



HAL
open science

Questions on surface braid groups

Paolo Bellingeri, Eddy Godelle

► **To cite this version:**

| Paolo Bellingeri, Eddy Godelle. Questions on surface braid groups. 2005. hal-00004548

HAL Id: hal-00004548

<https://hal.science/hal-00004548>

Preprint submitted on 29 Mar 2005

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

QUESTIONS ON SURFACE BRAID GROUPS

PAOLO BELLINGERI AND EDDY GODELLE

ABSTRACT. We provide new group presentations for surface braid groups which are positive. We study some properties of such presentations and we solve the conjugacy problem in a particular case.

1. INTRODUCTION AND MOTIVATION

Let $\Sigma_{g,p}$ be an orientable surface of genus g with p boundary components. For instance, $\Sigma_{0,0}$ is the 2-sphere, $\Sigma_{1,0}$ is the torus, and $\Sigma_{0,1}$ corresponds to the disk.

A *geometric braid* on $\Sigma_{g,p}$ based at \mathcal{P} is a collection $B = (\psi_1, \dots, \psi_n)$ of $n \geq 2$ paths from $[0, 1]$ to $\Sigma_{g,p}$ such that $\psi_i(0) = P_i$, $\psi_i(1) \in \mathcal{P}$ and $\{\psi_1(t), \dots, \psi_n(t)\}$ are distinct points for all $t \in [0, 1]$. Two braids are considered as equivalent if they are isotopic. The usual product of paths defines a group structure on the equivalence classes of braids. This group does not depend, up to isomorphism, on the choice of \mathcal{P} . It is called *the (surface) braid group on n strands* on $\Sigma_{g,p}$ and denoted by $B_n(\Sigma_{g,p})$. The group $B_n(\Sigma_{0,1})$ is the classical braid group B_n on n strings. Some elements of $B_n(\Sigma_{g,p})$ are shown in Figure 1. The braid σ_i corresponds to the standard generator of B_n and it can be represented by a geometric braid on $\Sigma_{g,p}$ where all the strands are trivial except the i -th one and the $(i+1)$ -th one. The i -th strand goes from P_i to P_{i+1} and the $(i+1)$ -th strand goes from P_{i+1} to P_i according to Figure 1. The loops $\delta_1, \dots, \delta_{p+2g-1}$ based on P_1 in Figure 1 represent standard generators of $\pi_1(\Sigma_{g,p})$. By its definition, $B_1(\Sigma_{g,p})$ is isomorphic to $\pi_1(\Sigma_{g,p})$. We can also consider $\delta_1, \dots, \delta_{p+2g-1}$ as braids on n strands on Σ , where last $n-1$ strands are trivial.

It is well known since E. Artin ([2]) that the braid group B_n has a positive presentation (see for instance [17] Chapter 2 Theorem 2.2), i.e. a group presentation which involves only generators and not their inverses. Hence one can associate a (braid) monoid B_n^+ with the same presentation, but as a monoid presentation. It turns out that the braid monoid B_n^+ is a Garside monoid (see [8]), that is a monoid with a good divisibility structure, and that the braid group B_n is the group of fractions of the monoid B_n^+ . As a consequence, the natural morphism of monoids from B_n^+ to B_n is onto, and we can solve the word problem, the conjugacy problem and obtain normal forms in B_n (see [4, 6, 8, 10, 12]). These results extend to Artin-Tits groups of spherical type which are a well-known algebraic generalization of the braid group B_n ([4, 6, 8, 10]).

In the case of surface braid groups $B_n(\Sigma_{g,p})$, some group presentations are known but they are not positive. Furthermore, questions as the conjugacy problem are not solved in the general case. The word problem in surface braid groups is known to be solvable (see [14]) even if algorithms are not as efficient as the ones proposed for the braid group B_n .

In this note we provide positive presentations for $B_n(\Sigma_{g,p})$ and we address questions related to the conjugacy problem of surface braid groups. We do not discuss the case of $B_n(\Sigma_{0,0})$, the braid group on the 2-sphere; this is a particular case with specific properties. For instance if Σ is an oriented surface, the surface braid group $B_n(\Sigma)$ has torsion elements only and only if Σ is the

2-sphere (see [13] page 277, [11] page 255, and [19] proposition 1.5).

In Section 2 and 3 we focus on braid groups on surfaces with boundary components and without boundary components respectively. In Section 4, we investigate the special case of $B_2(\Sigma_{1,0})$ and we solve the word problem and the conjugacy problem for this group.

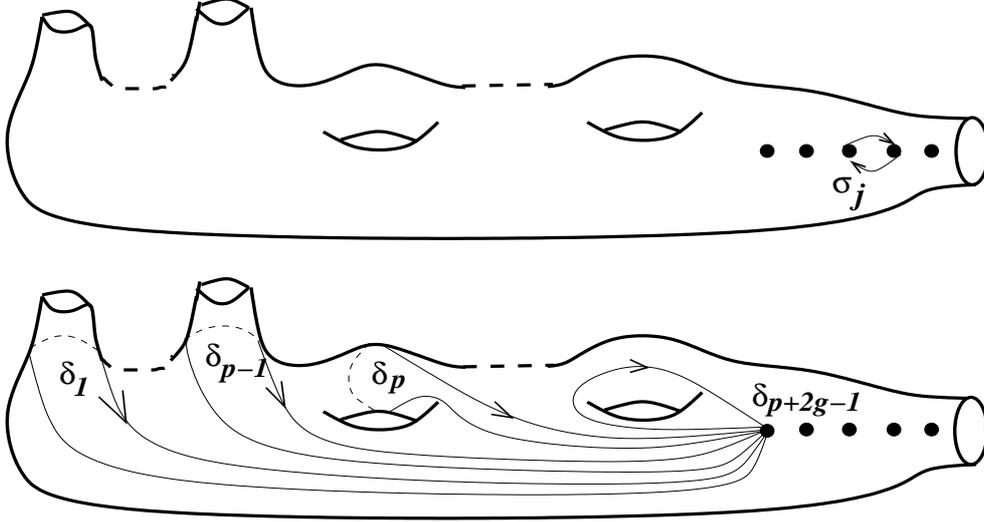


Figure 1: some braid elements

2. BRAID GROUPS ON SURFACES WITH BOUNDARY COMPONENTS

In this section we investigate braid groups on oriented surfaces with a positive number of boundary components. Our first objective is to prove Theorem 2.1:

Theorem 2.1. *Let n and p be positive integers. Let g be a non negative integer. Then, the group $B_n(\Sigma_{g,p})$ admits the following group presentation:*

- *Generators:* $\sigma_1, \dots, \sigma_{n-1}, \delta_1, \dots, \delta_{2g+p-1}$;

- *Relations:*

- *Braid relations:*

$$(BR1) \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| \geq 2;$$

$$(BR2) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } 1 \leq i \leq n - 1.$$

- *Commutative relations between surface braids:*

$$(CR1) \quad \delta_r \sigma_i = \sigma_i \delta_r \quad \text{for } i \neq 1; 1 \leq r \leq 2g + p - 1;$$

$$(CR2) \quad \delta_r \sigma_1 \delta_r \sigma_1 = \sigma_1 \delta_r \sigma_1 \delta_r \quad 1 \leq r \leq 2g + p - 1;$$

$$(CR3) \quad \delta_r \sigma_1 \delta_r \delta_s \sigma_1 = \sigma_1 \delta_r \delta_s \sigma_1 \delta_r \quad \text{for } 1 \leq r < s \leq 2g + p - 1 \text{ with } (r, s) \neq (p + 2i, p + 2i + 1), 0 \leq i \leq g - 1.$$

- *Skew commutative relations on the handles:*

$$(SCR1) \quad \sigma_1 \delta_{r+1} \sigma_1 \delta_r \sigma_1 = \delta_r \sigma_1 \delta_{r+1} \quad \text{for } r = p + 2i \text{ where } 0 \leq i \leq g - 1.$$

The above presentation can be compared to the presentation of $B_{g,n}$ given in [15] page 18.

Proof. Let us denote by $\widetilde{B}_n(\Sigma_{g,p})$ the group defined by the presentation given in Theorem 2.1. We prove that the group $B_n(\Sigma_{g,p})$ is isomorphic to the group $\widetilde{B}_n(\Sigma_{g,p})$ using the presentation given in

Theorem A.1. Let $\psi : \{\sigma_1, \dots, \sigma_{n-1}, a_1, \dots, a_g, b_1, \dots, b_g, z_1, \dots, z_{p-1}\} \rightarrow \{\sigma_1, \dots, \sigma_{n-1}, \delta_1, \dots, \delta_{2g+p-1}\}$ be the set-map defined by $\psi(\sigma_i) = \sigma_i$ for $i = 1, \dots, n-1$, $\psi(a_r) = \delta_{p+2(r-1)}^{-1}$, $\psi(b_r) = \delta_{p+2(r-1)+1}^{-1}$ for $r = 1, \dots, g$ and $\psi(z_j) = \delta_j^{-1}$ for $j = 1, \dots, p-1$. We claim that ψ extends to a homomorphism of groups $\psi : B_n(\Sigma_{g,p}) \rightarrow \widetilde{B_n(\Sigma_{g,p})}$. We have to verify that the image by ψ of the braid relations and of the relations of type (R1)-(R8) are true in $\widetilde{B_n(\Sigma_{g,p})}$. It is enough to verify that

$$(\dagger) \quad \sigma_1^{-1} \delta_r^{-1} \sigma_1 \delta_s^{-1} = \delta_s^{-1} \sigma_1^{-1} \delta_r^{-1} \sigma_1$$

for $1 \leq r < s \leq 2g+p-1$ and $(r, s) \neq (p+2k, p+2k+1)$ which corresponds to the image by ψ in $B_n(\Sigma_{g,p})$ of the relations of type (R3), (R6) and (R7); the other cases are true as they are relations of the presentation of $\widetilde{B_n(\Sigma_{g,p})}$.

The relations of type (CR3) can be written $\delta_r \sigma_1 \delta_r \delta_s = \sigma_1 \delta_r \delta_s \sigma_1 \delta_r \sigma_1^{-1}$. From the relations of type (CR1) we deduce that, in $\widetilde{B_n(\Sigma_{g,p})}$, the equalities $\delta_r \sigma_1 \delta_r \delta_s = \delta_r \sigma_1 \delta_r \sigma_1 \sigma_1^{-1} \delta_s = \sigma_1 \delta_r \sigma_1 \delta_r \sigma_1^{-1} \delta_s$ holds. Hence we obtain $\sigma_1 \delta_r \delta_s \sigma_1 \delta_r \sigma_1^{-1} = \sigma_1 \delta_r \sigma_1 \delta_r \sigma_1^{-1} \delta_s$. From this equality, we derive that $\delta_s \sigma_1 \delta_r \sigma_1^{-1} = \sigma_1 \delta_r \sigma_1^{-1} \delta_s$, and finally we get the relations (\dagger) .

On the other hand, consider $\overline{\psi}$ the set-map defined from $\{\sigma_1, \dots, \sigma_{n-1}, \delta_1, \dots, \delta_{2g+p-1}\}$ to $\{\sigma_1, \dots, \sigma_{n-1}, a_1, \dots, a_g, b_1, \dots, b_g, z_1, \dots, z_{p-1}\}$ by $\overline{\psi}(\sigma_i) = \sigma_i$ for $i = 1, \dots, n-1$, $\overline{\psi}(\delta_j) = z_j^{-1}$ for $j = 1, \dots, p-1$, $\overline{\psi}(\delta_{p+2(r-1)}) = a_r^{-1}$ and $\overline{\psi}(\delta_{p+2(r-1)+1}) = b_r^{-1}$ for $r = 1, \dots, g$. We prove that $\overline{\psi}$ extends to an homomorphism of groups from $B_n(\Sigma_{g,p})$ to $B_n(\Sigma_{g,p})$. Since braid relations and the images by $\overline{\psi}$ of the relations of type (ER), (CR1) and (CR2) are verified, it suffices to check that the equalities corresponding to relations of type (CR3) hold in $B_n(\Sigma_{g,p})$. We verify that the equality $a_r^{-1} \sigma_1 a_r^{-1} a_s^{-1} \sigma_1 = \sigma_1 a_r^{-1} a_s^{-1} \sigma_1 a_r^{-1}$ for $(1 \leq r > s \leq g)$ holds in $B_n(\Sigma_{g,p})$. The other cases can easily be verified by the reader. From the relations of type (R2), it follows that $a_r^{-1} \sigma_1 a_r^{-1} = \sigma_1 a_r^{-1} \sigma_1 a_r^{-1} \sigma_1^{-1}$; thus we have $a_r^{-1} \sigma_1 a_r^{-1} a_s^{-1} \sigma_1 = \sigma_1 a_r^{-1} \sigma_1 a_r^{-1} \sigma_1^{-1} a_s^{-1} \sigma_1$. Applying relations of type (R3) we deduce that $a_r^{-1} \sigma_1^{-1} a_s^{-1} \sigma_1 = \sigma_1^{-1} a_s^{-1} \sigma_1 a_r^{-1}$ and therefore the equalities $a_r^{-1} \sigma_1 a_r^{-1} a_s^{-1} \sigma_1 = \sigma_1 a_r^{-1} a_s^{-1} \sigma_1 a_r^{-1}$ hold in $B_n(\Sigma_{g,p})$. Then, the morphism $\overline{\psi}$ from $\widetilde{B_n(\Sigma_{g,p})}$ to $B_n(\Sigma_{g,p})$ is well defined and it is the inverse of ψ . Hence, $B_n(\Sigma_{g,p})$ is isomorphic to $\widetilde{B_n(\Sigma_{g,p})}$. \square

We remark that the presentation given in Theorem 2.1 is positive and has less types of relations than the presentation given in Theorem A.1.

Lemma 2.2. *Let G be a group and let σ, δ, δ' be in G .*

(i) *If (a) $\delta(\sigma\delta\delta'\sigma) = (\sigma\delta\delta'\sigma)\delta$, (b) $\sigma\delta\sigma\delta = \delta\sigma\delta\sigma$ and (c) $\sigma\delta'\sigma\delta' = \delta'\sigma\delta'\sigma$ then $\delta'(\sigma\delta\delta'\sigma) = (\sigma\delta\delta'\sigma)\delta'$.*

(ii) *If (a) $\delta(\sigma\delta\delta'\sigma) = \sigma(\delta\delta'\sigma\delta)$, (b) $\delta'(\sigma\delta\delta'\sigma) = (\sigma\delta\delta'\sigma)\delta'$ and (c) $\sigma\delta'\sigma\delta' = \delta'\sigma\delta'\sigma$ then $\sigma\delta\sigma\delta = \delta\sigma\delta\sigma$.*

(iii) *In the presentation of Theorem 2.1, we can replace relation (CR3) by:*

$$(CR3') \quad \delta_s \sigma_1 \delta_r \delta_s \sigma_1 = \sigma_1 \delta_r \delta_s \sigma_1 \delta_s$$

for $1 \leq r < s \leq 2g+p-1$ with $(r, s) \neq (p+2i, p+2i+1)$, $0 \leq i \leq g-1$.

Proof. (i) Assume (a) $\delta\sigma\delta\delta'\sigma = \sigma\delta\delta'\sigma\delta$, (b) $\sigma\delta\sigma\delta = \delta\sigma\delta\sigma$ and (c) $\sigma\delta'\sigma\delta' = \delta'\sigma\delta'\sigma$. Then, $\delta'\sigma\delta\delta'\sigma = \delta^{-1}\sigma^{-1}\sigma\delta\delta'\sigma\delta\delta'\sigma = \delta^{-1}\sigma^{-1}\delta\sigma\delta\delta'\sigma\delta'\sigma = \delta^{-1}\sigma^{-1}\delta\sigma\delta\sigma\delta'\sigma\delta' = \delta^{-1}\sigma^{-1}\sigma\delta\sigma\delta\delta'\sigma\delta' = \sigma\delta\delta'\sigma\delta'$.

(ii) Assume (a) $\delta(\sigma\delta\delta'\sigma) = \sigma(\delta\delta'\sigma\delta)$, (b) $\delta'(\sigma\delta\delta'\sigma) = (\sigma\delta\delta'\sigma)\delta'$ and (c) $\sigma\delta'\sigma\delta' = \delta'\sigma\delta'\sigma$. Then

$$\begin{aligned} \sigma\delta\sigma\delta &= \sigma\delta\delta'\sigma\delta(\delta'\sigma\delta)^{-1}\sigma\delta = \delta\sigma\delta\delta'\sigma\delta^{-1}\sigma^{-1}\delta'^{-1}\sigma\delta = \delta\sigma\delta(\sigma\delta'\sigma\delta'\sigma^{-1}\delta'^{-1})\delta^{-1}\sigma^{-1}\delta'^{-1}\sigma\delta = \\ &= \delta\sigma\delta\sigma\delta'\sigma\delta'\delta'^{-1}\sigma^{-1}\delta'^{-1}\delta^{-1}\sigma^{-1}\sigma\delta = \delta\sigma\delta\sigma. \end{aligned}$$

(iii) is a consequence of (i). \square

Since the relations of the presentation of $B_n(\Sigma_{g,p})$ are positive, one can define a monoid with the same presentation but as a monoid presentation. It is easy to see that the monoid we obtain doesnot inject in $B_n(\Sigma_{g,p})$, even if we add the relations of type $(CR3')$ to the presentation given in Theorem 2.1. In fact the following relations,

$$(CR3)_k \quad \delta_r\sigma_1\delta_r\delta_s^k\sigma_1 = \sigma_1\delta_r\delta_s^k\sigma_1\delta_r$$

for $1 \leq r < s \leq 2g + p - 1$ with $(r, s) \neq (p + 2i, p + 2i + 1)$, $0 \leq i \leq g - 1$ and $k \in \mathbb{N}^*$, and

$$(CR3')_k \quad \delta_s\sigma_1\delta_r^k\delta_s\sigma_1 = \sigma_1\delta_r^k\delta_s\sigma_1\delta_s$$

for $1 \leq r < s \leq 2g + p - 1$ with $(r, s) \neq (p + 2i, p + 2i + 1)$, $0 \leq i \leq g - 1$ and $k \in \mathbb{N}^*$, are true in $B_n(\Sigma_{g,p})$ for each positive integer k , but they are false in the monoid for k greater than 1: no relation of the presentation can be applied to the left side of the equalities. Then starting from the left side of the equality for $k > 1$, we cannot obtain the right side of the equality by using the relations of the monoid presentation only.

Question 1. *Let $B_n^*(\Sigma_{g,p})$ be the monoid defined by the presentation of Theorem 2.1 with the extra relations $(CR3)_k$, $(CR3')_k$ for $k \in \mathbb{N}^*$. Is the canonical homomorphism φ from $B_n^*(\Sigma_{g,p})$ to $B_n(\Sigma_{g,p})$ into ?*

We remark that we can define a length function ℓ on $B_n^*(\Sigma_{g,p})$: if F^* is the free monoid based on $\sigma_1 \cdots, \sigma_{n-1}, \delta_1, \dots, \delta_{2g}$, if $l : F^* \rightarrow \mathbb{N}$ is the canonical length function and if $w \mapsto \bar{w}$ is the canonical morphism from F^* onto $B_n^*(\Sigma_{g,p})$ then, for each g in $B_n^*(\Sigma_{g,p})$, one has $\sup\{l(w) \mid w \in F^* ; \bar{w} = g\} < +\infty$; furthermore if we set $\ell(g) = \sup\{l(w) \mid w \in F^*\}$, then for g_1, g_2 in $B_n^*(\Sigma_{g,p})$ we have $\ell(g_1g_2) \leq \ell(g_1) + \ell(g_2)$.

Now, let us consider the particular case of planar surfaces.

Proposition 2.3. *Let n, p be positive integers with $n \geq p - 1$. Let $I \subset \{1, \dots, n\}$ with $\text{Card}(I) = p - 1$.*

Then $B_n(\Sigma_{0,p})$ admits the following presentation:

• *Generators : $\sigma_1, \dots, \sigma_{n-1}$ and ρ_i for $i \in I$;*

• *Relations:*

$$\begin{array}{ll} (BR1) & \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } 1 \leq i, j \leq n - 1 \text{ with } |i - j| \geq 2; \\ (BR1)' & \rho_r \rho_s = \rho_s \rho_r & r, s \in I, r \neq s; \\ (BR1)'' & \rho_r \sigma_i = \sigma_i \rho_r & r \in I; 1 \leq i \leq n - 1, i \neq r - 1, r. \\ (BR2) & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } 1 \leq i \leq n - 1; \\ (BR3) & \sigma_i \rho_r \sigma_i \rho_r = \rho_r \sigma_i \rho_r \sigma_i & r \in I, i = r, r - 1; \\ (BR3)' & (\sigma_{r-1} \sigma_r) \rho_r \sigma_{r-1} \rho_r = \rho_r (\sigma_{r-1} \sigma_r) \rho_r \sigma_{r-1} & r \in I ; r \neq 1, n. \end{array}$$

Proof. Consider the presentation of Theorem 2.1. Let $I = \{r_1 < r_2 \cdots < r_{p-1}\}$ and set $\rho_{r_j} = (\sigma_{r-1} \cdots \sigma_1) \delta_{p-j} (\sigma_{r-1} \cdots \sigma_1)^{-1}$. Relations $(BR1)''$ are equivalent to relations $(CR1)$ with $i \neq r$, by using braid relations. Using $(CR2)$, we get $(BR1)' \iff (CR3)$. Using braid relations $(BR1)$ and $(BR2)$, the relations $(CR2)$ are equivalent to relations $(BR3)$ by conjugation by $(\sigma_{r-1} \sigma_r) \cdots (\sigma_1 \sigma_2)$ and $(\sigma_{r-2} \sigma_{r-1}) \cdots (\sigma_1 \sigma_2)$ when $i = r-1$ and $i = r$ respectively. Now consider the relation $(CR1)$ for $i = r$. By conjugation by $\sigma_{r-1} \cdots \sigma_1$, we get $\sigma_{r-1} \sigma_r \sigma_{r-1}^{-1} \rho_r = \rho_r \sigma_{r-1} \sigma_r \sigma_{r-1}^{-1}$ and, then $\sigma_{r-1} \sigma_r \rho_r \sigma_{r-1} = \sigma_r \rho_r \sigma_{r-1} \sigma_r$. It follows that the relations of type $(CR1)$ for $i = r$ is equivalent to the relation $\sigma_{r-1} \sigma_r \rho_r \sigma_{r-1} = \sigma_r \rho_r \sigma_{r-1} \sigma_r$. This last relation is equivalent to relation $(BR3)'$ using relation $(BR3)$:

$$\begin{aligned} \sigma_{r-1} \sigma_r \rho_r \sigma_{r-1} &= \sigma_r \rho_r \sigma_{r-1} \sigma_r \iff \sigma_{r-1} \sigma_r \rho_r \sigma_{r-1} \rho_r \sigma_{r-1} = \sigma_r \rho_r \sigma_{r-1} \sigma_r \rho_r \sigma_{r-1} \iff \\ \sigma_{r-1} \sigma_r \sigma_{r-1} \rho_r \sigma_{r-1} \rho_r &= \sigma_r \rho_r \sigma_{r-1} \sigma_r \rho_r \sigma_{r-1} \iff \sigma_r \sigma_{r-1} \sigma_r \rho_r \sigma_{r-1} \rho_r = \sigma_r \rho_r \sigma_{r-1} \sigma_r \rho_r \sigma_{r-1} \iff \\ \sigma_{r-1} \sigma_r \rho_r \sigma_{r-1} \rho_r &= \rho_r \sigma_{r-1} \sigma_r \rho_r \sigma_{r-1}. \end{aligned}$$

□

Corollary 2.4. ([1] Table 1.1)

$B_n(\Sigma_{0,3})$ is isomorphic to the Affine Artin group of type $\tilde{B}(n+1)$ for $n \geq 2$.

Proof. We apply Proposition 2.3 with $I = \{1, n\}$. □

Recall that a monoid M is cancellative if the property “ $\forall x, y, z, t \in M, (xyz = xtz) \Rightarrow (y = t)$ ” holds in M .

Question 2. Let $B_n^+(\Sigma_{0,p})$ the monoid defined by the presentation given in Proposition 2.3, considered as a monoid presentation.

- (i) Is the monoid $B_n^+(\Sigma_{0,p})$ cancellative ?
- (ii) is the natural homomorphism from $B_n^+(\Sigma_{0,p})$ to $B_n(\Sigma_{0,p})$ injective ?

For $p = 1$ and $p = 2$, the groups $B_n(\Sigma_{0,p})$ are isomorphic to the braid group B_n and the Artin-Tits group of type B respectively. Hence, the answer to above questions are positive. In the case of $B_n(\Sigma_{0,3})$, the answers are also positive (see [7] and [18]). Note that the relations of the presentation of $B_n(\Sigma_{0,p})$ are homogeneous. Therefore we can define a length function ℓ on $B_n^+(\Sigma_{0,p})$ such that $\ell(g_1 g_2) = \ell(g_1) + \ell(g_2)$ for every $g_1, g_2 \in B_n^+(\Sigma_{0,p})$.

3. BRAID GROUPS ON CLOSED SURFACES

In this section, we consider braid groups on closed surfaces, that is without boundary components. In particular, we prove Corollaries 3.2 and 3.3.

Proposition 3.1. Let n, g be positive integers. The group $B_n(\Sigma_{g,0})$ admits the following presentation:

- Generators: $\sigma_1 \cdots, \sigma_{n-1}, \delta_1, \cdots, \delta_{2g}$;

- *Relations*

-Braid relations: $(BR1) \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| \geq 2.$
 $(BR2) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad 1 \leq i \leq n - 1;$

-Commutative relation between surface braids:

$$(CR1) \quad \sigma_i \delta_r = \delta_r \sigma_i \quad 3 \leq i \leq n - 1; 1 \leq r \leq 2g;$$

$$(CR4) \quad \sigma_1 \delta_r^2 = \delta_r^2 \sigma_1 \quad 1 \leq r \leq 2g;$$

$$\sigma_2 \delta_{2r-1} \sigma_2 = \delta_{2r-1} \sigma_2 \sigma_1 \quad 1 \leq r \leq 2g;$$

$$\sigma_1 \delta_{2r} \sigma_2 = \sigma_2 \delta_{2r} \sigma_1 \quad 1 \leq r \leq 2g;$$

-Skew commutative relations on the handles:

$$(SCR2) \quad \sigma_1 \delta_r \delta_{r+2s} \sigma_1 = \delta_{r+2s} \delta_r \quad 1 \leq r < r + 2s \leq 2g;$$

$$(SCR3) \quad \delta_{2r} \sigma_1 \delta_{2s-1} \delta_{2r} \sigma_1 = \delta_{2s-1} \delta_{2r}^2 \quad 1 \leq s \leq r \leq g;$$

$$\sigma_1 \delta_{2s} \delta_{2r-1} \sigma_1 \delta_{2s} = \delta_{2s}^2 \delta_{2r-1} \quad 1 \leq s < r \leq g;$$

-Relation associated to the fundamental group of the surface:

$$(FGR) \quad (\sigma_2 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_2) \sigma_1 \delta_1 \delta_2 \cdots \delta_{2g} \sigma_1 = \delta_{2g} \cdots \delta_2 \delta_1.$$

Proof. Starting from the presentation of Theorem A.2, we set $\sigma_i = \theta_i^{-1}$, $\delta_{2r} = b_{2r} \theta_1^{-1}$ and $\delta_{2r-1} = \theta_1 b_{2r-1}^{-1}$; we obtain easily the required presentation. \square

Corollary 3.2. *Let n and g be positive integers with $g \geq 2$. Then, the group $B_n(\Sigma_{g,0})$ admits the following group presentation:*

- *Generators:* $\sigma_1, \dots, \sigma_{n-1}, \delta_1, \dots, \delta_{2g};$

- *Relations:*

-Braid relations:

$$(BR1) \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } 1 \leq i, j \leq n - 1 \text{ with } |i - j| \geq 2;$$

$$(BR2) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } 1 \leq i \leq n - 1.$$

-Commutative relations between surface braids:

$$(CR1) \quad \delta_r \sigma_i = \sigma_i \delta_r \quad \text{for } 2 < i; 1 \leq r \leq 2g;$$

$$(CR4) \quad \delta_r^2 \sigma_1 = \sigma_1 \delta_r^2 \quad \text{for } 1 \leq r \leq 2g;$$

$$\sigma_2 \delta_{2r-1} \sigma_2 = \delta_{2r-1} \sigma_2 \sigma_1;$$

$$\sigma_1 \delta_{2r} \sigma_2 = \sigma_2 \delta_{2r} \sigma_1;$$

$$(CR5) \quad (\delta_{2r} \sigma_1)(\delta_{2s-1} \delta_{2s}) = (\delta_{2s-1} \delta_{2s})(\delta_{2r} \sigma_1) \quad 1 \leq s < r \leq g;$$

$$(\sigma_1 \delta_{2r})(\delta_{2s} \delta_{2s-1}) = (\delta_{2s} \delta_{2s-1})(\sigma_1 \delta_{2r}) \quad 1 \leq r < s \leq g;$$

$$(\delta_{2r} \sigma_1)(\delta_{2r-1} \delta_{2s}) = (\delta_{2r-1} \delta_{2s})(\delta_{2r} \sigma_1) \quad 1 \leq s < r \leq g;$$

$$(\sigma_1 \delta_{2r-1})(\delta_{2s-1} \delta_{2r}) = (\delta_{2s-1} \delta_{2r})(\sigma_1 \delta_{2r-1}) \quad 1 \leq r < s \leq g;$$

-Skew commutative relations on the handles

$$(SCR2) \quad \sigma_1 \delta_r \delta_{r+2s} \sigma_1 = \delta_{r+2s} \delta_r \quad 1 \leq r < r + 2s \leq 2g;$$

-Relation associated to the fundamental group of the surface

$$(FGR) \quad (\sigma_2 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_2) \sigma_1 \delta_1 \delta_2 \cdots \delta_{2g} \sigma_1 = \delta_{2g} \cdots \delta_2 \delta_1.$$

Corollary 3.3. *Let n be positive integer. Then, the group $B_n(\Sigma_{1,0})$ admits the following group presentation:*

- *Generators:* $\sigma_1, \dots, \sigma_{n-1}, \delta_1, \delta_2;$

• *Relations:*

-*Braid relations:*

$$(BR1) \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| \geq 2;$$

$$(BR2) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } 1 \leq i \leq n - 1.$$

-*Commutative relations between surface braids:*

$$(CR1) \quad \delta_r \sigma_i = \sigma_i \delta_r \quad \text{for } 2 < i; r = 1, 2;$$

$$(CR4) \quad \delta_r^2 \sigma_1 = \sigma_1 \delta_r^2 \quad r = 1, 2;$$

$$\sigma_2 \delta_1 \sigma_2 = \delta_1 \sigma_2 \sigma_1;$$

$$\sigma_1 \delta_2 \sigma_2 = \sigma_2 \delta_2 \sigma_1;$$

-*Skew commutative relations on the handles:*

$$(SCR4) \quad \delta_2 \sigma_1 \delta_1 \delta_2 \sigma_1 = \delta_1 \delta_2^2;$$

-*Relation associated to the fundamental group of the surface:*

$$(FGR) \quad (\sigma_2 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_2) \sigma_1 \delta_1 \delta_2 \sigma_1 = \delta_2 \delta_1.$$

Corollary 3.3 is a special case of Proposition 3.1 when $g = 1$. When $g \geq 2$, Corollary 3.2 follows from Proposition 3.1 using Lemma 3.4 below.

Lemma 3.4. (i) Let G be a group and $\sigma, \delta_1, \delta_2, \delta'_1, \delta'_2$ be in G such that a) $\sigma \delta_i^2 = \delta_i^2 \sigma$ for $i = 1, 2$;

b) $\sigma \delta'_i = \delta'_i \sigma$ for $i = 1, 2$; c) $g \delta'_2 \delta_2 g = \delta_2 \delta'_2$ and d) $g \delta'_1 \delta_1 g = \delta_1 \delta'_1$. Then,

$$(i) \quad \delta_2 \sigma \delta'_1 \delta_2 \sigma = \delta'_1 \delta_2^2 \iff \delta_2 \sigma \delta'_1 \delta'_2 = \delta'_1 \delta'_2 \delta_2 \sigma \iff \delta_1 \delta_2 \sigma \delta'_1 = \sigma \delta'_1 \delta_1 \delta_2.$$

$$(ii) \quad \delta_2 \sigma \delta_1 \delta_2 \sigma = \delta_1 \delta_2^2 \iff \delta_2 \sigma \delta_1 \delta'_2 = \delta_1 \delta'_2 \delta_2 \sigma.$$

$$(iii) \quad \delta'_2 \sigma \delta'_1 \delta'_2 \sigma = \delta'_1 \delta'^2_2 \iff \delta_1 \delta'_2 \sigma \delta'_1 = \sigma \delta'_1 \delta_1 \delta'_2.$$

Proof. We prove under the hypothesis that $\delta_2 \sigma \delta'_1 \delta_2 \sigma = \delta'_1 \delta_2^2 \iff \delta_2 \sigma \delta'_1 \delta'_2 = \delta'_1 \delta'_2 \delta_2 \sigma$. The other cases are similar.

$$\delta_2 \sigma \delta'_1 \delta_2 \sigma = \delta'_1 \delta_2^2 \iff \delta_2 \sigma \delta'_1 \delta_2 = \delta'_1 \delta_2^2 \sigma^{-1} \iff \delta_2 \sigma \delta'_1 \delta_2 = \delta'_1 \sigma^{-1} \delta_2^2 \iff \delta_2 \sigma \delta'_1 = \delta'_1 \sigma^{-1} \delta_2 \iff$$

$$\delta_2 \sigma \delta'_1 \delta'_2 = \delta'_1 \sigma^{-1} \delta_2 \delta'_2 \iff \delta_2 \sigma \delta'_1 \delta'_2 = \delta'_1 \delta'_2 \delta_2 \sigma.$$

□

Lemma 3.5. Let n, g be positive integers. Consider the group $B_n(\Sigma_{g,0})$ and the presentation of Proposition 3.1. Then, for every $1 \leq r, s \leq g$,

$$\delta_{2s-1} \delta_{2r}^2 \delta_{2s-1} = \delta_{2r} \delta_{2s-1}^2 \delta_{2r}.$$

Proof. Let $1 \leq s \leq r \leq g$. From the relations of type (SCR3) and the relations of the first type of (CR4), it follows $\delta_{2s-1} \delta_{2r} \sigma_1 \delta_{2s-1} \delta_{2r} \sigma_1 = \delta_{2s-1}^2 \delta_{2r}^2 = \sigma_1 \delta_{2s-1} \delta_{2r} \sigma_1 \delta_{2s-1} \delta_{2r}$. Hence, $\delta_{2s-1}^2 \delta_{2r} \delta_{2s-1}^{-1} = \sigma_1 \delta_{2s-1} \delta_{2r} \sigma_1 = \delta_{2r}^{-1} \delta_{2s-1} \delta_{2r}^2$ and then $\delta_{2s-1} \delta_{2r}^2 \delta_{2s-1} = \delta_{2r} \delta_{2s-1}^2 \delta_{2r}$.

If $1 \leq r < s \leq g$ then we proceed in the same way, using that $\sigma_1 \delta_{2s} \delta_{2r-1} \sigma_1 \delta_{2s} = \delta_{2s}^2 \delta_{2r-1}$. □

Question 3. Consider the monoids defined by the presentation given in Theorem 3.1, Corollary 3.2 or in Corollary 3.3. Are they cancellative? do they embed in $B_n(\Sigma_{g,0})$?

4. BRAID GROUP ON TWO STRANDS ON THE TORUS

4.1. The word problem and the conjugacy problem. In this section we solve the word problem and the conjugacy problem for the special case when $g = 1$ and $n = 2$ by using a presentation derived from the one obtained in the previous section.

As a consequence of Corollary 3.3 we have:

Corollary 4.1. $B_2(\Sigma_{1,0})$ admits the group presentation:

$$B_2(\Sigma_{1,0}) = \langle \sigma_1, \delta_1, \delta_2 \mid \delta_1^2 \sigma_1 = \sigma_1 \delta_1^2; \delta_2^2 \sigma_1 = \sigma_1 \delta_2^2; \delta_2 \sigma_1 \delta_1 \delta_2 \sigma_1 = \delta_1 \delta_2^2; \sigma_1 \delta_1 \delta_2 \sigma_1 = \delta_2 \delta_1 \rangle.$$

Lemma 4.2. Let G be group and $\sigma_1, \delta_1, \delta_2$ be in G such that a) $\delta_1^2 \sigma_1 = \sigma_1 \delta_1^2$; b) $\delta_2^2 \sigma_1 = \sigma_1 \delta_2^2$; c) $\sigma_1 \delta_1 \delta_2 \sigma_1 = \delta_2 \delta_1$. Then $\delta_2 \sigma_1 \delta_1 \delta_2 \sigma_1 = \delta_1 \delta_2^2 \iff (\delta_1^2 \delta_2 = \delta_2 \delta_1^2 \text{ and } \delta_2^2 \delta_1 = \delta_1 \delta_2^2)$.

Corollary 4.3. ([17], Chapter 11, Exercises 5.2 and 6.3) The group $B_2(\Sigma_{1,0})$ admits the two following group presentations:

$$B_2(\Sigma_{1,0}) = \langle \sigma_1, \delta_1, \delta_2 \mid \delta_1^2 \sigma_1 = \sigma_1 \delta_1^2; \delta_2^2 \sigma_1 = \sigma_1 \delta_2^2; \delta_1^2 \delta_2 = \delta_2 \delta_1^2; \delta_2^2 \delta_1 = \delta_1 \delta_2^2; \sigma_1 \delta_1 \delta_2 \sigma_1 = \delta_2 \delta_1 \rangle.$$

$$(\ddagger) \quad B_2(\Sigma_{1,0}) = \langle a, b, c \mid a^2 b = b a^2; b^2 a = a b^2; a^2 c = c a^2; b^2 c = c b^2; a^2 b^2 = c^2 \rangle.$$

Proof. (i) follows from Corollary 4.1 and Lemma 4.2. For ii), we set $a = \delta_2$, $b = \delta_1$ and $c = \delta_2 \delta_1 \sigma_1^{-1}$ as suggested in [17] Chapter 11, Exercise 6.3. \square

Using the presentation (\ddagger) , we are able to solve the word problem and the conjugacy problem in $B_2(\Sigma_{1,0})$. Considering (\ddagger) , for $x = a$, or $x = b$ we can define a weight homomorphism of groups $\ell_x : B_2(\Sigma_{1,0}) \rightarrow \mathbb{Z}$ such that $\ell_x(x) = 0$ and $\ell_x(y) = 1$ for $y \in \{a, b, c\}$ and $y \neq x$.

In the following we denote by $F(a, b, c)$ the free group based on $\{a, b, c\}$. We denote by $W(a, b, c)$ the Coxeter group associated to $F(a, b, c)$ and defined by $W(a, b, c) = \langle a, b, c \mid a^2 = b^2 = c^2 = 1 \rangle$. If w is in $F(a, b, c)$ we denote by \bar{w} its image in $B_2(\Sigma_{1,0})$. Considering (\ddagger) , there exists a morphism $p : B_2(\Sigma_{1,0}) \rightarrow W(a, b, c)$ that sends $x \in \{a, b, c\}$ on x . Note that the canonical morphism from $F(a, b, c)$ onto $W(a, b, c)$ factorises through p .

We denote by $L_{a,b}$ the set-map from $B_2(\Sigma_{1,0})$ to $F(a, b, c)$ defined by $L_{a,b}(g) = a^{\ell_b(g)} b^{\ell_a(g)}$ for g in $B_2(\Sigma_{1,0})$. If w is in $F(a, b, c)$, we write, by abuse of notation, $p(w)$ and $L_{a,b}(w)$ for $p(\bar{w})$ and $L_{a,b}(\bar{w})$ respectively.

Proposition 4.4. (i) The center $Z(B_2(\Sigma_{1,0}))$ of the group $B_2(\Sigma_{1,0})$ is a free Abelian group based on a^2 and b^2 . Furthermore, for each element g of $Z(B_2(\Sigma_{1,0}))$, the word $L_{a,b}(g)$ is a representing element of g .

(ii) The group $B_2(\Sigma_{1,0})$ is a central extension of $W(a, b, c)$.

In other words, the sequence $1 \rightarrow Z(B_2(\Sigma_{1,0})) \rightarrow B_2(\Sigma_{1,0}) \rightarrow W(a, b, c) \rightarrow 1$ is exact.

Proof. We remark that the presentation (\ddagger) implies that a^2 , b^2 and c^2 are in $Z(B_2(\Sigma_{1,0}))$ and that $W(a, b, c) = B_2(\Sigma_{1,0})/a^2 = b^2 = 1$. Since the center of $W(a, b, c)$ is trivial, (ii) follows. As a

consequence, we get that a^2 and b^2 generated the Abelian group $Z(B_2(\Sigma_{1,0}))$. Now, let g belong to $Z(B_2(\Sigma_{1,0}))$. Each word w that represents g and written on the letters a^2 , b^2 , and their inverses, can be modify in order to obtain $L_{a,b}(g)$ by using the relations $a^2b^2 = b^2a^2$ and the relations of a^2 and b^2 with their respective inverses. Hence, $Z(B_2(\Sigma_{1,0}))$ is a free Abelian group based on a^2 and b^2 . □

Corollary 4.5. *Let w be in $F(a, b, c)$; then $\bar{w} = 1 \iff (L_{a,b}(w) = 1 \text{ and } p(w) = 1)$.*

Proof. Assume $L_{a,b}(w) = 1$ and $p(w) = 1$. Since $p(\bar{w}) = 1$, the element \bar{w} is in the center of $B_2(\Sigma_{1,0})$. But in that case, $L_{a,b}(w) = 1$ represents \bar{w} . Then $\bar{w} = 1$. □

Corollary 4.6. *The word problem in $B_2(\Sigma_{1,0})$ is solvable.*

Proof. The word problem is solvable in the free group $F(a, b, c)$ and in the Coxeter group $W(a, b, c)$. Then the claim follows from Corollary 4.5. □

Corollary 4.7. *Let g, h be in $F(a, b, c)$; then, $\bar{g} = \bar{h} \iff p(g) = p(h) \text{ and } L_{a,b}(g) = L_{a,b}(h)$.*

Proof. Assume $p(g) = p(h)$ and $L_{a,b}(g) = L_{a,b}(h)$. Then the element $\bar{g}\bar{h}^{-1}$ is in the center of $B_2(\Sigma_{1,0})$. Since $L_{a,b}(g) = L_{a,b}(h)$ it follows that $\ell_{\hat{a}}(g) = \ell_{\hat{a}}(h)$ and $\ell_{\hat{b}}(g) = \ell_{\hat{b}}(h)$. Therefore $L_{a,b}(g\bar{h}^{-1}) = 1$ and thus $\bar{g} = \bar{h}$. □

We denote by $F^+(a, b, c)$ the free monoid based on $\{a, b, c\}$. It is a submonoid of $F(a, b, c)$. If w is in $W(a, b, c)$, there exists a unique element $[w]$ in $F^+(a, b, c)$ of minimal length such that its image in $W(a, b, c)$ is w . By construction, for each w in $W(a, b, c)$ the element $p(\overline{[w]})$ is equal to w . As a consequence, the map sending each w in $W(a, b, c)$ on $\overline{[w]}$ is injective. For short, we will write $[w]$ for $\overline{[w]}$.

Corollary 4.8. *(i) Let g, h be in $B_2(\Sigma_{1,0})$; then,*

$$(\exists r \in B_2(\Sigma_{1,0}), rgr^{-1} = h) \iff (L_{a,b}(g) = L_{a,b}(h) \text{ and } \exists w \in W(a, b, c), wp(g)w^{-1} = p(h)).$$

Furthermore, if the right side holds, then $[w]g[w]^{-1} = h$.

Proof. The side “ \Rightarrow ” is clear with $w = p(r)$. Assume conversely that $L_{a,b}(g) = L_{a,b}(h)$ and $\exists w \in W(a, b, c)$, $wp(g)w^{-1} = p(h)$. Since $wp(g)w^{-1} = p(h)$, we have $p([w]g[w]^{-1}) = p(h)$. But $L_{a,b}([w]g[w]^{-1}) = L_{a,b}(g) = L_{a,b}(h)$ and $[w]g[w]^{-1} = h$ by Corollary 4.7. □

Corollary 4.9. *The conjugacy problem in $B_2(\Sigma_{1,0})$ is solvable.*

Proof. The conjugacy problem is solvable in each Coxeter group (see [16]). □

4.2. The Garside method and complete presentation. In order to solve the word problem and the conjugacy problem in $B_2(\Sigma_{1,0})$, we can try to use the method used by Garside to solve the word problem and the conjugacy problem, that is to find a Garside structure for $B_2(\Sigma_{1,0})$. Let us remark that surface braid groups on surfaces of genus greater than 1 have trivial center (see [19]) and then they cannot be Garside groups. Recall that in a monoid M we say that a left-divides b if $b = ac$ for some c in M . We say, in a similar way, that a right-divides b when $b = ca$ for some c in M . An element Δ of M is said to be balanced when its set of left-divisors is equal to its set of right-divisors. We denote by $B_2^+(\Sigma_{1,0})$ the monoid defined by the presentation (\ddagger) , but considered as a monoid presentation. Then in $B_2^+(\Sigma_{1,0})$ the element c^2 is balanced. Furthermore its set $D(c^2)$ of divisors generates $B_2^+(\Sigma_{1,0})$. Nevertheless, $B_2^+(\Sigma_{1,0})$ fails to be a Garside monoid with c^2 for Garside element (see [8] for a definition) because it is not a lattice for left-divisibility: a and b have two distinct minimal common multiples, namely ab^2 and ba^2 . Anyway, as shown in [8] Section 8, part of the results established for Garside groups still hold, as we will see in Lemma 4.10.

Let $\iota : B_2^+(\Sigma_{1,0}) \rightarrow B_2(\Sigma_{1,0})$ be the canonical homomorphism of monoids. By abuse of notation, we denote by $\ell_{\hat{a}}$ and $\ell_{\hat{b}}$ the morphisms $\ell_{\hat{a}} \circ \iota$ and $\ell_{\hat{b}} \circ \iota$ respectively. We remark that $z \mapsto \bar{z}$ factorises through ι . By abuse of notation, we denote by $z \mapsto \bar{z}$ and by $w \mapsto \overline{[w]}$ the factorizations. Then we have $\bar{w} = \iota(\overline{[w]})$. As before, we write $[w]$ for $\overline{[w]} \in B_2^+(\Sigma_{1,0})$. We remark that Corollary 4.5 and 4.6 still hold if we consider \bar{w} in $B_2^+(\Sigma_{1,0})$ and g, h in $B_2^+(\Sigma_{1,0})$. As a consequence, we have the following result:

Lemma 4.10. (i) $B_2^+(\Sigma_{1,0})$ is cancellative and the canonical morphism $\iota : B_2^+(\Sigma_{1,0}) \rightarrow B_2(\Sigma_{1,0})$ is into.

(ii) $\forall G \in B_2(\Sigma_{1,0}), \exists ! j \in \mathbb{Z}, \exists ! g \in B_2^+(\Sigma_{1,0})$ such that $G = c^{-2j}\iota(g)$ and c^2 does not divide g .

Proof. (i) Let h, g_1, g_2 be in $B_2^+(\Sigma_{1,0})$ such that $hg_1 = hg_2$. Then $\ell_{\hat{a}}(hg_1) = \ell_{\hat{a}}(h) + \ell_{\hat{a}}(g_1)$ and $\ell_{\hat{a}}(hg_2) = \ell_{\hat{a}}(h) + \ell_{\hat{a}}(g_2)$. Hence we have $\ell_{\hat{a}}(g_1) = \ell_{\hat{a}}(g_2)$. In the same way we get $\ell_{\hat{b}}(g_1) = \ell_{\hat{b}}(g_2)$, and also $p(g_1) = p(g_2)$ in the group $W(a, b, c)$. Then $g_1 = g_2$. We proceed in the same way if $g_1h = g_2h$. The other results are consequences of the Garside like structure as proved in Proposition 8.10 of [8]. \square

In the following, we identify $B_2^+(\Sigma_{1,0})$ with its image in $B_2(\Sigma_{1,0})$. In order to solve the word problem in $B_2(\Sigma_{1,0})$, it is then enough to solve the words problem in $B_2^+(\Sigma_{1,0})$. Then, using the following proposition, we obtain another solution to the word problem.

Proposition 4.11. Let g be in $B_2^+(\Sigma_{1,0})$; then, there exist a unique pair (h, l) in \mathbb{N}^2 , and a unique w in $W(a, b, c)$ such that $g = a^{2h}b^{2l}[w]$.

Proof. Since $c^2 = a^2b^2$ and that a^2, b^2 are in the center of $Z(B_2^+(\Sigma_{1,0}))$, it follows that we can write $g = a^{2k}b^{2l}[w]$ with k, l in \mathbb{N} and w in $W(a, b, c)$. Assume that $g = a^{2i}b^{2j}[z]$ for some i, j in

\mathbb{N} and z in $W(a, b, c)$. We have (1) $w = p(g)z$ and then $[w] = [z]$; (2) $k = \frac{\ell_b(g) - \ell_b([w])}{2} = i$; (3) $l = \frac{\ell_a(g) - \ell_a([w])}{2} = j$. Then the decomposition of $g = a^{2k}b^{2l}[w]$ is unique. \square

If we want to solve the conjugacy problem by using the idea of Garside, we need to understand the normal form of $B_2^+(\Sigma_{1,0})$ as defined in Definition 7.2 of [8]. This lead us to the notion of complete presentation as defined in [9]. Let S be a finite set, and S^* be the free monoid based on S . We denote by ϵ the empty word. Let B be a monoid with presentation (S, \mathcal{R}) . We write $w \equiv w'$ if the word w, w' of S^* have the same image in B . Let w, w' be two words in $(S \cup S^{-1})^*$, where S^{-1} is a disjoint copy of S . We say that w reverses in w' , and write $w \curvearrowright w'$ if w' is obtained from w by a finite sequence of the following steps: deleting some $u^{-1}u$ for some $u \in S$ or replacing some subword $u^{-1}v$ where u, v are in S , with a a word $v'u'^{-1}$ such that $uv' = vu'$ is a relation of \mathcal{R} .

Definition 1 ([9] Definition 2.1 and Proposition 3.3). *Let B be a monoid with presentation (S, \mathcal{R}) . We say that the presentation (S, \mathcal{R}) is complete if*

$$\forall u, v \in S^*, (u \equiv v \iff u^{-1}v \curvearrowright \epsilon).$$

For instance the classical presentation of each Artin-Tits monoid is complete. The definition of complete presentation is easy to understand. Nevertheless, it is not easy to verify that a given presentation is complete. In [9], Dehornoy gives a semi-algorithmical method in order to decide if a given presentation is complete. Semi-algorithmical means that when the process finishes, it gives an answer, but it is possible that it doesnot finish. We do not explain this technical method, named *the cube condition*, but refer to Definition 3.1 and Figure 3.1 of [9].

Applying the cube condition process, it is quiet clear that the presentation (\ddagger) of the monoid $B_2(\Sigma_{0,1})$ is not complete and that we must add to the presentation the relation “ $b^2a^2 = c^2$ ” if we want to expect that the presentation is complete.

Question 4. *Is the presentation*

$$(\S) \quad \langle a, b, c \mid a^2b = ba^2; b^2a = ab^2; a^2c = ca^2; b^2c = cb^2; a^2b^2 = c^2; b^2a^2 = c^2 \rangle^+$$

a complete presentation of the monoid $B_2(\Sigma_{1,0})$? In other words, does this presentation verify the cube condition ?

Question 5. *Is the monoid presentation $\langle a, b \mid ab^2 = b^2a; ba^2 = a^2b \rangle^+$ complete ? In other words, does this presentation verify the cube condition ?*

A positive answer to Question 5 seems to be crucial in order to state the interest of the method of completeness.

APPENDIX A. PRESENTATIONS OF SURFACE BRAID GROUPS

Theorem A.1 ([3] Theorem 1.1). *Let n, p be positive integers and g a non negative integer. Let g be a non negative integer. The group $B_n(\Sigma_{g,p})$ admits the following group presentation:*

- *Generators:* $\sigma_1, \dots, \sigma_{n-1}, a_1, \dots, a_g, b_1, \dots, b_g, z_1, \dots, z_{p-1}$.
- *Relations:*
 - *braid relations:*

$$(BR1) \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| \geq 2.$$

$$(BR2) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad 1 \leq i \leq n - 1;$$
 - *mixed relations:*

$$(R1) \quad a_r \sigma_i = \sigma_i a_r \quad \text{for } 1 \leq r \leq g; i \neq 1;$$

$$b_r \sigma_i = \sigma_i b_r \quad \text{for } 1 \leq r \leq g; i \neq 1;$$

$$(R2) \quad \sigma_1^{-1} a_r \sigma_1^{-1} a_r = a_r \sigma_1^{-1} a_r \sigma_1^{-1} \quad \text{for } 1 \leq r \leq g;$$

$$\sigma_1^{-1} b_r \sigma_1^{-1} b_r = b_r \sigma_1^{-1} b_r \sigma_1^{-1} \quad \text{for } 1 \leq r \leq g;$$

$$(R3) \quad \sigma_1^{-1} a_s \sigma_1 a_r = a_r \sigma_1^{-1} a_s \sigma_1 \quad \text{for } s < r;$$

$$\sigma_1^{-1} b_s \sigma_1 b_r = b_r \sigma_1^{-1} b_s \sigma_1 \quad \text{for } s < r;$$

$$\sigma_1^{-1} a_s \sigma_1 b_r = b_r \sigma_1^{-1} a_s \sigma_1 \quad \text{for } s < r$$

$$\sigma_1^{-1} b_s \sigma_1 a_r = a_r \sigma_1^{-1} b_s \sigma_1 \quad \text{for } s < r$$

$$(R4) \quad \sigma_1^{-1} a_r \sigma_1^{-1} b_r = b_r \sigma_1^{-1} a_r \sigma_1 \quad \text{for } 1 \leq r \leq g;$$

$$(R5) \quad z_j \sigma_i = \sigma_i z_j \quad \text{for } i \neq 1, j = 1, \dots, p - 1;$$

$$(R6) \quad \sigma_1^{-1} z_i \sigma_1 a_r = a_r \sigma_1^{-1} z_i \sigma_1 \quad \text{for } 1 \leq r \leq g; i = 1, \dots, p - 1;$$

$$\sigma_1^{-1} z_i \sigma_1 b_r = b_r \sigma_1^{-1} z_i \sigma_1 \quad \text{for } 1 \leq r \leq g; i = 1, \dots, p - 1$$

$$(R7) \quad \sigma_1^{-1} z_j \sigma_1 z_l = z_l \sigma_1^{-1} z_j \sigma_1 \quad \text{for } j = 1, \dots, p - 1, j < l;$$

$$(R8) \quad \sigma_1^{-1} z_j \sigma_1^{-1} z_j = z_j \sigma_1^{-1} z_j \sigma_1^{-1} \quad \text{for } j = 1, \dots, p - 1.$$

Theorem A.2 ([3] Theorem A.4). *Let $\Sigma_{0,g}$ be a closed surface of positive genus g . Then $B_n(\Sigma_{0,g})$ admits the following presentation:*

- *Generators:* $\theta_1, \dots, \theta_{n-1}, b_1, \dots, b_{2g}$;
- *Relations:*

$$(BR1) \quad \theta_i \theta_j = \theta_j \theta_i \quad \text{for } 2 \leq |i - j|.$$

$$(BR1) \quad \theta_i \theta_{i+1} \theta_i = \theta_{i+1} \theta_i \theta_{i+1} \quad \text{for } 1 \leq i \leq n - 2.$$

$$(R1) \quad b_r \theta_i = \theta_i b_r \quad 1 \leq r \leq 2g; i \neq 1;$$

$$(R2) \quad b_s \theta_1^{-1} b_r \theta_1^{-1} = \theta_1 b_r \theta_1^{-1} b_s \quad 1 \leq s < r \leq 2g;$$

$$(R3) \quad b_r \theta_1^{-1} b_r \theta_1^{-1} = \theta_1^{-1} b_r \theta_1^{-1} b_r \quad 1 \leq r \leq 2g;$$

$$(TR) \quad b_1 b_2^{-1} \cdots b_{2g-1} b_{2g}^{-1} b_1^{-1} b_2 \cdots b_{2g-1} b_{2g} = \theta_1 \cdots \theta_{n-2} \theta_{n-1}^2 \theta_{n-2} \cdots \theta_1.$$

REFERENCES

- [1] D. Allcock, *Braid pictures for Artin groups*, Trans. Amer. Math. Soc. **354** (2002) 3455-3474.
- [2] E. Artin, *Theorie der Zöpfe*, Hamburg Abh. **4** (1925) 47-72.
- [3] P. Bellingeri, *On presentation of surface braids*, J. Algebra **274** (2004) 543-563.
- [4] E. Brieskorn and K. Saito, *Artin-Gruppen und Coxeter-Gruppen*, Invent. Math. **17** (1972) 245-271.
- [5] J. S. Birman, *Braids, Links, and Mapping Class Groups*, Ann. of Math. Studies, No. 82, Princeton Univ. Press, Princeton, N. J., 1974.

- [6] R. Charney, *Geodesic automation and growth function for Artin groups of finite type*, Math. Ann. **301** (1995) 307–324.
- [7] R. Corran, *Solving the word problem in the singular braid group*, J. Algebra **223** (2000) 256–282.
- [8] P. Dehornoy, *Groupe de Garside*, Ann. Sc. Ec. Norm. Sup. **35** (2002) 267–306.
- [9] P. Dehornoy, *Complete positive group presentation*, J. Algebra **268** (2003) 156–197.
- [10] P. Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math. **17** (1972), 273–302.
- [11] E. Fadell and J.M. Van Buskirk, *The Braid groups of E^2 and S^2* , Duke Math. J. **29** (1962) 243–257.
- [12] F.A Garside, *The Braid group and other groups*, Quarterly J. Math. Oxford **20** (1969) 235–254.
- [13] R. Gillette and J.M. Van Buskirk, *The Word problem and consequences for the braid groups and mapping class groups of the 2-sphere*, Trans. Amer. Math. Soc. **131** (1968) 277–296.
- [14] J. González-Meneses, *New presentations of surface braid groups*, J. Knot Theory Ramification, **10** (2001) 431–451.
- [15] R. Haering-Oldenburg and S. Lambropoulou *Knot theory in handlebodies*, J. Knot Theory and Ramifications, **11** (2002) 921–943.
- [16] D. Kramer, *The conjugacy problem for Coxeter groups*, Universiteit Utrecht, 1994.
- [17] K. Murasugi and B.I. Kurpita, *A study of braid*, Mathematics and its Applications, N 484, Kluwer Academic Publishers, The Netherlands, 1999.
- [18] L. Paris *Artin monoids inject in their groups*, Comment. Math. Helv. **77** (2002), 609–637.
- [19] L. Paris and D. Rolfsen, *Geometric subgroups of surface braid groups*, Ann. Inst. Fourier, **49** (1999) 417–472.
- [20] J.M. Van Buskirk, *Braid groups of compact 2-manifolds with elements of finite order*, Trans. Amer. Math. Soc. **122** (1966) 81–97.

UNIV. PISA, DIPARTIMENTO DI MATEMATICA, 56205 PISA, ITALY
E-mail address: bellingeri@mail.dm.pisa.it

UNIV. CAEN, LABORATOIRE DE MATHÉMATIQUES NICOLAS ORESME, 14032 CAEN, FRANCE
E-mail address: Eddy.Godelle@math.unicaen.fr