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Hypocoercivity and exponential time decay for the linear inhomogeneous relaxation Boltzmann equation

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Abstract

We consider an inhomogeneous linear Boltzmann equation, with an external confining potential. The collision operator is a simple relaxation toward a local Maxwellian, therefore without diffusion. We prove the exponential time decay toward the global Maxwellian, with an explicit rate of decay. The methods are based on hypoelliptic methods transposed here to get spectral information. They were inspired by former works on the Fokker-Planck equation and the main feature of this work is that they are relevant although the equation itself has no regularizing properties.

1 Introduction.

This article is devoted to the study of the long time behavior of the solutions of the following kinetic equation in $\mathbb{R}^{1+2d}$ of unknown $f$

$$\begin{cases}
\partial_t f + v \cdot \partial_x f - \partial_x V(x) \cdot \partial_v f = Q(f), \\
|t=0 = f_0.
\end{cases}$$

The right-hand side is a simple linear model for the Boltzmann operator

$$Q(f) = \gamma (\rho \mu_\infty - f), \quad \rho(t,x) = \int f(t,x,v) dv,$$

where $\mu_\infty$ is the Maxwellian in the velocity direction

$$\mu_\infty(v) = \frac{e^{-v^2/2}}{(2\pi)^{d/2}}.$$ 

This equation describes a system of large number of particles submitted to an external force deriving from a potential $V(x)$, and for which the collision operator in the right-hand side is a simple relaxation toward the local Maxwellian $\rho \mu_\infty$. In particular there is no diffusion.

We suppose that the derivatives of $V$ of order 2 or more are bounded, and also that $e^{-V} \in L^1$, which implies that there is a unique steady state. In this case we say that $V$ is a confining potential (anyway the adaptation in the case when $e^{-V} \notin L^1$ is straightforward,

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see remark 4.2). It can be useful introduce the spatial Maxwellian and global Maxwellian
defined respectively by
\[
\rho_\infty(x) = \frac{e^{-V(x)}}{\int e^{-V(x)} dx}, \quad \mathcal{M}(x, v) = \rho_\infty(x) \mu_\infty(v).
\]
All steady states in \(S'(\mathbb{R}^{2d})\) are proportional to the Maxwellian \(\mathcal{M}\). In order to study
the exponential decay we now introduce an additionnal operator,
\[
\Lambda^2 = -\gamma \partial_v (\partial_v + v) - \gamma \partial_x (\partial_x + \partial_x V) + 1.
\]
This operator has nice properties in the following weighted space
\[
B^2 = \left\{ f \in \mathcal{D}' \text{ s. t. } f/M^{1/2} \in L^2(dxdv) \right\},
\]
with the natural norm defined by \(\|f\|_{B^2}^2 = \int \|f\|^2 \mathcal{M}^{-1} dxdv\). Indeed the closure from \(C_0^\infty\) of 
\(\Lambda^2 - 1\) in \(B^2\) is maximal accretive (see [4]) and has 0 as single eigenvalue associated with the
eigenfunction \(\mathcal{M}\). We shall assume the following:
\[
\text{Operator } \Lambda^2 - 1 \text{ has a spectral gap } \alpha > 0 \text{ in } B^2. \tag{5}
\]
Recall that the spectral gap is defined has the infimum of the spectrum except the lowest
eigenvalue. We mention now some simple cases when it happens. For example when \(\text{Hess} V \geq \lambda \text{Id}\) then \(\alpha = \lambda\). It is a special case of the one when \(|V'(x)|\) goes to infinity with \(x\), which implies that \(\Lambda^2\) is with compact resolvent in \(B^2\) and that (5) is also satisfied. We refer to [5]
or [4] and reference therein for complementary information about it.

Now about the collision operator \(Q\), we just mention here that it is mass and positivity
preserving and "dissipative" in the sense non-negative in \(B^2\) (see [1]). We shall study it more
carefully later, and refer to remark 4.3 here for complements.

It is easy to verify that \(-v \partial_x + \partial_x V(x) \partial_v + Q\) is also mass and positivity preserving
and dissipative, and that its closure in \(B^2\) from \(C_0^\infty\) generates a semi-group of contraction in \(B^2\).
The Cauchy problem (1) is therefore well posed and it was proven in [1] that under regularity
assumptions and bounds on the solution \(f\) of (1), \(f(t)\) tends to \(\mathcal{M}\) when \(t\) goes to infinity
faster than any inverse power of \(t\). We now state our main result:

**Theorem 1.1** There exists a constant \(A > 0\) depending only on the second and third order
derivatives of \(V\), such that for all \(L^1\) normalized function \(f_0 \in B^2\), we have the following
\[
\|f(t, \cdot) - f_\infty\|_{B^2} \leq 3 \|f_0 - f_\infty\|_{B^2} e^{-\alpha^2 t/A}
\]
here \(f_\infty = \mathcal{M}\), and \(f\) is the unique solution of equation (1).

As a direct consequence we also obtain the decrease of the so-called relative entropy:

**Corollary 1.2** Under the hypothesis of the preceding Theorem, and assuming in addition
that \(f_0 \geq 0\), we have
\[
0 \leq H(f, f_\infty)(t) \overset{\text{def}}{=} \int \int f(t) \ln \left(\frac{f(t)}{f_\infty}\right) dxdv \leq 3 \|f_0\|_{B^2} \|f_0 - f_\infty\|_{B^2} e^{-\alpha^2 t/A}.
\]
This study is motivated by proving the validity of some new tools and ideas, namely the one called hypocoercivity, appeared in a few recent articles in order to prove exponential time-decay convergence for some inhomogeneous (mostly linear) kinetic equations such as Fokker-Planck [5][4][8][6], Vlasov-Poisson-Fokker-Planck [7], chains of anharmonic oscillators [8]. It is well-known that in the homogeneous case the exponential decay can be easily obtained by spectral methods if we assume (in the linear case) some coercivity of the collision operator. In the inhomogeneous case, the global coercivity is false in general, but can be obtained in a modified but norm-equivalent Hilbert space ($B^2$ in this work). This property can serve as a definition of hypocoercivity. Several tools can be used for this, essentially inspired by hypoelliptic ideas, that’s why this name was introduced very recently in [8]. We mention some of them: The use of Kohn’s method to get simultaneously hypocoercivity and hypoellipticity [5], [4], [8], in the case of the Fokker-Planck operator; the use of analytic dilation and complex FBI-Bargmann transform [6], a method of multiplier via pseudodifferential operators [6] or functional analysis using harmonic oscillators and Witten Laplacian (linear part of [7]).

About the trend to equilibrium, this has been studied for the long time, and we only want to quote the entropy dissipating methods introduced by Villani and Desvillettes to prove arbitrary and explicit algebraic time decay. It was used for the Fokker-Planck equation [2], for the model studied here [1], and in its main achievement for the full Boltzmann equation [3]. Now the question naturally arose whether the exponential decay, obtained via hypoelliptic tools is also true for non-hypoelliptic operators. In this work we choose a simple example of collision operator which has no regularity property. It appears that Lie techniques (those also in the core of the hypoelliptic theory) also give sufficient information on the spectrum and in particular the spectral gap (in some modified $L^2$ space) to get hypocoercivity and then exponential decay. We therefore hope this techniques to be applied in the future to other inhomogeneous kinetic equations with linear or non-linear collision operators with or without regularity properties (see the review [9] for examples).

Eventually our result answer a question raised by Cáceres, Carillo and Goudon in [1] about the applicability of hypoelliptic techniques of [5] to obtain explicit exponential decay of the model studied here.

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2 Notations and functional analysis

For a Hilbert-space type analysis, it is more convenient to work directly on the equation obtained after conjugating with $\mathcal{M}^{1/2}$. Therefore we pose $u = f/\mathcal{M}^{1/2}$ which is now supposed
to vary in $L^2$, and the equation satisfied by $u$ is then

$$\begin{aligned}
\begin{cases}
\partial_t u + Ku = 0 \\
u_{|t=0} = u_0,
\end{cases}
\end{aligned}$$

(6)

where we introduced the notations

$$\begin{aligned}
\begin{cases}
K = X_0 + \gamma \left( \text{Id} - \Pi_1 \right), \\
X_0 = v \partial_x - \partial_x V(x), \partial_v, \\
\Pi_1 u(x, v) = \left( \int u(x, v') \mu^{1/2}(v') dv' \right) \mu^{1/2}(v) \quad \text{for } u \in L^2.
\end{cases}
\end{aligned}$$

(7)

It is immediate to see that $\Pi_1$ is an orthogonal projector onto

$$E_1 = \left\{ u \text{ s.t. exists } \rho \in L^2(dx) \text{ with } u = \rho \mu^{1/2} \right\}$$

which is a closed subspace of $L^2$. We therefore directly get that in the new formulation in $L^2$, the collision operator, which is now $-(\text{Id} - \Pi_1)$ is dissipative. Of course we also have

$$P_1 M^{1/2} = M^{1/2},$$

so that the square root of Maxwellian is in the kernel of the (new) collision operator. It is clear that $K$ and its adjoint $K^* = -X_0 + \gamma(\text{Id} - \Pi_1)$ are well defined in $S(\mathbb{R}^{2d})$, in $S'(\mathbb{R}^{2d})$ and as (non closed) operators in $L^2(\mathbb{R}^{2d})$ with domain $D(K) = D(K^*) = S(\mathbb{R}^{2d})$. We denote by the same later their maximal closure in $L^2$. For $j = 1, \ldots, d$, we introduce the differential operators $a_j$, the annihilation operator $b_j$:

$$a_j = \gamma^{1/2} \left( \partial_{x_j} + \partial_x V(x)/2 \right) \quad b_j = \gamma^{1/2} \left( \partial_{v_j} + v_j/2 \right),$$

and their formal adjoints

$$a_j^* = \gamma^{1/2} \left( -\partial_{x_j} + \partial_x V(x)/2 \right) \quad b_j^* = \gamma^{1/2} \left( -\partial_{v_j} + v_j/2 \right).$$

For the sake of conciseness, the letters $a$ and $b$ denote the vectors

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix}$$

while $a^*$ and $b^*$ are the forms

$$a^* = (a_1^*, \ldots, a_d^*) \quad b^* = (b_1^*, \ldots, b_d^*).$$

Up to the factor $\gamma$, the non negative operator $\Lambda^2 - 1$ is nothing but the sum of the Witten Laplacian on 0-forms $\gamma^{-1} a^* a = -\Delta_x + |\partial_x V(x)|^2 / 4 - \Delta V(x)/2$ and of the harmonic oscillator $\gamma^{-1} b^* b = -\Delta_v + v^2 / 4 - d/2$. Under the hypothesis on $V$, $\Lambda^2 - 1$ is a nonnegative, $S(\mathbb{R}^{2d})$ is a core and $\Lambda^r$ is well-defined for $r \in \mathbb{R}$. Moreover the kernel of $\Lambda^2 - 1$ is $E_0 = C M^{1/2}$. For the following we shall denote $\Pi_0$ the orthogonal projector onto $E_0$,

$$\Pi_0 u = (u, M^{1/2})_{L^2} M^{1/2}$$

which also extends to $u \in S'(\mathbb{R}^d)$. Let us now recall some relations involving the operators $a$, $b$ and $X_0$, that can be found for example in section 1 of [5]. We mention here that since
the operators are continuous in \( S \) and \( S' \) there is no problem of defining their commutators. First recall the Canonical Commutation Relations (CCR) for \( b \) and their counterparts for \( a \)

\[
[b_j, b_k] = [b_j^*, b_k^*] = 0 \quad [b_j, b_k^*] = \gamma \delta_{jk} \quad [a_j, a_k] = [a_k, a_j] = 0 \quad [a_j, a_k^*] = \gamma \partial^2_{x_j x_k} V.
\] (8)

It is also clear that the \( a \)'s and \( a^* \)'s commute with the \( b \)'s and \( b^* \)'s. The main remark is that the \( a_j \)'s, \( a_j^* \)'s are in the Lie algebra generated by the \( b_j \)'s, \( b_j^* \)'s and the vector field \( X_0 \):

\[
[b_j, X_0] = a_j \quad [b_j^*, X_0] = a_j^*.
\] (9)

Similarly, the \( b_j \)'s and \( b_j^* \)'s can be derived from the \( a_j \)'s, \( a_j^* \)'s and \( X_0 \)

\[
[a_j, X_0] = -\sum_{k=1}^d \left( \partial^2_{x_j x_k} V \right) b_k \quad [a_j^*, X_0] = -\sum_{k=1}^d b_k^* \left( \partial^2_{x_k x_j} V \right).
\] (10)

The relations (9) and (10) are summarized by

\[
[b, X_0] = a, \quad [b^*, X_0] = a^* , \quad [a, X_0] = -\text{Hess}Vb \quad \text{and} \quad [a^*, X_0] = -b^*\text{Hess}V.
\] (11)

By combination we have the useful formulas:

\[
[\Lambda^2, X_0] = -b^*(\text{Hess}V - \text{Id})a - a^*(\text{Hess}V - \text{Id})b, \quad b^*(a^*a) = (a^*a)b^* \quad a^*(a^*a) = (a^*a)a^* - \gamma a^*\text{Hess}V, \quad a^*(b^*b) = (b^*b)a^* \quad b^*(b^*b) = (b^*b)b^* - \gamma b^*.
\] (12) (13) (14)

and their adjoint relations hold as equality of continuous operators in \( S(\mathbb{R}^{2d}) \) and \( S'(\mathbb{R}^{2d}) \).

3 Hypocoercivity for operator \( K \).

In this section we continue to work with operator \( K \) defined in the preceding section. We shall prove that operator \( K \) is hypocoercitive, i.e. coercitive in \( L^2 \) with a modified scalar product. For this we introduce an additional operator \( L = \Lambda^{-2}a^*b = \Lambda^{-2}(\sum_j a_j^*b_j) \).

We shall see later that this operator is explicitly bounded in terms of the second and third derivatives of \( V \).

Proposition 3.1 Let \( \alpha \) be defined in (5). Then there exists \( \varepsilon, A > 0 \) such that for all \( S \ni u \perp \mathcal{M}^{1/2} \) we have

\[
\text{Re} \left( Ku, (\text{Id} + \varepsilon(L + L^*))u \right) \geq \frac{\alpha^2}{A} \|u\|^2,
\]

where \( A \) can be chosen to depend explicitly on \( \gamma \) and the second and third derivatives of \( V \), and \( \|\varepsilon L\| \leq 1 \).
Proof. Let us take $u \in L^2$ and $\varepsilon > 0$. We write

$$
\text{Re} \left( K u, (\text{Id} + \varepsilon(L + L^*))u \right)
= \text{Re} \gamma \left( (\text{Id} - \Pi_1)u, (\text{Id} + \varepsilon(L + L^*))u \right) + \text{Re} \left( X_0 u, (\text{Id} + \varepsilon(L + L^*))u \right)
= \gamma \| (\text{Id} - \Pi_1)u \|^2 + \varepsilon \gamma \text{Re} \left( (\text{Id} - \Pi_1)u, (L + L^*)u \right) + \varepsilon \text{Re} (X_0 u, (L + L^*)u)
= I + II + III,
$$

where in the last term we used the fact that $X_0$ is skew-adjoint. We first study the two first terms. Using the Cauchy-Schwartz inequality we can write

$$
I + II \geq \frac{\gamma}{2} \| (\text{Id} - \Pi_1)u \|^2 - \frac{\gamma}{2} \varepsilon^2 \| (L + L^*)u \|^2 \geq \frac{\gamma}{2} \| (\text{Id} - \Pi_1)u \|^2 - \varepsilon^2 \gamma \| L \|^2 \| u \|^2.
$$

Now we study more carefully the third one

$$
III = \varepsilon \text{Re} (X_0 u, (L + L^*)u) = \varepsilon \text{Re} \left( [L, X_0]u, u \right),
$$

again since $X_0$ is skewadjoint. Recalling that $L = \Lambda^{-2}a^*b$ we can write using the equalities (12)

$$
[L, X_0] = [\Lambda^{-2}a^*b, X_0]
= [\Lambda^{-2}X_0]a^*b + \Lambda^{-2}[a^*, X_0]b + \Lambda^{-2}a^*[b, X_0]
= -\Lambda^{-2}[\Lambda^2, X_0]a^*b - \Lambda^{-2}b^*Hess V b + \Lambda^{-2}a^*a.
$$

Here we used the fact that for $A$, $B$, and $B^{-1}$ continuous on $S$ and $S'$ we have $[A, B^{-1}] = -B^{-1}[A, B]B^{-1}$. Let us denote

$$
\mathcal{A} = -\Lambda^{-2}[\Lambda^2, X_0]a^*b - \Lambda^{-2}b^*Hess V b
$$

We postpone to the end of this section the proof of following lemma concerning operators $\mathcal{A}$ and $L$.

Lemma 3.2 Operator $\mathcal{A}$ and $L$ are bounded on $L^2$. Besides their norms of can be explicitly bounded in terms of $\gamma$ and the second and third order derivatives of $V$.

Now it is clear that denoting $\mathcal{H}^{0,-1} = \{bu \text{ for } u \in L^2 \}$, operator $b$ considered as an operator from $L^2$ into $\mathcal{H}^{0,-1}$ satisfies $E_1 \subset \text{Ker}(b)$. In fact this comes from the fact that $b$ is then the annihilation operator in the velocity variable. Since it appears only in the right in the expression of $\mathcal{A}$ we therefore get

$$
E_1 \subset \text{Ker}(\mathcal{A}).
$$

As a consequence we can write $\mathcal{A} = \mathcal{A}(I - \Pi_1)$ and therefore

$$
[L, X_0] = \mathcal{A}(I - \Pi_1) + \Lambda^{-2}a^*a
$$

Putting this in the expression of the term III appearing in (16) yields

$$
III = \varepsilon \text{Re} \left( \mathcal{A}(I - \Pi_1)u, u \right) + \varepsilon \text{Re} \left( \Lambda^{-2}a^*au, u \right)
\geq -\frac{\gamma}{4} \| (\text{Id} - \Pi_1)u \|^2 - \frac{\varepsilon^2}{\gamma} \| \mathcal{A} \|^2 \| u \|^2 + \varepsilon \text{Re} \left( \Lambda^{-2}a^*au, u \right).
$$
Now it is also clear that $\Lambda^2$, $a^*a$ and $P_1$ commute together, and we can therefore write for the second term appearing in the last inequality
\[
\varepsilon \text{Re} \left( \Lambda^{-2} a^* a u, u \right) = \varepsilon \text{Re} \left( \Lambda^{-2} a^* a \Pi_1 u, u \right) + \varepsilon \text{Re} \left( \Lambda^{-2} a^* a (1 - \Pi_1) u, u \right)
= \varepsilon \text{Re} \left( \Lambda^{-2} a^* a \Pi_1 u, \Pi_1 u \right) + \varepsilon \text{Re} \left( \Lambda^{-2} a^* a (1 - \Pi_1) u, (1 - \Pi_1) u \right)
\geq \varepsilon \text{Re} \left( \Lambda^{-2} a^* a \Pi_1 u, \Pi_1 u \right) - \varepsilon \| (1 - \Pi_1) u \|^2.
\] (19)
for the last inequality we simply used the fact that $a^*a \leq \Lambda^2$, which implies easily that the norm of $a\Lambda^{-1}$ is bounded by 1.

Now we can take into account the spectral gap property of $\Lambda^2$ together with the fact that $u$ is supposed to orthogonal to $M^{1/2}$. We write it as a lemma

**Lemma 3.3** Recalling that $u \perp M^{1/2}$ we have $\text{Re} \left( \Lambda^{-2} a^* a \Pi_1 u, \Pi_1 u \right) \geq \frac{\alpha}{1 + \gamma} \| u \|^2$

**Proof.** We first notice that $\Lambda^2$ and $a^*a$ leave $E_1$ invariant, and that
\[
\Lambda^2 |_{E_1} = 1 + a^*a
\]
is essentially the Witten Laplacian in the spatial direction. We define now $\tau$ to be the spectral gap for $a^*a$. Now for $w \in S(\mathbb{R}^d)$ such that $w \perp \rho^{1/2}$ we have
\[
(a^*aw, w)_{L^2(\mathbb{R}^d)} \geq \tau \| w \|_{L^2(\mathbb{R}^d)}^2.
\]
Since $(a^*a + 1)^{-1/2}w \perp \rho^{1/2}$ also we get
\[
(a^*a(a^*a + 1)^{-1/2}w, (a^*a + 1)^{-1/2}w)_{L^2(\mathbb{R}^d)} \geq \frac{\tau}{1 + \tau} \| w \|_{L^2(\mathbb{R}^d)}^2.
\]
Now the following inequalities are clear:
\[
\frac{\tau}{1 + \tau} \geq \frac{\alpha}{1 + \alpha} \geq \frac{\alpha}{1 + \gamma}.
\]
Indeed $\tau \geq \alpha$ from the definition of $\alpha$ and $\alpha \leq \gamma$ because of the harmonic part of $\Lambda^2 - 1$. Now since $\Pi_1 u \in E_1 \cap E_0^c$ we get the result of the lemma from the preceding study by applying it to the function defined for a.e. $v$ by $x \mapsto \Pi_1 u(x, v)$.

**End of the proof of Proposition 3.1.** Now we can put the result of the lemma in (19) and we get
\[
\varepsilon \text{Re} \left( \Lambda^{-2} a^* a u, u \right) \geq \varepsilon \frac{\alpha}{1 + \gamma} \| \Pi_1 u \|^2 - \varepsilon \| (1 - \Pi_1) u \|^2.
\] (20)
We obtain the following lower bound for the term $\text{III}$ from (15)
\[
\text{III} \geq -\frac{\gamma}{4} \| (\text{Id} - \Pi_1) u \|^2 - \frac{\varepsilon^2}{\gamma} \| A \|^2 \| u \|^2 + \varepsilon \frac{\alpha}{1 + \gamma} \| \Pi_1 u \|^2 - \varepsilon \| (1 - \Pi_1) u \|^2.
\] (21)
Eventually putting together the estimate on $\text{I} + \Pi_1$ and $\text{III}$ we get
\[
\text{Re} \left( Ku, (\text{Id} + \varepsilon(L + L^*))u \right) \geq \frac{\gamma}{2} \| (\text{Id} - \Pi_1) u \|^2 - \varepsilon^2 \gamma \| L \|^2 \| u \|^2
- \frac{\gamma}{4} \| (\text{Id} - \Pi_1) u \|^2 - \frac{\varepsilon^2}{\gamma} \| A \|^2 \| u \|^2 + \varepsilon \frac{\alpha}{\delta^2 + \alpha} \| \Pi_1 u \|^2 - \varepsilon \| (1 - \Pi_1) u \|^2.
\]
\[
\geq \frac{\gamma}{8} \| (\text{Id} - \Pi_1) u \|^2 + \varepsilon \frac{\alpha}{1 + \gamma} \| \Pi_1 u \|^2 - \varepsilon \left( \gamma^{-1} \| A \|^2 + \gamma \| L \|^2 \right) \| u \|^2.
\]
by taking $\varepsilon \leq \gamma/8$. Now we use the fact that $\Pi_1$ is an orthogonal projector and that $\varepsilon \leq \gamma/8$:

$$\text{Re} \ (Ku, (\text{Id} + \varepsilon(L + L^*))u) \geq \left(\varepsilon \frac{\alpha}{1 + \gamma} - \varepsilon^2 \left(\gamma^{-1} \|A\|^2 + \gamma \|L\|^2\right)\right) \|u\|^2.$$  

For $\varepsilon/\alpha$ sufficiently small, but depending only on $\gamma$ and the second and third order derivatives of $V$ via lemma 3.2, we obtain

$$\text{Re} \ (Ku, (\text{Id} + \varepsilon(L + L^*))u) \geq \frac{\alpha^2}{A} \|u\|^2$$

where $A$ satisfies the hypothesis of the Proposition. The proof is then complete. \hspace{1cm} \Box

**Proof of lemma 3.2.** Recall that $L = \Lambda^{-2}a^*b$. Now for $A$ we use the expression of $[\Lambda^2, X_0]$ in (12) and we get

$$A = \Lambda^{-2}b^*(\text{Hess}(V) - \text{Id})a\Lambda^{-2}a^*b + \Lambda^{-2}a^*(\text{Hess}(V) - \text{Id})b\Lambda^{-2}a^*b - \Lambda^{-2}b^*\text{Hess}(V)b. \quad (22)$$

We therefore see that it is sufficient to prove that for any $d \times d$ real matrix $M(x)$ depending only on $x$, bounded and with first derivative bounded, the following operators

$$\Lambda^{-2}b^*M(x)a, \quad \Lambda^{-2}b^*M(x)b, \quad \Lambda^{-2}a^*M(x)b$$

are bounded as operators on $L^2$. We give the proof for the first one since for the remaining ones, the proof is similar and easier. We shall prove the result for its adjoint $a^*M(x)b\Lambda^{-2}$. We write for $u \in \mathcal{S}$,

$$\|a^*M(x)b\Lambda^{-2}u\| \leq \sum_{j,k} \|a_j^*M_{j,k}(x)b_k\Lambda^{-2}u\|$$

$$\leq \sum_{j,k} \|M_{j,k}(x)a_j^*b_k\Lambda^{-2}u\| + \sum_{j,k} \gamma^{1/2} \|(\partial_{x_j} M_{j,k})(x)b_k\Lambda^{-2}u\| \quad (23)$$

$$\leq \left(\|M\|_{L^\infty} + \gamma^{1/2} \|\partial_x M\|_{L^\infty}\right) \sum_{j,k} (\|a_j^*b_k\Lambda^{-2}u\| + \|b_k\Lambda^{-2}u\|)$$

where we used the fact that $[a_j^*, M_{j,k}] = -\partial_{x_j} M_{j,k}$. Now this is straightforward to check that $\|b_k\Lambda^{-2}u\| \leq \|u\|$ since $b^*b \leq A^2$ and $1 \leq A^2$. For the other term $\|a_j^*b_k\Lambda^{-2}u\|$ in the last inequality of (23), we write

$$\|a_j^*b_k\Lambda^{-2}u\|^2 = (a_j^*a_j b_k\Lambda^{-2}u, b_k\Lambda^{-2}u)$$

$$\leq (a_j^*a_j b_k\Lambda^{-2}u, b_k\Lambda^{-2}u) + \left(\gamma (\partial_{x_j} V)b_k\Lambda^{-2}u, b_k\Lambda^{-2}u\right)$$

$$\leq (\Lambda^2 b_k\Lambda^{-2}u, b_k\Lambda^{-2}u) + \gamma \|\text{Hess}V\|_{L^\infty} (b_k\Lambda^{-2}u, b_k\Lambda^{-2}u) \quad (24)$$

Now using the fact that $[\Lambda^2, b_k] = -\gamma b_k$ we can continue the preceding series of inequalities:

$$\leq (\Lambda^2 b_k\Lambda^{-2}u, b_k\Lambda^{-2}u) + \gamma \|\text{Hess}V\|_{L^\infty} (b_k\Lambda^{-2}u, b_k\Lambda^{-2}u)$$

$$\leq (b_ku, b_k\Lambda^{-2}u) + \gamma (\|\text{Hess}V\|_{L^\infty} + 1) (b_k\Lambda^{-2}u, b_k\Lambda^{-2}u)$$

$$\leq \gamma (\|\text{Hess}V\|_{L^\infty} + 2) \|u\|^2$$
again since $b_k^* b_k \leq \Lambda^2$ and $1 \leq \Lambda^2$. Therefore the term $\left\| a_j^* b_k \Lambda^{-2} u \right\|$ in the last inequality of (23) is also bounded by $C \| u \|$, where $C$ depends only on $\gamma$ and the second and third derivatives of $V$. The proof of lemma 3.2 is therefore complete.

4 Proof of the Theorem and comments

We go on studying operator $K$ defined in the preceding sections. We first quote an easy result from [7] relying hypocoercivity to exponential decay

Lemma 4.1 (lemma A6 in [7]) Let $K$ be the infinitesimal generator of a semigroup of contraction on a Hilbert space $H$ and suppose that there exist a constant $\delta > 0$ and a bounded operator $\mathcal{L}$ with norm bounded by $C_L \geq 1$ such that

$$\forall u \in D(K), \quad \delta \| u \|^2 \leq \text{Re} (Ku, u) + \text{Re} (Ku, (\mathcal{L} + \mathcal{L}^*) u)$$

then for all $u_0 \in H$ and $t \geq 0$ we have $\left\| e^{-tK} u_0 \right\| \leq 3e^{-\frac{\delta t}{C_L}} \| u_0 \|$.

Proof of Theorem 1.1. We first consider operator $K$ defined in the preceding sections. The result for $K$ in the Hilbert space $(\mathcal{M}^{1/2})^\perp$ is then a direct consequence of Proposition 3.1. Indeed it suffices to apply the preceding lemma when considering $K$ in place of $\mathcal{K}$, $\varepsilon L$ in place of $\mathcal{L}$, $C_L = 1$ and $\alpha^2/A$ in place of $\delta$. The result of the theorem is then a simple transcription in terms of $f = \mathcal{M}^{1/2} u$ and $f - f_\infty = \mathcal{M}^{1/2} (u - \Pi_0 u)$ (of course $(u - \Pi_0 u) \perp \mathcal{M}^{1/2}$, which is an other way to say that $\iint (f - f_\infty) dxdv = 0$). The proof is complete.

Proof of Corollary 1.2. This is then a direct consequence of the main Theorem, and the proof follows exactly the one in [5, corollary 0.2]: Let $f_0$ be a $L^1$-normalized function which belongs to $\mathcal{M}^{1/2} L^2$ and let $f(t)$ be the solution of (1), (it stays non-negative for all time). The non-negativity of the relative entropy is clear from the fact that $f$ stays $L^1$ normalized. For the other side, we write for $t \geq 0$, with $f_\infty = \mathcal{M}$:

$$H(f(t)|\mathcal{M}) = \iint f(t) \log \left( \frac{f(t)}{\mathcal{M}} \right) dxdv = \iint \frac{f(t)}{\mathcal{M}^{1/2}} \mathcal{M}^{1/2} \log \left( \frac{f(t)}{\mathcal{M}} \right) dxdv.$$  

Applying first $\ln(x) \leq x - 1$ and then the Cauchy-Schwarz inequality for $t \geq 0$, we get

$$H(f(t)|\mathcal{M}) \leq \iint \frac{f(t)}{\mathcal{M}^{1/2}} \mathcal{M}^{1/2} \left( \frac{f(t)}{\mathcal{M}} - 1 \right) dxdv \leq \left\| \frac{f(t)}{\mathcal{M}^{1/2}} \right\| \left\| f(t) - \mathcal{M} \right\|_{B^2} \left\| f(t) - \mathcal{M} \right\|_{B^2}.$$ 

(26)

Hence the exponential decay of the relative entropy is a consequence of Theorem 1.1.

We end this work by making some remarks about the main result:
Remark 4.2 In the case $e^{-V} \not\in L^1$, Theorem 1.1 remains true when replacing $f_\infty$ by 0. A careful study of the proof shows that it is exactly the same with the following adaptations: $\rho_\infty(x) = e^{-V(x)}$ is not anymore in $L^1(dx)$ and neither does $\mathcal{M}(x,v) = \rho_\infty(x)\mu_\infty(v)$ in $L^1(dx dv)$. Then $\alpha$ is the bottom of the spectrum of operator $\Lambda^2 - 1$, and in the proof $E_0 = \{0\}$ and the corresponding projector is $\Pi_0 = 0$. In this case Theorem 1.1 has to be understood has a vanishing.

Remark 4.3 A natural question is to understand the common features between for example the Fokker-Planck operator (and also the chains of anaharmonic oscillators) and the one studied here. Let us see this after conjugating by the square root of the Maxwellian and only in $L^2$ for the velocity variable (homogeneous case). The collision operators are respectively

$$Q_{F_{FP}} = b^*b \quad \text{(Fokker-Planck)}, \quad Q_{lib} = \text{Id} - \Pi_1 \quad \text{(Linear Inhom. Boltzmann)}.$$ 

Here $\Pi_1$ is the orthogonal projection on the space spanned by the Maxwellian in the velocity variable $\mu_\infty^{1/2}$. A simple remark can be made using the Hermite decomposition of functions (in the velocity variable), which we denote $H_k$. We know that the annihilation operators $b$ and its conjugate $b^*$ (creation operator) have both a nice description, since for all $k \geq 1$,

$$bH_k = \sqrt{k}H_{k-1}, \quad b^*H_{k-1} = \sqrt{k}H_k$$

and $bH_0 = 0$ (recall that $H_0 = \mu_\infty^{1/2}$). Therefore $b$ and $b^*$ can be represented by respectively an upperdiagonal and a subdiagonal infinite matrices with coefficients $\sqrt{k}$. Immediately we get the expression of the harmonic oscillator $b^*b$, for which the Hermite polynomials are an Hilbertian base. Now using this decomposition we can build operators $c$ and its adjoint $c^*$ by imposing

$$cH_k = H_{k-1}, \quad c^*H_{k-1} = H_k$$

and $cH_0 = 0$. Then $c$ and $c^*$ have the same representation as matrices than $b$ and $b^*$ respectively (note that they are bounded). It is immediate to check that

$$c^*c = \text{Id} - \Pi_1 = Q_{lib}.$$

As a conclusion, and transferring this in the inhomogeneous case, we see that the Fokker-Planck and the linear inhomogeneous Boltzmann models have the same structure, explaining (a bit) the efficiency of hypoelliptic methods in the last case.

References


