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An algebraic approach of Pólya processes

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Abstract

Pólya processes are natural generalization of Pólya-Eggenberger urn models. This article presents a new approach of their asymptotic behaviour *via* moments, based on the spectral decomposition of a suitable finite difference operator on polynomial functions. Especially, it provides new results for processes whose replacement matrices have large eigenvalues in a sense to be defined.

Contents

1	Introduction	2
2	Pólya processes, definition and notations	5
2.1	Definition of a Pólya process	5
2.2	Jordan basis of linear forms of the process; notations	7
2.3	Semiprincipality, large and small projections	10
2.4	Examples	11
3	Transition operator of a Pólya process	13
3.1	Transition operator	13
3.2	Rough actions on polynomials	15
3.3	Reduced polynomials and reduced moments of the process	17
3.4	Fundamental cone	22
3.5	Reduced coordinates of joint moments polynomials	25
4	Asymptotic results on large processes	29
4.1	Asymptotic joint moments of projections	30
4.2	Large and principally semisimple processes	32
4.3	Large and principally nonsemisimple processes	35

1 Introduction

Take an urn (with infinite capacity) containing first finitely many balls of s different colours named $1, \dots, s$. This initial composition of the urn can be described by an s -dimensional vector U_1 , the k -th coordinate of U_1 being the number of balls of colour k at time 1. Proceed then to successive draws of one ball at random in the urn, any ball being at any time equally likely drawn. After each draw, inspect the colour of the ball, put it back into the urn and add new balls following at any time the same rule. This rule, summed up by the so called *replacement matrix* $R = (r_{i,j})_{1 \leq i,j \leq s} \in \mathcal{M}_s(\mathbb{Z})$ consists in adding (algebraically), for any $j \in \{1, \dots, s\}$, $r_{i,j}$ balls of colour j when a ball of colour i has been drawn. In particular, a negative entry of R corresponds to subtraction of balls from the urn, when it is possible. The urn process is the sequence $(U_n)_{n \geq 1}$ of random vectors with nonnegative integer coordinates, the k -th coordinate of U_n being the number of balls of colour k at time n , *i.e.* after the $(n-1)$ -st draw.

Such urn models appear for the first time in [7]. In 1931, in its original article *Sur quelques points de la théorie des probabilités* ([17]), G. Pólya makes a complete study of the two-colour urn process having $S \cdot \text{Id}_2$ as replacement matrix.

An urn process will be called **Pólya-Eggenberger** when it is *balanced*, in reference to the work of these authors; this means that all rows of R have a constant entries' sum, say S . Under this assumption, the number of added balls is S at any time, so that the total number of balls at time n is non random. Furthermore, in order to avoid extinction of the process, we will only consider replacement matrices having nonnegative off-diagonal entries, with an extra classical arithmetical assumption on the column of any negative diagonal entry (see definition 1 and related comments).

A Pólya-Eggenberger urn process can be viewed as a Markovian random walk in the first quadrant of \mathbb{R}^s with finitely many possible increments (the rows of R), the conditional transition probabilities between times n and $n+1$ being linear functions of the space at time n . This point of view leads to the following natural generalization: we will name **Pólya process** such a random walk in \mathbb{R}^s or in any finite dimensional vector space, even if it does not come from an urn process (see section 2 for a precise definition).

The present text deals with Pólya processes, so that all its results are valid for Pólya-Eggenberger urn processes. Such a process being given, different natural questions arise: what is the distribution of the vector at any time n ? Can the random vector be renormalized to get convergence? What kind (and speed) of convergence is obtained? What is the asymptotic distribution of the process?

Since the work of Pólya and Eggenberger, many authors have considered such models, sometimes with more general hypotheses, often with restrictive assumptions. Direct combinatoric attacks in some particular cases were first intended

([17], [7], [10] for example). They have recently be considerably refined by analytic considerations on generating functions in low dimensions by much more general methods ([9], [19]). A second approach was first introduced in [1] and developed in [14] and [15], viewing such urns as multitype branching processes. It consists in embedding the process in continuous time, using martingale arguments and coming back at discrete time. This method provides convergence results. The limits laws are normal when the process is small; when the process is large, the method does not tell anything on asymptotic distributions (see below for a definition of small and large). It requires some irreducibility-type assumptions and is still valid if the replacement matrix has random entries. One can find in [9], [19] and [14] good surveys and references on the subject.

In the present paper, we deal principally with asymptotics of Pólya processes; the method also leads to results on distributions at finite time (exact expressions for moments for example) but we do not focus on this point of view. We present a new approach based on spectral decomposition of the vector-valued process, asymptotics of (polynomial) joint moments and discrete martingale arguments. The adjective algebraic used in the title refers to a fine study of coefficients of joint moments of the process' coordinates in a suitable basis of polynomials, which requires some geometric considerations in the space of monomials' exponents (section 3 and more particularly subsection 3.4).

Take a Pólya process named $(X_n)_n$ in an s -dimensional real vector space V . With such a process is associated a canonical endomorphism A of V , that corresponds naturally to the replacement matrix in the urn version of the process (see subsection 2.2). The choice of a Jordan basis of A gives rise to a system of "spectral coordinates" $(u_1(X_n), \dots, u_s(X_n))$ of the process. The results of this article come from the computation and the asymptotic expansion of the joint moments of these coordinates, say the expectations

$$E \mathbf{u}^\alpha(X_n),$$

the notation \mathbf{u}^α denoting the product $\prod_{k=1}^s u_k^{\alpha_k}$ for any s -uple $\alpha = (\alpha_1, \dots, \alpha_s)$ of nonnegative integers.

When f is any (measurable) function, the expectation of $f(X_n)$ can be expressed in terms of the initial state of the process and of the image of f by a polynomial function of the so-called **transition operator** (see subsection 3.1). It turns out that this operator stabilizes finite dimensional polynomial subspaces. An appropriate reduction of the transition operator's restrictions to these polynomial subspaces leads to existence and unicity of a basis $(Q_\alpha)_{\alpha \in (\mathbb{Z}_{>0})^s}$ of polynomials in s variables, the "**reduced polynomials**" of the process, that has good spectral properties with regard to the transition operator (each Q_α generates a monogenic $\mathbb{C}[t]$ -module for the module structure induced on the space of polynomials of degree $\leq \deg Q_\alpha$ by the transition operator). Relatively to the Q_α 's scale induced

by the degree-antialphabetical order on s -uples, the expansion of any \mathbf{u}_α in the Q_β 's-basis takes the form

$$\mathbf{u}^\alpha = Q_\alpha + \sum_{\beta < \alpha} q_{\alpha,\beta} Q_\beta$$

(the $q_{\alpha,\beta}$'s are complex numbers) where, for any s -uple β of nonnegative integers, there exist a complex number c_β and a nonnegative integer ν_β such that

$$EQ_\beta(X_n) \sim c_\beta n^{\langle \beta, \lambda \rangle} \log^{\nu_\beta} n$$

as n tends to infinity; in this last formula, $\lambda = (\lambda_1, \dots, \lambda_s)$ denotes the s -uple of eigenvalues of the associated endomorphism A and $\langle \beta, \lambda \rangle$ the standard product $\sum_{k=1}^s \beta_k \lambda_k$.

The determination of coefficients $q_{\alpha,\beta}$ that necessarily vanish leads to the intervention of a rational polyhedral cone Σ of \mathbb{R}^s (the "fundamental cone", universal, *i.e.* defined independently from the process) as follows: the above expansion can be refined as

$$\mathbf{u}^\alpha = Q_\alpha + \sum_{\beta \in \alpha - \Sigma} q_{\alpha,\beta} Q_\beta \quad (1)$$

if one assumes the endomorphism A to be semisimple; if one does not assume A to be semisimple, the expansion is of the same type, but slightly more complicated. Furthermore, geometrical characteristics of Σ contain the "1/2-phase transition" already discovered in the irreducible case by Athreya and Karlin (see [1]), and give an answer to the determination of the $\langle \beta, \lambda \rangle$'s that have the largest real part in formula (1). This leads to asymptotic expansions of any joint moment $E \mathbf{u}^\alpha(X_n)$.

A process will be called "large" when 1 is a multiple eigenvalue of A , or when its second greatest eigenvalue's real part is $> 1/2$. This study leads to asymptotic results for large processes, for which the fundamental cone plays as power-trap; their generic asymptotic form can be stated as follows. We name here λ_2 the eigenvalue of A having the second largest eigenvalue supposed for a while to be generic, that is to say simple, with real part in $]1/2, 1[$. Note that the expectation EX_n is always equivalent to n times some projection of the initial value X_1 as n tends to infinity (see end of subsection 3.1).

Asymptotics of generic large Pólya processes: *there exists some complex random vector Λ in V such that*

$$X_n = EX_n + \Re(n^{\lambda_2} \Lambda) + o(n^{\Re \lambda})$$

the small o being almost sure and in any L^p , $p \geq 1$. Furthermore, any joint moment of the vectors Λ and its complex conjugate $\bar{\Lambda}$ is computed in a simple way in terms of reduced polynomials and of the initial value X_1 of the process.

For a general statement of the result, see Theorem 18 and 19 that are the main theorems of the paper. The proof of this result is based on the asymptotics of joint moments $E \mathbf{u}^\alpha(X_n)$; the second order term that contains the random limit Λ (or W_k 's for the general form of (43) and (45)) is obtained by martingale techniques (quadratic variation, Burkholder inequality), the small o rest being a consequence of Borel-Cantelli lemma. Speed of convergence (refinement of the o) can also easily be obtained from further expansion of joint moments' asymptotics, even if it is not much developed in this paper.

Definition of Pólya processes and principal other definitions and notations used to state the results and their proofs are established in section 2. Section 3 is devoted to the algebraic kernel of the method, together with its immediate asymptotic consequences on the process; it is centered around the transition operator, the reduced polynomials and the fundamental cone. Statement and proof of the main theorems take place in section 4.

2 Pólya processes, definition and notations

2.1 Definition of a Pólya process

Let V be a real vector space of finite dimension $s \geq 1$, and $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ its complexified space.

Definition 1 Let X_1, w_1, \dots, w_s be vectors of V and $(l_k)_{1 \leq k \leq s}$ be a free family of linear forms on V satisfying the following assumptions:

i- (initialization hypothesis)

$$X_1 \neq 0 \quad \text{and} \quad \forall k \in \{1, \dots, s\}, \quad l_k(X_1) \geq 0; \quad (2)$$

ii- (balance hypothesis) for all $k \in \{1, \dots, s\}$,

$$\sum_{j=1}^s l_j(w_k) = 1; \quad (3)$$

iii- (sufficient conditions of tenability) for all $k, k' \in \{1, \dots, s\}$,

$$\begin{cases} k \neq k' \implies l_k(w_{k'}) \geq 0, & (4.a) \\ l_k(w_k) \geq 0 \quad \text{or} \quad l_k(X_1)\mathbb{Z} + \sum_{j=1}^s l_k(w_j)\mathbb{Z} = l_k(w_k)\mathbb{Z}. & (4.b) \end{cases} \quad (4)$$

The (discrete and finite dimensional) **Pólya process** associated with these data is the V -valued random walk $(X_n)_{n \in \mathbb{Z}_{\geq 1}}$ with increments in the finite set $\{w_1, \dots, w_s\}$, defined by X_1 and the induction: for every $n \geq 1$ and $k \in \{1, \dots, s\}$,

$$\text{Prob}(X_{n+1} = X_n + w_k | X_n) = \frac{l_k(X_n)}{n + \tau_1 - 1} \quad (5)$$

where τ_1 is the positive real number defined by

$$\tau_1 = \sum_{k=1}^s l_k(X_1). \quad (6)$$

The process is defined on the space of all trajectories of $X_1 + \sum_{1 \leq k \leq s} \mathbb{Z}_{\geq 0} w_k$ endowed with the natural filtration $(\mathcal{F}_n)_{n \geq 0}$ where \mathcal{F}_n is the σ -field generated by X_1, \dots, X_n . It is Markovian¹ and the transition conditional probabilities between times n and $n + 1$ depend **linearly** on the state at time n , as stated in equations (5). Conditions (2) and (3) are necessary and sufficient for the random vector X_2 to be well defined by relation (5); an elementary induction shows the deterministic relation

$$\forall n \geq 1, \quad \sum_{k=1}^s l_k(X_n) = n + \tau_1 - 1. \quad (7)$$

Condition (4) is sufficient to guarantee that the process is well defined, *i.e.* that the numbers $l_k(X_n)$ do not become negative so that the process does not extinguish as can be checked by an elementary induction. The arithmetical assumption (4.b), which has become classical (compare with [11], [14], [9] for urns) is equivalent to the following one: $l_k(w_k)$ is nonnegative, or it divides $l_k(X_1)$ and all the $l_k(w_j)$ as real numbers. In fact, if conditioned on non extinction, all the results about Pólya processes in this article remain valid when conditions (4) are removed from the definition.

This definition is invariant after any linear change of coordinates: if $P \in \text{GL}(V)$ and if $(X_n)_n$ is the Pólya process defined by vectors X_1 , $(w_k)_k$ and forms $(l_k)_k$, then $(PX_n)_n$ is the Pólya process defined by the vectors PX_1 , $(Pw_k)_k$ and the forms $(l_k \circ P^{-1})_k$ that satisfy all the required assumptions. In the same way, if one extends the definition of a Pólya process to complex ones (allowing X_1 and the w_k to be in $V_{\mathbb{C}}$, only the numbers $l_k(X_1)$ and $l_j(w_k)$ having to be real numbers), the definition of Pólya process is invariant after any complex linear change of coordinates.

Pólya processes are natural generalizations of Pólya-Eggenberger urns in the following sense (see [2], [11], [9], [19] for base references on Pólya-Eggenberger urns). Take a Pólya-Eggenberger s -colour urn process having replacement matrix R and vector U_1 as initial composition; let S be the common sum of R 's rows, assumed to be nonzero. The data consisting in taking the rows of $\frac{1}{S}R$ as vectors w_k 's, the coordinate forms as forms l_k 's and $X_1 = \frac{1}{S}U_1$ as initial vector define a Pólya process $(X_n)_n$, the random vector X_n being $1/S$ times the $1 \times s$ matrix whose entries are the numbers of balls of different colours after $n - 1$ draws. We will name this process *standardized urn process*. Conversely, if one considers the forms l_k of a Pólya process as being the coordinate forms of V (choice of a basis of

¹The time-homogeneity of the process is more explicit when one reads condition (5) with denominator $\sum_k l_k(X_n)$ instead of $n + \tau_1 - 1$.

V), the matrix whose rows are the coordinates of the w_k 's satisfies all hypotheses of a Pólya-Eggenberger urns' replacement matrix, except that its entries are not integers but real numbers. Note that the balanced property of the urn version is expressed in relation (3).

2.2 Jordan basis of linear forms of the process; notations

Computation of the conditional expectation of the vector at time $n+1$ with regard to the state at time n leads in a natural way to define the so-called associated endomorphism that will be denoted by A in the whole paper.

Definition 2 *If $(X_n)_n$ is a Pólya process with notations as above, its **associated endomorphism** is the endomorphism $A = \sum_{1 \leq k \leq s} l_k \otimes w_k \in V^* \otimes V \simeq \text{End}(V)$, defined as*

$$A(v) = \sum_{1 \leq k \leq s} l_k(v)w_k$$

for every v in V .

In the urn version of a Pólya process, the matrix of A in the dual basis of $(l_k)_k$ is the transpose of the normalized urn's replacement matrix $\frac{1}{S}R$ (notations as above). If $P \in \text{GL}(V)$, the endomorphism associated with the process $(PX_n)_n$ is PAP^{-1} .

With this definition, the announced conditional expectation can be written as

$$E^{\mathcal{F}_n} X_{n+1} = \left(\text{Id} + \frac{A}{n + \tau_1 - 1} \right) (X_n).$$

Let $\gamma_{\tau_1, n} \in \mathbb{R}[t]$ be the polynomial with real coefficients defined for every integer $n \geq 1$ by $\gamma_{\tau_1, 1}(t) = 1$ and

$$\forall n \geq 2, \quad \gamma_{\tau_1, n}(t) = \prod_{k=1}^{n-1} \left(1 + \frac{t}{k + \tau_1 - 1} \right). \quad (8)$$

Note that, as soon as the terms are defined,

$$\gamma_{\tau_1, n}(t) = \frac{\Gamma(\tau_1)}{\Gamma(\tau_1 + t)} \frac{\Gamma(n + \tau_1 - 1 + t)}{\Gamma(n + \tau_1 - 1)}. \quad (9)$$

The expectation of the process is straightforwardly expressed in terms of the endomorphism $\gamma_{\tau_1, n}(A)$ and the initial value of the process as

$$EX_n = \gamma_{\tau_1, n}(A)(X_1). \quad (10)$$

One of the first tools used to describe the asymptotics of a Pólya process is the reduction of the associated endomorphism A (or its transpose on the dual vector

space of V). Because of condition (3), the linear form $u_1 = \sum_{k=1}^s l_k$ satisfies $u_1 \circ A = u_1$, which shows that **1 is always eigenvalue of A** . Because of the whole assumptions (2),(3) and (4), one can say more on A 's spectral decomposition. Even if these properties can be seen in the field of Perron-Frobenius', we give a proof's hint of Proposition 1.

Proposition 1 *Any complex eigenvalue λ of A equals 1 or satisfies $\Re\lambda < 1$. Moreover, $\dim \ker(A - 1)$ equals the multiplicity of 1 as eigenvalue of A .*

PROOF. Replace A by its matrix in the dual basis of $(l_k)_k$. Suppose first that all entries of A are nonnegative. The space of all $s \times s$ matrices having nonnegative entries and columns with entries' sum 1 is bounded (for the norms' topology) and stable for multiplication. This forces the sequence $(A^n)_{n \geq 0}$ to be bounded, which implies both results. If A has at least one negative diagonal entry, apply the results to $(A + a)/(1 + a)$ for any positive a such that $A + a$ has nonnegative entries. ■

In the whole paper, the endomorphism A being given, we will denote by σ_2 the real number ≤ 1 defined by

$$\sigma_2 = \begin{cases} 1 & \text{if 1 is multiple eigenvalue of } A; \\ \max\{\Re\lambda, \lambda \in \text{Sp}(A), \lambda \neq 1\} & \text{otherwise,} \end{cases} \quad (11)$$

where $\text{Sp}(A)$ is the set of eigenvalues of A .

Definition 3 *If $(X_n)_n$ is a Pólya process of dimension s , a basis $(u_k)_{1 \leq k \leq s}$ of linear forms on $V_{\mathbb{C}}$ is called a **Jordan basis of linear forms of the process** or **shortlier a Jordan basis of the process** when*

1- $u_1 = \sum_{1 \leq k \leq s} l_k$;

2- $u_k \circ A = \lambda_k u_k + \varepsilon_k u_{k-1}$ for all $k \geq 2$, where the λ_k are complex numbers (necessarily eigenvalues of A) and where the ε_k are numbers in $\{0, 1\}$ that satisfy $\lambda_k \neq \lambda_{k-1} \implies \varepsilon_k = 0$.

In other words, the matrix of the transposed endomorphism tA in a Jordan basis of linear forms has a block-diagonal form $\text{Diag}(1, J_{p_1}(\lambda_{k_1}), \dots, J_{p_t}(\lambda_{k_t}))$ where $J_p(z)$ denotes the p -dimensional square matrix

$$J_p(z) = \begin{pmatrix} z & 1 & & & \\ & z & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & z \end{pmatrix}.$$

A (real or complex) linear form u_k will be called **eigenform** of the process when $A \circ u_k = \lambda_k u_k$, i.e. when $\varepsilon_k = 0$. An eigenform of the process is an eigenvector of

tA ; some authors call these linear forms left eigenvectors of A , referring to matrix operations.

Definition 4 A Jordan basis of linear forms being chosen with notations as above, a subset $J \subseteq \{1, \dots, s\}$ is called a **monogenic block of indices** when J has the form $J = \{m, m+1, \dots, m+r\}$ ($r \geq 0, m \geq 1, m+r \leq s$) with $\varepsilon_m = 0, \varepsilon_k = 1$ for every $k \in \{m+1, \dots, m+r\}$ and J is maximal for this property. Any monogenic block of indices J is associated with a unique eigenvalue of A that will be denoted by $\lambda(J)$.

In other words, J is monogenic when $\text{Vect}\{u_j, j \in J\}$ is A -stable and when the matrix of the endomorphism of $\text{Vect}\{u_j, j \in J\}$ induced by tA in the Jordan basis is one of the Jordan blocks mentioned above with number $\lambda(J)$ on its diagonal. The adjective *monogenic* has been chosen because this means that the subspace $\text{Vect}\{u_j, j \in J\} = \mathbb{C}[{}^tA].u_{m+r}$ is a monogenic sub- $\mathbb{C}[t]$ -module of the dual space $V_{\mathbb{C}}^*$ for the usual $\mathbb{C}[t]$ -module structure induced by tA .

Definition 5 A monogenic block of indices J is called a **principal block** when $\Re\lambda(J) = \sigma_2$ and J has maximal size among the monogenic blocks J' such that $\Re\lambda(J') = \sigma_2$.

A Jordan basis $(u_k)_{1 \leq k \leq s}$ of linear forms of the process being chosen,

$$(v_k)_{1 \leq k \leq s} \tag{12}$$

will denote its dual basis, made of the vectors of $V_{\mathbb{C}}$ that satisfy $u_k(v_l) = \delta_{kl}$ (Kronecker notation) for any k and l , and

$$\lambda = (\lambda_1, \dots, \lambda_s) \tag{13}$$

the s -uple of eigenvalues (distinct or not) respectively associated with u_1, \dots, u_s (or v_1, \dots, v_s). In particular, $\lambda_1 = 1$ for any Jordan basis of linear forms. The eigenvalues $\lambda_1, \dots, \lambda_s$ of A are called **roots of the process**. For any k , we also denote by π_k the projection on the line $\mathbb{C}v_k$ relative to the decomposition $V_{\mathbb{C}} = \bigoplus_{1 \leq l \leq s} \mathbb{C}v_l$; these projections satisfy the relations

$$\text{Id} = \sum_{1 \leq k \leq s} \pi_k \text{ and } \pi_k = u_k.v_k. \tag{14}$$

Note that the π_k do not commute neither with A nor with each other. Nevertheless, A commutes with $\sum_{j \in J} \pi_j$, the sum being extended to any monogenic block of indices J (these sums are polynomials in A). This fact will be used in the proofs of Theorems 18 and 19.

The lines spanned by the vectors v_k can be seen as principal directions of the process the word principal being used in physicists' sense. Random variables $u_k(X_n)$ are then suitable coordinates for the study of the vector-valued process. Our method is based on the computation of their polynomial joint moments.

2.3 Semiprincipality, large and small projections

For every Jordan basis $(u_k)_{1 \leq k \leq s}$ of linear forms, and for every $\alpha = (\alpha_k)_{1 \leq k \leq s} \in \mathbb{Z}^s$, we adopt the notations

$$|\alpha| = \sum_k \alpha_k$$

$$\langle \alpha, \lambda \rangle = \sum_{1 \leq k \leq s} \alpha_k \lambda_k$$

and, when all the α_k are nonnegative integers

$$\mathbf{u}^\alpha = \prod_{1 \leq k \leq s} u_k^{\alpha_k},$$

\mathbf{u}^α being a homogeneous polynomial function of degree $|\alpha|$.

Given a Jordan basis $(u_k)_{1 \leq k \leq s}$ of linear forms of the process, we adopt the following definitions.

Definition 6 A Pólya process is said to be **semisimple** when its associated endomorphism A is semisimple, i.e. when A admits a basis of eigenvectors in $V_{\mathbb{C}}$ (this means that all the u_k are real or complex eigenforms of A). The process is said to be **principally semisimple** when all principal blocks have size one (for any choice of a Jordan basis).

The four following assertions are equivalent:

- i) the process is principally semisimple;
- ii) for any $k \in \{1, \dots, s\}$, $(\Re \lambda_k = \sigma_2 \implies u_k \text{ is eigenform of the process})$;
- iii) the induced endomorphism $(\sum_{\{k, \Re \lambda_k = \sigma_2\}} \pi_k) A$ is semisimple;
- iv) if $r \geq 1$ and if $\{\lambda_k, k \geq r+1\}$ are the roots of the process having a real part $< \sigma_2$, the matrix of ${}^t A$ in the Jordan basis has a block-diagonal form $\text{Diag}(1, \lambda_2, \dots, \lambda_r, J_{p_1}(\lambda_{k_1}), \dots, J_{p_t}(\lambda_{k_t}))$.

Note that Proposition 1 asserts that any u_k associated with root 1 is eigenform of A .

Definition 7 A root of the process is said to be **small** when its real part is $\leq 1/2$; otherwise, it is said **large**. The process is said to be **small** when $\sigma_2 \leq 1/2$, which means that 1 is simple root and all other roots are small; when the process is not small, it is called **large**.

Definition 8 Let $\alpha = (\alpha_1, \dots, \alpha_s) \in (\mathbb{Z}_{\geq 0})^s$.

1- α is called **power of large projections** when \mathbf{u}^α is a product of linear forms associated with large roots, i.e. when for all $k \in \{1, \dots, s\}$, $(\alpha_k \neq 0 \implies \Re \lambda_k > 1/2)$.

2- α is called **power of small projections** when \mathbf{u}^α is a product of linear forms associated with small roots, *i.e.* when for all $k \in \{1, \dots, s\}$, $(\alpha_k \neq 0 \implies \Re \lambda_k \leq 1/2)$.

3- α is called **semisimple power** when \mathbf{u}^α is a product of eigenforms, *i.e.* when for all $k \in \{1, \dots, s\}$, $(\alpha_k \neq 0 \implies u_k \text{ is eigenform of the process})$.

4- α is called **monogenic power** when its support is contained in a monogenic block of indices.

In the whole text, the canonical basis of \mathbb{Z}^s (or of \mathbb{R}^s) will be denoted by

$$(\delta_k)_{1 \leq k \leq s} \quad (15)$$

and the symbol

$$\alpha \leq \beta \quad (16)$$

on s -uples of nonnegative integers will denote the **degree-antialphabetical** (total) **order**, defined by $\alpha = (\alpha_1, \dots, \alpha_s) < \beta = (\beta_1, \dots, \beta_s)$ when $(|\alpha| < |\beta|)$ or $(|\alpha| = |\beta| \text{ and } \exists r \in \{1, \dots, s\} \text{ such that } \alpha_r < \beta_r \text{ and } \alpha_t = \beta_t \text{ for any } t > r)$. For this order, $\delta_1 < \delta_2 < \dots < \delta_s$.

2.4 Examples

1- Pólya-Eggenberger urns

As stated in subsection 2.1, any Pólya-Eggenberger urn is a Pólya process after standardization, *i.e.* after division by S in order to get balance equal to 1. For further developments of examples on the general two dimensional urn process, on some generic examples in dimension 5 and on the so-called *s*-dimensional *cyclic urn* whose (semisimple) replacement matrix is

$$\begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & 0 & & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix},$$

see [18] (the cyclic urn defines a small Pólya process if and only if $s \leq 6$ because $\sigma_2 = \cos(2\pi/s)$). In the present article, see (29) for some developments on the general triangular urn with two colours; other considerations are made on the same subject in [19].

2- *m*-ary search trees

The space-requirements vector of an m -ary search tree is an $m - 1$ -dimensional Pólya process as can be seen in [4]. The associated endomorphism A is semisimple and the process is large whenever $m \geq 27$. One can find further developments on it in [18]. See [5], [14] and [8] for different treatments of the subject.

3- Random 2 – 3 trees

The repartition of external nodes of a random 2 – 3 trees having 1 or 2 sisters is the two-dimensional Pólya-Eggenberger urn process with initial condition $X_1 = {}^t(2, 0)$ and replacement matrix $\begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}$. This process is small ($\sigma_2 = -6$) and principally semisimple. It follows from [14] that its second order term has normal distribution. This example is the base example of [9].

If one goes one step further, one can distinguish external nodes of a random 2 – 3 tree with regard to the shape of the descendants-tree of their grand-mothers. This process is a 10-dimensional principally semisimple small urn process with $\sigma_2 = 0$. Its replacement matrix R contains negative off-diagonal entries. This does not prevent the urn to be tenable (for “physical” reasons, as first argument!), the columns of R containing these negative entries being coupled in the following sense: if $j \neq k$ and $r_{j,k} < 0$, then the columns of $r_{j,k}$ and $r_{k,j}$ are proportional. It turns out that it can be reduced to a 7-dimensional Pólya process. This study shows for instance that, if n is the number of external nodes of the tree, the average number of their grand-mothers is $\sim 0.182n$, that on average $\sim 21\%$ (resp. $\sim 24\%$) of external nodes have grand-mothers having themselves 4 (resp. 5) grand-children *etc.*

Patient readers can go still a step further, looking at the fourth level of genealogical trees of external nodes. This leads to the study of a 76-dimensional urn process.

4- Congruence in binary search trees

The following example is mentioned in [6] as a private unpublished idea of S. Janson. Take a binary search tree and an integer $s \geq 2$. Consider the random vector of \mathbb{R}^s whose k -th coordinate is the number of leaves whose depth is $\equiv k \pmod{s}$. This defines an s -colour urn process with (semisimple) replacement matrix

$$\begin{pmatrix} -1 & 2 & & & \\ & -1 & 2 & & \\ & & -1 & & \\ & & & \ddots & 2 \\ 2 & & & & -1 \end{pmatrix};$$

the balance is one and $\sigma_2 = -1 + 2 \cos(2\pi/s)$, so that the urn is small if and only if $s \leq 8$. As it is irreducible, it can be deduced from [14] that its second order term has normal distribution when $s \leq 8$. When $s \geq 9$, the process is large and its asymptotics is described by Theorem 18.

3 Transition operator of a Pólya process

The study of the asymptotic distribution of a Pólya process, in particular the study of the asymptotic distributions of its normalized projections leads to consider the expectation of $f(X_n)$ for bounded and continuous functions f . This expectation depends on the initial value of the process X_1 and of the value at f of some iteration polynomial of the so-called transition operator of the process.

3.1 Transition operator

Let $(X_n)_n$ be a Pólya process, defined by vectors X_1, w_1, \dots, w_s and linear forms l_1, \dots, l_s as in section 2. Let Φ be the **transition operator** associated with the process, defined on the space of all functions $f : V \rightarrow \mathbb{R}$ (or more generally on the space of all functions $f : V \rightarrow W$ where W is any real vector space) by: $\forall v \in V$,

$$\Phi(f)(v) = \sum_{1 \leq k \leq s} l_k(v) \left(f(v + w_k) - f(v) \right). \quad (17)$$

Proposition 2 expresses the expectation of any $f(X_n)$ in terms of f , the transition operator Φ and the initial value of the process. Polynomials $\gamma_{\tau_1, n}$ were defined at (8).

Proposition 2 *If $f : V \rightarrow W$ is any measurable function into some real (or complex) vector space W , then for all $n \geq 1$,*

$$Ef(X_n) = \gamma_{\tau_1, n}(\Phi)(f)(X_1) \quad (18)$$

PROOF. The conditional expectation of $f(X_{n+1})$ with respect to the state at time n is

$$\begin{aligned} E^{\mathcal{F}_n} f(X_{n+1}) &= \sum_{1 \leq k \leq s} \frac{1}{n + \tau_1 - 1} l_k(X_n) f(X_n + w_k) \\ &= f(X_n) + \frac{1}{n + \tau_1 - 1} \sum_{1 \leq k \leq s} l_k(X_n) \left(f(X_n + w_k) - f(X_n) \right). \end{aligned}$$

With the help of the transition operator, this formula can be written as

$$E^{\mathcal{F}_n} f(X_{n+1}) = \left(\text{Id} + \frac{1}{n + \tau_1 - 1} \Phi \right) (f)(X_n); \quad (19)$$

taking the expectation leads to the result after a straightforward induction. ■

It follows from Proposition 2 that the asymptotic weak behaviour of the process, or at least the asymptotic behaviour of its moments is reachable by decompositions of the operator Φ on suitable function spaces. Corollary 3 is the first step

in this direction, stating the result for functions that belong to finite dimensional stable subspaces. It is the starting point of the decomposition of Φ on polynomial subspaces.

Corollary 3 *Let $f : V \rightarrow W$ be a measurable function into some real (or complex) vector space W .*

1- If f is an eigenfunction of Φ associated with the (real or complex) eigenvalue z , that is if $\Phi(f) = zf$, then

$$Ef(X_n) = n^z \frac{\Gamma(\tau_1)}{\Gamma(\tau_1 + z)} f(X_1) + O(n^{z-1})$$

as n tends to infinity (Γ is Euler's function).

2- Assume that f belongs to some Φ -stable subspace \mathcal{S} of measurable functions $V \rightarrow W$ and that the operator induced by Φ on \mathcal{S} is a sum $z \text{Id}_{\mathcal{S}} + \Phi_N$, where Φ_N is a nonzero nilpotent operator on \mathcal{S} and z a complex number. Let ν be the positive integer such that $\Phi_N^\nu(f) \neq 0$ and $\Phi_N^{\nu+1}(f) = 0$. Then,

$$Ef(X_n) = \frac{n^z \log^\nu n}{\nu!} \frac{\Gamma(\tau_1)}{\Gamma(\tau_1 + z)} \Phi_N^\nu(f)(X_1) + O(n^z \log^{\nu-1} n)$$

as n tends to infinity.

PROOF. 1- It follows from Proposition 2 that $Ef(X_n) = \gamma_{\tau_1, n}(z) \times f(X_1)$; the result comes from Stirling formula (see (9)).

2- Taylor expansion of $\gamma_{\tau_1, n}(z \text{Id} + \Phi_N)$ leads to

$$Ef(X_n) = \sum_{p \geq 0} \frac{1}{p!} \gamma_{\tau_1, n}^{(p)}(z) \Phi_N^p(f)(X_1)$$

(finite sum), where $\gamma_{\tau_1, n}^{(p)}$ denotes the p -th derivative of $\gamma_{\tau_1, n}$. Besides, if p is any positive integer,

$$\gamma_{\tau_1, n}^{(p)}(z) = n^z \log^p n \frac{\Gamma(\tau_1)}{\Gamma(\tau_1 + z)} + O(n^z \log^{p-1} n) \quad (20)$$

when n tends to infinity, as can be shown by Stirling formula (see (9)) and an elementary induction starting from the computation of $\gamma_{\tau_1, n}$'s logarithmic derivative. These two facts imply the result. \blacksquare

Remark 1 As it is written, Corollary 3 is valid only if $\tau_1 + z$ is not a nonpositive integer. If one admits the convention $1/\Gamma(w) = 0$ when $w \in \mathbb{Z}_{\leq 0}$, this corollary is valid in all cases. Nevertheless, we will only use it when $\tau_1 + z \notin \mathbb{Z}_{\leq 0}$.

Remark 2 If $f : V \rightarrow W$ is linear, formula (17) implies that $\Phi(f) = f \circ A$. In this particular case, formula (18) gives $Ef(X_n) = f \circ \gamma_{\tau_1, n}(A)(X_1)$.

The notations about A 's spectral decomposition being chosen, one can state Proposition 4, first result (already given in [1] when 1 is simple root) on the average case study of the process.

Proposition 4 *If $\Pi_1 = \sum_{k, \lambda_k=1} \pi_k$ denotes the projection on the eigensubspace $\ker(A - 1)$, then, as n tends to infinity,*

$$E(X_n) = n\Pi_1(X_1) + o(n).$$

PROOF. Thanks to relation (14), $EX_n = \sum_{1 \leq k \leq s} Eu_k(X_n).v_k$. If J is any monogenic block of indices, the subspace $\text{Vect}\{u_k, \bar{k} \in J\}$ is Φ -stable (see remark 2 just above: the restriction of Φ to such a space is the one of tA). It follows from the definition of Jordan basis $(u_k)_k$ that this subspace satisfies the assumption of Corollary 3. Thus $Eu_k(X_n) \in o(n)$ when $\lambda_k \neq 1$ and $Eu_k(X_n) = n\pi_k(X_1) + o(n)$ when $\lambda_k = 1$ (see Proposition 1) and the proposition is proven. ■

3.2 Rough actions on polynomials

Because of condition (3) in the definition of a Pólya process, none of the vectors w_k is zero. For any k , if f is a function defined on V , we denote by $\partial f / \partial w_k$, when it exists, the derivative of f along the direction carried by the vector w_k . With this notation, we associate with the finite difference operator Φ the **differential operator** Φ_∂ defined by

$$\Phi_\partial(f)(v) = \sum_{1 \leq k \leq s} l_k(v) \frac{\partial f}{\partial w_k}(v) \quad (21)$$

for every function f defined on V and derivable at each point along the directions carried by the vectors w_k 's.

Remark 3 When the w_k 's constitute a basis of V , the differential operator can be written as $\Phi_\partial(f)(v) = Df_v.Av$ for any differentiable function f , where Df_v denotes the differential of f at point v .

Remark 4 As for the operator Φ (see remark 2 at the end of subsection 3.1), if $f : V \rightarrow W$ is linear, then $\Phi_\partial(f) = f \circ A$.

Proposition 5 (Rough action of Φ_∂ on the \mathbf{u}^α 's) *For any choice of a Jordan basis $(u_k)_{1 \leq k \leq s}$ of linear forms of the process,*

1- *for every $\alpha \in (\mathbb{Z}_{\geq 0})^s$,*

$$\Phi_\partial(\mathbf{u}^\alpha) - \langle \alpha, \lambda \rangle \mathbf{u}^\alpha \in \text{Vect}\{\mathbf{u}^\beta, \beta < \alpha\};$$

2- *if $\alpha \in (\mathbb{Z}_{\geq 0})^s$ is a semisimple power, then $\Phi_\partial(\mathbf{u}^\alpha) = \langle \alpha, \lambda \rangle \mathbf{u}^\alpha$.*

PROOF. Φ_∂ is a derivation which has the same behaviour with respect to composition of applications as usual derivation. In particular, for any $\alpha \in (\mathbb{Z}_{\geq 0})^s$, one has the formula

$$\Phi_\partial(\mathbf{u}^\alpha) = \sum_{k=1}^s \alpha_k \mathbf{u}^{\alpha - \delta_k} \Phi_\partial(u_k). \quad (22)$$

Besides, as any u_k is linear, $\Phi_\partial(u_k) = u_k \circ A$. The properties of a Jordan basis of linear forms lead directly to the results (the degree-antialphabetical order on s -uples is defined in (16) at the end of subsection 2.3). ■

Remark 5 One can formally extend the result of 2- in Proposition 5 to any family of *complex* numbers $\alpha_1, \dots, \alpha_s$ defining a semisimple power. This gives other eigenfunctions of Φ_∂ , defined on suitable open subsets of V or $V_{\mathbb{C}}$ (usual topology).

Corollary 6 (Rough action of Φ on the \mathbf{u}^α 's) *For any choice of a Jordan basis $(u_k)_{1 \leq k \leq s}$ of linear forms of the process and for every $\alpha \in (\mathbb{Z}_{\geq 0})^s$,*

$$\Phi(\mathbf{u}^\alpha) - \langle \alpha, \lambda \rangle \mathbf{u}^\alpha \in \text{Vect}\{\mathbf{u}^\beta, \beta < \alpha\}.$$

PROOF. If F is the subspace $F = \text{Vect}\{\mathbf{u}^\beta, \beta < \alpha\}$, then $(\Phi - \Phi_\partial)(\mathbf{u}^\alpha) \in F$ because of Taylor formula (the family $(\mathbf{u}^\beta)_{|\beta| \leq |\alpha| - 1}$ constitutes a basis of polynomials of degree $\leq |\alpha| - 1$) and $\Phi_\partial(\mathbf{u}^\alpha) - \langle \alpha, \lambda \rangle \mathbf{u}^\alpha \in F$ because of Proposition 5. ■

In the sequel, a more refined study of the action of Φ_∂ on the \mathbf{u}^α 's will be needed. This is the subject of the end of current subsection 3.2. In particular, the set A_α , defined just below, will play a key role in the spectral decomposition of Φ on polynomials spaces in the nonsemisimple case.

Definition 9 *Let $(u_k)_{1 \leq k \leq s}$ be a Jordan basis of linear forms of the process. For every $\alpha \in (\mathbb{Z}_{\geq 0})^s$, we define the set of powers A_α as*

$$A_\alpha = \min \{ B \subseteq (\mathbb{Z}_{\geq 0})^s, \alpha \in B \text{ and } \text{Vect}\{\mathbf{u}^\beta, \beta \in A\} \text{ is } \Phi_\partial\text{-stable} \}, \quad (23)$$

the min being understood for the inclusion of subsets of $(\mathbb{Z}_{\geq 0})^s$.

In other words, A_α is the set of powers $\beta \in (\mathbb{Z}_{\geq 0})^s$ such that \mathbf{u}^β appears as a \mathbf{u} -monomial somewhere in the (stationary) sequence $\mathbf{u}^\alpha, \Phi_\partial(\mathbf{u}^\alpha), \Phi_\partial^2(\mathbf{u}^\alpha), \dots$. It is a finite set. In particular, we focus on the following assertions: for every $\alpha \in (\mathbb{Z}_{\geq 0})^s$,

$$\alpha \in A_\alpha, \quad (24)$$

$$\text{Vect}\{\mathbf{u}^\beta, \beta \in A_\alpha\} \text{ is } \Phi_\partial\text{-stable} \quad (25)$$

(and A_α is minimal for these two properties among all subsets of $(\mathbb{Z}_{\geq 0})^s$).

Proposition 7 (Properties of A_α) Let $(u_k)_{1 \leq k \leq s}$ be a Jordan basis of linear forms of the process and $\alpha \in (\mathbb{Z}_{\geq 0})^s$. Then

- 1- A_α is the set of powers $\beta \in (\mathbb{Z}_{\geq 0})^s$ such that \mathbf{u}^β appears as a \mathbf{u} -monomial in the (almost zero) sequence $\mathbf{u}^\alpha, (\Phi_\partial - \langle \alpha, \lambda \rangle)(\mathbf{u}^\alpha), (\Phi_\partial - \langle \alpha, \lambda \rangle)^2(\mathbf{u}^\alpha), \dots$
- 2- $A_\alpha = (\alpha - D_\alpha) \cap (\mathbb{Z}_{\geq 0})^s$, where D_α is the set of $\mathbb{Z}_{\geq 0}$ -linear combinations of vectors $\delta_k - \delta_{k-1}$ such that $\alpha_k \geq 1$ and $\varepsilon_k = 1$, where $\alpha - D_\alpha = \{\alpha - d, d \in D_\alpha\}$.
- 3- $\langle \alpha', \lambda \rangle = \langle \alpha, \lambda \rangle$ for every $\alpha' \in A_\alpha$.

PROOF. Because of Jordan basis properties (see subsection 2.2), $\Phi_\partial(u_1) = u_1$ and $\Phi_\partial(u_k) = \lambda_k u_k + \varepsilon_k u_{k-1}$ if $k \geq 2$, where $\varepsilon_k \in \{0, 1\}$. Because of Leibnitz formula (Φ_∂ is a derivation), a more refined study of the computation in Proposition 5 and the expansion of the l_k 's in the $(u_k)_k$ basis show that for every β , $\Phi_\partial(\mathbf{u}^\beta) - \langle \beta, \lambda \rangle \mathbf{u}^\beta$ is linear combination of polynomials \mathbf{u}^γ , where $\gamma = \beta - \delta_k + \delta_{k-1}$ for integers $k \geq 2$ such that $\beta_k \geq 1$ and $\varepsilon_k = 1$ (hence $\lambda_k = \lambda_{k-1}$). Furthermore, $\langle \gamma, \lambda \rangle = \langle \beta, \lambda \rangle$ for such γ 's. This considerations are enough to prove 1-, 2- and 3-. Of course, the precise meaning of statement 1- in Proposition 7 is as above the following:

$A_\alpha = \min \{B \subseteq (\mathbb{Z}_{\geq 0})^s, \alpha \in B \text{ and } \text{Vect}\{\mathbf{u}^\beta, \beta \in B\} \text{ is } (\Phi_\partial - \langle \alpha, \lambda \rangle)\text{-stable}\}$.

■

Corollary 8 (Special values of A_α)

- 1- If α is a semisimple power, then $A_\alpha = \{\alpha\}$.
- 2- If α is a power of large (respectively small) projections, then every element of A_α is a power of large (resp. small) projections.

PROOF. If α is a semisimple power, Proposition 5 shows directly that $A_\alpha = \{\alpha\}$. If α is a power of large (respectively small) projections, the result can be deduced from the expression of A_α in terms of generators (2- of Proposition 7) by induction on α (degree-antialphabetical order): if $\alpha_k \geq 1$ and $\varepsilon_k = 1$, then $\alpha - \delta_k + \delta_{k-1}$ is $< \alpha$ and a power of large (resp. small) projections. ■

3.3 Reduced polynomials and reduced moments of the process

Let S be the algebra of polynomial functions on V (s variables). For every non-negative integer d , let S_d be the subspace of polynomial functions of degree $\leq d$.

If f is a polynomial of degree $d \geq 1$, both $\Phi(f)$ and $\Phi_\partial(f)$ are also polynomials of degree d . Furthermore, $\Phi_\partial(f)$ is homogeneous if f is. Consequently, the subspaces $(S_d)_{d \geq 0}$ form an increasing sequence of *finite-dimensional stable subspaces for both operators*. The sequel is based on a particular reduction of Φ (and Φ_∂) on these subspaces. We will use indistinctly the notation Φ to refer to Φ itself or to the endomorphism induced by Φ on S or on some stable subspace.

We begin with an elementary linear algebra lemma that will be used several times in the sequel (notations are chosen in order to make this adaptation more

comfortable). Its first application leads to existence and unicity of the *reduced polynomials* of the process.

Notation: if Ψ is an endomorphism of a (finite dimensional) vector space, we will denote by $\ker \Psi^\infty$ the characteristic space of Ψ associated with zero, that is

$$\ker \Psi^\infty = \bigcup_{p \geq 0} \ker \Psi^p.$$

Lemma 9 *Let E be a finite dimensional vector space (on any field) and Ψ an endomorphism of E . Suppose F is a Ψ -stable subspace of E and u a vector of $E \setminus F$ such that $\Psi(u) \in F$. Then, there exists a unique $Q \in E$ such that*

- i) $Q - u \in F \cap \bigoplus_{z \neq 0} \ker(\Psi - z)^\infty$;*
- ii) $\Psi(Q) \in F \cap \ker \Psi^\infty$.*

PROOF. We adopt the following notations: $F' = F \cap \ker \Psi^\infty$ and $F'' = F \cap \bigoplus_{z \neq 0} \ker(\Psi - z)^\infty$. These two spaces are supplementary Ψ -stable subspaces of F . Let v' and v'' be the respective projections of $\Psi(u)$ on F' and F'' with regard to the direct sum $F = F' \oplus F''$. Since the restriction of Ψ induces an automorphism of F'' , let w'' be the unique vector of F'' such that $\Psi(w'') = v''$. Then, the vector $Q = u - w''$ is the unique solution to the problem *i) and ii)* of the lemma because $\Psi(u - w'') = v' \in F'$ and the restriction of Ψ to F'' is injective. ■

Theorem 10 (Existence and unicity of reduced polynomials of the process) *For every choice of a Jordan basis $(u_k)_{1 \leq k \leq s}$ of linear forms of the process, there exists a unique family*

$$(Q_\alpha)_{\alpha \in (\mathbb{Z}_{\geq 0})^s}$$

of polynomials of S such that:

- 1- $Q_0 = 1$ and $Q_\alpha = \mathbf{u}^\alpha$ if $|\alpha| = 1$;*
- 2- for all α , $Q_\alpha - \mathbf{u}^\alpha$ belongs to $\text{Vect}\{Q_\beta, \beta < \alpha, \langle \beta, \lambda \rangle \neq \langle \alpha, \lambda \rangle\}$;*
- 3- for all α , $\Phi(Q_\alpha) - \langle \alpha, \lambda \rangle Q_\alpha$ belongs to $\text{Vect}\{Q_\beta, \beta < \alpha, \langle \beta, \lambda \rangle = \langle \alpha, \lambda \rangle\}$.*

Furthermore,

- 4- for every $\alpha \in (\mathbb{Z}_{\geq 0})^s$, $\text{Vect}\{Q_\beta, \beta \leq \alpha\} = \text{Vect}\{\mathbf{u}^\beta, \beta \leq \alpha\}$;*
- 5- for every $z \in \mathbb{C}$, $\ker(\Phi - z)^\infty = \text{Vect}\{Q_\alpha, \langle \alpha, \lambda \rangle = z\}$.*

PROOF. By induction on α for the degree-antialphabetical order, we prove points 1- to 4- together with $Q_\alpha \in \ker(\Phi - \langle \alpha, \lambda \rangle)^\infty$. Point 5- is then a direct consequence, the equality being satisfied by the restriction of Φ to any finite dimensional subspace S_d .

If $|\alpha| = 0$ or 1 , there is nothing to prove. Suppose that $|\alpha| \geq 2$ and denote $E = \text{Vect}\{\mathbf{u}^\beta, \beta \leq \alpha\}$ and $F = \text{Vect}\{\mathbf{u}^\beta, \beta < \alpha\}$. Corollary 6 implies that E and F are Φ -stable subspaces and that $(\Phi - \langle \alpha, \lambda \rangle)(\mathbf{u}^\alpha) \in F$. The application of Lemma 9 to these spaces E and F with $\Psi = \Phi - \langle \alpha, \lambda \rangle$ (more exactly the endomorphism of E induced by this Ψ) and $u = \mathbf{u}^\alpha$ shows the existence of a unique Q_α such that

$Q_\alpha - \mathbf{u}^\alpha \in F \cap \bigoplus_{z \neq \langle \alpha, \lambda \rangle} \ker(\Phi - z)^\infty$ and $\Phi(Q_\alpha) - \langle \alpha, \lambda \rangle Q_\alpha \in F \cap \ker(\Phi - \langle \alpha, \lambda \rangle)^\infty$. The induction hypotheses are enough to ensure that $F \cap \bigoplus_{z \neq \langle \alpha, \lambda \rangle} \ker(\Phi - z)^\infty = \text{Vect}\{Q_\beta, \beta < \alpha, \langle \alpha, \lambda \rangle \neq \langle \beta, \lambda \rangle\}$ and that $F \cap \ker(\Phi - \langle \alpha, \lambda \rangle)^\infty = \text{Vect}\{Q_\beta, \beta < \alpha, \langle \alpha, \lambda \rangle = \langle \beta, \lambda \rangle\}$ and this implies points 2- and 3-. Point 4- is true thanks to point 2- and Q_α belongs to $\ker(\Phi - \langle \alpha, \lambda \rangle)^\infty$ because its image by $\Phi - \langle \alpha, \lambda \rangle$ does. ■

Definition 10 For every $\alpha \in (\mathbb{Z}_{\geq 0})^s$, the polynomial Q_α is the α -th **reduced polynomial of the process** $(X_n)_n$ (relative to the choice of a Jordan basis of linear forms of the process). The expectation of $Q_\alpha(X_n)$ will also occasionally be named α -th **reduced moment of** X_n .

Until the end of section 3, a Jordan basis $(u_k)_{1 \leq k \leq s}$ of linear forms of the process is fixed and the reduced polynomials $(Q_\alpha)_{\alpha \in (\mathbb{Z}_{\geq 0})^s}$ are relative to it.

Proposition 11 (Asymptotics of reduced moments)

For every $\alpha \in (\mathbb{Z}_{\geq 0})^s$, let ν_α be the nonnegative integer defined by

$$\nu_\alpha = \max\{p \geq 0, (\Phi - \langle \alpha, \lambda \rangle)^p(Q_\alpha) \neq 0\}. \quad (26)$$

1- If $\nu_\alpha = 0$, i.e. if Q_α is eigenfunction of Φ , then, as n tends to infinity,

$$EQ_\alpha(X_n) = n^{\langle \alpha, \lambda \rangle} \frac{\Gamma(\tau_1)}{\Gamma(\tau_1 + \langle \alpha, \lambda \rangle)} Q_\alpha(X_1) + O(n^{\langle \alpha, \lambda \rangle - 1}).$$

2- If $\nu_\alpha \geq 1$, then, as n tends to infinity,

$$EQ_\alpha(X_n) = \frac{n^{\langle \alpha, \lambda \rangle} \log^{\nu_\alpha} n}{\nu_\alpha!} \frac{\Gamma(\tau_1)}{\Gamma(\tau_1 + \langle \alpha, \lambda \rangle)} (\Phi - \langle \alpha, \lambda \rangle)^{\nu_\alpha}(Q_\alpha)(X_1) + O(n^{\langle \alpha, \lambda \rangle} \log^{\nu_\alpha - 1} n).$$

Note that ν_α is a finite number because the endomorphism induced by $\Phi - \langle \alpha, \lambda \rangle$ on $\ker(\Phi - \langle \alpha, \lambda \rangle)^\infty$ is nilpotent, and Q_α belongs to this subspace. We give a way to reach these numbers for powers of large projections at the end of subsection 3.5 (Lemma 16).

PROOF. Q_α belongs to the Φ -stable subspace $\mathcal{S}_\alpha = \ker(\Phi - \langle \alpha, \lambda \rangle)^\infty$ and the operator induced by Φ on \mathcal{S}_α is the sum of $\langle \alpha, \lambda \rangle \text{Id}_{\mathcal{S}_\alpha}$ and of the nilpotent operator induced on \mathcal{S}_α by $\Phi - \langle \alpha, \lambda \rangle$. This fact being considered, Proposition 11 is a consequence of Corollary 3. ■

The reduced polynomials have been chosen after a suitable reduction of the transition operator on polynomials spaces so that the asymptotics of reduced moments are computable (and the expansions in Proposition 11 could be made as

precise as one wants). The joint moments of the “Jordan” coordinates of the process $(u_k(X_n))_{1 \leq k \leq s}$ we attend to reach asymptotically are the expectations of the $\mathbf{u}^\alpha(X_n)$. It results directly from their definition that the reduced polynomials of degree $\leq d$ form a basis of S_d for any $d \geq 1$, so that it is natural to expand the polynomials \mathbf{u}^α as linear combinations of reduced polynomials. The complex numbers $q_{\alpha,\beta}$ defined by the formulae

$$\mathbf{u}^\alpha = Q_\alpha + \sum_{\beta < \alpha} q_{\alpha,\beta} Q_\beta \quad (27)$$

(see Theorem 10 point 2-) will be named **reduced coordinates of joint moments polynomials**. Joint moments of the process are thus related to reduced ones by

$$E \mathbf{u}^\alpha(X_n) = EQ_\alpha(X_n) + \sum_{\beta < \alpha} q_{\alpha,\beta} EQ_\beta(X_n). \quad (28)$$

Because of Proposition 11 that asserts that any $EQ_\beta(X_n)$ has order of magnitude $n^{\langle\beta,\lambda\rangle} \log^{\nu_\beta} n$ as n tends to infinity, equalities (27) and (28) let two natural questions arise about the asymptotics of the joint moments:

- 1- which $q_{\alpha,\beta}$ are zero in relation (27) ?
- 2- For a given α , for which $\beta < \alpha$ among those who have a nonzero $q_{\alpha,\beta}$ is $\Re\langle\beta,\lambda\rangle$ maximal ?

This is the subject of the following two subsections, subsection 3.4 being the self-contained study of a polyhedral cone Σ of \mathbb{Z}^s that will play a central role in determinating the nonzero $q_{\alpha,\beta}$ in (27) and the dominating powers $\langle\beta,\lambda\rangle$ in (28).

Remark 6 Computation of $Q_{p\delta_1}$'s.

An immediate computation shows that the reduced polynomials corresponding to powers of u_1 is the same one for any Pólya process (and this has to be related to the non random drift, consequence of (7): $\forall n, u_1(X_n) = n + \tau_1 - 1$): for any integer $p \geq 0$, $Q_{p\delta_1} = u_1(u_1 + 1) \dots (u_1 + p - 1)$ and $\Phi(Q_{p\delta_1}) = pQ_{p\delta_1}$. The powers of u_1 are thus expanded in the reduced polynomials basis (the $Q_{p\delta_1}$'s are enough) by means of Stirling numbers of the second kind (see [12] for example), which gives the corresponding reduced coordinates:

$$u_1^p = \sum_{k=1}^p (-1)^{p-k} \left\{ \begin{matrix} p \\ k \end{matrix} \right\} Q_{k\delta_1}.$$

Remark 7 Triangular urns with two types of balls.

The general (generalized to real numbers) two-dimensional balanced triangular Pólya urn has the following R as replacement matrix:

$$R = \begin{pmatrix} 1 & 0 \\ 1 - \ell & \ell \end{pmatrix}, \quad (29)$$

where ℓ is any real number ≤ 1 . Remember that in the Pólya process version, this means that l_1 and l_2 are the coordinate forms, $w_1 = {}^t(1, 0)$ and $w_2 = {}^t(1 - \ell, \ell)$. If one chooses $u_2(x, y) = y$ as second form for a Jordan basis, a straightforward computation shows that for any integer $p \geq 0$, one has $Q_{p\delta_2} = u_2(u_2 + \ell) \dots (u_2 + (p - 1)\ell)$ and $\Phi(Q_{p\delta_2}) = p\ell Q_{p\delta_2}$. Reversing this last formula leads, for any integer $p \geq 0$, to

$$u_2^p = \sum_{k=1}^p (-\ell)^{p-k} \left\{ \begin{matrix} p \\ k \end{matrix} \right\} Q_{k\delta_2}. \quad (30)$$

In particular, if $\ell > 0$, since the order of magnitude of $EQ_{p\delta_2}(X_n)$ is $n^{p\ell}$ (Proposition 11), $Eu_2(X_n/n^\ell)^p$ tends to $\ell^p \times \Gamma(x_1 + y_1) / \Gamma(x_1 + y_1 + p\ell) \times \Gamma(y_1/\ell + p) / \Gamma(y_1/\ell)$ as n tends to infinity, where $X_1 = {}^t(x_1, y_1)$ is the initial composition of the urn. This shows the convergence in distribution of $(X_n - nv_1)/n^\ell = u_2(X_n/n^\ell)v_2$ to the law having the written above expression as p -th moment (the asymptotics of the computed p -th moment as p tends to infinity by means of Stirling formula shows that the limit law is determined by its moments, showing the convergence in law; see for example [3] for relations between convergence of moments and convergence in distribution). For description of this limit law in some very particular cases of parameters X_1 and ℓ in terms of stable laws or Mittag-Leffler distribution, one can refer to [19] or [15].

It will be shown later that this convergence is almost sure and in any L^p , $p \geq 1$ when $\ell > 1/2$.

The case $\ell = 0$ is degenerate: the process is deterministic. When $\ell < 0$, as $EQ_{p\delta_2}(X_n) \in O(n^{p\ell})$ (in all cases, even if $\tau_1 + p\ell$ is a nonpositive integer), one sees from (30) that $Eu_2(X_n)^p = O(n^\ell)$ for any p . Borel-Cantelli lemma asserts thus that $u_2(X_n) = X_n - nv_1$ converges almost surely to zero as n tends to infinity (as it was to be foreseen, because balls of the second type can never be added).

One can compare this to the results of [19] and [15]. It can easily be generalized to some classes of triangular urns of higher dimension, principally semisimple or not (with enough zero entries, see [18] for examples).

Remark 8 Inductive computation of Q_α 's.

In the general case, the numbers $q_{\alpha,\beta}$, defined by

$$\mathbf{u}^\alpha - Q_\alpha = \sum_{\beta < \alpha} q_{\alpha,\beta} Q_\beta = \sum_{\beta < \alpha, \langle \beta, \lambda \rangle \neq \langle \alpha, \lambda \rangle} q_{\alpha,\beta} Q_\beta$$

and the numbers $p_{\alpha,\beta}$ defined by

$$(\Phi - \langle \alpha, \lambda \rangle)(Q_\alpha) = \sum_{\beta < \alpha} p_{\alpha,\beta} Q_\beta = \sum_{\beta < \alpha, \langle \beta, \lambda \rangle = \langle \alpha, \lambda \rangle} p_{\alpha,\beta} Q_\beta$$

can easily be inductively computed (and implemented) the following way. We

denote by $r_{\alpha,\beta}$ the complex numbers defined by

$$(\Phi - \langle \alpha, \lambda \rangle)(\mathbf{u}^\alpha) = \sum_{\beta < \alpha} r_{\alpha,\beta} Q_\beta, \quad (31)$$

that can be deduced by computation of $(\Phi - \langle \alpha, \lambda \rangle)(\mathbf{u}^\alpha)$ and its expansion in the $(Q_\beta)_{\beta < \alpha}$ basis with the help of formula (27), the corresponding numbers $q_{\beta,\gamma}$ being known by induction. Write two expressions of $(\Phi - \langle \alpha, \lambda \rangle)(\mathbf{u}^\alpha)$ with formulae (27) and (31) and identify the coordinates in the (Q_β) basis. This gives the equations with $p_{\alpha,\beta}$ and $q_{\alpha,\beta}$ as unknowns:

$$\begin{cases} \langle \beta, \lambda \rangle = \langle \alpha, \lambda \rangle & \implies r_{\alpha,\beta} = p_{\alpha,\beta} \\ \langle \beta, \lambda \rangle \neq \langle \alpha, \lambda \rangle & \implies r_{\alpha,\beta} = (\langle \beta, \lambda \rangle - \langle \alpha, \lambda \rangle) q_{\alpha,\beta} + \sum_{\beta < \gamma < \alpha} q_{\alpha,\gamma} p_{\gamma,\beta}. \end{cases}$$

The expansion of any Q_α in the (\mathbf{u}^α) basis can be obtained by reversing the triangular system written in (27). All these computations can be handled by means of symbolic computation.

3.4 Fundamental cone

In this subsection, we define a polyhedral cone of \mathbb{R}^s named *fundamental cone* and denoted by Σ . It will appear in a natural way in the determination of the reduced coordinates of joint moments polynomials (see Theorem 15 below). Proposition 14's interest is directly related to Theorem 15 but can be stated as soon as Σ 's first properties are known.

Notations: if $I \subseteq \{1, \dots, s\}$ and $(i, j) \in \{1, \dots, s\}^2$, we adopt the notations

$$\begin{aligned} \delta_I &= \sum_{1 \leq i \leq s} \delta_i + \sum_{i \in I} \delta_i \in \mathbb{R}^s & \text{and} & \quad \delta_I^* = \sum_{1 \leq i \leq s} dx_i + \sum_{i \in I} dx_i \in \mathbb{R}^{s*}, \\ \delta_{(i,j)} &= 2\delta_i - \delta_j \in \mathbb{R}^s & \text{and} & \quad \delta_{(i,j)}^* = 2dx_i - dx_j \in \mathbb{R}^{s*} \end{aligned}$$

where dx_i denotes the i -th coordinate form $(x_1, \dots, x_s) \mapsto x_i$ in the dual space \mathbb{R}^{s*} (and, remember, δ_i the i -th vector of the canonical basis of \mathbb{R}^s).

Definition 11 We denote by Σ and name **fundamental cone** the polyhedral cone of \mathbb{R}^s spanned by the $s(s-1)$ vectors $\delta_{(i,j)}$ for all ordered pairs (i, j) of distinct elements, i.e.

$$\Sigma = \sum_{(i,j) \in \{1, \dots, s\}, i \neq j} \mathbb{R}_{\geq 0} \delta_{(i,j)}. \quad (32)$$

As usual, we define the dual cone $\check{\Sigma}$ of Σ as

$$\check{\Sigma} = \{x \in \mathbb{R}^s, \forall y \in \Sigma, \langle x, y \rangle \geq 0\},$$

identified to the cone of all linear forms on \mathbb{R}^s that are nonnegative on Σ , via the bijective linear application $x \in \mathbb{R}^s \mapsto \langle x, \cdot \rangle \in \mathbb{R}^{s*}$ (the symbol $\langle x, y \rangle$ denotes the standard scalar product of x and y on \mathbb{R}^s).

Lemma 12 describes the dual cone $\check{\Sigma}$ as intersection of hyperplanes (faces) and gives a system of minimal generators (edges). Corollary 13 just transcribes Lemma 12 in the Σ -side and gives the equations of the faces of Σ .

Lemma 12 (faces and edges of $\check{\Sigma}$)

$$\check{\Sigma} = \bigcap_{(i,j) \in \{1, \dots, s\}, i \neq j} \{x \in \mathbb{R}^s, \delta_{(i,j)}^*(x) \geq 0\} = \sum_{\substack{I \subseteq \{1, \dots, s\} \\ 1 \leq \#I \leq s-1}} \mathbb{R}_{\geq 0} \delta_I.$$

PROOF. The first equality that describes the faces of $\check{\Sigma}$ comes directly from (32). For every permutation $w \in \mathfrak{S}_s$, let τ_w be the simplicial cone defined by

$$\tau_w = \{x \in \mathbb{R}^s, x_{w(s)} \leq x_{w(s-1)} \leq \dots \leq x_{w(1)} \leq 2x_{w(s)}\}.$$

The cones τ_w provide a subdivision of $\check{\Sigma}$ in $s!$ simplicial cones -this subdivision is the intersection of $\check{\Sigma}$ with the barycentric subdivision of the first quadrant of \mathbb{R}^s . Each τ_w is the image of $\tau_1 = \tau_{\text{id}}$ by the permutation of coordinates induced by w (and τ_1 is a fundamental domain for the group action of \mathfrak{S}_s on $\check{\Sigma}$ by permutations of coordinates). Because of the elementary computation

$$\begin{aligned} (x_1, x_2, x_3, x_4) &= (2x_4 - x_1)(1, 1, 1, 1) \\ &\quad + (x_1 - x_2)(2, 1, 1, 1) \\ &\quad + (x_2 - x_3)(2, 2, 1, 1) \\ &\quad + (x_3 - x_4)(2, 2, 2, 1) \end{aligned}$$

that can be straightforwardly generalized in all dimensions, one sees that the edges of τ_1 are spanned by $(1, \dots, 1) = \sum_{1 \leq k \leq s} \delta_k$ and $\delta_{\{1\}}, \delta_{\{1,2\}}, \dots, \delta_{\{1, \dots, s-1\}}$. The images of these last vectors under permutations of coordinates are exactly the δ_I , where $I \neq \emptyset$ and $I \neq \{1, \dots, s\}$. This completes the proof. ■

Corollary 13 (faces of Σ) *The cone Σ has $2^s - 2$ faces of dimension $s - 1$, described as*

$$\Sigma = \bigcap_{\substack{I \subseteq \{1, \dots, s\} \\ 1 \leq \#I \leq s-1}} \{x \in \mathbb{R}^s, \delta_I^*(x) \geq 0\}. \quad (33)$$

In dimensions two, Σ is spanned by $(2, -1)$, and $(-1, 2)$ and $\check{\Sigma}$ by the forms $2dx_1 + dx_2$ and $dx_1 + 2dx_2$. In dimension three, Σ is spanned by $(2, -1, 0)$, $(-1, 2, 0)$,

$(2, 0, -1)$, $(-1, 0, 2)$, $(0, 2, -1)$ and $(0, -1, 2)$ and the coordinates of the spanning forms of $\tilde{\Sigma}$ are $(2, 1, 1)$, $(1, 2, 1)$, $(1, 1, 2)$, $(2, 2, 1)$, $(2, 1, 2)$ and $(1, 2, 2)$ in the canonical basis (dx_1, dx_2, dx_3) . The numbers of edges of Σ and $\tilde{\Sigma}$ coincide only in dimensions 2 and 3. Figures 1 and 2 give pictures of Σ in dimensions 2 and 3.

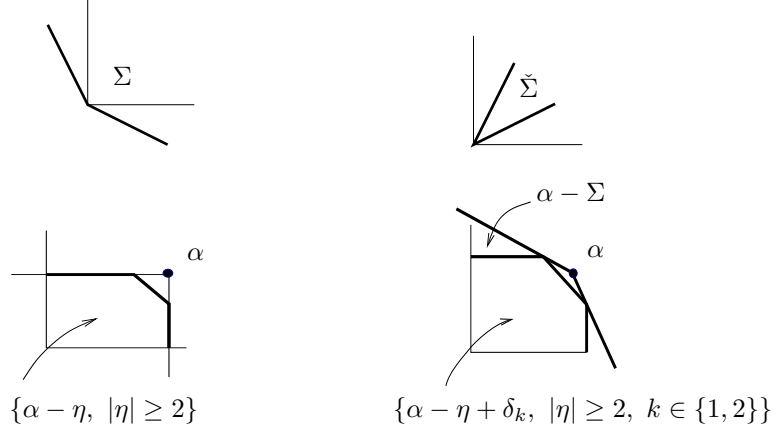


Figure 1: fundamental cone Σ in dimension 2

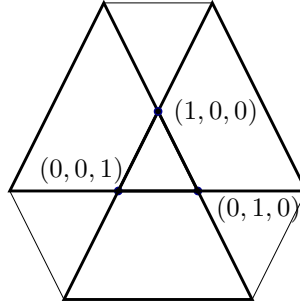


Figure 2: trace of Σ (convex hull) and of $\{\eta - \delta_k, |\eta| \geq 2, k \in \{1, 2, 3\}\}$ (union of three triangles) on the hyperplane $\{x_1 + x_2 + x_3 = 1\}$ of \mathbb{R}^3

For any $\alpha \in \mathbb{R}^s$, we denote

$$\alpha - \Sigma = \{\alpha - \sigma, \sigma \in \Sigma\}.$$

Theorem 15, that specifies the terms of formula (28) that do not give necessarily a zero contribution, leads to compare the real parts of all numbers $\langle \beta, \lambda \rangle$ when β belongs to $\alpha - \Sigma$ for a given α . This is the virtue of Proposition 14.

Proposition 14 Let $\alpha \in (\mathbb{Z}_{\geq 0})^s$ and $\beta \in (\mathbb{Z}_{\geq 0})^s \cap (\alpha - \Sigma)$.

- 1- If α is a power of large projections, then $\alpha = \beta$ or $\Re\langle\beta, \lambda\rangle < \Re\langle\alpha, \lambda\rangle$.
- 2- If α is a power of small projections, then $\Re\langle\beta, \lambda\rangle \leq \frac{1}{2}|\alpha|$.

PROOF. Without loss of generality, one can assume that $1/2 < \Re(\lambda_k) \leq 1$ for all $k \in \{1, \dots, r\}$ and that $\Re(\lambda_k) \leq 1/2$ for all $k \in \{r+1, \dots, s\}$ where $r \geq 1$. We denote $\sigma = \alpha - \beta \in \Sigma$, $I = \{k \leq r, \sigma_k \leq 0\}$ and $J = \{k \leq r, \sigma_k > 0\}$.

1- If α is a power of large projections, then $\alpha_{r+1} = \dots = \alpha_s = 0$. Thus $\sigma_k \leq 0$ if $k \geq r+1$ and

$$\Re\langle\sigma, \lambda\rangle \geq \frac{1}{2} \sum_{k \in J} \sigma_k + \sum_{k \in I} \sigma_k + \frac{1}{2} \sum_{k \geq r+1} \sigma_k = \frac{1}{2} \delta_I^*(\sigma). \quad (34)$$

Since σ lies in Σ , this number is nonnegative. Besides, since α is a power of large projections, $\sigma_k \leq 0$ if $k \geq r+1$; hence there exists some $k \leq r$ such that $\sigma_k > 0$ because the only point of Σ with only nonpositive coordinates is 0, as can be seen on the equations of Σ . Thus $J \neq \emptyset$ and the inequality of (34) is strict.

2- If α is a power of small projections, then $\alpha_1 = \dots = \alpha_r = 0$. Thus

$$\langle\beta, \lambda\rangle \leq \frac{1}{2} \sum_{k \geq r+1} (\alpha_k - \sigma_k) - \frac{1}{2} \sum_{k \in J} \sigma_k - \sum_{k \in I} \sigma_k = \frac{1}{2}|\alpha| - \frac{1}{2} \delta_I^*(\sigma).$$

Since σ lies in Σ , this number is $\leq |\alpha|/2$. ■

Remark 9 One can show that α is a power of large projections *if and only if* $\Re\langle\beta, \lambda\rangle \leq \Re\langle\alpha, \lambda\rangle$ for every $\beta \in (\mathbb{Z}_{\geq 0})^s \cap (\alpha - \Sigma)$. This gives another interpretation of the fundamental cone in terms of asymptotics of large moments (see relation (28), Theorem 15 and proof of Theorem 17).

3.5 Reduced coordinates of joint moments polynomials

Definition and properties of the cone Σ being established, Theorem 15 just below, key point for the final results, can be stated. It refines the action of Φ on polynomials (subsection 3.2) and gives an (optimal) condition for the reduced coordinates $q_{\alpha, \beta}$ to vanish (see (27)). Note that point 2- of Theorem 15 can be viewed as an algebraic or geometric explanation of the “1/2-phase transition” mentioned for example in [1] or [5]; this expression is used to point out the transition between normal behaviours for small processes to some nonnormal ones for large processes (in the irreducible case).

When Q_β is eigenvector of Φ , its contribution to the asymptotics in (28) has magnitude $n^{\langle\beta, \lambda\rangle}$ (Proposition 11); for instance, this happens always when the roots λ_k are incommensurable, that is when they admit no linear relation with rational coefficients. When a *contrario* Q_β is not eigenfunction, a power ν_β of $\log n$ appears in this contribution. Proposition 16 shows how to compute these numbers

ν_α working with the differential operator Φ_∂ , which is much more adapted for calculations, and gives a formula for the monogenic case.

For any $B \subseteq \mathbb{R}^s$, we denote

$$B - \Sigma = \bigcup_{b \in B} (b - \Sigma) = \{b - \sigma, b \in B, \sigma \in \Sigma\}.$$

Definition and properties of the set A_α can be found at the end of previous subsection 3.2; it is characterized by relations (24) and (25).

Theorem 15 (Coefficients lemma)

1- For any $\alpha \in (\mathbb{Z}_{\geq 0})^s$, one has

$$\mathbf{u}^\alpha \in \text{Vect}\{Q_\beta, \beta \in A_\alpha - \Sigma\}; \quad (35)$$

Furthermore,

$$\text{Vect}\{Q_\beta, \beta \in A_\alpha - \Sigma\} \text{ is } \Phi\text{-stable} \quad (36)$$

and

$$\text{Vect}\{\mathbf{u}^\beta, \beta \in A_\alpha - \Sigma\} = \text{Vect}\{Q_\beta, \beta \in A_\alpha - \Sigma\} \quad (37)$$

2- If α is a power of large projections, then

$$\Phi(Q_\alpha) \in \text{Vect}\{Q_\beta, \beta \in A_\alpha\}.$$

Consequently, relations (27) and (31) can be refined in the general case with the help of (35), (36) and Theorem 10 as

$$\mathbf{u}^\alpha = Q_\alpha + \sum_{\substack{\beta \in A_\alpha - \Sigma \\ \langle \beta, \lambda \rangle \neq \langle \alpha, \lambda \rangle}} q_{\alpha, \beta} Q_\beta \quad (38)$$

and

$$\Phi(Q_\alpha) = \langle \alpha, \lambda \rangle Q_\alpha + \sum_{\substack{\beta \in A_\alpha - \Sigma \\ \langle \beta, \lambda \rangle = \langle \alpha, \lambda \rangle}} r_{\alpha, \beta} Q_\beta. \quad (39)$$

PROOF. 1- For every $\alpha \in (\mathbb{Z}_{\geq 0})^s$, let E_α be the subspace

$$E_\alpha = \text{Vect}\{\mathbf{u}^\beta, \beta \in A_\alpha - \Sigma\}.$$

We first prove that E_α is Φ -stable. Let $\beta \in A_\alpha - \Sigma$; let $\alpha' \in A_\alpha$ and $\sigma \in \Sigma$ such that $\beta = \alpha' - \sigma$. We show that both $\Phi_\partial(\mathbf{u}^\beta)$ and $(\Phi - \Phi_\partial)(\mathbf{u}^\beta)$ belong to E_α .

• As in the beginning of Proposition 7's proof, $\Phi_\partial(\mathbf{u}^\beta) - \langle \beta, \lambda \rangle \mathbf{u}^\beta$ is a linear combination of polynomials \mathbf{u}^γ , where $\gamma = \beta - \delta_k + \delta_{k-1}$ for integers $k \geq 2$ such

that $\beta_k \geq 1$ and $\varepsilon_k = 1$ (hence $\lambda_k = \lambda_{k-1}$). If γ is such a power, we claim that $\gamma \in A_\alpha - \Sigma$, which shows that $\Phi_\partial(\mathbf{u}^\beta) \in E_\alpha$. If $\alpha'_k \geq 1$, just write $\gamma = \alpha'' - \sigma$ where $\alpha'' = \alpha' - \delta_k + \delta_{k-1} \in A_\alpha$. If $\alpha'_k = 0$, this α'' is not in A_α because it is not nonnegative; in this case, write $\gamma = \alpha' - \sigma'$ where $\sigma' = \sigma + \delta_k - \delta_{k-1}$. It only remains to show that $\sigma' \in \Sigma$. Let I a proper subset of $\{1, \dots, s\}$, that gives the equation of a face of Σ (see Corollary 13). If $k \in I$ or $k-1 \notin I$, then $\delta_I^*(\sigma') \geq 0$ because $\delta_I^*(\sigma) \geq 0$. If $k \notin I$ and $k-1 \in I$, then $\delta_I^*(\sigma') = \delta_I^*(\sigma) - 1 = \delta_{I \cup \{k\}}^*(\sigma) - \sigma_k - 1$; but $\sigma_k = -\beta_k \leq -1$ because $\beta_k \geq 1$. Thus $\delta_I^*(\sigma') \geq \delta_{I \cup \{k\}}^*(\sigma) \geq 0$ since $\sigma \in \Sigma$ (note that this last inequality is true even if $I \cup \{k\} = \{1, \dots, s\}$ because $(1, \dots, 1) \in \check{\Sigma}$ as can be seen from Lemma 12).

• Taylor formula implies that $(\Phi - \Phi_\partial)(\mathbf{u}^\beta)$ is linear combination of polynomials $\mathbf{u}^\gamma = \mathbf{u}^{\beta - \eta + \delta_k}$ with $1 \leq k \leq s$, $\eta \geq 0$, $|\eta| \geq 2$, $\beta - \eta \geq 0$ (the η -terms correspond to partial derivatives of order ≥ 2 of \mathbf{u}^β , the δ_k -terms come from the expansion of the linear forms l_k of the process in the Jordan basis $(u_k)_k$). If $\gamma = \beta - \eta + \delta_k$ is such a power and if δ_I^* is the equation of any one of the defining hyperplanes of Σ where I is a proper subset of $\{1, \dots, s\}$ (see Corollary 13), then

$$\delta_I^*(\beta - \gamma) = \delta_I^*(\eta - \delta_k) = |\eta| - 1 + \sum_{k \in I} \eta_k - \delta_I^*(k) \geq 1 - \delta_I^*(k) + \sum_{i \in I} \eta_k \geq 0.$$

This proves that $(\Phi - \Phi_\partial)(\mathbf{u}^\beta)$ belongs to $\text{Vect}\{\mathbf{u}^\gamma, |\gamma| \leq |\beta| - 1, \gamma \in \beta - \Sigma\}$. If $\gamma = \beta - \sigma' \in \beta - \Sigma$, then $\gamma = \alpha' - (\sigma + \sigma') \in \alpha' - \Sigma$ because the cone Σ is stable under addition. This shows that $(\Phi - \Phi_\partial)(\mathbf{u}^\beta) \in E_\alpha$ (see Figures 1 and 2).

Thus, $E_\alpha \subseteq \text{Vect}\{\mathbf{u}^\beta, \beta \leq \alpha\}$ is a Φ -stable subspace of polynomials. As intersection of stable subspaces, $F_\alpha = E_\alpha \cap \text{Vect}\{\mathbf{u}^\beta, \beta < \alpha\}$ is a stable subspace of E_α ; furthermore, $\mathbf{u}^\alpha \in E_\alpha$ and $(\Phi - \langle \alpha, \lambda \rangle)(\mathbf{u}^\alpha) \in F_\alpha$ (see Corollary 6). The unicity in Lemma 9 applied to this situation for the endomorphism of E_α induced by $\Phi - \langle \alpha, \lambda \rangle$ and the definition of the reduced polynomials imply (35) and (36). In particular, $E_\alpha = \text{Vect}\{Q_\beta, \beta \in A_\alpha - \Sigma\}$ for any α .

2- Combine property (36) and Theorem 10 point 3- to obtain

$$(\Phi - \langle \alpha, \lambda \rangle)(Q_\alpha) \in \text{Vect}\{Q_\beta, \alpha < \beta, \beta \in A_\alpha - \Sigma, \langle \beta, \lambda \rangle \neq \langle \alpha, \lambda \rangle\}.$$

Conclude with Corollary 8 and Proposition 14 point 1-. ■

Remark 10 On the fundamental cone.

The fundamental cone Σ appears in a natural way in the proof of Theorem 15, to ensure the Φ -stability of a (minimal) subspace that contains some given \mathbf{u}^α . Indeed, suppose for simplicity that α is a semisimple power. Then $\Phi(\mathbf{u}^\alpha)$ is the sum of $\langle \alpha, \lambda \rangle \mathbf{u}^\alpha$ and of a linear combination of polynomials $\mathbf{u}^{\alpha - \eta + \delta_k}$ where $|\eta| \geq 2$ and $1 \leq k \leq s$. The iterations of Φ on such polynomials forces to consider the least (for inclusion) set of powers that contains these $\eta - \delta_k$ and that is stable under addition (and contains zero); this least set is the fundamental cone. For an

illustration of this fact, see Figure 1 and 2. If α is not semisimple, the situation is complicated by powers α' of same degree and leads to consider the set A_α .

The properties of large projections that have just been established allow one to get more precisions on the asymptotics of reduced moments (see Proposition 11). Numbers ν_α , that has been defined in Proposition 11 as

$$\nu_\alpha = \max\{p \geq 0, (\Phi - \langle \alpha, \lambda \rangle)^p(Q_\alpha) \neq 0\}$$

intervene in the asymptotics of reduced moments. We end this subsection with a lemma that computes ν_α for powers of large projections. It is a consequence of Theorem 15.

Lemma 16 (Computation of ν_α)

1- If α is a power of large projections, then

$$\nu_\alpha = \max\{p \geq 0, (\Phi_\partial - \langle \alpha, \lambda \rangle)^p(\mathbf{u}^\alpha) \neq 0\}.$$

2- If $\alpha = (\alpha_1, \dots, \alpha_s)$ is a monogenic power of large projections whose support is contained in the monogenic block of indices $J = \{m, \dots, m+r\}$ ($r \geq 0$), then

$$\nu_\alpha = \sum_{k=0}^r k \alpha_{m+k}$$

and $\nu_\alpha > \nu_\beta$ for every $\beta \in A_\alpha \setminus \{\alpha\}$.

PROOF. 1- If $\alpha \in (\mathbb{Z}_{\geq 0})^s$ is not a semisimple power, we denote by k_α the index $k_\alpha = \min\{k \geq 3, \alpha_k \geq 1, \varepsilon_k = 1\}$ and $p(\alpha)$ the power $< \alpha$ of A_α defined by $p(\alpha) = \alpha - \delta_{k_\alpha} + \delta_{k_\alpha-1}$ ($p(\alpha)$ is the predecessor of α for the degree-antialphabetical order restricted to A_α). As shows a direct computation of $\Phi_\partial(\mathbf{u}^\alpha)$ (see (22) in the proof of Proposition 5), combined with A_α 's definition,

$$(\Phi_\partial - \langle \alpha, \lambda \rangle)(\mathbf{u}^\alpha) - \alpha_{k_\alpha} \mathbf{u}^{p(\alpha)} \in \text{Vect}\{\mathbf{u}^\beta, \beta \in A_\alpha, \beta < p(\alpha)\}. \quad (40)$$

We claim that $(\Phi - \langle \alpha, \lambda \rangle)(Q_\alpha) - \alpha_{k_\alpha} Q_{p(\alpha)} \in \text{Vect}\{Q_\beta, \beta < p(\alpha)\}$ for any non semisimple α (proof just below). With coefficients lemma (Theorem 15), this implies, when α is moreover power of large projections, that

$$(\Phi - \langle \alpha, \lambda \rangle)(Q_\alpha) - \alpha_{k_\alpha} Q_{p(\alpha)} \in \text{Vect}\{Q_\beta, \beta \in A_\alpha, \beta < p(\alpha)\}. \quad (41)$$

Assertions (40) and (41) are then enough to show that

$$\max\{p \geq 0, (\Phi - \langle \alpha, \lambda \rangle)^p(Q_\alpha) \neq 0\} = \max\{p \geq 0, (\Phi_\partial - \langle \alpha, \lambda \rangle)^p(\mathbf{u}^\alpha) \neq 0\}$$

when α is a power of large projections, this number being respectively equal to

$$\min\{q \geq 0, p^{[q]}(\alpha) \text{ is a semisimple power}\}$$

(the notation $p^{[q]}$ denotes the composition $p \circ \dots \circ p$ iterated q times and $p^{[0]}(\alpha) = \alpha$).

It remains to prove that $(\Phi - \langle \alpha, \lambda \rangle)(Q_\alpha) - \alpha_{k_\alpha} Q_{p(\alpha)} \in \text{Vect}\{Q_\beta, \beta < p(\alpha)\}$ for any non semisimple α to obtain 1-. Note first that $|p(\alpha)| = |\alpha|$, so that $\beta < p(\alpha)$ as soon as $|\beta| \leq |\alpha| - 1$. As for Landau's $o - O$, the notation $x = y + F$ will mean that $x - y \in F$ (for vectors x and y and a subspace F). As $\Phi - \Phi_\partial$ let the degree fall down (Taylor formula), one has

$$(\Phi - \langle \alpha, \lambda \rangle)(Q_\alpha) = (\Phi_\partial - \langle \alpha, \lambda \rangle)(Q_\alpha) + \text{Vect}\{\mathbf{u}^\beta, |\beta| \leq |\alpha| - 1\}.$$

Because of coefficients lemma (Theorem 15), and because the only point of Σ having a nonpositive degree ($|\cdot|$) is zero,

$$Q_\alpha = \mathbf{u}^\alpha + \text{Vect}\{\mathbf{u}^\beta, |\beta| \leq |\alpha| - 1\} + \text{Vect}\{\mathbf{u}^\beta, \beta < \alpha, \beta \in A_\alpha\}.$$

Taking the image by $\Phi_\partial - \langle \alpha, \lambda \rangle$ of this last relation leads to

$$(\Phi - \langle \alpha, \lambda \rangle)(Q_\alpha) = (\Phi_\partial - \langle \alpha, \lambda \rangle)(\mathbf{u}^\alpha) + \text{Vect}\{\mathbf{u}^\beta, \beta < p(\alpha)\}.$$

Because of assertion (40),

$$(\Phi - \langle \alpha, \lambda \rangle)(Q_\alpha) = \alpha_{k_\alpha} \mathbf{u}^{p(\alpha)} + \text{Vect}\{\mathbf{u}^\beta, \beta < p(\alpha)\}.$$

The conclusion follows then from Theorem 10 (points 2- and 4-):

$$(\Phi - \langle \alpha, \lambda \rangle)(Q_\alpha) = \alpha_{k_\alpha} Q_{p(\alpha)} + \text{Vect}\{Q_\beta, \beta < p(\alpha)\}.$$

2- Because of 1-, the computation of ν_α can be made with the help of the action of Φ_∂ on the \mathbf{u}^α 's. The degree $|\alpha|$ being fixed, we proceed by induction on α . If α is semisimple, $\alpha = |\alpha| \delta_m$ and $\nu_\alpha = 0$; there is nothing to prove. If α is not semisimple, the computation of $(\Phi_\partial - \langle \alpha, \lambda \rangle)(\mathbf{u}^\alpha)$ shows that

$$\nu_\alpha = 1 + \max\{\nu_{\alpha - \delta_k + \delta_{k-1}}, m + 1 \leq k \leq m + r, \alpha_k \geq 1\}.$$

All these $\alpha - \delta_k + \delta_{k-1}$ are $< \alpha$ and have degree $|\alpha|$; the induction hypothesis implies that they all have the same ν , and that this number is $-1 + \sum_{0 \leq k \leq r} k \alpha_{m+k}$. The formula for ν_α is proven. It shows in particular that $\nu_{p(\alpha)} = \nu_\alpha - 1$ which is enough to prove the last assertion. ■

4 Asymptotic results on large processes

It follows from section 3 that the random variables $u_k(X_n)$ are natural coordinates of the Pólya process $(X_n)_n$, well adapted to its asymptotic study. Their

joint moments $\mathbf{u}^\alpha(X_n)$ have natural asymptotic expansions in terms of reduced polynomials (Theorem 17), coming from a suitable reduction of the transition operator Φ (subsection 4.1). Thus, martingale arguments applied to the spectral decomposition of the process are made possible as shown in subsections 4.2 and 4.3.

4.1 Asymptotic joint moments of projections

Theorem 17 (Joint moments of small or large projections)

Let $\alpha \in (\mathbb{Z}_{\geq 0})^s$ and $(u_k)_{1 \leq k \leq s}$ be a Jordan basis of linear forms of the process.

1- If α is a power of small projections, then there exists some nonnegative integer ν such that

$$E \mathbf{u}^\alpha(X_n) \in O\left(n^{|\alpha|/2} \log^\nu n\right)$$

as n tends to infinity.

2- If α is a power of large projections, then there exists a nonnegative integer ν and a complex number c such that

$$E \mathbf{u}^\alpha(X_n) = cn^{(\alpha, \lambda)} \log^\nu n + o\left(n^{\Re(\alpha, \lambda)} \log^\nu n\right)$$

as n tends to infinity. If moreover α is a monogenic power, then

$$E \mathbf{u}^\alpha(X_n) = cn^{(\alpha, \lambda)} \log^{\nu_\alpha} n + o\left(n^{\Re(\alpha, \lambda)} \log^{\nu_\alpha} n\right)$$

as n tends to infinity.

3- If α is a semisimple power of large projections, then

$$E \mathbf{u}^\alpha(X_n) = n^{(\alpha, \lambda)} \frac{\Gamma(\tau_1)}{\Gamma(\tau_1 + \langle \alpha, \lambda \rangle)} Q_\alpha(X_1) + o\left(n^{\Re(\alpha, \lambda)}\right)$$

as n tends to infinity, where Q_α is the α -th reduced polynomial of the process relative to the Jordan basis $(u_k)_{1 \leq k \leq s}$.

PROOF. Taking the expectation of the value at X_n in the expansion (27) leads to $E \mathbf{u}^\alpha(X_n) = EQ_\alpha(X_n) + \sum_{\beta < \alpha} q_{\alpha, \beta} EQ_\beta(X_n)$. Besides, Proposition 11 asserts that for all $\beta \leq \alpha$, there exists $\nu \geq 0$ such that $EQ_\beta(X_n) \in O\left(n^{\Re(\beta, \lambda)} \log^\nu n\right)$.

1- If α is a power of small projections, then $\mathbf{u}^\alpha = \sum_{\beta \in A_\alpha - \Sigma} q_{\alpha, \beta} Q_\beta$ (see Theorem 15) and any $\alpha' \in A_\alpha$ is a power of small projections that satisfies $|\alpha'| = |\alpha|$ (Corollary 8). Hence $n^{(\beta, \lambda)} \in O\left(n^{|\alpha|/2}\right)$ if $\beta \in \alpha' - \Sigma$, as can be deduced from Proposition 14.

2- If α is a power of large projections, then $\mathbf{u}^\alpha = \sum_{\beta \in A_\alpha - \Sigma} q_{\alpha, \beta} Q_\beta$ and any $\alpha' \in A_\alpha$ is power of large projections that satisfies $\langle \alpha', \lambda \rangle = \langle \alpha, \lambda \rangle$. Hence for every $\beta \in \alpha' - \Sigma$, $\alpha' = \beta$ or $\Re(\beta, \lambda) < \Re(\alpha, \lambda)$ (Proposition 14). Thus

$E \mathbf{u}^\alpha(X_n) = \sum_{\alpha' \in A_\alpha} q_{\alpha, \alpha'} EQ_{\alpha'}(X_n) + o(n^{\Re(\alpha, \lambda)})$ has the required asymptotics as can be deduced from Proposition 11. The assertion on the monogenic case comes from Lemma 16 point 2-.

3- If α is a semisimple power of large projections, then $\mathbf{u}^\alpha = \sum_{\beta \in \alpha - \Sigma} q_{\alpha, \beta} Q_\beta$ (see Theorem 15) and $\Re(\beta, \lambda) < \Re(\alpha, \lambda)$ for every $\beta \in (\alpha - \Sigma) \setminus \{\alpha\}$ (see Proposition 14). Thus $E \mathbf{u}^\alpha(X_n) = EQ_\alpha(X_n) + o(n^{\Re(\alpha, \lambda)})$. But $\nu_\alpha = 0$ as can be seen with Lemma 16. Proposition 11 thus gives the required asymptotics. ■

Remark 11 More precision on the small o of point 3- in Theorem 17 can be deduced from its proof: one can replace it by $O(n^a)$ where

$$a = \max \{ \{ \Re(\beta, \lambda), \beta \neq \alpha, \beta \in \alpha - \Sigma \} \cup \{ \Re(\alpha, \lambda) - 1 \} \}.$$

Remark 12 The constant c of point 2- in Theorem 17 is a polynomial function of degree $|\alpha|$ of X_1 . The integer ν is roughly bounded by the dimension of the vector space $S_{|\alpha|}$ and tends possibly to infinity when $|\alpha|$ does. These facts can be deduced from the proof of the theorem, and easy to refine.

Remark 13 Definition of numbers ν_α were given in (26) and a way to compute them for powers of large projections can be found in Lemma 16. The numbers ν that appear in Theorem 17 can be refined in terms of ν_β 's for suitable powers β , as can be checked from part 1- and 2- of the above proof.

Remark 14 When the process is small, only the powers of u_1 correspond to powers of large projections. Refinement of part 1- Theorem 17 requires more detailed knowledge on the process (on A). This comes from expansion $E \mathbf{u}^\alpha(X_n) = EQ_\alpha(X_n) + \sum_{\beta < \alpha} q_{\alpha, \beta} EQ_\beta(X_n)$: if α is a power of small projection, the term in the sum having the highest order of magnitude as n tends to infinity is not necessarily $EQ_\alpha(X_n)$ any more, but $EQ_\alpha(X_n)$ can nevertheless be the winner. We just give the case of two-colour urns as example (see [18] for more details). Take the general two-dimensional Pólya process, and choose coordinates such that the forms l_k are the coordinates forms in \mathbb{R}^2 ; the endomorphism A is then represented by some matrix $\begin{pmatrix} 1-a & b \\ a & 1-b \end{pmatrix}$ where a and b are nonnegative reals (with restrictive conditions (4) if one of them is > 1). The process is small whenever $a + b \geq 1/2$ because $\sigma_2 = 1 - a - b$. We assume for the example that $a + b > 1/2$. Computation of the first reduced polynomials shows that

$$u_2^2 = Q_{(0,2)} - (a-b)(1-a-b)Q_{(0,1)} + \frac{ab(1-a-b)^2}{2(a+b)-1}Q_{(1,0)} \quad (42)$$

for the choice $u_2 = ax - by$. The term of $Eu_2^2(X_n)$ having the highest order of magnitude is $EQ_{(1,0)}(X_n) = nQ_{(1,0)}(X_1)$, but its coefficient is zero if a or b vanish. Such considerations justify the fact that the study of small triangular urns has to be done separately in terms of asymptotics and limit laws (see [14], [15], [19]).

Theorem 17 on joint moments being established, one can derive results on the asymptotics of **large** Pólya processes. Although it is not formally necessary, we split our description in two different cases, presented in the following subsections.

4.2 Large and principally semisimple processes

Theorem 18 (Asymptotics of large and principally semisimple Pólya processes) *Suppose that the Pólya process $(X_n)_n$ is large and principally semisimple. Fix a Jordan basis $(u_k)_{1 \leq k \leq s}$ of linear forms of the process such that u_2, \dots, u_r ($2 \leq r \leq s$) are all the eigenforms of the basis that are associated with roots $\lambda_2, \dots, \lambda_r$ having σ_2 as real part.*

Then, with notations (12) and (13) of section 2, there exist unique (complex) random variables W_2, \dots, W_r such that

$$X_n = nv_1 + \sum_{2 \leq k \leq r} n^{\lambda_k} W_k v_k + o(n^{\sigma_2}), \quad (43)$$

the small o being almost sure and in L^p for every $p \geq 1$. Furthermore, if one denotes by $(Q_\alpha)_{\alpha \in (\mathbb{Z}_{\geq 0})^s}$ the reduced polynomials of the process relative to the Jordan basis $(u_k)_{1 \leq k \leq s}$, all joint moments of the random variables W_2, \dots, W_r exist and are given by: for all $\alpha_2, \dots, \alpha_r \in \mathbb{Z}_{\geq 0}$,

$$E \left(\prod_{2 \leq k \leq r} W_k^{\alpha_k} \right) = \frac{\Gamma(\tau_1)}{\Gamma(\tau_1 + \langle \alpha, \lambda \rangle)} Q_\alpha(X_1)$$

where $\alpha = \sum_{2 \leq k \leq r} \alpha_k \delta_k = (0, \alpha_2, \dots, \alpha_r, 0, \dots)$.

Remark 15 As can be seen in the proof, for any $k \in \{2, \dots, r\}$, the random variable W_k is defined as the limit of the process $u_k(X_n)/n^{\lambda_k}$ as n tends to infinity. This convergence is almost sure and in any L^p , $p \geq 1$ and obtained by martingale techniques.

Remark 16 To know whether W_k is zero or not, it is enough to check the nullity of $EW_k^2 = \Gamma(\tau_1)Q_{2\delta_k}(X_1)/\Gamma(\tau_1 + 2\lambda_k)$ when W_k is real-valued (that is when λ_k is real), or of $E|W_k|^2 = \Gamma(\tau_1)Q_{\delta_k + \delta_{k'}}(X_1)/\Gamma(\tau_1 + 2\Re\lambda_k)$ when W_k is not real valued (*i.e.* when $\lambda_k \in \mathbb{C} \setminus \mathbb{R}$), where k' is such that $\overline{u_k} = u_{k'}$.

Remark 17 In the asymptotic almost sure expansion (43), σ_2 and the complex numbers λ_k depend only on the conjugacy class of A . On the contrary, the distributions of the random variables W_k depend on the increment vectors w_k and on the linear forms l_k (and on initial condition X_1): two processes having conjugate associated endomorphisms have the same asymptotic form (43), but in general different limit laws W_k . For example, the two standardized large urns having

$A = \begin{pmatrix} 1 & 9/20 \\ 0 & 11/20 \end{pmatrix}$ and $A' = \begin{pmatrix} 3/4 & 1/5 \\ 1/4 & 4/5 \end{pmatrix}$ as (conjugate) replacement matrices have respective second reduced polynomials $Q_{(0,2)} = u_2(u_2 + 11/20)$ (see (30)) and $Q'_{(0,2)} = u_2'^2 - \frac{11}{400}u_2' + \frac{121}{800}u_1'$ (see (42), evident notations). The algebraic relations satisfied by the moments of W and W' are not of the same kind.

A natural question arises: are the limit laws of two processes having conjugate associated endomorphisms connected by some functional relation?

PROOF. We adopt the notations of section 2. Let's denote $\pi = \sum_{2 \leq k \leq r} \pi_k$ and $\pi' = \sum_{k \geq r+1} \pi_k$; the random vector X_n splits into the sum

$$X_n = \pi_1 X_n + Y_n + Z_n, \quad (44)$$

where $Y_n = \pi X_n$ and $Z_n = \pi' X_n$.

- First term $\pi_1 X_n$

Because of relation (7), $\pi_1 X_n = nv_1 + O(1)$ as n tends to infinity, the big O being non random.

- Second term Y_n

As follows from (14), $Y_n = \sum_{k=2}^r u_k(X_n)v_k$. Take any $k \in \{2, \dots, r\}$. The computation of the conditional expectation of $u_k(X_{n+1})$ with regard to the state at time n gives $E^{\mathcal{F}^n} u_k(X_{n+1}) = (1 + \lambda_k/(n + \tau_1 - 1))u_k(X_n)$ for any positive integer n (see (19), u_k is eigenform of the process); this implies that $(\gamma_{\tau_1, n}(\lambda_k)^{-1}u_k(X_n))_n$ is a martingale (one can divide by $\gamma_{\tau_1, n}(\lambda_k)$ because λ_k is not a negative integer). Let p be any integer ≥ 1 and let $l \in \{2, \dots, r\}$ such that $u_l = \overline{u_k}$ (complex conjugacy). Then, thanks to Theorem 17, as $p \times (\delta_k + \delta_l)$ is a semisimple power (the process is principally semisimple),

$$E|u_k^{2p}(X_n)| = E u_k^p(X_n) u_l^p(X_n) = n^{2p\sigma_2} \frac{\Gamma(\tau_1)}{\Gamma(\tau_1 + 2p\sigma_2)} Q_{p(\delta_k + \delta_l)}(X_1) + o(n^{2p\sigma_2}).$$

Note that this is valid even if λ_k is real. The martingales $\gamma_{\tau_1, n}(\lambda_k)^{-1}u_k(X_n)$ are consequently all convergent in every L^p space, $p \geq 1$.

For every $k \in \{2, \dots, r\}$, let W_k be the (complex) random variable defined by

$$W_k = \lim_{n \rightarrow +\infty} \frac{u_k(X_n)}{\gamma_{\tau_1, n}(\lambda_k)} \frac{\Gamma(\tau_1)}{\Gamma(\tau_1 + \lambda_k)} = \lim_{n \rightarrow +\infty} u_k(X_n/n^{\lambda_k})$$

the second equality coming from Stirling's asymptotics as n tends to infinity:

$$\gamma_{\tau_1, n}(\lambda)\Gamma(\tau_1 + \lambda) = \Gamma(\tau_1)n^\lambda(1 + o(1)),$$

for every $\lambda \notin -\tau_1 + \mathbb{Z}_{\leq 0}$. This shows that $Y_n = \sum_{2 \leq k \leq r} n^{\lambda_k} W_k v_k + o(n^{\sigma_2})$, the small o being almost sure and in every L^p space for any $p \geq 1$.

Computation of the limits of joint moments: if $\alpha = (0, \alpha_2, \dots, \alpha_r, 0, \dots) \in (\mathbb{Z}_{\geq 0})^s$, α is a semisimple power of large projections and if one denotes $W^\alpha = \prod_k W_k^{\alpha_k}$, Theorem 17 implies that

$$EW^\alpha = \lim_{n \rightarrow +\infty} \frac{1}{n^{\langle \alpha, \lambda \rangle}} E \mathbf{u}^\alpha(X_n) = \frac{\Gamma(\tau_1)}{\Gamma(\tau_1 + \langle \alpha, \lambda \rangle)} Q_\alpha(X_1).$$

• Third term Z_n

$Z_n = \sum_{k \geq r+1} u_k(X_n) v_k$. We show that $n^{-\sigma_2} u_k(X_n)$ converges to zero almost surely and in every L^p space ($p \geq 1$), for every $k \geq r+1$. Take any $k \geq r+1$, and $l \geq r+1$ such that $u_k = \overline{u_l}$ (complex conjugacy; note that $\Re \lambda_k < 1$). Because of Theorem 17, for a suitable nonnegative integer ν , one has $E|u_k^{2p}(X_n)| = E u_k^p(X_n) u_l^p(X_n) \in O(n^{2p\Re \lambda_k} \log^\nu n)$ if $\Re \lambda_k > 1/2$ and $E|u_k^{2p}(X_n)| \in O(n^p \log^\nu n)$ if $\Re \lambda_k \leq 1/2$. In each case $E|u_k^{2p}(X_n)| \in o(n^{2p\sigma_2})$, which gives the L^p convergences. Furthermore, let p be any positive integer such that $1/p \leq 2(\sigma_2 - \Re \lambda_k)$ if $\Re \lambda_k > 1/2$ or such that $1/p \leq 2\sigma_2 - 1$ if not; for such a p , the series

$$\sum_n E \left| \frac{1}{n^{\sigma_2}} u_k(X_n) \right|^{2p}$$

converges. The almost sure convergence to zero of $n^{-\sigma_2} u_k(X_n)$ follows thus from Borel-Cantelli lemma, and the proof of Theorem 18 is complete. \blacksquare

Remark 18 Processes having 1 as multiple root.

If a Pólya process $(X_n)_n$ has 1 as multiple root, the way to use Theorem 18 to determinate the almost sure limit law of X_n/n suggests to abandon our convention $u_1 = \sum_{k=1}^s l_k$; this does not change the validity of the whole result. Using the so-called graph of the l_k 's and w_k 's (or the graph of the replacement matrix in the urn version), one finds a basis (u_1, \dots, u_r) ($r \geq 2$ is the multiplicity of 1 as eigenvalue of A) of eigenforms of the process with root 1 and a partition I_1, \dots, I_r of $\{1, \dots, s\}$ such that for any $k \in \{1, \dots, r\}$, $u_k(w_j) = 1$ if $j \in I_k$ and $u_k(w_j) = 0$ if $j \notin I_k$ (see [11]). For such a basis, $\sum_{k=1}^s l_k = \sum_{k=1}^r u_k$. This property of u_k 's implies in particular that for any $\alpha = (\alpha_1, \dots, \alpha_r, 0, \dots)$,

$$Q_\alpha = \prod_{k=1}^r u_k(u_k + 1) \dots (u_k + \alpha_k - 1)$$

and Q_α is eigenfunction for Φ , associated with the eigenvalue $|\alpha| = \sum_{k=1}^r \alpha_k$. It follows then from Theorem 18 that X_n/n converges almost surely and in any L^p , $p \geq 1$ to a random vector $\sum_{k=1}^r W_k v_k$, where the joint moments of the real random variables W_1, \dots, W_r are given by

$$EW^\alpha = \frac{\Gamma(\tau_1)}{\Gamma(\tau_1 + |\alpha|)} \prod_{k=1}^r \frac{\Gamma(u_k(X_1) + \alpha_k)}{\Gamma(u_k(X_1))}.$$

One recognizes here the moments of a Dirichlet distribution with parameters $u_1(X_1), \dots, u_r(X_1)$ whose density on the simplex $\{x_1 \geq 0, \dots, x_r \geq 0, \sum_{k=1}^r x_k = 1\}$ of \mathbb{R}^r is given by

$$(x_1, \dots, x_r) \mapsto \Gamma\left(\sum_{k=1}^r u_k(X_1)\right) \prod_{k=1}^r \frac{x_k^{u_k(X_1)}}{\Gamma(u_k(X_1))}.$$

(see [11]). In reference to the original paper of Pólya, the name **essentially Pólya** for processes under this assumption could be given.

Remark 19 Drift when 1 is simple root.

It can be deduced from Theorem 17, decomposition (44) and Borel-Cantelli lemma that X_n/n converges almost surely and in $L^{\geq 1}$ to the non random vector v_1 when 1 is simple root, even for small processes. This result is valid without any irreducibility-type condition on the process (compare with [14]).

4.3 Large and principally nonsemisimple processes

Theorem 19 (Asymptotics of large and principally nonsemisimple Pólya processes) *Suppose that the Pólya process $(X_n)_n$ is large and principally nonsemisimple. Fix a Jordan basis $(u_k)_{1 \leq k \leq s}$ of linear forms of the process; let J_2, \dots, J_r be the principal blocks of indices² and $\nu + 1$ the common size of the J_k 's ($\nu \geq 1$).*

Then, with notations (12) and (13) of section 2, there exist unique (complex) random variables W_2, \dots, W_r such that

$$X_n = nv_1 + \frac{1}{\nu!} \log^\nu n \sum_{2 \leq k \leq r} n^{\lambda(J_k)} W_k v_{\max J_k} + o(n^{\sigma_2} \log^\nu n), \quad (45)$$

the small o being almost sure and in L^p for every $p \geq 1$. Furthermore, if one denotes by $(Q_\alpha)_{\alpha \in (\mathbb{Z}_{\geq 0})^s}$ the reduced polynomials of the process relative to the Jordan basis $(u_k)_{1 \leq k \leq s}$, all joint moments of the random variables W_2, \dots, W_r exist and are given by: for all $\alpha_2, \dots, \alpha_r \in \mathbb{Z}_{\geq 0}$,

$$E \left(\prod_{2 \leq k \leq r} W_k^{\alpha_k} \right) = \frac{\Gamma(\tau_1)}{\Gamma(\tau_1 + \langle \alpha, \lambda \rangle)} Q_\alpha(X_1)$$

where $\alpha = \sum_{2 \leq k \leq r} \alpha_k \delta_{\min J_k}$.

²In other words, if J is any Jordan block of A in the u_k 's basis, J is 1 or one of the J_k 's, or the size of J is $\leq \nu$, or the root of J has a real part $< \sigma_2$. See definition 5 (of a principal block).

Remark 20 As can be seen in the proof, for any $k \in \{2, \dots, r\}$, the random variable W_k is defined as the limit of $u_{\min J_k}(X_n)/n^{\lambda(J_k)}$ as n tends to infinity. This convergence is almost sure and in any L^p , $p \geq 1$ and obtained by martingale techniques (quadratic variation, discrete Burkholder inequality).

Remark 21 As in remark 17, σ_2 and the numbers $\lambda(J_k)$ and ν depend only on the conjugacy class of A .

PROOF. We adopt the notations of section 2. For any monogenic block of indices J , we denote by π_J the projection $\pi_J = \sum_{k \in J} \pi_k$.

• **Claim** *If J is a monogenic block of indices associated with a root λ having a real part $\sigma > 1/2$, then $\gamma_{\tau_1, n}(\pi_J A)$ is invertible and $\gamma_{\tau_1, n}(\pi_J A)^{-1} \pi_J X_n$ is a martingale that converges in L^p for every $p \geq 1$ (thus almost surely). If M_J denotes the limit of this martingale and if $r = \#J - 1$, then*

$$\pi_J X_n = \frac{n^\lambda \log^r n}{r!} \frac{\Gamma(\tau_1)}{\Gamma(\tau_1 + \lambda)} u_{\min J}(M_J) v_{\max J} + o(n^\sigma \log^r n) \quad (46)$$

as n tends to infinity, the small o being almost sure and in L^p for every $p \geq 1$. Furthermore, almost surely and in L^p for every $p \geq 1$,

$$u_{\min J}(M_J) = \frac{\Gamma(\tau_1 + \lambda)}{\Gamma(\tau_1)} \lim_{n \rightarrow \infty} \frac{u_{\min J}(X_n)}{n^\lambda}. \quad (47)$$

$\gamma_{\tau_1, n}(\pi_J A)$ is invertible because every $\text{Id} + \pi_J A / (k + \tau_1 - 1)$ is (its unique eigenvalue has a real part > 1). Since J is a monogenic block of indices, A and π_J commute. Thus $M_n = \gamma_{\tau_1, n}(\pi_J A)^{-1} \pi_J X_n$ is a martingale (see (19) with $f = \pi_J$ and remark 2 in subsection 3.1). We show that for any $k \in J$, the quadratic variation of the martingale $u_k(M_n)$ is almost surely bounded, which is enough, thanks to Burkholder's inequality for discrete time martingales (see [13] for example), to ensure that the projection $u_k(M_n)$ is bounded in L^p for every $p \geq 1$, hence the validity of the convergence part of the claim.

Without loss of generality, we can assume for simplicity that $J = \{2, \dots, r+2\}$. If one denotes $N = \pi_J(A - \lambda)$, then N commutes with A and satisfies $N^r \neq 0$ and $N^{r+1} = 0$; furthermore, elementary considerations on A , the u_k 's and the v_k 's show that for any nonnegative integer q and for any $k \in \{2, \dots, r+2\}$, one has $N^q \pi_k = u_k v_{k+q}$ if $k+q \leq r+2$ and $N^q \pi_k = 0$ if $k+q \geq r+3$. In particular, for any q , one can write $N^q = N^q (\sum_{k \in J} \pi_k) = \sum_{q+2 \leq k \leq r+2} u_{k-q} v_k$ (with the convention $N^0 = \pi_J$). Hence, if $\beta_n = 1/\gamma_{\tau_1, n}$ (as formal series or rational fraction; we omit the parameter τ_1 for simplicity of notation), Taylor formula leads to

$$M_n = \sum_{q=0}^r \frac{1}{q!} \beta_n^{(q)}(\lambda) N^q X_n = \sum_{k=2}^{r+2} \left(\sum_{q=0}^{k-2} \frac{1}{q!} \beta_n^{(q)}(\lambda) u_{k-q}(X_n) \right) v_k. \quad (48)$$

Thus, for any $k \in \{2, \dots, r+2\}$, one has $u_k(M_n) = \sum_{q=0}^{k-2} \frac{1}{q!} \beta_n^{(q)}(\lambda) u_{k-q}(X_n)$ and

$$u_k(M_{n+1}) - u_k(M_n) = \sum_{q=0}^{k-2} \frac{1}{q!} \beta_{n+1}^{(q)}(\lambda) \left[u_{k-q}(X_{n+1}) - \frac{\beta_n^{(q)}(\lambda)}{\beta_{n+1}^{(q)}(\lambda)} u_{k-q}(X_n) \right]. \quad (49)$$

One can write

$$\begin{aligned} u_{k-q}(X_{n+1}) - \frac{\beta_n^{(q)}(\lambda)}{\beta_{n+1}^{(q)}(\lambda)} u_{k-q}(X_n) &= u_{k-q}(X_{n+1} - X_n) \\ &+ \left[1 - \frac{\beta_n^{(q)}(\lambda)}{\beta_{n+1}^{(q)}(\lambda)} \right] u_{k-q}(X_n). \end{aligned}$$

The relation $\beta_n(\lambda) = (1 + \lambda/(n + \tau_1 - 1))\beta_{n+1}(\lambda)$ implies, with Leibnitz formula, that

$$1 - \frac{\beta_n^{(q)}(\lambda)}{\beta_{n+1}^{(q)}(\lambda)} \in O\left(\frac{1}{n}\right).$$

Besides, the definition of the process $(X_n)_n$ itself (definition 1) implies that $X_{n+1} - X_n \in \{w_1, \dots, w_s\}$ is almost surely $O(1)$ and that X_n is almost surely $O(n)$ as n tends to infinity (elementary induction). Hence

$$u_{k-q}(X_{n+1}) - \frac{\beta_n^{(q)}(\lambda)}{\beta_{n+1}^{(q)}(\lambda)} u_{k-q}(X_n) \in O(1) \quad (50)$$

almost surely, as n tends to infinity. With the same tools as for the derivatives of $\gamma_{\tau_1, n}$ (see (20)), for every nonnegative integer q ,

$$\beta_n^{(q)}(\lambda) = \frac{\log^q n \Gamma(\tau_1 + \lambda)}{n^\lambda \Gamma(\tau_1)} + o\left(\frac{\log^q n}{n^{\Re \lambda}}\right) \quad (51)$$

as n tends to infinity. Thus (49), (50) and (51) allows one to conclude that

$$u_k(M_{n+1}) - u_k(M_n) \in O\left(\frac{\log^{k-2} n}{n^{\Re \lambda}}\right)$$

almost surely as n tends to infinity. In particular, $|u_k(M_{n+1}) - u_k(M_n)|^2$ is almost surely the general term of a convergent series: the quadratic variation of the martingale $u_k(M_n)$ is almost surely bounded and the convergence part of the claim is proved.

Almost surely and in L^p for every $p \geq 1$,

$$\pi_J X_n = \gamma_{\tau_1, n}(\pi_J A) [\gamma_{\tau_1, n}(\pi_J A)^{-1} \pi_J X_n] = \gamma_{\tau_1, n}(\pi_J A)(M_J + o(1))$$

as n tends to infinity. As for equation (48), one has

$$\gamma_{\tau_1, n}(\pi_J A) = \sum_{k=2}^{r+2} \left(\sum_{q=0}^{k-2} \frac{1}{q!} \gamma_{\tau_1, n}^{(q)}(\lambda) u_{k-q} \right) v_k$$

and the asymptotics of derivatives of $\gamma_{\tau_1, n}$ (see (20)) implies

$$\pi_J X_n = \frac{n^\lambda \log^r n}{r!} \frac{\Gamma(\tau_1)}{\Gamma(\tau_1 + \lambda)} u_2(M_J) v_{r+2} + o(n^\sigma \log^r n)$$

which is the expected result (46) on $\pi_J X_n$. Equation (48) shows that $u_2(M_n) = \beta_n(\lambda) u_2(X_n)$ and allows one to conclude with the help of (51).

- As in the proof of the large and principally semisimple case (Theorem 18), $\pi_1 X_n = (n + \tau_1 - 1) v_1$, and the process splits into the sum

$$X_n = n v_1 + \sum_{k=2}^r \pi_{J_k} X_n + Y_n + Z_n$$

where $Y_n = \sum_J \pi_J X_n$ the sum being extended to all monogenic blocks of indices different of any J_k that correspond to roots having real parts $> 1/2$ and $Z_n = \sum_{\{k, \Re \lambda_k \leq 1/2\}} \pi_k X_n$. We study the behaviour of the terms of this sum separately.

- Because of Theorem 17 part 1- and Borel-Cantelli lemma, as in the proof of the large and principally semisimple case, $Z_n \in o(n^{\sigma_2} \log^\nu n)$ almost surely and in L^p for every $p \geq 1$ (remember that $\pi_k X_n = u_k(X_n) v_k$).
- Every J in the definition of Y_n satisfies the assumption of the claim with a root's real part $< \sigma_2$ or a cardinality $\leq \nu$. Thus almost surely and in L^p for every $p \geq 1$,

$$Y_n = o(n^{\sigma_2} \log^\nu n)$$

as n tends to infinity.

- For every $k \in \{2, \dots, r\}$, J_k satisfies the assumption of the claim and if one denotes

$$W_k = \lim_{n \rightarrow \infty} \frac{u_{\min J_k}(X_n)}{n^{\lambda(J_k)}},$$

one obtains almost surely and in L^p for every $p \geq 1$

$$\pi_{J_k} X_n = \frac{1}{\nu!} n^{\lambda(J_k)} \log^\nu n W_k v_{\max J_k} + o(n^{\sigma_2} \log^\nu n)$$

almost surely and in L^p for every $p \geq 1$, which completes the proof of (45). Note that $u_{\min J_k}$ is an eigenform of A and that $\gamma_{\tau_1, n}(\lambda(J_k))^{-1} u_{\min J_k}(X_n)$ is an $L^{\geq 1}$ -convergent complex martingale.

- Take any $\alpha_2, \dots, \alpha_r \in \mathbb{Z}_{\geq 0}$. Then $\alpha = \sum_{2 \leq k \leq r} \alpha_k \delta_{\min J_k}$ is a semisimple power of large projections, and

$$E \left(\prod_{2 \leq k \leq r} W_k^{\alpha_k} \right) = \lim_{n \rightarrow \infty} \frac{1}{n^{\langle \alpha, \lambda \rangle}} E \mathbf{u}^\alpha(X_n) = \frac{\Gamma(\tau_1)}{\Gamma(\tau_1 + \langle \alpha, \lambda \rangle)} Q_\alpha(X_1)$$

as can be deduced from Theorem 17 point 3-. This completes the proof of the theorem. ■

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