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