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Fractal first order partial differential equations

Jérôme Droniou*, Cyril Imbert*†
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Abstract

The present paper is concerned with semilinear partial differential equations involving a particular pseudo-differential operator. It investigates both fractal conservation laws and non-local Hamilton-Jacobi equations. The starting point is a new formula for the operator. It permits to prove the key *a priori* estimate that stands for the scalar conservation law *and* the Hamilton-Jacobi equation. The smoothing effect of the operator is also put in light and used to solve both equations. As far as Hamilton-Jacobi equations are concerned, a non-local vanishing viscosity method is used to construct a (viscosity) solution when existence of regular solutions fails, and a rate of convergence is provided. Turning to conservation laws, global-in-time existence and uniqueness are established. We also show that our formula allows to obtain entropy inequalities for the non-local conservation law, and thus to prove the convergence of the solution, as the non-local term vanishes, toward the entropy solution of the pure conservation law.

Mathematical subject classifications: 35B45, 35B65, 35A35, 35S30
Keywords: Lévy operator, fractal conservation laws, non-local Hamilton-Jacobi equations, smoothing effect, *a priori* estimates, global-in-time existence, rate of convergence.

1 Introduction

In this paper, we are interested in solving semilinear partial differential equations involving the fractal operator (or Lévy operator) defined, for $\varphi \in \mathcal{S}(\mathbb{R}^N)$, by

$$g_\lambda[\varphi] = \mathcal{F}^{-1}\left(|\cdot|^\lambda \mathcal{F}(\varphi)\right) \qquad \text{with } 0 < \lambda < 2 \tag{1.1}$$

($\mathcal{F}$ is the Fourier transform). The study of PDE's involving g_λ is motivated by a number of physical problems, such as overdriven detonations in gases [9] or anomalous diffusion in semiconductor growth [26], and by mathematical models in finance (see below for references). We consider perturbations by g_λ of Hamilton-Jacobi equations or scalar conservation laws, that is to say

$$\begin{cases} \partial_t u(t,x) + g_\lambda[u(t,\cdot)](x) = F(t,x,u(t,x),\nabla u(t,x)) & t>0\,,\ x\in\mathbb{R}^N\,, \\ u(0,x) = u_0(x) & x\in\mathbb{R}^N \end{cases} \tag{1.2}$$

or

$$\begin{cases} \partial_t u(t,x) + \operatorname{div}(f(t,x,u(t,x))) + g_\lambda[u(t,\cdot)](x) = h(t,x,u(t,x)) & t>0\,,\ x\in\mathbb{R}^N\,, \\ u(0,x) = u_0(x) & x\in\mathbb{R}^N. \end{cases} \tag{1.3}$$

These kinds of equations have already been studied. As far as scalar conservation laws are concerned, some recent papers have investigated them. One of the first works on this subject is probably [4], which deals with (1.3) when $h = 0$ and $f(t,x,u) = f(u)$; using energy estimates, it states some local-in-time existence and uniqueness results of weak solutions if f has a polynomial growth. This result is strengthened in [14], where a splitting method is used to prove global-in-time existence and uniqueness of a regular solution if $\lambda > 1$.

To our best knowledge, Hamilton-Jacobi equations of type (1.2) first appeared in the context of mathematical finance as Bellman equations of optimal control of jump diffusion processes [23]. See also [24, 22, 1] and more recently [6, 7, 8]. A general theory for non-linear integro-partial differential equations is developed by Jakobsen and Karlsen [18, 19]. Some of the ideas of [14] are adapted in [17] to prove that

*Département de Mathématiques, UMR CNRS 5149, CC 051, Université Montpellier II, Place Eugène Bataillon, 34095 Montpellier cedex 5, France. E-mails: droniou@math.univ-montp2.fr, imbert@math.univ-montp2.fr
†Polytech'Montpellier, Université Montpellier II, Montpellier, France

(1.2) has regular solutions if $\lambda > 1$. It adapts previous viscosity solution theories to the equation (1.2) (existence via Perron's method, comparison results, stability) and use techniques of [14] to obtain further regularity.

The preceding methods to handle (1.2) and (1.3) are somewhat incompatible: the splitting method for Hamilton-Jacobi equation (1.2) is less direct than Perron's one and it is well known that the notion of viscosity solution is inadequate to conservation laws such as (1.3). However, a scalar conservation law can always be formally written as a Hamilton-Jacobi equation (write $\operatorname{div}(f(u)) = f'(u) \cdot \nabla u$). In this paper, we present a way to simultaneously solve (1.2) and (1.3) by using this simple fact and the construction of regular solutions to the Hamilton-Jacobi equations under weak assumptions (that are satisfied by both (1.2) and (1.3)). The key estimate is given by Proposition 3.1. As we notice in the course of the proofs, the method we use is also valid for more general operators than g_λ.

The starting point of this work is an equivalent definition of the fractal operator: we give a formula for g_λ which precises and generalizes (to the case $\lambda \leq 1$) the formula in [17]. This formula permits to extend the operator from Schwartz functions to C_b^2 ones and is moreover used to establish what we call a "reverse maximum principle" that says, freely speaking, that $g_\lambda[\phi](x)$ is nonnegative if x is a maximum point of ϕ. This principle is the key point when proving the estimate for the regular solutions of (1.2). The formula for g_λ also has other applications that are mentioned below.
As far as Hamilton-Jacobi equations are concerned, the study of (1.2) in [17] is made for $\lambda > 1$ and by using Perron's method. We generalize here the results of this paper to the case $\lambda \in]0, 2[$ and we weaken the hypotheses on the Hamiltonian. We first prove that the fractal operator has a smoothing effect for $\lambda \in]1, 2[$ under very general (and natural) hypotheses on F; the idea to obtain a global solution is, roughly speaking, to study how $\sup_{\mathbb{R}^N} |u(t, \cdot)|$ evolves. We next treat the case $\lambda \in]0, 2[$ (recall that, contrary to the formula in [17], ours is valid for such λ) by solving (1.2) in the sense of viscosity solutions; as expected in this context (see (1.1)), these solutions are no more regular but only bounded and uniformly continuous. We use a non-local vanishing viscosity method (though we could have used Perron's method, see Remark 3.6): precisely, we add a vanishing fractal operator εg_μ with $\mu > 1$ and we pass to the limit $\varepsilon \to 0$. We also provide, for all $\mu \in]0, 2[$, a rate of convergence that is in some respect surprising, compared with the case $\mu > 1$ treated in [17]. Let us also mention that the reverse maximum principle and its main consequence, namely the key estimate given by Proposition 3.1, can be generalized to the framework of viscosity solutions: nonsmooth versions of both results are stated and proved in Appendix. We have chosen not to use these versions because, anywhere we can, we search for regular solutions and we turn to the notion of viscosity solution only if mandatory (that is if $\lambda \leq 1$; see Subsection 3.2).
The case of scalar hyperbolic equations (1.3) with $\lambda > 1$ is treated next. The *a priori* estimates on the solution follow from Proposition 3.1, the same proposition that gives the *a priori* estimates for (1.2), and the existence of a regular solution is as straightforward. The splitting method of [14] can be adapted to some cases where f and h depend on (t, x) (see [13]), but this is awfully technical; the technique we use here therefore presents a noticeable simplification in the study of (1.3) in the general case. The question of non-local vanishing viscosity regularization (multiplying $g_\lambda[u]$ by ε and letting $\varepsilon \to 0$) is treated in [12], still using the splitting method (and for $h = 0$, $f(t, x, u) = f(u)$); at the end of the present paper, we quickly indicate how the formula for g_λ allows to significantly simplify the corresponding proofs for general f and h, in particular the proof of the entropy inequalities for (1.3).

The paper is organised as follows. Section 2 is devoted to establish another formula for g_λ (Theorem 2.1). It also contains the "reverse maximum principle" (Theorem 2.2) we presented above. In Section 3, we study Hamilton-Jacobi equations. We first present the smoothing effect of the fractal operator on (1.2) for $\lambda \in]1, 2[$ (Theorem 3.1); next, we construct viscosity solutions for $\lambda \in]0, 2[$ by a non-local vanishing viscosity method (Theorem 3.3) and we prove a rate of convergence (Theorem 3.4). Section 4 contains the resolution of (1.3) and the (short) proof of the entropy inequalities associated with the perturbed conservation law (Subsection 4.2). Some appendixes in Section 5 conclude the paper. In particular, the reader can find there a generalization of the "reverse maximum principle" and of the key estimate to the viscosity framework.

Notations. Throughout the paper, B_r (resp. $B_r(x)$) denotes the ball of $\mathbb{R}^N$ centered at the origin (resp. at x) and of radius r. Euler's function is denoted by Γ.

2 Another formula for g_λ

The main result of this section is a new formula for g_λ, which generalizes Lemma 1 in [17]. As a consequence, we extend the definition of g_λ from Schwartz functions to C_b^2 functions and we prove what we call a "reverse maximum principle": roughly speaking, it says that at a maximum point of a C_b^2 function φ, we have $g_\lambda[\varphi] \geq 0$. This result is the crucial argument when proving the key estimate stated in Proposition 3.1.

Theorem 2.1 *If $\lambda \in]0,2[$, then, for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$, all $x \in \mathbb{R}^N$ and all $r > 0$,*

$$g_\lambda[\varphi](x) = -c_N(\lambda)\left(\int_{B_r} \frac{\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z}{|z|^{N+\lambda}}\,dz + \int_{\mathbb{R}^N\setminus B_r} \frac{\varphi(x+z)-\varphi(x)}{|z|^{N+\lambda}}\,dz\right) \tag{2.1}$$

where $c_N(\lambda) = \frac{\lambda\Gamma(\frac{N+\lambda}{2})}{2\pi^{\frac{N}{2}+\lambda}\Gamma(1-\frac{\lambda}{2})}$. We can generalize this formula in two cases:

i) If $\lambda \in]0,1[$, we can take $r = 0$:

$$g_\lambda[\varphi](x) = -c_N(\lambda)\int_{\mathbb{R}^N} \frac{\varphi(x+z)-\varphi(x)}{|z|^{N+\lambda}}\,dz.$$

ii) If $\lambda \in]1,2[$, we can take $r = +\infty$:

$$g_\lambda[\varphi](x) = -c_N(\lambda)\int_{\mathbb{R}^N} \frac{\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z}{|z|^{N+\lambda}}\,dz.$$

Before proving these formulae, let us state some of their consequences. We first notice that (2.1) allows to define $g_\lambda[\varphi] \in C_b(\mathbb{R}^N)$ for $\varphi \in C_b^2(\mathbb{R}^N)$. In fact, this gives a continuous extension of g_λ in the following sense.

Proposition 2.1 *Let $\lambda \in]0,2[$ and $\varphi \in C_b^2(\mathbb{R}^N)$. If $(\varphi_n)_{n\geq 1} \in C_b^2(\mathbb{R}^N)$ is bounded in $L^\infty(\mathbb{R}^N)$ and $D^2\varphi_n \to D^2\varphi$ locally uniformly on $\mathbb{R}^N$, then $g_\lambda[\varphi_n] \to g_\lambda[\varphi]$ locally uniformly on $\mathbb{R}^N$.*

Remark 2.1 *We could also define g_λ on Hölder spaces of functions (depending on λ), and state an equivalent of Proposition 2.1 in this framework.*

Proof of Proposition 2.1
The operator g_λ being linear, we can assume that $\varphi = 0$. Since $(\varphi_n)_{n\geq 1}$ is bounded in $L^\infty(\mathbb{R}^N)$, the second integral term of (2.1) applied to $\varphi = \varphi_n$ is small, uniformly for $n \geq 1$ and $x \in \mathbb{R}^N$, if r is large. By Taylor's formula, for $|z| \leq r$ and $|x| \leq R$ we have $|\varphi_n(x+z)-\varphi_n(x)-\nabla\varphi_n(x)\cdot z| \leq ||D^2\varphi_n||_{L^\infty(B_{r+R})}|z|^2$; hence, with r fixed, the first integral term of (2.1) applied to $\varphi = \varphi_n$ is small, uniformly for $x \in B_R$, if n is large. ■

From (2.1) it is obvious that, if x is a global maximum of φ, then $g_\lambda[\varphi](x) \geq 0$, with equality if and only if φ is constant (notice that $c_N(\lambda) > 0$ for $\lambda \in]0,2[$). We have a generalization of this property, which will be the key argument to study first order perturbations of $\partial_t + g_\lambda$.

Theorem 2.2 *Let $\lambda \in]0,2[$ and $\varphi \in C_b^2(\mathbb{R}^N)$. If $(x_n)_{n\geq 1}$ is a sequence of $\mathbb{R}^N$ such that $\varphi(x_n) \to \sup_{\mathbb{R}^N}\varphi$ as $n \to \infty$, then $\lim_{n\to\infty}\nabla\varphi(x_n) = 0$ and $\liminf_{n\to\infty} g_\lambda[\varphi](x_n) \geq 0$.*

Proof of Theorem 2.2
Since the second derivative of φ is bounded, there exists C such that, for all $n \geq 1$ and all $z \in \mathbb{R}^N$,

$$\sup_{\mathbb{R}^N} \varphi \geq \varphi(x_n + z) \geq \varphi(x_n) + \nabla\varphi(x_n) \cdot z - C|z|^2. \tag{2.2}$$

Up to a subsequence, we can assume that $\nabla\varphi(x_n) \to p$ (this sequence is bounded). Passing to the limit $n \to \infty$ in (2.2) gives $0 \geq p \cdot z - C|z|^2$; choosing then $z = tp$ and letting $t \to 0^+$ shows that $p = 0$, which proves that $\lim_{n\to\infty} \nabla\varphi(x_n) = 0$ (the only adherence value of the bounded sequence $(\nabla\varphi(x_n))_{n\geq 1}$ is 0). Since $\varphi(x_n + z) - \varphi(x_n) \leq \sup_{\mathbb{R}^N} \varphi - \varphi(x_n) \to 0$, we deduce that, for all $z \in \mathbb{R}^N$,

$$\begin{aligned} &\limsup_{n\to\infty} (\varphi(x_n + z) - \varphi(x_n)) \leq 0 \\ &\limsup_{n\to\infty} (\varphi(x_n + z) - \varphi(x_n) - \nabla\varphi(x_n) \cdot z) \leq 0. \end{aligned} \tag{2.3}$$

We also have

$$\frac{|\varphi(x_n + z) - \varphi(x_n)|}{|z|^{N+\lambda}} \leq \frac{2||\varphi||_{L^\infty(\mathbb{R}^N)}}{|z|^{N+\lambda}} \in L^1(\mathbb{R}^N \backslash B_r)$$

and

$$\frac{|\varphi(x_n + z) - \varphi(x_n) - \nabla\varphi(x_n) \cdot z|}{|z|^{N+\lambda}} \leq \frac{||D^2\varphi||_{L^\infty(\mathbb{R}^N)}|z|^2}{|z|^{N+\lambda}} \in L^1(B_r).$$

Hence, by (2.3) and Fatou's Lemma,

$$0 \geq \int_{\mathbb{R}^N \backslash B_r} \limsup_{n\to\infty} \frac{\varphi(x_n + z) - \varphi(x_n)}{|z|^{N+\lambda}}\, dz \geq \limsup_{n\to\infty} \int_{\mathbb{R}^N \backslash B_r} \frac{\varphi(x_n + z) - \varphi(x_n)}{|z|^{N+\lambda}}\, dz$$

and

$$0 \geq \int_{B_r} \limsup_{n\to\infty} \frac{\varphi(x_n + z) - \varphi(x_n) - \nabla\varphi(x_n) \cdot z}{|z|^{N+\lambda}}\, dz \geq \limsup_{n\to\infty} \int_{B_r} \frac{\varphi(x_n + z) - \varphi(x_n) - \nabla\varphi(x_n) \cdot z}{|z|^{N+\lambda}}\, dz.$$

Combining these inequalities and (2.1) permits to achieve the proof of the theorem. ■

Remark 2.2 *This theorem is also true for $\lambda = 2$, that is to say $g_2 = -4\pi^2\Delta$, provided that $\varphi \in C_b^3(\mathbb{R}^N)$.*

We now conclude this section by proving the formula given for g_λ.

Proof of Theorem 2.1
Step 1: a preliminary formula.
We first assume that $\lambda \in]1, 2[$. We have $g_\lambda[\varphi] = \mathcal{F}^{-1}(|\cdot|^\lambda \mathcal{F}(\varphi))$; but $\mathcal{F}(\Delta\varphi) = -4\pi^2|\cdot|^2\mathcal{F}(\varphi)$ and therefore, for $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$g_\lambda[\varphi] = \frac{1}{-4\pi^2}\mathcal{F}^{-1}(|\cdot|^{\lambda-2}\mathcal{F}(\Delta\varphi)). \tag{2.4}$$

Since $\lambda \in]1, 2[$, we have $\lambda - 2 \in]-N, 0[$; hence $|\cdot|^{\lambda-2}$ is locally integrable and is in $\mathcal{S}'(\mathbb{R}^N)$. The inverse Fourier transform of $|\cdot|^{\lambda-2}$ is a distribution with radial symmetry and homogeneity of order $-N-(\lambda-2)$; we deduce that there exists $C_N(\lambda)$ such that

$$\mathcal{F}^{-1}(|\cdot|^{\lambda-2}) = C_N(\lambda)|\cdot|^{-N-(\lambda-2)} \tag{2.5}$$

in $\mathcal{D}'(\mathbb{R}^N \backslash \{0\})$; since $|\cdot|^{-N-(\lambda-2)}$ is locally integrable, it is quite easy to see that (2.5) also holds in $\mathcal{S}'(\mathbb{R}^N)$. We compute $C_N(\lambda)$ by taking the test function $\gamma(x) = e^{-\pi|x|^2}$, which is its own inverse Fourier transform:

$$\int_{\mathbb{R}^N} |x|^{\lambda-2} e^{-\pi|x|^2}\, dx = \langle \mathcal{F}^{-1}(|\cdot|^{\lambda-2}), \gamma\rangle_{\mathcal{S}'(\mathbb{R}^N),\mathcal{S}(\mathbb{R}^N)} = C_N(\lambda) \int_{\mathbb{R}^N} |x|^{-N-(\lambda-2)} e^{-\pi|x|^2}\, dx.$$

Using polar coordinates, we deduce $\int_0^\infty r^{N+\lambda-4}e^{-\pi r^2}\,rdr = C_N(\lambda)\int_0^\infty r^{-\lambda}e^{-\pi r^2}\,rdr$ and the change of variable $\tau = \pi r^2$ implies

$$\pi^{-(N+\lambda-4)/2}\int_0^\infty \tau^{(N+\lambda-4)/2}e^{-\tau}\,\frac{d\tau}{2\pi} = C_N(\lambda)\pi^{\lambda/2}\int_0^\infty \tau^{-\lambda/2}e^{-\tau}\,\frac{d\tau}{2\pi}\,,$$

that is to say $C_N(\lambda) = \Gamma(\frac{N+\lambda}{2}-1)/[\pi^{\frac{N}{2}+\lambda-2}\Gamma(1-\frac{\lambda}{2})]$. With this value of $C_N(\lambda)$, (2.4) and (2.5) give, for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$g_\lambda[\varphi] = -\frac{\Gamma\left(\frac{N+\lambda}{2}-1\right)}{4\pi^{\frac{N}{2}+\lambda}\Gamma\left(1-\frac{\lambda}{2}\right)}|\cdot|^{-N-(\lambda-2)} * \Delta\varphi. \tag{2.6}$$

Step 2: proof of (2.1) for $\lambda \in]1,2[$.
Let $r > 0$, $\varphi \in \mathcal{S}(\mathbb{R}^N)$, $x \in \mathbb{R}^N$ and define $\phi_x(z) = \varphi(x+z) - \varphi(x) - \nabla\varphi(x)\cdot z\,\theta(z)$, where $\theta \in C_c^\infty(\mathbb{R}^N)$ is even and equal to 1 on B_r. We have $\Delta\phi_x(z) = \Delta\varphi(x+z) - \nabla\varphi(x)\cdot\Delta(z\,\theta(z))$, and thus, with $\beta = -N-(\lambda-2) \in]-N,0[$,

$$|\cdot|^{-N-(\lambda-2)} * \Delta\varphi(x) = \int_{\mathbb{R}^N}|z|^\beta\Delta\varphi(x+z)\,dz = \int_{\mathbb{R}^N}|z|^\beta\Delta\phi_x(z)\,dz + \nabla\varphi(x)\cdot\int_{\mathbb{R}^N}|z|^\beta\Delta(z\,\theta(z))\,dz$$

(all these functions are integrable since $\Delta\varphi(x+z)$ and $\Delta(z\,\theta(z))$ are both Schwartz functions). But $z \mapsto z\,\theta(z)$ is odd, so $z \mapsto |z|^\beta\Delta(z\,\theta(z))$ is also odd and its integral on $\mathbb{R}^N$ vanishes. Hence,

$$|\cdot|^{-N-(\lambda-2)} * \Delta\varphi(x) = \int_{\mathbb{R}^N}|z|^\beta\Delta\phi_x(z)\,dz = \lim_{\varepsilon\to 0}\int_{C_\varepsilon}|z|^\beta\Delta\phi_x(z)\,dz \tag{2.7}$$

where $C_\varepsilon = \{\varepsilon \le |z| \le 1/\varepsilon\}$. By Green's formula,

$$\int_{C_\varepsilon}|z|^\beta\Delta\phi_x(z)\,dz = \int_{C_\varepsilon}\Delta(|z|^\beta)\phi_x(z)\,dz + \int_{\partial C_\varepsilon}\left[|z|^\beta\nabla\phi_x(z)\cdot\mathbf{n}(z) - \phi_x(z)\nabla(|z|^\beta)\cdot\mathbf{n}(z)\right]d\sigma_\varepsilon(z) \tag{2.8}$$

where σ_ε is the $(N-1)$-dimensional measure on $\partial C_\varepsilon = S_\varepsilon \cup S_{1/\varepsilon}$ (with $S_a = \{|z| = a\}$) and $\mathbf{n}$ is the outer unit normal to C_ε. On a neighbourhood of 0, we have $\phi_x(z) = \varphi(x+z) - \varphi(x) - \nabla\varphi(x)\cdot z$, and thus $\phi_x(z) = \mathcal{O}(|z|^2)$, $\nabla\phi_x(z) = \mathcal{O}(|z|)$; using $|\nabla(|z|^\beta)| = |\beta|\,|z|^{\beta-1}$, we deduce, since $N+\beta = 2-\lambda > 0$,

$$\left|\int_{S_\varepsilon}\left(|z|^\beta\nabla\phi_x(z)\cdot\mathbf{n}(z) - \phi_x(z)\nabla(|z|^\beta)\cdot\mathbf{n}(z)\right)d\sigma_\varepsilon(z)\right| \le C\varepsilon^{N-1}\varepsilon^{\beta+1} \to 0 \qquad \text{as } \varepsilon \to 0. \tag{2.9}$$

Since $-(N-1)-(\beta-1) = 2-(N+\beta) = \lambda > 0$ and, at infinity, $\phi_x(z) = \varphi(x+z) - \varphi(x)$ is bounded and $\nabla\phi_x(z) = \nabla\varphi(x+z)$ is rapidly decreasing, we obtain

$$\begin{aligned}&\left|\int_{S_{1/\varepsilon}}\left(|z|^\beta\nabla\phi_x(z)\cdot\mathbf{n}(z) - \phi_x(z)\nabla(|z|^\beta)\cdot\mathbf{n}(z)\right)d\sigma_\varepsilon(z)\right| \\ &\le\ C\left(\frac{1}{\varepsilon}\right)^{N-1+\beta}\sup_{S_{1/\varepsilon}}|\nabla\varphi(x+\cdot)| + C\varepsilon^{-(N-1)}\varepsilon^{-(\beta-1)} \to 0 \qquad \text{as } \varepsilon \to 0.\end{aligned} \tag{2.10}$$

An easy computation gives

$$\Delta(|z|^\beta) = \operatorname{div}(\beta|z|^{\beta-2}z) = \beta\left(N|z|^{\beta-2} + (\beta-2)|z|^{\beta-3}\frac{z}{|z|}\cdot z\right) = \beta(N+\beta-2)|z|^{\beta-2}$$

and therefore

$$\int_{C_\varepsilon}\Delta(|z|^\beta)\phi_x(z)\,dz = (N+\lambda-2)\lambda\int_{C_\varepsilon}|z|^{-N-\lambda}\phi_x(z)\,dz. \tag{2.11}$$

Since $\phi_x(z) = \mathcal{O}(|z|^2)$ on a neighbourhood of 0 and ϕ_x is bounded on $\mathbb{R}^N$, the function $|\cdot|^{-N-\lambda}\phi_x$ is integrable on $\mathbb{R}^N$ and we can pass to the limit as $\varepsilon \to 0$ in the right-hand side of (2.11).

Combining (2.7), (2.8), (2.9), (2.10) and (2.11) yields

$$|\cdot|^{-N-(\lambda-2)} * \Delta\varphi(x) = \lambda(N+\lambda-2)\int_{\mathbb{R}^N} \frac{\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z\,\theta(z)}{|z|^{N+\lambda}}\,dz.$$

Since $\theta = 1$ on B_r, this gives

$$\begin{aligned} |\cdot|^{-N-(\lambda-2)} * \Delta\varphi(x) &= \lambda(N+\lambda-2)\int_{B_r} \frac{\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z}{|z|^{N+\lambda}}\,dz \\ &\quad +\lambda(N+\lambda-2)\int_{\mathbb{R}^N\setminus B_r} \frac{\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z\,\theta(z)}{|z|^{N+\lambda}}\,dz. \end{aligned}$$

But $|z|^{-N-\lambda}(\varphi(x+z)-\varphi(x))$ and $|z|^{-N-\lambda}z\,\theta(z)$ are integrable on $\mathbb{R}^N\setminus B_r$, and thus

$$\int_{\mathbb{R}^N\setminus B_r} \frac{\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z\,\theta(z)}{|z|^{N+\lambda}}\,dz = \int_{\mathbb{R}^N\setminus B_r} \frac{\varphi(x+z)-\varphi(x)}{|z|^{N+\lambda}}\,dz - \nabla\varphi(x)\cdot\int_{\mathbb{R}^N\setminus B_r} \frac{z\,\theta(z)}{|z|^{N+\lambda}}\,dz.$$

Since $z \mapsto |z|^{-N-\lambda}z\,\theta(z)$ is odd, this last integral vanishes and we deduce

$$\begin{aligned} |\cdot|^{-N-(\lambda-2)} * \Delta\varphi(x) &= \lambda(N+\lambda-2)\int_{B_r} \frac{\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z}{|z|^{N+\lambda}}\,dz \\ &\quad +\lambda(N+\lambda-2)\int_{\mathbb{R}^N\setminus B_r} \frac{\varphi(x+z)-\varphi(x)}{|z|^{N+\lambda}}\,dz. \end{aligned}$$

Using this formula in (2.6) and taking into account $(N+\lambda-2)\Gamma(\frac{N+\lambda}{2}-1) = 2(\frac{N+\lambda}{2}-1)\Gamma(\frac{N+\lambda}{2}-1) = 2\Gamma(\frac{N+\lambda}{2})$, we obtain (2.1) if $\lambda \in]1,2[$.

Notice that, up to now, the reasoning is also valid for any $\lambda \in]0,2[$ if $N \geq 2$. To prove (2.1) in the general case, we must use a holomorphy argument.

Step 3: conclusion.
We now obtain (2.1) in the case $\lambda \in]0,1]$. Let $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$. Since $\mathcal{F}(\varphi) \in \mathcal{S}(\mathbb{R}^N)$, we have, for all λ in the strip $E = \{\lambda \in \mathbb{C} \mid 0 < \mathrm{Re}(\lambda) < 2\}$,

$$|\,|\cdot|^\lambda \mathcal{F}(\varphi)\,| = |\cdot|^{\mathrm{Re}(\lambda)}|\mathcal{F}(\varphi)| \leq (1+|\cdot|^2)|\mathcal{F}(\varphi)| \in L^1(\mathbb{R}^N).$$

Hence, by holomorphy under the integral sign, the function

$$\lambda \mapsto g_\lambda[\varphi](x) = \int_{\mathbb{R}^N} e^{2i\pi x\cdot\xi}|\xi|^\lambda \mathcal{F}(\varphi)(\xi)\,d\xi$$

is holomorphic on E. For all $0 < a \leq \mathrm{Re}(\lambda) \leq b < 2$, the integrands in (2.1) are bounded by integrable functions which only depend on a and b: if $z \in B_r$, we have $|\,|z|^\lambda\,| = |z|^{\mathrm{Re}(\lambda)} \geq r^{\mathrm{Re}(\lambda)-b}|z|^b \geq c_{r,b}|z|^b$ and, if $z \notin B_r$, then $|\,|z|^\lambda| \geq r^{\mathrm{Re}(\lambda)-a}|z|^a \geq c'_{r,a}|z|^a$; hence, the two integral terms in this formula are also holomorphic with respect to $\lambda \in E$. Since $\lambda \mapsto c_N(\lambda)$ is holomorphic on E (Γ is holomorphic in the half-plane $\{\mathrm{Re} > 0\}$ and has no zero), all the functions of λ in (2.1) are holomorphic on E; this equality being satisfied for all real λ in $]1,2[$, it holds in fact for any $\lambda \in E$. In particular, this proves (2.1) if $\lambda \in]0,2[$.

The special cases i) and ii) of the theorem are easy consequences of (2.1). Indeed, $\varphi(x+z)-\varphi(x) = \mathcal{O}(|z|)$ on a neighbourhood of 0; thus, if $\lambda < 1$, $|z|^{-N-\lambda}(\varphi(x+z)-\varphi(x))$ is integrable on $\mathbb{R}^N$ and we can pass to the limit $r \to 0$ in (2.1). The function $z \mapsto \varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z$ has a linear growth at infinity; therefore, if $\lambda > 1$, $|z|^{-N-\lambda}(\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z)$ is integrable on $\mathbb{R}^N$ and we conclude by letting $r \to \infty$ in (2.1). ■

3 Fractal Hamilton-Jacobi equations

3.1 A smoothing effect for $\lambda \in]1,2[$

We assume here that $\lambda \in]1,2[$ and we consider the Cauchy problem

$$\begin{cases} \partial_t u(t,x) + g_\lambda[u(t,\cdot)](x) = F(t,x,u(t,x),\nabla u(t,x)) & t>0\,,\ x\in\mathbb{R}^N\,, \\ u(0,x) = u_0(x) & x\in\mathbb{R}^N\,, \end{cases} \tag{3.1}$$

where $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ and $F \in C^\infty([0,\infty[\times\mathbb{R}^N \times \mathbb{R}\times\mathbb{R}^N)$ satisfies

$$\begin{gathered} \forall T>0\,,\ \forall R>0\,,\ \forall k\in\mathbb{N},\ \exists\, C_{T,R,k} \text{ such that,} \\ \text{for all } (t,x,s,\xi)\in[0,T]\times\mathbb{R}^N\times[-R,R]\times B_R \text{ and all } \alpha\in\mathbb{N}^{2N+2} \text{ satisfying } |\alpha|\le k, \\ |\partial^\alpha F(t,x,s,\xi)| \le C_{T,R,k}. \end{gathered} \tag{3.2}$$

We also assume that

$$\begin{gathered} \forall T>0, \text{ there exists } \Lambda_T : [0,+\infty[\mapsto]0,+\infty[\text{ continuous nondecreasing} \\ \text{such that } \int_0^\infty \tfrac{1}{\Lambda_T(a)}\,da = +\infty \text{ and, for all } (t,x,s)\in[0,T]\times\mathbb{R}^N\times\mathbb{R}, \\ \operatorname{sgn}(s)F(t,x,s,0) \le \Lambda_T(|s|)\,, \end{gathered} \tag{3.3}$$

$$\begin{gathered} \forall T>0, \forall R>0, \text{ there exists } \Gamma_{T,R} : [0,+\infty[\mapsto]0,+\infty[\text{ continuous nondecreasing} \\ \text{such that } \int_0^\infty \tfrac{1}{\Gamma_{T,R}(a)}\,da = +\infty \text{ and, for all } (t,x,s,\xi)\in[0,T]\times\mathbb{R}^N\times[-R,R]\times\mathbb{R}^N, \\ |\xi|\partial_s F(t,x,s,\xi) \le \Gamma_{T,R}(|\xi|)\,, \quad |\nabla_x F(t,x,s,\xi)| \le \Gamma_{T,R}(|\xi|) \end{gathered} \tag{3.4}$$

and we define

$$\mathcal{L}_T(a) = \int_0^a \frac{1}{\Lambda_T(b)}\,db \quad \text{and} \quad \mathcal{G}_{T,R}(a) = \int_0^a \frac{1}{2N\Gamma_{T,R}(b)}\,db. \tag{3.5}$$

By the assumptions on Λ_T and $\Gamma_{T,R}$, the functions $\mathcal{L}_T$ and $\mathcal{G}_{T,R}$ are nondecreasing C^1-diffeomorphisms from $[0,\infty[$ to $[0,\infty[$. Our main result concerning (3.1) is the following.

Theorem 3.1 *Let $\lambda\in]1,2[$, $u_0\in W^{1,\infty}(\mathbb{R}^N)$ and F satisfy (3.2), (3.3) and (3.4). There exists a unique solution u to (3.1) in the following sense: for all $T>0$,*

$$u\in C_b(]0,T[\times\mathbb{R}^N)\,,\ \nabla u\in C_b(]0,T[\times\mathbb{R}^N)^N \ \text{ and, for all } a\in]0,T[,\ u\in C_b^\infty(]a,T[\times\mathbb{R}^N)\,, \tag{3.6}$$

$$u \text{ satisfies the PDE of (3.1) on }]0,T[\times\mathbb{R}^N\,, \tag{3.7}$$

$$u(t,\cdot)\to u_0 \text{ uniformly on } \mathbb{R}^N, \text{ as } t\to 0. \tag{3.8}$$

We also have the following estimates on the solution: for all $0<t<T<\infty$,

$$||u(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \le (\mathcal{L}_T)^{-1}\left(t+\mathcal{L}_T(||u_0||_{L^\infty(\mathbb{R}^N)})\right)\,, \tag{3.9}$$

$$||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \le (\mathcal{G}_{T,R})^{-1}\left(t+\mathcal{G}_{T,R}(||Du_0||_{L^\infty(\mathbb{R}^N)})\right)\,, \tag{3.10}$$

where $\mathcal{L}_T$ and $\mathcal{G}_{T,R}$ are defined by (3.5), $||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)} = \sum_{i=1}^N ||\partial_i u(t,\cdot)||_{L^\infty(\mathbb{R}^N)}$ and R is any upper bound of $||u||_{L^\infty(]0,T[\times\mathbb{R}^N)}$.

Remark 3.1 *The uniqueness holds under weaker assumptions (see Corollary 3.1) and, with the technique used in Section 4, it can also be proved if the uniform convergence in (3.8) is replaced by a L^∞ weak-$*$ convergence.*

3.1.1 Discussion of the assumptions

We assume that F is regular because we look here for regular solutions to (3.1) (we relax this in Subsection 3.2); in this framework, (3.2) is restricting only in the sense that it imposes bounds which are uniform with respect to $x \in \mathbb{R}^N$, but this is natural since we want solutions that also satisfy such global bounds.
Assumption (3.3) is used to bound the solution, and (3.4) to bound its gradient. As a simple particular case of these assumptions, we can take $\Lambda_T(a) = K_T(1+a)$ and $\Gamma_{T,R}(a) = M_{T,R}(1+a)$ with K_T and $M_{T,R}$ constants (see (3.20), (3.21) and Remark 3.4). With these choices, (3.9) and (3.10) read

$$||u(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \leq (1 + ||u_0||_{L^\infty(\mathbb{R}^N)})e^{K_T t} - 1$$
$$||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \leq (1 + ||Du_0||_{L^\infty(\mathbb{R}^N)})e^{2NM_{T,R}t} - 1\,,$$

which are quite classical estimates. Note that if this choice of $\Gamma_{T,R}$ in (3.4) is usual (see [11] for $\lambda = 2$, [17] for $\lambda \in]1,2[$ and [3] for the pure Hamilton-Jacobi equation — *i.e.* without g_λ), Assumption (3.3) is, even with the preceding choice of Λ_T, less usual in the framework of Hamilton-Jacobi equations. To ensure global existence, this hypothesis is in general replaced by a bound on $F(t,x,0,0)$ and by the assumption that F is nonincreasing with respect to s (see the preceding references). Assumption (3.3) with $\Lambda_T(a) = K_T(1+a)$ however appears in [15] in the case of parabolic equations (*i.e.* $\lambda = 2$).
In their general form, Assumptions (3.3) and (3.4) do not seem common in the literature; however, they are completely natural with respect to the technique we use here. They allow to consider, for example, $F(t,x,u,\nabla u) = u^2 \ln(1 + |\nabla u|^2)$.

We now turn to the proof of Theorem 3.1.

3.1.2 L^∞ estimates and uniqueness

The following proposition gives the key estimate, both for Hamilton-Jacobi equations and for scalar conservation laws; it relies on Theorem 2.2 in an essential way. This estimate still holds true in a more general case (precisely, in the framework of viscosity solution, with less regular solutions; see Subsection 3.2 for a definition): see Proposition 5.1 in Appendix. We choose to present below the estimate in the smooth case because we look here for regular solutions.

Proposition 3.1 *Let $\lambda \in]0,2[$, $T > 0$ and $G \in C(]0,T[\times\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ be such that, for all $R > 0$, $\nabla_\xi G$ is bounded on $]0,T[\times\mathbb{R}^N \times [-R,R] \times B_R$. We also assume that*

$$\begin{gathered}\text{there exists } h : [0,\infty[\mapsto]0,\infty[\text{ continuous nondecreasing} \\ \text{such that } \int_0^\infty \tfrac{1}{h(a)}\,da = +\infty \text{ and, for all } (t,x,s) \in]0,T[\times\mathbb{R}^N \times \mathbb{R}, \\ \operatorname{sgn}(s)G(t,x,s,0) \leq h(|s|).\end{gathered} \tag{3.11}$$

If $u \in C_b^2(]a,T[\times\mathbb{R}^N)$ for all $a \in]0,T[$ and satisfies

$$\partial_t u(t,x) + g_\lambda[u(t,\cdot)](x) = G(t,x,u(t,x),\nabla u(t,x)) \quad \text{on} \quad]0,T[\times\mathbb{R}^N\,, \tag{3.12}$$

then, defining $\mathcal{H}(a) = \int_0^a \frac{1}{h(b)}\,db$, we have, for all $0 < t' < t < T$,

$$||u(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \leq \mathcal{H}^{-1}\left(t - t' + \mathcal{H}(||u(t',\cdot)||_{L^\infty(\mathbb{R}^N)})\right). \tag{3.13}$$

Proof of Proposition 3.1
Let $a \in]0,T[$. Since $\partial_t^2 u$ is bounded on $]a/2,T[\times\mathbb{R}^N$ (say by C_a), we have, for all $t \in]a,T[$, all $0 < \tau < a/2$ and all $x \in \mathbb{R}^N$,

$$\begin{aligned} u(t,x) &\leq u(t-\tau,x) + \tau\partial_t u(t,x) + C_a\tau^2 \\ &\leq \sup_{\mathbb{R}^N} u(t-\tau,\cdot) + \tau G(t,x,u(t,x),\nabla u(t,x)) - \tau g_\lambda[u(t,\cdot)](x) + C_a\tau^2. \end{aligned} \tag{3.14}$$

Fix $t > a$ and assume that $\sup_{\mathbb{R}^N} u(t,\cdot) > 0$. Let $(x_n)_{n\geq 1} \in \mathbb{R}^N$ be a sequence such that $u(t,x_n) \to \sup_{\mathbb{R}^N} u(t,\cdot)$. We have

$$G(t,x_n,u(t,x_n),\nabla u(t,x_n)) \leq G(t,x_n,u(t,x_n),0) + M_t|\nabla u(t,x_n)|$$

where $M_t = \sup\{|\nabla_\xi G(t,x,s,\xi)|\,,\ (x,s,\xi) \in \mathbb{R}^N \times [-R_t,R_t] \times B_{R_t}\}$ with R_t an upper bound of $u(t,\cdot)$ and $\nabla u(t,\cdot)$. For n large enough, $u(t,x_n) > 0$ and thus, by (3.11),

$$G(t,x_n,u(t,x_n),\nabla u(t,x_n)) \leq h(u(t,x_n)) + M_t|\nabla u(t,x_n)| \leq h\left(\sup_{\mathbb{R}^N} u(t,\cdot)\right) + M_t|\nabla u(t,x_n)|.$$

Injected in (3.14), this gives, for all $t \in]a,T[$ and all $0 < \tau < a/2$,

$$u(t,x_n) \leq \sup_{\mathbb{R}^N} u(t-\tau,\cdot) + \tau h\left(\sup_{\mathbb{R}^N} u(t,\cdot)\right) + \tau M_t|\nabla u(t,x_n)| - \tau g_\lambda[u(t,\cdot)](x_n) + C_a\tau^2.$$

By Theorem 2.2, we have $\liminf_{n\to\infty} g_\lambda[u(t,\cdot)](x_n) \geq 0$ and $\nabla u(t,x_n) \to 0$ as $n \to \infty$. Hence, taking the $\limsup_{n\to\infty}$ of the preceding inequality leads to

$$\sup_{\mathbb{R}^N} u(t,\cdot) \leq \sup_{\mathbb{R}^N} u(t-\tau,\cdot) + \tau h\left(\sup_{\mathbb{R}^N} u(t,\cdot)\right) + C_a\tau^2.$$

This has been obtained under the condition that $\sup_{\mathbb{R}^N} u(t,\cdot) > 0$; defining $\Phi(t) = \max(\sup_{\mathbb{R}^N} u(t,\cdot),0)$, we deduce, whatever the sign of $\sup_{\mathbb{R}^N} u(t,\cdot)$ is, that $\Phi(t) \leq \Phi(t-\tau) + \tau h(\Phi(t)) + C_a\tau^2$, that is to say, for $t \in]a,T[$ and $0 < \tau < a/2$,

$$\frac{\Phi(t)-\Phi(t-\tau)}{\tau} \leq h(\Phi(t)) + C_a\tau.$$

As $\partial_t u$ is bounded on $]a,T[\times\mathbb{R}^N$, it is easy to see that Φ is Lipschitz continuous on $]a,T[$ and this inequality therefore implies $\Phi' \leq h(\Phi)$ almost everywhere on $]0,T[$. Hence, the derivative of the locally Lipschitz continuous function $t \in]0,T[\mapsto \mathcal{H}(\Phi(t))$ (notice that $\mathcal{H}$ is C^1 on $[0,\infty[$) is bounded from above by 1 and, for all $0 < t' < t < T$, $\mathcal{H}(\Phi(t)) \leq t - t' + \mathcal{H}(\Phi(t'))$. Since $\mathcal{H}$ is a nondecreasing bijection $[0,\infty[\mapsto [0,\infty[$, we deduce

$$\sup_{\mathbb{R}^N} u(t,\cdot) \leq \Phi(t) \leq \mathcal{H}^{-1}\left(t-t'+\mathcal{H}(\Phi(t'))\right) \leq \mathcal{H}^{-1}\left(t-t'+\mathcal{H}(||u(t',\cdot)||_{L^\infty(\mathbb{R}^N)})\right).$$

The same reasoning applied to $-u$ (solution to (3.12) with $(t,x,s,\xi) \mapsto -G(t,x,-s,-\xi)$, which also satisfies (3.11), instead of G) gives an upper bound on $\sup_{\mathbb{R}^N}(-u(t,\cdot)) = -\inf_{\mathbb{R}^N} u(t,\cdot)$ and concludes the proof. ■

We deduce from this proposition the following corollary, which implies the uniqueness stated in Theorem 3.1.

Corollary 3.1 *Let $\lambda \in]0,2[$, $T > 0$ and $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. If F satisfies (3.2), then there exists at most one function defined on $]0,T[\times\mathbb{R}^N$ which satisfies (3.6), (3.7) and (3.8).*

Proof of Corollary 3.1
Assume that u and v are two such functions. The difference $w = u - v$ is in $C_b^2(]a,T[\times\mathbb{R}^N)$ for all $a \in]0,T[$ and satisfies

$$\partial_t w(t,x) + g_\lambda[w(t,\cdot)](x) = G(t,x,w(t,x),\nabla w(t,x)) \quad \text{on} \quad]0,T[\times\mathbb{R}^N$$

with

$$\begin{aligned} G(t,x,s,\xi) \;=\; & \left(\int_0^1 \partial_s F(t,x,\tau u(t,x)+(1-\tau)v(t,x),\nabla u(t,x))\,d\tau\right) s \\ & + \left(\int_0^1 \nabla_\xi F(t,x,v(t,x),\tau\nabla u(t,x)+(1-\tau)\nabla v(t,x))\,d\tau\right)\cdot\xi. \end{aligned}$$

By (3.2) and the hypotheses on u and v, G is continuous on $]0,T[\times\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ and $\nabla_\xi G$ is bounded on $]0,T[\times\mathbb{R}^N \times [-R,R] \times B_R$ for all $R > 0$. Moreover, G satisfies (3.11) with $h(a) = C(\kappa + a)$ where κ

is any positive number (added so that $h > 0$ on $\mathbb{R}^+$) and C only depends on u, v and the constants in (3.2). For this h, we have $\mathcal{H}(a) = \frac{1}{C}(\ln(\kappa + a) - \ln(\kappa))$ and $\mathcal{H}^{-1}(a) = \kappa e^{Ca} - \kappa$; hence, by Proposition 3.1 we find, for $0 < t' < t < T$, $||w(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \leq e^{C(t-t')}(\kappa + ||w(t',\cdot)||_{L^\infty(\mathbb{R}^N)}) - \kappa$. Since $u(t',\cdot) \to u_0$ and $v(t',\cdot) \to u_0$ uniformly on $\mathbb{R}^N$ as $t' \to 0$, we have $||w(t',\cdot)||_{L^\infty(\mathbb{R}^N)} \to 0$ as $t' \to 0$ and we conclude, letting $t' \to 0$ and $\kappa \to 0$ in the preceding inequality, that $w(t,\cdot) = 0$ for all $t \in]0,T[$. ■

3.1.3 $W^{1,\infty}$ estimates and existence

To prove the existence of a solution to (3.1), we first introduce another definition of solution, in the spirit of [25, chapter 15] or [14].

Definition 3.1 *Let $\lambda \in]1,2[$, $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, $T > 0$ and F satisfy (3.2). A* weak solution *to (3.1) on $[0,T]$ is a function $u \in L^\infty(]0,T[\times\mathbb{R}^N)$ such that $\nabla u \in L^\infty(]0,T[\times\mathbb{R}^N)^N$ and, for a.e. $(t,x) \in]0,T[\times\mathbb{R}^N$,*

$$u(t,x) = K_\lambda(t,\cdot) * u_0(x) + \int_0^t K_\lambda(t-s,\cdot) * F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))(x)\, ds\,, \tag{3.15}$$

where K_λ is the kernel associated with g_λ.

The kernel associated with g_λ is $K_\lambda(t,x) = \mathcal{F}^{-1}(\xi \mapsto e^{-t|\xi|^\lambda})(x)$. It is defined so that the solution to $\partial_t v + g_\lambda[v] = 0$ is given by $v(t,x) = K_\lambda(t,\cdot) * v(0,\cdot)(x)$ and (3.15) is simply Duhamel's formula on (3.1). Let us recall the main properties of K_λ (valid for $\lambda \in]0,2[$) which allow in particular to see that each term in (3.15) is well-defined.

$$\begin{gathered} K_\lambda \in C^\infty(]0,\infty[\times\mathbb{R}^N) \text{ and } (K_\lambda(t,\cdot))_{t\to 0} \text{ is an approximate unit} \\ \text{(in particular, } K_\lambda \text{ is nonnegative and, for all } t > 0,\ ||K_\lambda(t,\cdot)||_{L^1(\mathbb{R}^N)} = 1), \\ \forall t > 0\,,\ \forall t' > 0\,,\ K_\lambda(t+t',\cdot) = K_\lambda(t,\cdot) * K_\lambda(t',\cdot)\,, \\ \exists \mathcal{K} > 0\,,\ \forall t > 0\,,\ ||\nabla K_\lambda(t,\cdot)||_{L^1(\mathbb{R}^N)} \leq \mathcal{K} t^{-1/\lambda} \end{gathered} \tag{3.16}$$

(notice that the nonnegativity of K_λ can be proved from Theorem 2.2 by using the same technique as in the proof of Proposition 3.1). Using the Banach fixed point theorem, it is quite simple to prove the local existence (and uniqueness) of a weak solution to (3.1); its regularity is obtained by the same means. We give in Appendix ideas for the proof of the following theorem and let the reader check the details (see, for example, [14] and [17]).

Theorem 3.2 *Let $\lambda \in]1,2[$, $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ and F satisfy (3.2).*

i) For all $T > 0$, there exists at most one weak solution to (3.1) on $[0,T]$.

ii) A weak solution to (3.1) on $[0,T]$ satisfies (3.6), (3.7) and (3.8).

iii) Let $M \geq ||u_0||_{W^{1,\infty}(\mathbb{R}^N)}$. There exists $T > 0$, only depending on M and the constants in Hypothesis (3.2), such that (3.1) has a weak solution on $[0,T]$.

We now obtain estimates on the gradient of the weak solution, and conclude the proof of Theorem 3.1.

Proposition 3.2 *Let $\lambda \in]1,2[$ and $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. Assume that F satisfies (3.2) and (3.4). If u is a weak solution to (3.1) on $[0,T]$ and $R \geq ||u||_{L^\infty(]0,T[\times\mathbb{R}^N)}$ then (3.10) holds for all $t \in]0,T[$.*

Proof of Proposition 3.2
The proof is very similar to the proof of Proposition 3.1. If $\varphi \in C_b^3(\mathbb{R}^N)$, then a derivation under the integral sign on (2.1) shows that $\partial_i g_\lambda[\varphi] = g_\lambda[\partial_i\varphi]$. Since u satisfies (3.6) and (3.7) (Theorem 3.2), we deduce that

$$\begin{aligned} \partial_t(\partial_i u)(t,x) + g_\lambda[\partial_i u(t,\cdot)](x) \quad = \quad & \partial_{x_i} F(t,x,u(t,x),\nabla u(t,x)) + \partial_s F(t,x,u(t,x),\nabla u(t,x))\partial_i u(t,x) \\ & + \nabla_\xi F(t,x,u(t,x),\nabla u(t,x)) \cdot \nabla(\partial_i u)(t,x). \end{aligned}$$

Let $a \in]0,T[$; the function $\partial_t^2 \partial_i u$ is bounded on $]a/2,T[\times\mathbb{R}^N$ (say by $C_{a,i}$) and thus, for $t \in]a,T[$, $0<\tau<a/2$ and $x\in\mathbb{R}^N$,

$$\begin{aligned}\partial_i u(t,x) &\le \partial_i u(t-\tau,x)+\tau\partial_t\partial_i u(t,x)+C_{a,i}\tau^2\\ &\le \sup_{\mathbb{R}^N}\partial_i u(t-\tau,\cdot)+\tau\partial_{x_i}F(t,x,u(t,x),\nabla u(t,x))+\tau\partial_s F(t,x,u(t,x),\nabla u(t,x))\partial_i u(t,x)\\ &\quad +\tau\nabla_\xi F(t,x,u(t,x),\nabla u(t,x))\cdot\nabla(\partial_i u)(t,x)-\tau g_\lambda[\partial_i u(t,\cdot)](x)+C_{a,i}\tau^2. \end{aligned}\tag{3.17}$$

Assume that $\sup_{\mathbb{R}^N}\partial_i u(t,\cdot)>0$ and take a sequence $(x_n)_{n\ge1}\in\mathbb{R}^N$ such that $\partial_i u(t,x_n)\to\sup_{\mathbb{R}^N}\partial_i u(t,\cdot)$. Since $\partial_i u(t,\cdot)\in C_b^2(\mathbb{R}^N)$, Theorem 2.2 gives $\liminf_{n\to\infty} g_\lambda[\partial_i u(t,\cdot)](x_n)\ge 0$ and $\lim_{n\to\infty}\nabla(\partial_i u)(t,x_n)=0$. For n large enough, $\partial_i u(t,x_n)>0$ and we can apply (3.17) to $x=x_n$, use (3.4) and (3.2) to bound the terms involving F and then take the $\limsup_{n\to\infty}$ of the resulting inequality; we find

$$\sup_{\mathbb{R}^N}\partial_i u(t,\cdot)\le\sup_{\mathbb{R}^N}\partial_i u(t-\tau,\cdot)+2\tau\Gamma_{T,R}\left(\sup_{\mathbb{R}^N}|\nabla u(t,\cdot)|\right)+C_{a,i}\tau^2$$

(recall that R is an upper bound of $||u||_{L^\infty(]0,T[\times\mathbb{R}^N)}$). As in the proof of Proposition 3.1, to obtain an inequality which holds whatever the sign of $\sup_{\mathbb{R}^N}\partial_i u(t,\cdot)$ is, we define $w_{i,+}(t)=\max(\sup_{\mathbb{R}^N}\partial_i u(t,\cdot),0)$ and we have, for all $t\in]a,T[$ and all $0<\tau<a/2$,

$$w_{i,+}(t)\le w_{i,+}(t-\tau)+2\tau\Gamma_{T,R}\left(||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)}\right)+C_{a,i}\tau^2.$$

This reasoning applied to $-u$ (the function $-F(\cdot,\cdot,-\cdot,-\cdot)$ satisfies (3.2) and (3.4)) leads to the same inequality for $w_{i,-}(t)=\max(\sup_{\mathbb{R}^N}(-\partial_i u(t,\cdot)),0)=\max(-\inf_{\mathbb{R}^N}\partial_i u(t,\cdot),0)$. This inequality is therefore also satisfied by $\max(w_{i,+}(t),w_{i,-}(t))=||\partial_i u(t,\cdot)||_{L^\infty(\mathbb{R}^N)}$ and, summing on $i=1,\dots,N$, we deduce that, for all $t\in]a,T[$ and all $0<\tau<a/2$,

$$||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)}\le||Du(t-\tau,\cdot)||_{L^\infty(\mathbb{R}^N)}+2N\tau\Gamma_{T,R}\left(||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)}\right)+\sum_{i=1}^N C_{a,i}\tau^2.$$

Since $t\mapsto||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)}$ is locally Lipschitz continuous (because $\partial_t\partial_i u$ is bounded on $]a,T[\times\mathbb{R}^N$ for all $a\in]0,T[$), we infer as in the proof of Proposition 3.1 that, for $0<t'<t<T$,

$$||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)}\le(\mathcal{G}_{T,R})^{-1}\left(t-t'+\mathcal{G}_{T,R}(||Du(t',\cdot)||_{L^\infty(\mathbb{R}^N)})\right).\tag{3.18}$$

The function u_0 being Lipschitz continuous, the definition of the derivative and the dominated convergence theorem show that $\partial_i(K(t',\cdot)*u_0)=K(t',\cdot)*\partial_i u_0$. Thanks to Lemma 5.1 in Appendix and (3.16), we have $\partial_i[K_\lambda(t'-s,\cdot)*F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))]=\partial_i K_\lambda(t'-s,\cdot)*F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))$, which is bounded independently of $x\in\mathbb{R}^N$ by an integrable function of $s\in]0,t'[$; we can therefore derivate (3.15) under the integral sign to find

$$\partial_i u(t',x)=K_\lambda(t',\cdot)*\partial_i u_0(x)+\int_0^{t'}\partial_i K_\lambda(t'-s,\cdot)*F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))(x)\,ds\tag{3.19}$$

and, still using (3.16), we obtain

$$||\partial_i u(t',\cdot)||_{L^\infty(\mathbb{R}^N)}\le||\partial_i u_0||_{L^\infty(\mathbb{R}^N)}+||F(\cdot,\cdot,u,\nabla u)||_\infty\frac{\mathcal{K}t'^{1-\frac{1}{\lambda}}}{1-\frac{1}{\lambda}}.$$

This shows that $\limsup_{t'\to0}||Du(t',\cdot)||_{L^\infty(\mathbb{R}^N)}\le||Du_0||_{L^\infty(\mathbb{R}^N)}$ and we conclude the proof by letting $t'\to0$ in (3.18). ■

The proof of the existence and estimates in Theorem 3.1 is then straightforward. Indeed, take u a weak solution to (3.1) on $[0,T]$ given by Theorem 3.2. By (3.3), F satisfies (3.11) with $h=\Lambda_T$; since u satisfies (3.6) and (3.7), Proposition 3.1 shows that, for $0<t'<t<T$, $||u(t,\cdot)||_{L^\infty(\mathbb{R}^N)}\le(\mathcal{L}_T)^{-1}(t-t'+\mathcal{L}_T(||u(t',\cdot)||_{L^\infty(\mathbb{R}^N)}))$; but (3.8) holds for u, and we can therefore let $t'\to0$ to deduce that (3.9) is valid. We have (3.10) by Proposition 3.2. These estimates (3.9) and (3.10) show that the $W^{1,\infty}$ norm of $u(t,\cdot)$

does not explode in finite time; item iii) in Theorem 3.2 then allows to indefinitely extend u ([1]), which gives a global weak solution to (3.1), and thus a global solution in the sense of Theorem 3.1.

Remark 3.2 *As a by-product of this proof of existence, we see that the solution to (3.1) given by Theorem 3.1 also satisfies (3.15), which was not obvious from (3.6)—(3.8).*

Remark 3.3 *The preceding technique also works if g_λ is replaced by a more general operator, provided that it satisfies Theorem 2.2 (in fact, this theorem is only needed for $\varphi \in C_b^\infty(\mathbb{R}^N)$) and that its kernel satisfies (3.16) (for small t and some $\lambda > 1$ in the estimate of the gradient) and (5.1) in Appendix. As interesting and simple examples of such operators, we can mention:*

1) *The laplace operator $-\Delta$ (which corresponds, up to a multiplicative constant, to g_2). Or course, the preceding results are known for semi-linear parabolic equations (at least for classical choices of Λ_T and $\Gamma_{T,R}$).*

2) *Multifractal operators such as in [5], that is to say $\sum_{j=1}^l \alpha_j g_{\lambda_j}$ with $\alpha_j > 0$, $\lambda_j \in]0,2]$ and $\lambda_1 \in]1,2]$. The kernel of this operator is $K_{\lambda_1}(\alpha_1 t, \cdot) * \cdots * K_{\lambda_l}(\alpha_l t, \cdot)$, and it satisfies (3.16) with $\lambda = \lambda_1$.*

3) *Anisotropic operators of the kind $A[\varphi] = \mathcal{F}^{-1}(\sum_{j=1}^N |\xi_j|^{\gamma_j} \mathcal{F}(\varphi)(\xi))$ with $\gamma_j \in]1,2]$ (it comes to take a "γ_j-th derivative" in the j-th direction). This operator is the sum of 1-dimensional operators g_{γ_j} acting on each variable, and thus a formula of the kind of (2.1) can be established, which proves that Theorem 2.2 holds. The kernel of A is $\prod_{j=1}^N k_{\gamma_j}(t, x_j)$, where k_{γ_j} is the kernel of g_{γ_j} in dimension $N = 1$, and it thus satisfies (3.16) with $\lambda = \inf_j(\gamma_j)$.*

We refer the reader to [16] for the kernel properties of other pseudo-differential operators.

3.2 Existence and uniqueness results for $\lambda \in]0,2[$

We show here that, under weaker regularity (but stronger behaviour) assumptions on F and for $\lambda \in]0,2[$, we can still solve (3.1), albeit in the viscosity sense. We assume, in the following, that the Hamiltonian F is continuous with respect to (t,x,s,ξ), locally Lipschitz continuous with respect to (x,s,ξ) and satisfies (3.3). We replace (3.4) by

$$\begin{gathered}\forall T > 0\,,\ \forall R > 0\,,\ \exists\, \Theta_{T,R} > 0 \text{ such that, for all } (t,x,s,\xi) \in [0,T] \times \mathbb{R}^N \times [-R,R] \times \mathbb{R}^N\,, \\ \partial_s F(t,x,s,\xi) \leq \Theta_{T,R}\end{gathered} \tag{3.20}$$

$$\begin{gathered}\forall T > 0\,,\ \forall R > 0\,,\ \exists\, \Xi_{T,R} > 0 \text{ such that, for all } (t,x,s,\xi) \in [0,T] \times \mathbb{R}^N \times [-R,R] \times \mathbb{R}^N\,, \\ |\nabla_x F(t,x,s,\xi)| \leq \Xi_{T,R}(1 + |\xi|).\end{gathered} \tag{3.21}$$

and (3.2) is relaxed to

$$\begin{gathered}\forall T > 0, \forall R > 0, \exists C_{T,R} > 0 \text{ such that, for all } (t,x,s,\xi) \in [0,T] \times \mathbb{R}^N \times [-R,R] \times B_R, \\ |F(t,x,s,\xi)| \leq C_{T,R}\,,\ |\partial_s F(t,x,s,\xi)| \leq C_{T,R}\,,\ |\nabla_\xi F(t,x,s,\xi)| \leq C_{T,R}.\end{gathered} \tag{3.22}$$

In all the preceding inequalities, the derivatives of F are to be understood as the a.e. derivatives of a Lipschitz continuous function; these hypotheses therefore state bounds on F and its Lipschitz constants.

Remark 3.4 *Assumptions (3.20) and (3.21) are stronger than (3.4); they imply this last assumption with, for example, $\Gamma_{T,R}(a) = (\Theta_{T,R} + \Xi_{T,R})(1 + a)$.*

Let us first briefly recall the definition of a viscosity solution to (3.1) (an immediate generalization of the definition given in [17] in the case $\lambda > 1$).

[1] It is the semi-group property of K_λ in (3.16) which tells that, if we glue to $u_{|\,[0,t_0]}$ a weak solution with initial time t_0 and initial data $u(t_0, \cdot)$, then we construct another weak solution.

Definition 3.2 *Let $\lambda \in]0,2[$, $u_0 \in C_b(\mathbb{R}^N)$ and $F : [0,T[\times\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ be continuous. A function $u : [0,T[\times\mathbb{R}^N \mapsto \mathbb{R}$ is a* viscosity subsolution *to (3.1) if it is bounded upper semi-continuous, if $u(0,\cdot) \leq u_0$ and if, for all $(t,x) \in]0,T[\times\mathbb{R}^N$ and all $(\alpha,p) \in \mathbb{R}\times\mathbb{R}^N$ such that there exists $\sigma > 0$ and $r_0 > 0$ satisfying*

$$u(s,y) \leq u(t,x) + \alpha(s-t) + p\cdot(y-x) + \sigma|y-x|^2 + o(s-t) \quad \text{for } y \in B_{r_0}(x) \text{ and } s \in [0,T[, \tag{3.23}$$

we have, for all $r > 0$,

$$\alpha - c_N(\lambda)\int_{B_r} \frac{u(t,x+z)-u(t,x)-p\cdot z}{|z|^{N+\lambda}}\,dz - c_N(\lambda)\int_{\mathbb{R}^N\setminus B_r} \frac{u(t,x+z)-u(t,x)}{|z|^{N+\lambda}}\,dz \leq F(t,x,u(t,x),p). \tag{3.24}$$

We similarly define the notion of supersolution *for bounded lower semi-continuous functions by reversing the inequalities (and the sign of σ). A function is a* viscosity solution *of (3.1) if it is both a sub- and a supersolution of (3.1).*

Remark 3.5

1. *Since $u(t,x+z)-u(t,x)-p\cdot z \leq \sigma|z|^2$ on a neighbourhood of 0, the first integral term of (3.24) is defined in $[-\infty,+\infty[$ (the inequality in fact forbids the case where this term is $-\infty$); the second integral term is defined in $\mathbb{R}$, because u is bounded. Moreover, since $\int_{B_a\setminus B_b} \frac{p\cdot z}{|z|^{N+\lambda}}\,dz = 0$ for all $a > b > 0$, the quantities in (3.24) in fact do not depend on $r > 0$; in particular, if $\lambda < 1$ or $\lambda > 1$, we can take $r = 0$ or $r = +\infty$.*

2. *In [17], a couple $(\alpha,p) \in \mathbb{R}\times\mathbb{R}^N$ satisfying (3.23) is called a supergradient. The set of all such couples is denoted $\partial^P u(t,x)$ and is referred to as the superdifferential of u at (t,x). It is the projection on $\mathbb{R}\times\mathbb{R}^N$ of the upper jet of u at (t,x) (see [11] for a definition of semi-jets).*

We can now state our existence and uniqueness result for Lipschitz continuous Hamiltonians and $\lambda \in]0,2[$.

Theorem 3.3 *Let $\lambda \in]0,2[$ and F be continuous and such that (3.3), (3.20), (3.21) and (3.22) hold true. If $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, then there exists a unique viscosity solution of (3.1). Moreover, this solution is Lipschitz continuous with respect to x and satisfies, for $0 < t < T < \infty$, (3.9) and*

$$||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \leq (1 + ||Du_0||_{L^\infty(\mathbb{R}^N)})e^{2N(\Theta_{T,R}+\Xi_{T,R})t} - 1 \tag{3.25}$$

for any $R \geq ||u||_{L^\infty(]0,T[\times\mathbb{R}^N)}$.

Remark 3.6

1. *This result can be extended to initial conditions that are merely bounded and uniformly continuous. It suffices to adapt the classical method used for instance in [11]. Notice that, in this case, the Lipschitz continuity of the solution is no longer true.*

2. *As in Remark 3.3, this theorem also holds for more general operators g_λ.*

3. *Estimate (3.25) is exactly (3.10) when we take, as in Remark 3.4, $\Gamma_{T,R}(a) = (\Theta_{T,R}+\Xi_{T,R})(1+a)$.*

4. *The conclusions of this theorem are the same as the ones of Theorem 3 and Lemma 2 in [17], but the assumptions are more general; see the discussion following Theorem 3.1. Moreover, the $W^{1,\infty}$ estimate (3.25) involves a norm that is slightly different from the one used in [17].*

5. *Perron's method. As mentioned in the introduction, Theorem 3.3 can be proved by using Perron's method. In this case, "natural" barriers are to be considered for large time, namely $t \mapsto \pm(\mathcal{L}_T)^{-1}(t+\mathcal{L}_T(\|u_0\|_{L^\infty(\mathbb{R}^N)}))$. For small time, in order to ensure that the initial condition is fulfilled, classical barriers of the form $t \mapsto u_0(x) \pm Ct$ can be used for C large enough.*

Proof of Theorem 3.3
We prove the existence result by regularizing F and using a vanishing viscosity method based on the solution given by Theorem 3.1.

Step 1: regularization of F.
Let $h_\varepsilon : \mathbb{R}^d \to \mathbb{R}^d$ be defined by $h_\varepsilon(z) = \max(1 - \frac{\varepsilon}{|z|}, 0)z$. h_ε is at distance ε of the identity function, null on B_ε and $|h_\varepsilon| \leq |\cdot| - \varepsilon$ on $\mathbb{R}^d \backslash B_\varepsilon$; in particular,

$$\text{for all } z \in \mathbb{R}^d \text{ and } |z'| \leq \varepsilon, \quad |h_\varepsilon(z - z')| \leq |z|. \tag{3.26}$$

We define $F_\varepsilon : \mathbb{R}^{2N+2} \to \mathbb{R}$ by $F_\varepsilon(t, x, s, \xi) = F(h_\varepsilon(t)^+, x, h_{2\varepsilon}(s), h_\varepsilon(\xi))$; let $(t, x, s, \xi) \mapsto \rho_\varepsilon(t, x, s, \xi)$ be a classical regularizing kernel such that $\text{supp}(\rho_\varepsilon) \subset B_\varepsilon$ and define $\tilde{F}_\varepsilon = F_\varepsilon * \rho_\varepsilon$. In dimension $d = 1$, $h'_{2\varepsilon}$ takes its values in $[0, 1]$, and thus $\tilde{F}_\varepsilon$ satisfies (3.20) and (3.21) with the same constants as F (thanks to (3.26)). It therefore satisfies (3.4) with $\Gamma_{T,R}(a) = (\Theta_{T,R} + \Xi_{T,R})(1 + a)$ (see Remark 3.4). We also have (3.2) for $\tilde{F}_\varepsilon$ with $\tilde{C}_{T,R,k} = \sup_{|\alpha| \leq k} ||\partial^\alpha \rho_\varepsilon||_{L^1} \sup_{[0,T] \times \mathbb{R}^N \times [-R,R] \times B_R} |F|$ (this quantity, which depends on ε, is finite thanks to (3.22)).
Property (3.3) for $\tilde{F}_\varepsilon$ is slightly less obvious. Let $(t, x, s) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}$ and $|t'| \leq \varepsilon$, $|x'| \leq \varepsilon$, $|s'| \leq \varepsilon$, $|\xi'| \leq \varepsilon$. In the case $|s| \geq \varepsilon$, we have $\text{sgn}(s) = \text{sgn}(s - s') = \text{sgn}(h_{2\varepsilon}(s - s'))$ since $|s'| \leq \varepsilon$, and thus, by (3.3) and (3.26),

$$\begin{aligned}
\text{sgn}(s) F_\varepsilon(t - t', x - x', s - s', 0 - \xi') &= \text{sgn}(h_{2\varepsilon}(s - s')) F(h_\varepsilon(t - t')^+, x - x', h_{2\varepsilon}(s - s'), 0) \\
&\leq \Lambda_T(|h_{2\varepsilon}(s - s')|) \\
&\leq \Lambda_T(|s|).
\end{aligned}$$

In the case $|s| \leq \varepsilon$, (3.22) gives

$$\begin{aligned}
&|F_\varepsilon(t - t', x - x', s - s', 0 - \xi') - F(h_\varepsilon(t - t')^+, x - x', s, 0)| \\
&\qquad = |F(h_\varepsilon(t - t')^+, x - x', 0, 0) - F(h_\varepsilon(t - t')^+, x - x', s, 0)| \\
&\qquad \leq \varepsilon C_{T,1}
\end{aligned}$$

and therefore

$$\begin{aligned}
\text{sgn}(s) F_\varepsilon(t - t', x - x', s - s', 0 - \xi') &\leq \text{sgn}(s) F(h_\varepsilon(t - t')^+, x - x', s, 0) + \varepsilon C_{T,1} \\
&\leq \Lambda_T(|s|) + \varepsilon C_{T,1}.
\end{aligned}$$

In any cases, we have $\text{sgn}(s) F_\varepsilon(t - t', x - x', s - s', 0 - \xi') \leq \Lambda_T(|s|) + \varepsilon C_{T,1}$. Multiplying this inequality by $\rho_\varepsilon(t', x', s', \xi')$ and integrating on (t', x', s', ξ') shows that (3.3) holds for $\tilde{F}_\varepsilon$ with $\Lambda_T(a) + \varepsilon C_{T,1}$ instead of $\Lambda_T(a)$.

To sum up this step, we have found a regularization $\tilde{F}_\varepsilon$ of F which converges locally uniformly to F and satisfies (3.2), (3.4) with $\Gamma_{T,R}(a) = (\Theta_{T,R} + \Xi_{T,R})(1 + a)$ (independent of ε) and (3.3) with a function $\Lambda_T^\varepsilon = \Lambda_T + \varepsilon C_{T,1}$ which uniformly converges, as $\varepsilon \to 0$, to Λ_T.

Step 2: passing to the limit.
We take $\lambda \in]0, 2[$ and $\mu \in]1, 2[$. Applying Theorem 3.1 and Remark 3.3, we find a smooth solution u^ε of

$$\begin{cases} \partial_t u^\varepsilon(t, x) + g_\lambda[u^\varepsilon(t, \cdot)](x) + \varepsilon g_\mu[u^\varepsilon(t, \cdot)](x) = \tilde{F}_\varepsilon(t, x, u^\varepsilon(t, x), \nabla u^\varepsilon(t, x)) & t > 0\,,\ x \in \mathbb{R}^N\,, \\ u^\varepsilon(0, x) = u_0(x) & x \in \mathbb{R}^N \end{cases} \tag{3.27}$$

in the sense of (3.6), (3.7) and (3.8) (notice that, if $\lambda > 1$, there is no need to introduce the term εg_μ in this equation). Since, for $0 < \varepsilon \leq 1$, $\tilde{F}_\varepsilon$ satisfies (3.3) with $\Lambda_T(a) + C_{T,1}$ instead of $\Lambda_T(a)$, the theorem gives estimates on u^ε and ∇u^ε which do not depend on ε.

These estimates and (3.22) show that $\tilde{F}_\varepsilon(\cdot,\cdot,u^\varepsilon,\nabla u^\varepsilon)$ is bounded on $]0,T[\times\mathbb{R}^N$ independently of ε. The integral representation (3.15) reads here (see Remark 3.3)

$$\begin{aligned} u^\varepsilon(t,x) &= K_\lambda(t,\cdot)*K_\mu(\varepsilon t,\cdot)*u_0(x) \\ &\quad + \int_0^t K_\lambda(t-s,\cdot)*K_\mu(\varepsilon(t-s),\cdot)*\tilde{F}_\varepsilon(s,\cdot,u^\varepsilon(s,\cdot),\nabla u^\varepsilon(s,\cdot))(x)\,ds\,, \end{aligned}$$

and (3.16) thus gives $||u^\varepsilon(t,\cdot)-K_\lambda(t,\cdot)*K_\mu(\varepsilon t,\cdot)*u_0||_{L^\infty(\mathbb{R}^N)}\le Ct$ with C not depending on ε. Since $(K_\lambda(t,\cdot))_{t\to 0}$ and $(K_\mu(t,\cdot))_{t\to 0}$ are approximate units and $u_0\in W^{1,\infty}(\mathbb{R}^N)$, we easily see that $K_\lambda(t,\cdot)*K_\mu(\varepsilon t,\cdot)*u_0(x)\to u_0(x)$ as $t\to 0$, uniformly with respect to $x\in\mathbb{R}^N$ and $\varepsilon\in]0,1]$. Hence, $u^\varepsilon(t,x)\to u_0(x)$ as $t\to 0$, uniformly with respect to $(x,\varepsilon)\in\mathbb{R}^N\times]0,1]$, and the relaxed upper limit $\limsup^* u^\varepsilon(t,x)=\limsup_{\varepsilon\to 0,(s,y)\to(t,x)} u^\varepsilon(s,y)$ coincides with u_0 at $t=0$. So does the relaxed lower limit $\liminf_* u^\varepsilon = -\limsup^*(-u^\varepsilon)$.
Remark that u^ε is a viscosity solution of (3.27). Since $\tilde{F}_\varepsilon\to F$ locally uniformly, an easy adaptation of the stability theorem of [17] shows that $\limsup^* u^\varepsilon$ is a viscosity subsolution of (3.1) and that $\liminf_* u^\varepsilon$ is a viscosity supersolution of (3.1). The assumptions ensure that the comparison principle holds true for (3.1) (still a straightforward generalization of [17] to the case $\lambda\in]0,2[$). Thus, $\limsup^* u^\varepsilon(0,x)=u_0(x)=\liminf_* u^\varepsilon(0,x)$ implies $\limsup^* u^\varepsilon\le\liminf_* u^\varepsilon$ and we conclude that u^ε locally uniformly converges to $u=\limsup^* u^\varepsilon=\liminf_* u^\varepsilon$, a viscosity solution to (3.1); the estimates on u stated in the theorem are obtained by passing to the limit in the estimates on u^ε. To finish with, we recall that the comparison principle ensures that the solution we have just constructed is unique in the class of viscosity solutions which satisfy $u(0,\cdot)=u_0$. ■

Since we have proved in Step 2 that a vanishing regularization gives a solution to (3.1), we can now wonder if it is possible to obtain a rate of convergence. The next theorem answers this question.

Theorem 3.4 *Let $(\lambda,\mu)\in]0,2[$ and F be continuous and such that (3.3), (3.20), (3.21) and (3.22) hold true. Let $u_0\in W^{1,\infty}(\mathbb{R}^N)$, u be the viscosity solution of (3.1) and, for $\varepsilon>0$, u^ε be the viscosity solution of*

$$\begin{cases} \partial_t u^\varepsilon(t,x)+g_\lambda[u^\varepsilon(t,\cdot)](x)+\varepsilon g_\mu[u^\varepsilon(t,\cdot)](x)=F(t,x,u^\varepsilon(t,x),\nabla u^\varepsilon(t,x)) & t>0\,,\ x\in\mathbb{R}^N\,, \\ u^\varepsilon(0,x)=u_0(x) & x\in\mathbb{R}^N. \end{cases} \tag{3.28}$$

Then, for all $T>0$,

$$||u^\varepsilon-u||_{C_b([0,T]\times\mathbb{R}^N)}=\begin{cases} \mathcal{O}(\varepsilon) & \text{if } \mu<1\,, \\ \mathcal{O}(\varepsilon|\ln(\varepsilon)|) & \text{if } \mu=1\,, \\ \mathcal{O}(\varepsilon^{1/\mu}) & \text{if } \mu>1. \end{cases}$$

Remark 3.7 *1. As the preceding results, this theorem is valid for more general operators g_λ, and also for $\mu=2$. Moreover, the conclusion still holds if we remove g_λ from both equations (in this case, (3.1) is a pure Hamilton-Jacobi equation).*

2. These rates of convergence are optimal for any $\mu\in]0,2[$ (take $F=0$, remove g_λ, choose $u_0(x)=\max(1-|x|,0)$ and compare $u^\varepsilon(1,0)-u(1,0)=K_\mu(\varepsilon,\cdot)*u_0(0)-1$ thanks to the formula $K_\mu(\varepsilon,x)=\varepsilon^{-N/\mu}K_\mu(1,\varepsilon^{-1/\mu}x)$ and to the property $K_\mu(1,x)\sim C|x|^{-N-\mu}$ as $|x|\to\infty$).

Proof of Theorem 3.4
The proof relies on the same technique as in [17], with modifications due to the presence of g_λ and to the fact that μ can be equal to or less than 1.
By a classical change of unknown function, (3.20) allows to reduce to the case where F is nonincreasing with respect to s. Let $M=\sup_{[0,T[\times(\mathbb{R}^N)^2}\{u(t,x)-u^\varepsilon(t,y)-|x-y|^2/2\alpha-\beta|x|^2/2-\eta t-\gamma/(T-t)\}$, where α, β, η are positive and $\gamma\in]0,1]$. We want to prove that, for appropriate choices of η and γ, M is attained at $t=0$.

Let $\nu > 0$ and $M_\nu = \sup_{[0,T[^2\times(\mathbb{R}^N)^2}\{u(t,x)-u^\varepsilon(s,y)-|x-y|^2/2\alpha-|t-s|^2/2\nu-\beta|x|^2/2-\eta t-\gamma/(T-t)\}$. It is classical that M_ν is attained at some $(t_\nu, s_\nu, x_\nu, y_\nu)$ such that, up to a subsequence, $(t_\nu, s_\nu, x_\nu, y_\nu) \to (\overline{t}, \overline{t}, \overline{x}, \overline{y})$ as $\nu \to 0$, where $(\overline{t}, \overline{x}, \overline{y})$ realizes M. We now assume that $\overline{t} > 0$ and, with good choices of η and γ, we show that this leads to a contradiction.

If $\overline{t} > 0$ then, for ν small enough, $t_\nu > 0$ and $s_\nu > 0$. Let $p_\nu = (x_\nu - y_\nu)/\alpha$; by definition of M_ν, $(\gamma/(T-t_\nu)^2 + (t_\nu - s_\nu)/\nu + \eta, p_\nu + \beta x_\nu)$ is a supergradient of u at (t_ν, x_ν); since u is a subsolution of (3.1), we obtain

$$\begin{aligned}&\frac{\gamma}{(T-t_\nu)^2} + \frac{t_\nu - s_\nu}{\nu} + \eta\\ &-c_N(\lambda)\int_{B_r}\frac{u(t_\nu, x_\nu+z)-u(t_\nu,x_\nu)-(p_\nu+\beta x_\nu)\cdot z}{|z|^{N+\lambda}}\,dz - c_N(\lambda)\int_{\mathbb{R}^N\setminus B_r}\frac{u(t_\nu,x_\nu+z)-u(t_\nu,x_\nu)}{|z|^{N+\lambda}}\,dz\\ &\qquad\le F(t_\nu, x_\nu, u(t_\nu,x_\nu), p_\nu+\beta x_\nu). \end{aligned}\tag{3.29}$$

Similarly, by definition of M_ν we can use $((t_\nu - s_\nu)/\nu, p_\nu)$ in the equation at $(t,x) = (s_\nu, y_\nu)$ which states that u^ε is a supersolution of (3.28) and we obtain

$$\begin{aligned}&\frac{t_\nu - s_\nu}{\nu}\\ &-c_N(\lambda)\int_{B_r}\frac{u^\varepsilon(s_\nu,y_\nu+z)-u^\varepsilon(s_\nu,y_\nu)-p_\nu\cdot z}{|z|^{N+\lambda}}\,dz - c_N(\lambda)\int_{\mathbb{R}^N\setminus B_r}\frac{u^\varepsilon(s_\nu,y_\nu+z)-u^\varepsilon(s_\nu,y_\nu)}{|z|^{N+\lambda}}\,dz\\ &-\varepsilon c_N(\mu)\int_{B_R}\frac{u^\varepsilon(s_\nu,y_\nu+z)-u^\varepsilon(s_\nu,y_\nu)-p_\nu\cdot z}{|z|^{N+\mu}}\,dz - \varepsilon c_N(\mu)\int_{\mathbb{R}^N\setminus B_R}\frac{u^\varepsilon(s_\nu,y_\nu+z)-u^\varepsilon(s_\nu,y_\nu)}{|z|^{N+\mu}}\,dz\\ &\qquad\ge F(s_\nu, y_\nu, u^\varepsilon(s_\nu,y_\nu), p_\nu). \end{aligned}\tag{3.30}$$

We also have, still using the definition of M_ν,

$$u^\varepsilon(s_\nu,y_\nu+z)-u^\varepsilon(s_\nu,y_\nu)+u(t_\nu,x_\nu)-u(t_\nu,x_\nu+z) \ge \frac{\beta|x_\nu|^2}{2} - \frac{\beta|x_\nu+z|^2}{2} = -\beta x_\nu\cdot z - \frac{\beta|z|^2}{2}$$

and, by the estimate on ∇u^ε, $|u^\varepsilon(s_\nu,y_\nu+z)-u^\varepsilon(s_\nu,y_\nu)| \le C|z|$ (here and after, C stands for a positive real number which can change from one line to another but does not depend on ε, r, R, ν, α, β, η or γ). Hence, subtracting (3.29) from (3.30) and using the bounds we have on u and u^ε, we find (for $R \le 1$)

$$\begin{aligned}&-\frac{\gamma}{(T-t_\nu)^2} - \eta + c_N(\lambda)\frac{\beta}{2}\int_{B_r}\frac{|z|^2}{|z|^{N+\lambda}}\,dz\\ &+C\int_{\mathbb{R}^N\setminus B_r}\frac{1}{|z|^{N+\lambda}}\,dz - \varepsilon c_N(\mu)\int_{B_R}\frac{u^\varepsilon(s_\nu,y_\nu+z)-u^\varepsilon(s_\nu,y_\nu)-p_\nu\cdot z}{|z|^{N+\mu}}\,dz\\ &+C\varepsilon\int_{B_1\setminus B_R}\frac{|z|}{|z|^{N+\mu}}\,dz + C\varepsilon\int_{\mathbb{R}^N\setminus B_1}\frac{1}{|z|^{N+\mu}}\,dz\\ &\qquad\ge F(s_\nu,y_\nu,u^\varepsilon(s_\nu,y_\nu),p_\nu) - F(t_\nu,x_\nu,u(t_\nu,x_\nu),p_\nu+\beta x_\nu). \end{aligned}\tag{3.31}$$

Using once again the definition of M_ν, we write

$$u^\varepsilon(s_\nu,y_\nu+z)-u^\varepsilon(s_\nu,y_\nu)-p_\nu\cdot z \ge \frac{|x_\nu-y_\nu|^2-|x_\nu-y_\nu-z|^2}{2\alpha} - \frac{2(x_\nu-y_\nu)\cdot z}{2\alpha} = -\frac{|z|^2}{2\alpha}.$$

We can therefore bound the integral term containing u^ε in (3.31) and pass to the limit $\nu \to 0$ to obtain

$$\begin{aligned}&-\frac{\gamma}{(T-\overline{t})^2} - \eta + C\beta\int_{B_r}\frac{|z|^2}{|z|^{N+\lambda}}\,dz + C\int_{\mathbb{R}^N\setminus B_r}\frac{1}{|z|^{N+\lambda}}\,dz\\ &+C\frac{\varepsilon}{\alpha}\int_{B_R}\frac{|z|^2}{|z|^{N+\mu}}\,dz + C\varepsilon\int_{B_1\setminus B_R}\frac{|z|}{|z|^{N+\mu}}\,dz + C\varepsilon\\ &\qquad\ge F(\overline{t},\overline{y},u^\varepsilon(\overline{t},\overline{y}),\overline{p}) - F(\overline{t},\overline{x},u(\overline{t},\overline{x}),\overline{p}+\beta\overline{x}) \end{aligned}$$

where $\overline{p} = (\overline{x} - \overline{y})/\alpha$. Putting $t = 0$ and $x = y = 0$ in the definition of M, which is attained at $(\overline{t}, \overline{x}, \overline{y})$, we have $u(\overline{t}, \overline{x}) - u^\varepsilon(\overline{t}, \overline{y}) - \gamma/(T - \overline{t}) \geq M \geq -\gamma/T$, and thus $u(\overline{t}, \overline{x}) \geq u^\varepsilon(\overline{t}, \overline{y}) + \gamma/(T - \overline{t}) - \gamma/T \geq u^\varepsilon(\overline{t}, \overline{y})$; the function F being nonincreasing with respect to its third variable, we deduce

$$\begin{aligned} -\frac{\gamma}{T^2} - \eta + C\beta r^{2-\lambda} + Cr^{-\lambda} + C\frac{\varepsilon}{\alpha}R^{2-\mu} + C\varepsilon \int_{B_1 \setminus B_R} \frac{|z|}{|z|^{N+\mu}}\, dz + C\varepsilon \\ \geq F(\overline{t}, \overline{y}, u(\overline{t}, \overline{x}), \overline{p}) - F(\overline{t}, \overline{x}, u(\overline{t}, \overline{x}), \overline{p} + \beta\overline{x}). \end{aligned} \tag{3.32}$$

Using once again the definition of M_ν, we have $\beta|x_\nu|^2 \leq C$ (because $M_\nu \geq M \geq -\gamma/T \geq -1/T$), so that $\beta|\overline{x}| \leq C\sqrt{\beta}$. Moreover, since p_ν satisfies the reverse inequality of (3.23) with u^ε instead of u, and since we have a bound on the spatial Lipschitz constant of u^ε, we find $|p_\nu| \leq C$ and thus $|\overline{p}| \leq C$ and $|\overline{x} - \overline{y}| \leq C\alpha$. Assumptions (3.21) and (3.22) therefore give

$$-\frac{\gamma}{T^2} - \eta + C\beta r^{2-\lambda} + Cr^{-\lambda} + \frac{C\varepsilon}{\alpha}R^{2-\mu} + C\varepsilon \int_{B_1 \setminus B_R} \frac{|z|}{|z|^{N+\mu}}\, dz + C\varepsilon \geq -C\alpha - C\sqrt{\beta}$$

(we take $\beta \leq 1$). Choosing $\gamma = (C\sqrt{\beta} + C\beta r^{2-\lambda} + Cr^{-\lambda} + \beta)T^2$ (which is in $]0, 1]$ if r is large and β is small) and $\eta = C\alpha + \frac{C\varepsilon}{\alpha}R^{2-\mu} + C\varepsilon \int_{B_1 \setminus B_R} \frac{|z|}{|z|^{N+\mu}}\, dz + C\varepsilon$ leads to $-\beta \geq 0$, which is the contradiction we sought.

With these choices of γ and η, M is attained at $(0, \overline{x}, \overline{y})$ and, for all $(t, x) \in [0, T[\times\mathbb{R}^N$,

$$u(t, x) - u^\varepsilon(t, x) - \beta\frac{|x|^2}{2} - \eta t - \frac{\gamma}{T - t} \leq u_0(\overline{x}) - u_0(\overline{y}) - \frac{|\overline{x} - \overline{y}|^2}{2\alpha} \leq C\alpha$$

(we use the fact that u_0 is Lipschitz continuous). Thus,

$$\begin{aligned} u(t, x) \quad \leq \quad & u^\varepsilon(t, x) + \beta\frac{|x|^2}{2} + \left(C\alpha + \frac{C\varepsilon}{\alpha}R^{2-\mu} + C\varepsilon \int_{B_1 \setminus B_R} \frac{|z|}{|z|^{N+\mu}}\, dz + C\varepsilon\right) T \\ & + \frac{(C\sqrt{\beta} + C\beta r^{2-\lambda} + Cr^{-\lambda} + \beta)T^2}{T - t} + C\alpha. \end{aligned}$$

We now let $\beta \to 0$ and then $r \to +\infty$:

$$u(t, x) \leq u^\varepsilon(t, x) + C\alpha + \frac{C\varepsilon}{\alpha}R^{2-\mu} + C\varepsilon \int_{B_1 \setminus B_R} \frac{|z|}{|z|^{N+\mu}}\, dz + C\varepsilon.$$

If $\mu < 1$ (respectively $\mu = 1$, respectively $\mu > 1$), then $\int_{B_1 \setminus B_R} \frac{|z|}{|z|^{N+\mu}}\, dz$ is bounded by C (respectively $C|\ln(R)|$, respectively $CR^{1-\mu}$). A simple optimization with respect to R and then α leads to

$$u(t, x) \leq u^\varepsilon(t, x) + C \begin{cases} \varepsilon & \text{if } \mu < 1 \\ \varepsilon|\ln(\varepsilon)| & \text{if } \mu = 1 \\ \varepsilon^{1/\mu} & \text{if } \mu > 1 \end{cases}$$

and we obtain the reverse inequality by exchanging, from the beginning, the roles of u and u^ε. ■

4 Fractal scalar hyperbolic equations

4.1 Existence and uniqueness of a smooth solution

In this section, we come back to the case $\lambda \in]1, 2[$ and we handle

$$\begin{cases} \partial_t u(t, x) + \operatorname{div}(f(t, x, u(t, x))) + g_\lambda[u(t, \cdot)](x) = h(t, x, u(t, x)) & t > 0\,,\ x \in \mathbb{R}^N\,, \\ u(0, x) = u_0(x) & x \in \mathbb{R}^N\,, \end{cases} \tag{4.1}$$

where $u_0 \in L^\infty(\mathbb{R}^N)$, $f \in C^\infty([0,\infty[\times\mathbb{R}^N \times \mathbb{R})^N$, $h \in C^\infty([0,\infty[\times\mathbb{R}^N \times \mathbb{R})$ and

$$\begin{array}{c} \forall T > 0\,,\ \forall R > 0\,,\ \forall k \in \mathbb{N}, \exists C_{T,R,k} \text{ such that,} \\ \text{for all } (t,x,s) \in [0,T] \times \mathbb{R}^N \times [-R,R] \text{ and all } \alpha \in \mathbb{N}^{N+2} \text{ satisfying } |\alpha| \le k, \\ |\partial^\alpha f(t,x,s)| + |\partial^\alpha h(t,x,s)| \le C_{T,R,k}\,, \end{array} \tag{4.2}$$

$$\begin{array}{c} \forall T > 0, \text{ there exists } \Lambda_T : [0,+\infty[\to]0,+\infty[\text{ continuous nondecreasing} \\ \text{such that } \int_0^\infty \frac{1}{\Lambda_T(a)}\,da = +\infty \text{ and, for all } (t,x,s) \in [0,T] \times \mathbb{R}^N \times \mathbb{R}\,, \\ \mathrm{sgn}(s)\left(h(t,x,s) - \sum_{i=1}^N \partial_{x_i} f_i(t,x,s)\right) \le \Lambda_T(|s|). \end{array} \tag{4.3}$$

The term $h - \sum_{i=1}^N \partial_{x_i} f_i$ represents a source for (4.1), and an assumption on this source is not unexpected if we want global solutions; this hypothesis with $\Lambda_T(a) = K_T(1+a)$ (and K_T constant), as well as uniform spatial bounds such as in (4.2), also appear in [20] when dealing with the pure scalar conservation law (*i.e.* without g_λ). Here, we prove the following.

Theorem 4.1 *Let $\lambda \in]1,2[$ and $u_0 \in L^\infty(\mathbb{R}^N)$; assume that f and h satisfy (4.2) and (4.3). Then there exists a unique solution u to (4.1) in the sense: for all $T > 0$,*

$$u \in C_b(]0,T[\times\mathbb{R}^N) \text{ and, for all } a \in]0,T[\,,\ u \in C_b^\infty(]a,T[\times\mathbb{R}^N)\,, \tag{4.4}$$

$$u \text{ satisfies the PDE of (4.1) on }]0,T[\times\mathbb{R}^N\,, \tag{4.5}$$

$$u(t,\cdot) \to u_0 \text{ in } L^\infty(\mathbb{R}^N) \text{ weak-}*, \text{ as } t \to 0. \tag{4.6}$$

We also have Estimate (3.9) on the solution, that is to say: for all $0 < t < T < \infty$,

$$||u(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \le (\mathcal{L}_T)^{-1}\left(t + \mathcal{L}_T(||u_0||_{L^\infty(\mathbb{R}^N)})\right) \quad \text{with} \quad \mathcal{L}_T(a) = \int_0^a \frac{1}{\Lambda_T(b)}\,db.$$

Remark 4.1 *The proof of uniqueness shows that the solution to (4.1) also satisfies (4.7) below. As a consequence, the convergence in (4.6) also holds in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for all $p < \infty$.*

As for (3.1), the existence of a solution to (4.1) is obtained via a weak formulation based on Duhamel's formula.

Definition 4.1 *Let $\lambda \in]1,2[$, $u_0 \in L^\infty(\mathbb{R}^N)$, $T > 0$ and (f,h) satisfy (4.2). A weak solution to (4.1) on $[0,T]$ is a function $u \in L^\infty(]0,T[\times\mathbb{R}^N)$ such that, for a.e. $(t,x) \in]0,T[\times\mathbb{R}^N$,*

$$u(t,x) = K_\lambda(t,\cdot)*u_0(x) - \int_0^t \nabla K_\lambda(t-s,\cdot)*f(s,\cdot,u(s,\cdot))(x)\,ds + \int_0^t K_\lambda(t-s,\cdot)*h(s,\cdot,u(s,\cdot))(x)\,ds. \tag{4.7}$$

Thanks to (3.16), each term in (4.7) is well-defined. As before, a fixed point technique (see [14]) allows to prove the theorem stated below (Theorem 4.2); we leave the details to the interested reader (notice that once it has been proved that weak solutions to (4.1) have one continuous spatial derivative — which is a consequence of a result similar to Proposition 5.2 —, the full regularity of these weak solutions can be seen as a consequence of Theorem 3.2).

Theorem 4.2 *Let $\lambda \in]1,2[$, $u_0 \in L^\infty(\mathbb{R}^N)$ and (f,h) satisfy (4.2).*

i) For all $T > 0$, there exists at most one weak solution to (4.1) on $[0,T]$.

ii) A weak solution to (4.1) on $[0,T]$ satisfies (4.4), (4.5) and (4.6).

iii) Let $M \ge ||u_0||_{L^\infty(\mathbb{R}^N)}$. There exists $T > 0$, only depending on M and the constants in (4.2), such that (4.1) has a weak solution on $[0,T]$.

We can now prove the existence and uniqueness result for (4.1).

Proof of Theorem 4.1
Let u be a weak solution to (4.1) on $[0,T]$ in the sense of Definition 4.1. By Theorem 4.2, such a solution exists and satisfies (4.4), (4.5); hence, it satisfies (3.12) with

$$G(t,x,s,\xi) = h(t,x,s) - \sum_{i=1}^{N} \partial_{x_i} f_i(t,x,s) - \partial_s f(t,x,s)\cdot\xi.$$

Since (3.11) holds for G with $h = \Lambda_T$ given by Hypothesis (4.3), we deduce from Proposition 3.1 that, for all $0 < t' < t < T$,

$$||u(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \le (\mathcal{L}_T)^{-1}\left(t - t' + \mathcal{L}_T(||u(t',\cdot)||_{L^\infty(\mathbb{R}^N)})\right). \tag{4.8}$$

From (4.7) it is easy to see that $\limsup_{t'\to 0} ||u(t',\cdot)||_{L^\infty(\mathbb{R}^N)} \le ||u_0||_{L^\infty(\mathbb{R}^N)}$ (the last two terms of (4.7) tend to 0 in $L^\infty(\mathbb{R}^N)$ as $t\to 0$, thanks to (3.16)); hence, letting $t'\to 0$ in (4.8) shows that u satisfies (3.9). In particular, the L^∞ norm of $u(t,\cdot)$ does not explode in finite time and, by iii) in Theorem 4.2, we can indefinitely extend u; this proves the existence part of Theorem 4.1.

It remains to prove the uniqueness of the solution. Let u satisfy (4.4), (4.5) and (4.6) for all $T > 0$; take $t_0 > 0$. The function $u(t_0+\cdot,\cdot)$ belongs, for all $T > 0$, to $C_b^\infty([0,T[\times\mathbb{R}^N)$; hence, it satifies (3.6) and (3.8) with $u_0 = u(t_0,\cdot)$. Moreover, if we define

$$F(t,x,s,\xi) = h(t_0+t,x,u(t_0+t,x)) - \mathrm{div}(f(t_0+t,x,u(t_0+t,x)))$$

(in fact, F does not depend on s or ξ), the function $u(t_0+\cdot,\cdot)$ also satisfies (3.7). It is clear that this F satisfies (3.2), (3.3) and (3.4) (with Λ_T and $\Gamma_{T,R}$ constants) and, therefore, $u(t_0+\cdot,\cdot)$ is the unique solution to (3.1) given by Theorem 3.1; in particular, by Remark 3.2 we have, for all $t > 0$,

$$\begin{aligned} u(t_0+t,x) &= K_\lambda(t,\cdot) * u(t_0,\cdot)(x) \\ &\quad + \int_0^t K_\lambda(t-s,\cdot) * [h(t_0+s,\cdot,u(t_0+s,\cdot)) - \mathrm{div}(f(t_0+s,\cdot,u(t_0+s,\cdot)))](x)\,ds \\ &= K_\lambda(t,\cdot) * u(t_0,\cdot)(x) - \int_0^t \nabla K_\lambda(t-s,\cdot) * f(t_0+s,\cdot,u(t_0+s,\cdot))(x)\,ds \\ &\quad + \int_0^t K_\lambda(t-s,\cdot) * h(t_0+s,\cdot,u(t_0+s,\cdot))(x)\,ds. \end{aligned} \tag{4.9}$$

For $t > 0$ and $x\in\mathbb{R}^N$ fixed, by (4.6) we have $K_\lambda(t,\cdot)*u(t_0,\cdot)(x) \to K_\lambda(t,\cdot)*u_0(x)$ as $t_0\to 0$; using (4.4) and the dominated convergence theorem, we can let $t_0\to 0$ in the last two terms of (4.9) to see that u satisfies (4.7). Hence, u is a weak solution to (4.1) and, by i) in Theorem 4.2, is unique. ■

Remark 4.2 *Equation (4.1) can also be solved with more general operators g_λ, see Remark 3.3.*

4.2 About the vanishing regularization

Let us say a few things on the behaviour, as $\varepsilon\to 0^+$, of the solution to

$$\begin{cases} \partial_t u^\varepsilon(t,x) + \mathrm{div}(f(t,x,u^\varepsilon(t,x))) + \varepsilon g_\lambda[u^\varepsilon(t,\cdot)](x) = h(t,x,u^\varepsilon(t,x)) & t>0\,,\ x\in\mathbb{R}^N\,, \\ u^\varepsilon(0,x) = u_0(x) & x\in\mathbb{R}^N\,, \end{cases} \tag{4.10}$$

where we still take $\lambda\in]1,2[$, $u_0\in L^\infty(\mathbb{R}^N)$ and (f,h) satisfying (4.2) and (4.3). It has been proved in [12] that, if $h=0$ and f does not depend on (t,x), the solution u^ε to (4.10) converges, as $\varepsilon\to 0$, to the

entropy solution u of

$$\begin{cases} \partial_t u(t,x) + \mathrm{div}(f(t,x,u(t,x))) = h(t,x,u(t,x)) & t>0\,,\ x\in\mathbb{R}^N\,, \\ u(0,x) = u_0(x) & x\in\mathbb{R}^N\,. \end{cases} \tag{4.11}$$

The key argument is the obtention, via a splitting method, of entropy inequalities for (4.10); this method can be generalized to some cases where f and h depend on (t,x) (see [13]) but, in any cases, it is quite technical.

Thanks to formula (2.1), we have a trivial proof of these entropy inequalities, via the following lemma.

Lemma 4.1 *Let $\lambda\in]0,2[$, $\varphi\in C_b^2(\mathbb{R}^N)$ and $\eta\in C^2(\mathbb{R})$ be a convex function. Then $g_\lambda[\eta(\varphi)]\le\eta'(\varphi)g_\lambda[\varphi]$.*

Proof of Lemma 4.1
Since η is convex, we have $\eta(b)-\eta(a)\ge\eta'(a)(b-a)$. Hence,

$$\eta(\varphi(x+z))-\eta(\varphi(x))\ge\eta'(\varphi(x))(\varphi(x+z)-\varphi(x))$$

and

$$\eta(\varphi(x+z))-\eta(\varphi(x))-\nabla(\eta(\varphi))(x)\cdot z\ge\eta'(\varphi(x))(\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z).$$

The conclusion follows from these inequalities and (2.1). ■

Thus, if $\eta\in C^2(\mathbb{R})$ is convex and ϕ is such that $\partial_s\phi(t,x,s)=\eta'(s)\partial_s f(t,x,s)$, multiplying the PDE of (4.10) by $\eta'(u^\varepsilon(t,x))$ gives (recall that all the functions, including u^ε, are regular)

$$\begin{aligned} \partial_t(\eta(u^\varepsilon))(t,x)+\varepsilon g_\lambda[\eta(u^\varepsilon(t,\cdot))](x) \quad\le\quad & \eta'(u^\varepsilon(t,x))\left(h(t,x,u^\varepsilon(t,x))-\sum_{i=1}^N\partial_{x_i}f_i(t,x,u^\varepsilon(t,x))\right) \\ & -\mathrm{div}(\phi(t,x,u^\varepsilon(t,x)))+\sum_{i=1}^N\partial_{x_i}\phi_i(t,x,u^\varepsilon(t,x))\,, \end{aligned} \tag{4.12}$$

which is exactly the entropy inequality for (4.10). Once this inequality is established, the doubling variable technique of [20] (used in [12]) shows that, for all $T>0$ and as $\varepsilon\to0$, $u^\varepsilon\to u$ in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^N))$.

It is also possible, if the initial condition u_0 is in $L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)\cap BV(\mathbb{R}^N)$, to obtain a rate of convergence: $\mathcal{O}(\varepsilon^{1/\lambda})$ in $C([0,T];L^1(\mathbb{R}^N))$; this is well-known for $\lambda=2$ (see [21]) and has been done for $\lambda\in]1,2[$, $h=0$ and $f(t,x,u)=f(u)$ in [12]. However, to obtain such a rate of convergence we must first establish L^1 and BV estimates on u^ε, which demands additional hypotheses on f and h (some integrability properties with respect to x); we refer the reader to [13] for a set of suitable hypotheses. Once these estimates are established, the proof of the rate of convergence is made as in [21] or [12] by using (4.12).

5 Appendix

5.1 A technical Lemma

Lemma 5.1 *If $f\in C^1(\mathbb{R}^N)\cap W^{1,1}(\mathbb{R}^N)$ and $g\in L^\infty(\mathbb{R}^N)$, then $f*g\in C^1(\mathbb{R}^N)$ and $\nabla(f*g)=\nabla f*g$.*

Proof of Lemma 5.1
We have not assumed that $\nabla f(x-y)$ is bounded locally uniformly in x by some integrable function of y; hence, we cannot directly use a theorem of derivation under the integral sign.

Let $n \geq 1$ and define $g_n = g\mathbf{1}_{B_n}$; for all $x \in \mathbb{R}^N$, $f * g_n(x) = \int_{B_n} f(x-y)g(y)\,dy \to f * g(x)$ as $n \to \infty$. Since $f \in C^1(\mathbb{R}^N)$, a derivation under the integral sign shows that $f * g_n \in C^1(\mathbb{R}^N)$ with $\nabla(f * g_n) = \nabla f * g_n$. But, for all $|x| \leq R$,

$$|\nabla f * g_n(x) - \nabla f * g(x)| \leq ||g||_{L^\infty(\mathbb{R}^N)} \int_{\{|y| \geq n\}} |\nabla f(x-y)|\,dy \leq ||g||_{L^\infty(\mathbb{R}^N)} \int_{\{|z| \geq n-R\}} |\nabla f(z)|\,dz$$

(if $|x| \leq R$ and $|y| \geq n$, then $|x-y| \geq |y| - |x| \geq n - R$); hence, since $\nabla f \in L^1(\mathbb{R}^N)$, we have $\nabla(f * g_n) = \nabla f * g_n \to \nabla f * g$ locally uniformly on $\mathbb{R}^N$, which concludes the proof of the lemma. ■

5.2 Generalizations of Theorem 2.2 and Proposition 3.1

In this subsection, we state and prove generalizations of Theorem 2.2 and Proposition 3.1. Roughly speaking, we show that u needs not be in C_b^2 but only in C_b; in this case, the operator g_λ and Equation (3.12) must be understood in the viscosity sense.
For usc functions $\phi :]0,T[\to \mathbb{R}$, the notion of viscosity supergradient is used in order to define viscosity subsolutions of $\phi' = h(\phi)$. The notion of upper semi-continuous envelope of locally bounded functions is also used in the following. The definitions of viscosity supergradient, viscosity solution of $\phi' = h(\phi)$ and upper semi-continuous envelope can be found in [11].

Theorem 5.1 *Let $\lambda \in]0,2[$ and $v \in C_b(]0,T[\times\mathbb{R}^N)$. Let ϕ denote the upper semi-continuous envelope of the function $\sup_{x\in\mathbb{R}^N} v(\cdot,x)$. Then for any viscosity supergradient α of ϕ at $t \in]0,T[$, there exist $t_n \to t$, $\alpha_n \to \alpha$, and $x_n, p_n \in \mathbb{R}^N$ such that*

$$v(t_n,x_n) \to \phi(t) \quad \text{and} \quad (\alpha_n,p_n) \in \partial^P v(t_n,x_n) \quad \text{and} \quad p_n \to 0 \quad \text{and} \quad \liminf_{n\to\infty} g_\lambda[v(t_n,\cdot)](x_n) \geq 0.$$

Proof of Theorem 5.1
By definition of viscosity supergradient, there exists $\psi \in C^1(]0,T[)$ such that $\phi - \psi$ attains a global maximum at t and $\alpha = \psi'(t)$. Then for any $(s,x) \in]0,T[\times\mathbb{R}^N$, we have:

$$v(s,x) - \psi(s) \leq \phi(s) - \psi(s) \leq \phi(t) - \psi(t).$$

Next, for any $\varepsilon > 0$, consider $(t_\varepsilon, x_\varepsilon) \in]0,T[\times\mathbb{R}^N$ such that $\phi(t) < v(t_\varepsilon,x_\varepsilon) + \varepsilon/2$ and $t_\varepsilon \to t$ as $\varepsilon \to 0$. We can also ensure that $\psi(t) \geq \psi(t_\varepsilon) - \varepsilon/2$. Combining these facts yields:

$$\sup_{(s,x)\in]0,T[\times\mathbb{R}^N} (v(s,x) - \psi(s)) < v(t_\varepsilon,x_\varepsilon) - \psi(t_\varepsilon) + \varepsilon.$$

We then apply Borwein and Preiss' minimization principle (see for instance [10]) and get $(s_\varepsilon, y_\varepsilon)$ and $(r_\varepsilon, z_\varepsilon)$ such that

$$|(r_\varepsilon,z_\varepsilon) - (t_\varepsilon,x_\varepsilon)| < \varepsilon^{1/4} \quad \text{and} \quad |(s_\varepsilon,y_\varepsilon) - (r_\varepsilon,z_\varepsilon)| < \varepsilon^{1/4}$$
$$\text{and} \quad \sup_{(s,x)\in]0,T[\times\mathbb{R}^N} (v(s,x) - \psi(s)) \leq v(s_\varepsilon,y_\varepsilon) - \psi(s_\varepsilon) + \varepsilon$$

and such that $(s_\varepsilon, y_\varepsilon)$ is the unique point realizing the maximum of the perturbed function $(t,x) \mapsto v(t,x) - \psi(t) - \sqrt{\varepsilon}(t-r_\varepsilon)^2 - \sqrt{\varepsilon}|x-z_\varepsilon|^2$. This implies that $(\psi'(s_\varepsilon) + 2\sqrt{\varepsilon}(s_\varepsilon - r_\varepsilon), 2\sqrt{\varepsilon}(y_\varepsilon - z_\varepsilon)) \in \partial^P v(s_\varepsilon, y_\varepsilon)$. Define $\alpha_\varepsilon = \psi'(s_\varepsilon) + 2\sqrt{\varepsilon}(s_\varepsilon - r_\varepsilon)$ and $p_\varepsilon = 2\sqrt{\varepsilon}(y_\varepsilon - z_\varepsilon)$. They verify $\alpha_\varepsilon \to \alpha$ and $p_\varepsilon \to 0$ as $\varepsilon \to 0$. Moreover, $v(s_\varepsilon, y_\varepsilon) \to \phi(t)$ and $s_\varepsilon \to t$. It only remains to prove that $\liminf_{\varepsilon\to 0} g_\lambda[v(s_\varepsilon,\cdot)](y_\varepsilon) \geq 0$ by using Fatou's Lemma. First, notice that

$$v(s_\varepsilon, y_\varepsilon + z) - v(s_\varepsilon, y_\varepsilon) \leq \phi(s_\varepsilon) - v(s_\varepsilon, y_\varepsilon)$$

and since ϕ is upper semi-continuous and $v(s_\varepsilon, y_\varepsilon) \to \phi(t)$, the upper limit of the right-hand side is nonpositive. Secondly,

$$\frac{v(s_\varepsilon, y_\varepsilon + z) - v(s_\varepsilon, y_\varepsilon)}{|z|^{N+\lambda}} \leq \frac{2\|v\|_\infty}{|z|^{N+\lambda}} \in L^1(\mathbb{R}^N \setminus B_1),$$
$$\frac{v(s_\varepsilon, y_\varepsilon + z) - v(s_\varepsilon, y_\varepsilon) - p_\varepsilon \cdot z}{|z|^{N+\lambda}} \leq \frac{\sqrt{\varepsilon}}{|z|^{N+\lambda-2}} \in L^1(B_1).$$

Now choose $\varepsilon = 1/n$ and $(t_n, \alpha_n, x_n, p_n) = (s_{\varepsilon_n}, \alpha_{\varepsilon_n}, y_{\varepsilon_n}, p_{\varepsilon_n})$ satisfies the desired properties. ■

Proposition 5.1 *Let $\lambda \in]0,2[$, $T > 0$ and $G \in C(]0,T[\times\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ be such that (3.11) is satisfied and G is locally Lipschitz continuous w.r.t. ξ, locally in (t,s) and uniformly in x. Then any viscosity solution of (3.12) satisfies for any $0 < t' < t < T$:*

$$||u(t,\cdot)||^*_{L^\infty(\mathbb{R}^N)} \leq \mathcal{H}^{-1}\left(t - t' + \mathcal{H}\left(||u(t',\cdot)||^*_{L^\infty(\mathbb{R}^N)}\right)\right)$$

where $||u(s,\cdot)||^*_{L^\infty(\mathbb{R}^N)} = \limsup_{\tau\to s} ||u(\tau,\cdot)||_{L^\infty(\mathbb{R}^N)}$.

Proof of Proposition 5.1
Let us denote $\phi(t) = ||u(t,\cdot)||^*_{L^\infty(\mathbb{R}^N)}$. Suppose we have proved that ϕ is a viscosity subsolution of $w' = h(w)$ on $]0,T[$. Then the function $\mathcal{H}(\phi(t)) - t$ is a viscosity subsolution of $w' = 0$ (recall that $\mathcal{H}$ is C^1 and nondecreasing). This implies that $\mathcal{H}(\phi(t)) - t$ is nonincreasing and, since $\mathcal{H}$ is a nondecreasing bijection $[0,+\infty[\to[0,+\infty[$, we get the desired *a priori* estimate on u.
It remains to prove that ϕ is a viscosity subsolution of $w' = h(w)$ on $]0,T[$. It is a consequence of Theorem 5.1 applied to $v = |u|$. Let α be a viscosity supergradient of ϕ and consider $t_n \to t$, $\alpha_n \to \alpha$, and $x_n, p_n \in \mathbb{R}^N$ given by Theorem 5.1. We have to prove that $\alpha \leq h(\phi(t))$. We distinguish two cases. Suppose first that there exists a sequence $n_k \to \infty$ such that $v(t_{n_k}, x_{n_k}) = u(t_{n_k}, x_{n_k})$. Then $(\alpha_{n_k}, p_{n_k}) \in \partial^P u(t_{n_k}, x_{n_k})$ and, since u is a viscosity subsolution of (3.12), we get:

$$\alpha_{n_k} + g_\lambda[u(t_{n_k},\cdot)](x_{n_k}) \leq G(t_{n_k}, x_{n_k}, u(t_{n_k}, x_{n_k}), p_{n_k}).$$

As $k \to \infty$, we have $u(t_{n_k}, x_{n_k}) \to \phi(t)$ and $p_{n_k} \to 0$. We can use the local Lipschitz continuity of G with respect to ξ and find:

$$\alpha_{n_k} + g_\lambda[u(t_{n_k},\cdot)](x_{n_k}) \leq h(u(t_{n_k}, x_{n_k})) + M|p_{n_k}|$$

for M independent of k. As k goes to $+\infty$, we conclude in the first case that $\alpha \leq h(\phi(t))$ by using the fact that $u(t_{n_k}, x_{n_k}) \to \phi(t)$ and that $g_\lambda[u(t_{n_k},\cdot)](x_{n_k}) \geq g_\lambda[v(t_{n_k},\cdot)](x_{n_k})$ (because $v(t_{n_k}, x_{n_k}) = u(t_{n_k}, x_{n_k})$ and $v(t_{n_k}, x_{n_k} + z) \geq u(t_{n_k}, x_{n_k} + z)$), so that $\liminf_{k\to\infty} g_\lambda[u(t_{n_k},\cdot)](x_{n_k}) \geq 0$. In the second case, for n large enough, $v(t_n, x_n) = -u(t_n, x_n)$. Then $(-\alpha_n, -p_n) \in \partial_P u(t_n, x_n)$ and we can argue similarly, by using the fact that u is a viscosity supersolution of (3.12), to conclude that we also have $\alpha \leq h(\phi(t))$. ■

5.3 Ideas for the proof of Theorem 3.2

We need the following additional property on K_λ:

$$t \in]0,\infty[\mapsto K_\lambda(t,\cdot) \in L^1(\mathbb{R}^N) \text{ is continuous.} \tag{5.1}$$

This continuity is a consequence of the regularity of K_λ and of the homogeneity property $K_\lambda(t,x) = t^{-N/\lambda} K_\lambda(1, t^{-1/\lambda}x)$ which shows that, if A is a compact subset of $]0,\infty[$, then $(K_\lambda(t,\cdot))_{t\in A}$ is equi-integrable at infinity (that is to say, for all $\varepsilon > 0$, there exists $R > 0$ such that, for all $t \in A$, $\int_{\mathbb{R}^N\setminus B_R} |K_\lambda(t,x)|\,dx \leq \varepsilon$).

The most difficult task in the proof of Theorem 3.2 is the regularity of the weak solutions. The key result to prove this regularity is the following proposition.

Proposition 5.2 *Let $\lambda \in]1,2[$, $S>0$ and $G:(t,x,\zeta)\in]0,S[\times\mathbb{R}^N\times\mathbb{R}^N \to G(t,x,\zeta)\in\mathbb{R}$ be continuous; we suppose that $\partial_x G$, $\partial_\zeta G$, $\partial_\zeta\partial_x G$ and $\partial_\zeta\partial_\zeta G$ exist and are continuous on $]0,S[\times\mathbb{R}^N\times\mathbb{R}^N$; we also suppose that there exists $\omega:]0,\infty[\to\mathbb{R}^+$ such that, for all $L>0$, G and these derivatives are bounded on $]0,S[\times\mathbb{R}^N\times B_L$ by $\omega(L)$.*
Let $R_0>0$ and $R=(2+\mathcal{K})R_0$ where $\mathcal{K}$ is given by (3.16). Then there exists $T_0>0$ only depending on (λ,R_0,ω) such that, if $T=\inf(S,T_0)$ and $V_0\in L^\infty(\mathbb{R}^N)^N$ satisfies $||V_0||_{L^\infty(\mathbb{R}^N)^N}\le R_0$, there exists a unique $V\in C_b(]0,T[\times\mathbb{R}^N)^N$ bounded by R and such that

$$V(t,x)=K_\lambda(t,\cdot)*V_0(x)+\int_0^t \nabla K_\lambda(t-s,\cdot)*G(s,\cdot,V(s,\cdot))(x)\,ds. \tag{5.2}$$

Moreover, $\partial_x V\in C(]0,T[\times\mathbb{R}^N)^{N^2}$ and, for all $a\in]0,T[$, $||\partial_x V||_{C_b(]a,T[\times\mathbb{R}^N)^{N^2}}\le Ra^{-1/\lambda}$.

Sketch of the proof of Proposition 5.2
We define $E_T=\{V\in C_b(]0,T[\times\mathbb{R}^N)^N \mid t^{1/\lambda}\partial_x V\in C_b(]0,T[\times\mathbb{R}^N)^{N^2}\}$ and, for $V\in E_T$, $\Phi_T(V)$ as the right-hand side of (5.2).
Thanks to (5.1), the first term $K_\lambda(t,\cdot)*V_0(x)$ of $\Phi_T(V)$ is continuous in t uniformly with respect to x; since, for t fixed, it is also continuous in x (it is the convolution product of an integrable function and a bounded function), it is continuous in (t,x). The second term of $\Phi_T(V)$ is the convolution product in $\mathbb{R}\times\mathbb{R}^N$ of the integrable function $\nabla K_\lambda(t,x)\mathbf{1}_{]0,T[}(t)$ and the bounded function $G(t,x,V(t,x))\mathbf{1}_{]0,T[}(t)$: it is therefore continuous in (t,x). By Lemma 5.1 and (3.16), we have $\partial_x(K_\lambda(t,\cdot)*V_0)(x)=\partial_x K_\lambda(t,\cdot)*V_0(x)$; since $V\in E_T$, we can differentiate the second term of $\Phi_T(V)$ under the integral sign to obtain

$$\begin{aligned}\partial_x\Phi_T(V)(t,x) &= \partial_x K_\lambda(t,\cdot)*V_0(x)\\ &\quad+\int_0^t \nabla K_\lambda(t-s,\cdot)*\Big[\partial_x G(s,\cdot,V(s,\cdot))+\partial_\zeta G(s,\cdot,V(s,\cdot))\partial_x V(s,\cdot)\Big](x)\,ds.\end{aligned}$$

For $t_0>0$ and $t>0$, we have $\partial_x(K_\lambda(t_0+t,\cdot)*V_0)(x)=K_\lambda(t,\cdot)*(\partial_x K_\lambda(t_0,\cdot)*V_0)(x)$, which is continuous in (t,x) (same proof as the continuity of $K_\lambda(t,\cdot)*V_0(x)$); hence, the first term of $\partial_x\Phi_T(V)$ is continuous on $]0,T[\times\mathbb{R}^N$. The continuity the second term is proved first by replacing $\partial_x V(s,\cdot)$ with $\partial_x V(s,\cdot)\mathbf{1}_{[\delta,T[}(s)$ (since this function is bounded, the continuity is obtained as for the last term of (5.2)), and then by letting $\delta\to 0$ (the convergence is uniform in $(t,x)\in[t_0,T[\times\mathbb{R}^N$ for all $t_0>0$).
A simple application of (3.16) then allows to prove that, for T small enough, Φ_T is contracting from the ball in E_T of radius R into itself, which proves the existence of a solution to (5.2) in E_T. The uniqueness of the bounded solution comes from the fact that, if T is small, Φ_T is contracting on the ball in $L^\infty(]0,T[\times\mathbb{R}^N)$ of radius R. ■

The spatial regularity of any weak solution u to (3.1) is then quite easy. Indeed, from (3.15) and the fact that u and ∇u are bounded, we see as in the proof above that u is continuous on $]0,T[\times\mathbb{R}^N$; moreover, the gradient of u satisfies (3.19) which proves, still using the same technique, that it is continuous. Since $(u,\nabla u)\in C_b(]0,T[\times\mathbb{R}^N)$, these equations (3.19) can be written in the form of (5.2) (with G taking into account u); hence, Proposition 5.2 says that the second spatial derivative of u is continuous on $]0,T[\times\mathbb{R}^N$ and bounded far from $t=0$. We can also write an integral equation satisfied by this second derivative, provided that we begin at an initial time $t_0>0$ instead of 0; this equation is of the kind (5.2). An induction process, using Proposition 5.2 on the successive equations satisfied by the spatial derivatives of u, then proves that (3.6) holds for spatial derivatives (all the regularities and bounds we obtain are local in time, but since the time span on which they hold is controlled, we also obtain global bounds).
To prove that u is differentiable w.r.t. t, we first notice that, if $\varphi\in C_b^2(\mathbb{R}^N)$, then $t\mapsto K_\lambda(t,\cdot)*\varphi(x)$ is derivable and $\frac{d}{dt}(K_\lambda(t,\cdot)*\varphi(x))=-g_\lambda[K_\lambda(t,\cdot)*\varphi](x)$; this is quite obvious on (1.1) if $\varphi\in\mathcal{S}(\mathbb{R}^N)$ and can be deduced for general φ by a density argument (same technique as in the proof of Proposition 2.1). With this result, it is possible to derivate (3.15), written at an initial time $t_0>0$ and with initial data $u(t_0,\cdot)\in C_b^2(\mathbb{R}^N)$, with respect to t (to derivate the integral term, we first replace it by $\int_0^{t-\delta}$ and then

let $\delta \to 0$); this proves that u satisfies (3.7). The spatial regularity of u and (2.1) then show that u is also regular in time.
The proof of (3.8) is immediate on (3.15) (the integral term tends to 0 in $L^\infty(\mathbb{R}^N)$ as $t \to 0$, and since u_0 is bounded and uniformly continuous and $(K_\lambda(t,\cdot))_{t\to 0}$ is an approximate unit, $K_\lambda(t,\cdot) * u_0 \to u_0$ uniformly on $\mathbb{R}^N$ as $t \to 0$).

The uniqueness i) and existence iii) in Theorem 3.2 are straightforward applications of a contracting fixed point on (3.15) in the space $\{u \in L^\infty(]0,T[\times\mathbb{R}^N) \mid \nabla u \in L^\infty(]0,T[\times\mathbb{R}^N)^N\}$ (the uniqueness is first local in time, and can then be extented to any time interval in the same way global uniqueness for ODEs is proved).

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Annexes

A A propos d'holomorphie

A.1 Théorème d'holomorphie sous l'intégrale

Le théorème que nous utilisons pour prouver l'holomorphie des fonctions intervenant dans (2.1) est le suivant.

Théorème A.1 *Soit $(X, \mathcal{A}, \mu)$ un espace mesuré et Ω un ouvert de $\mathbb{C}$. Soit $f : \Omega \times X \to \mathbb{C}$ telle que:*

- *pour tout $z \in \Omega$, $f(z, \cdot) : X \to \mathbb{C}$ est mesurable,*
- *pour tout K compact de Ω, il existe $g_K \in L^1(X)$ telle que, pour presque tout $x \in X$ et pour tout $z \in K$, $|f(z, x)| \leq g_K(x)$.*
- *pour presque tout $x \in X$, $f(\cdot, x) : \Omega \to \mathbb{C}$ est holomorphe.*

Alors $F : z \in \Omega \to \int_X f(z, x)\, d\mu(x)$ est holomorphe sur Ω et $F'(z) = \int_X \frac{\partial f}{\partial z}(z, x)\, d\mu(x)$.

Preuve du théorème A.1
Soit $z_0 \in \Omega$ et $B(z_0, r)$ une boule compacte centrée en z_0 et incluse dans Ω. Par le théorème de Cauchy, on a, pour presque tout $x \in X$ et tout $z \in B(z_0, r/2)$,

$$\frac{\partial f}{\partial z}(z, x) = \frac{1}{2i\pi} \int_{\partial B(z_0, r)} \frac{f(w, x)}{w - z}\, dw$$

et donc

$$\sup_{z \in B(z_0, r/2)} \left| \frac{\partial f}{\partial z}(z, x) \right| \leq 2 g_{B(z_0, r)}(x).$$

Le théorème des accroissements finis donne donc, pour presque tout $x \in X$, lorsque $h \in B(0, r/2)$,

$$\left| \frac{f(z_0 + h, x) - f(z_0, x)}{h} \right| \leq 2 g_{B(z_0, r)}(x).$$

Puisque $\frac{f(z_0+h,x)-f(z_0,x)}{h} \to \frac{\partial f}{\partial z}(z_0, x)$ lorsque $h \to 0$, le théorème de convergence dominée montre alors que

$$\lim_{h \to 0} \frac{F(z_0 + h) - F(z_0)}{h} = \int_X \frac{\partial f}{\partial z}(z_0, x)\, d\mu(x)\,,$$

ce qui conclut la preuve. ■

A.2 Zeros de la fonction Γ

Par le théorème A.1, la fonction $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$ est holomorphe sur $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$. Nous désirons voir ici, le plus simplement possible, qu'elle n'a pas de zéro dans $\mathbb{C}^+$.

On commence par rappeler la formule bien connue: si on pose $\beta(a, b) = \int_0^1 \tau^{a-1}(1 - \tau)^{b-1}\, d\tau$, alors, pour tous $(a, b) \in \mathbb{C}^+$,

$$\Gamma(a)\Gamma(b) = \beta(a, b)\Gamma(a + b). \tag{A.1}$$

Pour cela, on écrit, par Fubini:

$$\Gamma(a)\Gamma(b) = \left(\int_0^\infty t^{a-1} e^{-t}\, dt \right) \left(\int_0^\infty s^{b-1} e^{-s}\, ds \right) = \int_{\mathbb{R}_+^2} t^{a-1} s^{b-1} e^{-(t+s)}\, dt ds.$$

Les changements de variables $(u,v)=(t,t+s)$ (de $\mathbb{R}^2_+$ dans $\{(u,v)\mid 0\le u\le v\}$) et $u=v\tau$ donnent

$$\begin{aligned}\Gamma(a)\Gamma(b) &= \int_0^\infty e^{-v}\int_0^v u^{a-1}(v-u)^{b-1}\,dudv\\ &= \int_0^\infty e^{-v}\int_0^1 v^{a-1}v^{b-1}\tau^{a-1}(1-\tau)^{b-1}\,vd\tau dv\\ &= \left(\int_0^\infty v^{a+b-1}e^{-v}\,dv\right)\left(\int_0^1 \tau^{a-1}(1-\tau)^{b-1}\,d\tau\right)\end{aligned}$$

ce qui conclut la preuve de (A.1).

Supposons que Γ ait un zéro $z_0\in\mathbb{C}^+$ et prenons $a\in]0,\mathrm{Re}(z_0)[$; alors $b=z_0-a$ est valide dans (A.1) et on a donc $\Gamma(a)\Gamma(z_0-a)=\beta(a,z_0-a)\Gamma(z_0)=0$. Comme a est réel strictement positif, $\Gamma(a)\neq 0$ et on en déduit $\Gamma(z_0-a)=0$. Ceci étant vrai pour tout $a\in]0,\mathrm{Re}(z_0)[$, on a prouvé que Γ s'annule sur le segment $]i\,\mathrm{Im}(z_0),z_0[$ de $\mathbb{C}^+$, ce qui prouve par holomorphie de Γ qu'elle est nulle sur $\mathbb{C}^+$ en entier et nous donne une contradiction.

B Preuve du théorème 3.2

La preuve de ce théorème est totalement identique à ce qui est fait dans [17] et [14]. Beaucoup de choses qui apparaissent ci-dessous sont des copiés-collés, avec des modification principalement syntaxiques, de [14].

B.1 i): Uniqueness

Lemma B.1 *Let $0\le t_0<T$, $u_0\in W^{1,\infty}(\mathbb{R}^N)$ and assume that F satisfies (3.2). If u is a weak solution to (3.1) on $[0,T]$, then $u(t_0+\cdot,\cdot)$ is a weak solution to (3.1) on $[0,T-t_0]$ with $u(t_0,\cdot)$ instead of u_0 and $F(t_0+\cdot,\cdot,\cdot,\cdot)$ instead of F.*

Proof of Lemma B.1
The proof relies on the semi-group formula $K_\lambda(t,\cdot)*K_\lambda(t',\cdot)=K_\lambda(t+t',\cdot)$ (which can be checked by Fourier transform). By (3.15) we have

$$u(t_0,\cdot)=K_\lambda(t_0,\cdot)*u_0+\int_0^{t_0}K_\lambda(t_0-s,\cdot)*F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))\,ds.$$

Thanks to (3.16), we have $|K_\lambda(t_0-s,\cdot)*F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))|\le ||F(\cdot,\cdot,u,\nabla u)||_{L^\infty(]0,T[\times\mathbb{R}^N)}$ and thus, still using (3.16), we see that the function $(s,y)\to K_\lambda(t,x-y)K_\lambda(t_0-s,\cdot)*F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))(y)$ is integrable on $]0,t_0[\times\mathbb{R}^N$ (x is fixed). Hence, by Fubini,

$$K_\lambda(t,\cdot)*u(t_0,\cdot)=K_\lambda(t_0+t,\cdot)*u_0+\int_0^{t_0}K_\lambda(t_0+t-s,\cdot)*F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))\,ds \tag{B.1}$$

We also have, for $t\in]0,T-t_0[$,

$$\int_0^t K_\lambda(t-s,\cdot)*F(t_0+s,\cdot,u(t_0+s,\cdot),\nabla u(t_0+s,\cdot))\,ds=\int_{t_0}^{t_0+t}K_\lambda(t_0+t-\tau,\cdot)*F(\tau,\cdot,u(\tau,\cdot),\nabla u(\tau,\cdot))\,d\tau$$

and therefore

$$\begin{aligned}&K_\lambda(t,\cdot)*u(t_0,\cdot)+\int_0^t K_\lambda(t-s,\cdot)*F(t_0+s,\cdot,u(t_0+s,\cdot),\nabla u(t_0+s,\cdot))\,ds\\ &\qquad= K_\lambda(t_0+t,\cdot)*u_0+\int_0^{t_0+t}K_\lambda(t_0+t-s,\cdot)*F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))\,ds\\ &\qquad= u(t_0+t,\cdot)\end{aligned}$$

which proves the lemma. ■

Theorem B.1 *Let $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ and $T > 0$. There exists at most one weak solution to (3.1) on $[0,T]$ in the sense of Definition 3.1.*

Proof of Theorem B.1
Step 1: we first prove a local uniqueness result.
Let $T_1 > 0$ and u, v be weak solutions to (3.1) on $[0,T_1]$; let R such that

$$||u||_{L^\infty(]0,T[\times\mathbb{R}^N)} \le R\,,\ ||v||_{L^\infty(]0,T[\times\mathbb{R}^N)} \le R\,,\ ||\nabla u||_{L^\infty(]0,T[\times\mathbb{R}^N)} \le R\,,\ ||\nabla v||_{L^\infty(]0,T[\times\mathbb{R}^N)} \le R.$$

By (3.15), (3.16) and (3.2), we have, for $t \in]0,T_1[$,

$$\begin{aligned} &||u(t,\cdot) - v(t,\cdot)||_{L^\infty(]0,T_1[\times\mathbb{R}^N)} \\ &\qquad \le \int_0^t ||F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot)) - F(s,\cdot,v(s,\cdot),\nabla v(s,\cdot))||_{L^\infty(\mathbb{R}^N)}\,ds \\ &\qquad \le C_{T,R,1}\left(||u-v||_{L^\infty(]0,T_1[\times\mathbb{R}^N)} + ||\nabla u - \nabla v||_{L^\infty(]0,T_1[\times\mathbb{R}^N)}\right) T_1. \end{aligned} \tag{B.2}$$

As in the proof of Proposition 3.2, we can use Lemma 5.1 and (3.16) to derivate (3.15) under the integral sign and find

$$\nabla u(t,x) = K_\lambda(t,\cdot) * \nabla u_0(x) + \int_0^T \nabla K_\lambda(t-s,\cdot) * F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))(x)\,ds$$

and thus, subtracting the same equation satisfied by ∇v,

$$\begin{aligned} &||\nabla u(t,\cdot) - \nabla v(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \\ &\qquad \le \int_0^t \mathcal{K}_1(t-s)^{-1/\lambda} ||F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot)) - F(s,\cdot,v(s,\cdot),\nabla v(s,\cdot))||_{L^\infty(\mathbb{R}^N)} \\ &\qquad \le C_{T,R,1}(||u-v||_{L^\infty(]0,T_1[\times\mathbb{R}^N)} + ||\nabla u - \nabla v||_{L^\infty(]0,T_1[\times\mathbb{R}^N)}) \frac{\mathcal{K}}{1-\frac{1}{\lambda}} T^{1-\frac{1}{\lambda}} \end{aligned} \tag{B.3}$$

where $\mathcal{K}$ is given by (3.16). Adding (B.2) and (B.3), we conclude that $u = v$ provided that

$$C_{T,R,1} T_1 + C_{T,R,1} \frac{\mathcal{K}_1}{1-\frac{1}{\lambda}} T_1^{1-\frac{1}{\lambda}} < 1\,,$$

i.e. that T_1 is small enough.

Step 2: proof of the uniqueness result.
Let u and v be two weak solutions to (3.1) on $[0,T]$. By Step 1, $u = v$ on $]0,T_1[\times\mathbb{R}^N$ for some $T_1 > 0$. Let $T' = \sup\{t \in]0,T[\mid u = v \text{ on }]0,t[\times\mathbb{R}^N\} \ge T_1$, and suppose that $T' < T$. By definition of T', and since u and v are continuous on $]0,T[\times\mathbb{R}^N$ (see the end of subsection B.2.1), we have $u(T',\cdot) = v(T',\cdot)$ on $\mathbb{R}^N$. By Lemma B.1, $u(T'+\cdot,\cdot)$ and $v(T'+\cdot,\cdot)$ are two weak solutions to (3.1) on $[0,T-T']$ with the same initial condition $u(T',\cdot) = v(T',\cdot)$ and $F(T'+\cdot,\cdot,\cdot,\cdot)$ instead of F. Step 1 shows then that $u(T'+\cdot,\cdot) = v(T'+\cdot,\cdot)$ on $]0,T_1'[\times\mathbb{R}^N$ for some $T_1' > 0$, which is a contradiction with the definition of T'. ■

B.2 ii) : Regularity

B.2.1 Spatial regularity

We need the following property on K_λ:

$$t \in]0,\infty[\to K_\lambda(t,\cdot) \in L^1(\mathbb{R}^N) \text{ is continuous.} \tag{B.4}$$

This is a consequence of the regularity of K_λ and of $K_\lambda(t,x) = t^{-N/\lambda}K_\lambda(1,t^{-1/\lambda}x)$ which shows that, if A is a compact subset of $]0,\infty[$, then $(K_\lambda(t,\cdot))_{t\in A}$ is equi-integrable at infinity (that is to say, for all $\varepsilon > 0$, there exists $R > 0$ such that, for all $t \in A$, $\int_{\mathbb{R}^N\setminus B_R} |K_\lambda(t,x)|\,dx \le \varepsilon$).

The key result to prove the regularity result of Theorem 3.2 is the following.

Proposition B.1 *Let $S > 0$ and $G : (t,x,\zeta) \in]0,S[\times\mathbb{R}^N \times \mathbb{R}^N \to G(t,x,\zeta) \in \mathbb{R}$ be continuous; we suppose that $\partial_x G$, $\partial_\zeta G$, $\partial_\zeta\partial_x G$ and $\partial_\zeta\partial_\zeta G$ exist and are continuous on $]0,S[\times\mathbb{R}^N \times \mathbb{R}^N$; we also suppose that there exists $\omega :]0,\infty[\to \mathbb{R}^+$ such that, for all $L > 0$, G and these derivatives are bounded on $]0,S[\times\mathbb{R}^N \times B_L$ by $\omega(L)$.*
Let $R_0 > 0$ and $R = (2+\mathcal{K}_1)R_0$ where $\mathcal{K}$ is given by (3.16). Then there exists $T_0 > 0$ only depending on (R_0,ω) such that, if $T = \inf(S,T_0)$ and $V_0 \in L^\infty(\mathbb{R}^N)^N$ satisfies $||V_0||_{L^\infty(\mathbb{R}^N)^N} \le R_0$, there exists a unique $V \in C_b(]0,T[\times\mathbb{R}^N)^N$ bounded by R and such that

$$V(t,x) = K_\lambda(t,\cdot) * V_0(x) + \int_0^t \nabla K_\lambda(t-s,\cdot) * G(s,\cdot,V(s,\cdot))(x)\,ds. \tag{B.5}$$

Moreover, $\partial_x V \in C(]0,T[\times\mathbb{R}^N)^{N^2}$ and, for all $a \in]0,T[$, $||\partial_x V||_{C_b(]a,T[\times\mathbb{R}^N)^{N^2}} \le Ra^{-1/\lambda}$.

Proof of Proposition B.1
The idea is to use a fixed point theorem. Let, for $T \in]0,S[$, $E_T = \{V \in C_b(]0,T[\times\mathbb{R}^N)^N \mid \partial_x V \in C(]0,T[\times\mathbb{R}^N)^{N^2}$ and $t^{1/\lambda}\partial_x V \in C_b(]0,T[\times\mathbb{R}^N)^{N^2}\}$, endowed with its natural norm $||V||_{E_T} = ||V||_\infty + ||t^{1/\lambda}\partial_x V||_\infty$. We define $\Psi_T : C_b(]0,T[\times\mathbb{R}^N)^N \to C_b(]0,T[\times\mathbb{R}^N)^N$ by

$$\Psi_T(V)(t,x) = K_\lambda(t,\cdot) * V_0(x) + \int_0^t \nabla K_\lambda(t-s,\cdot) * G(s,\cdot,V(s,\cdot))(x)\,ds.$$

We will see that Ψ_T is well defined.

Step 1: the first term of $\Psi_T(v)$ belongs to E_T.
By (B.4), the function $(t,x) \in]0,\infty[\times\mathbb{R}^N \to K_\lambda(t,\cdot) * V_0(x)$ is continuous in t, uniformly for $x \in \mathbb{R}^N$; since it is also continuous, for each $t \in]0,\infty[$, with respect to x (it is the convolution of an integrable function and a bounded function), we deduce that it is continuous with respect to its both variables. Young's inequality also gives, with (3.16),

$$||K_\lambda(t,\cdot) * V_0||_{L^\infty(\mathbb{R}^N)^N} \le ||V_0||_{L^\infty(\mathbb{R}^N)^N} \tag{B.6}$$

By Lemma 5.1 and (3.16), we see that $\partial_x(K_\lambda(t,\cdot) * V_0)(x) = \partial_x K_\lambda(t,\cdot) * V_0(x)$. For all $t_0 > 0$ and $t > 0$, we have $\partial_x K_\lambda(t_0+t,\cdot) = \partial_x(K_\lambda(t,\cdot) * K_\lambda(t_0,\cdot)) = K_\lambda(t,\cdot) * \partial_x K_\lambda(t_0,\cdot)$ (this is still Lemma 5.1, since $K_\lambda(t,\cdot)$ is bounded as the Fourier transform of an integrable function). Hence, $\partial_x(K_\lambda(t_0+\cdot,\cdot) * V_0) = K_\lambda(\cdot,\cdot) * (\partial_x K_\lambda(t_0,\cdot) * V_0)$ is continuous on $]0,\infty[\times\mathbb{R}^N$ (this is proved as at the beginning of the step, replacing V_0 by $\partial_x K_\lambda(t_0,\cdot) * V_0 \in L^\infty(\mathbb{R}^N)$). Since this is true for all $t_0 > 0$, we deduce that $(t,x) \to \partial_x(K_\lambda(t,\cdot) * V_0)(x)$ is continuous on $]0,\infty[\times\mathbb{R}^N$. With (3.16), we find

$$||\partial_x(K_\lambda(t,\cdot) * V_0)||_{L^\infty(\mathbb{R}^N)^{N^2}} \le \mathcal{K}t^{-1/\lambda}||V_0||_{L^\infty(\mathbb{R}^N)^N}. \tag{B.7}$$

This proves that $(t,x) \in]0,T[\times\mathbb{R} \to K_\lambda(t,\cdot) * V_0(x)$ belongs to E_T.

Step 2: if $V \in E_T$, the second term of $\Psi_T(V)$ belongs to E_T.
Define

$$Z(t,x) = \int_0^t \nabla K_\lambda(t-s,\cdot) * G(s,\cdot,V(s,\cdot))(x)\,ds = \int_0^t \int_{\mathbb{R}^N} \nabla K_\lambda(t-s,y)G(s,x-y,V(s,x-y))\,ds\,dy.$$

If we let $f(t,x) = \nabla K_\lambda(t,x)\mathbf{1}_{]0,T[}(t) \in L^1(\mathbb{R}\times\mathbb{R}^N)$ (see (3.16)) and $g(t,x) = G(t,x,V(t,x))\mathbf{1}_{]0,T[}(t) \in L^\infty(\mathbb{R}\times\mathbb{R}^N)$ (see the hypotheses on G), then $Z = f\star g$ on $]0,T[\times\mathbb{R}^N$, where $\star$ is the convolution product in $\mathbb{R}\times\mathbb{R}^N$. Since $f\star g$ is continuous (it is the convolution of an integrable function and a bounded function), we deduce that Z is continuous on $]0,T[\times\mathbb{R}^N$ and that

$$||Z||_{L^\infty(]0,T[\times\mathbb{R}^N)^N} \le ||f||_{L^1}||g||_{L^\infty} \le \mathcal{K}\frac{T^{1-\frac{1}{\lambda}}}{1-\frac{1}{\lambda}}||G(\cdot,\cdot,V(\cdot,\cdot))||_{L^\infty(]0,T[\times\mathbb{R}^N)}. \tag{B.8}$$

Let $t\in]0,T[$ and $s\in]0,t[$. The function $x\in\mathbb{R}^N \to G(s,x,V(s,x))$ is in $C^1_b(\mathbb{R}^N)$. We can thus differentiate under the integral sign to see that $\nabla K_\lambda(t-s,\cdot)*G(s,\cdot,V(s,\cdot))$ is C^1 with derivative $\nabla K_\lambda(t-s,\cdot)*(\partial_x G(s,\cdot,V(s,\cdot)) + \partial_\zeta G(s,\cdot,v(s,\cdot))\partial_x V(s,\cdot))$. Moreover, for all $x\in\mathbb{R}^N$,

$$\begin{aligned}&|\nabla K_\lambda(t-s,\cdot)*(\partial_x G(s,\cdot,V(s,\cdot)) + \partial_\zeta G(s,\cdot,V(s,\cdot))\partial_x V(s,\cdot))(x)|\\ \le\ & \frac{\mathcal{K}||\partial_x G(\cdot,\cdot,V(\cdot,\cdot))||_\infty}{(t-s)^{1/\lambda}} + \frac{\mathcal{K}||\partial_\zeta G(\cdot,\cdot,V(\cdot,\cdot))||_\infty||V||_{E_T}}{s^{1/\lambda}(t-s)^{1/\lambda}}.\end{aligned} \tag{B.9}$$

This last function is integrable with respect to $s\in]0,t[$, and we can thus apply the theorem of derivation under the integral sign to see that

$$\partial_x Z(t,x) = \int_0^t \nabla K_\lambda(t-s,\cdot)*(\partial_x G(s,\cdot,V(s,\cdot)) + \partial_\zeta G(s,\cdot,v(s,\cdot))\partial_x V(s,\cdot))(x)\,ds. \tag{B.10}$$

If $\partial_x V$ was bounded, the continuity of $\partial_x Z$ would be as simple as the continuity of Z. We thus approximate $\partial_x V$ by bounded functions to conclude. Take $0<\delta<T$ and define $W_\delta \in L^\infty(]0,T[\times\mathbb{R}^N)^N$ by

$$W_\delta(t,x) = \partial_x G(t,x,V(t,x)) + \partial_\zeta G(t,x,V(t,x))\partial_x V(t,x)\mathbf{1}_{[\delta,T[}(t).$$

Denoting $A_\delta(t,x) = \int_0^t \nabla K_\lambda(t-s,\cdot)*W_\delta(s,\cdot)(x)\,ds$, (B.10) allows to see that, for all $t_0\in]0,T[$, $A_\delta \to \partial_x Z$ uniformly on $[t_0,T[\times\mathbb{R}^N$ as $\delta\to 0$; since A_δ is continuous on $]0,T[\times\mathbb{R}^N$ (same proof as the continuity of Z, since $\partial_x V(t,x)\mathbf{1}_{[\delta,T[}(t)$ is bounded), we deduce that $\partial_x Z$ is continuous on $]0,T[\times\mathbb{R}$. Moreover, by (B.10) and (B.9) and the change of variable $s=t\tau$ in the integrals on $]0,t[$, we have, for all $(t,x)\in]0,T[\times\mathbb{R}^N$,

$$|\partial_x Z(t,x)| \le C_0\mathcal{K}\left(||\partial_x G(\cdot,\cdot,V(\cdot,\cdot))||_\infty t^{1-\frac{1}{\lambda}} + ||\partial_\zeta G(\cdot,\cdot,V(\cdot,\cdot))||_\infty||V||_{E_T}t^{1-\frac{2}{\lambda}}\right) \tag{B.11}$$

where $C_0 = \max(\int_0^1(1-\tau)^{-1/\lambda}\,d\tau, \int_0^1\tau^{-1/\lambda}(1-\tau)^{-1/\lambda}\,d\tau)$, which proves that $Z\in E_T$.
If V_0 is bounded by R_0 and V is bounded by R, the properties of G and (B.6), (B.7), (B.8) and (B.11) give

$$||\Psi_T(V)||_{E_T} \le R_0 + \mathcal{K}R_0 + \frac{\mathcal{K}}{1-\frac{1}{\lambda}}T^{1-\frac{1}{\lambda}}\omega(R) + C_0\mathcal{K}\omega(R)\left(T + T^{1-\frac{1}{\lambda}}||V||_{E_T}\right). \tag{B.12}$$

Step 3: fixed point.
Let $R = (2+\mathcal{K})R_0 > (1+\mathcal{K})R_0$. We denote, for $T>0$, $B_T(R)$ the closed ball in E_T of center 0 and radius R. Let $T_0>0$ be such that

$$R_0 + \mathcal{K}R_0 + \frac{\mathcal{K}}{1-\frac{1}{\lambda}}T_0^{1-\frac{1}{\lambda}}\omega(R) + C_0\mathcal{K}\omega(R)\left(T_0 + T_0^{1-\frac{1}{\lambda}}R\right) \le R \tag{B.13}$$

$$\mathcal{K}\frac{T_0^{1-\frac{1}{\lambda}}}{1-\frac{1}{\lambda}}\omega(R) + \mathcal{K}\omega(R)\left(\frac{T_0}{1-\frac{1}{\lambda}} + C_0T_0^{1-\frac{1}{\lambda}}R + C_0T_0^{1-\frac{1}{\lambda}}\right) < 1 \tag{B.14}$$

(by definition of R, such a T_0 exists and only depends on (R_0,ω)).

Let $T = \inf(S, T_0)$. Take $V_0 \in L^\infty(\mathbb{R}^N)^N$ bounded by R_0. Thanks to (B.12) and (B.13), Ψ_T sends $B_T(R)$ into $B_T(R)$. Let $(U, V) \in B_T(R)$. U and V are bounded by R and we have thus, for all $(t, x) \in]0, T[\times\mathbb{R}^N$, by (3.16) and the properties of G,

$$|\Psi_T(U)(t,x) - \Psi_T(V)(t,x)| \leq \mathcal{K}\frac{T^{1-\frac{1}{\lambda}}}{1-\frac{1}{\lambda}}\omega(R)||U-V||_\infty. \tag{B.15}$$

By (B.10) and the properties of G, we also have, for all $(t, x) \in]0, T[\times\mathbb{R}^N$,

$$\begin{aligned} & t^{1/\lambda}|\partial_x\Psi_T(U)(t,x) - \partial_x\Psi_T(V)(t,x)| \\ \leq \quad & \mathcal{K}\omega(R)\left(\frac{T}{1-\frac{1}{\lambda}} + C_0T^{1-\frac{1}{\lambda}}||U||_{E_T} + C_0T^{1-\frac{1}{\lambda}}\right)||U-V||_{E_T}. \end{aligned} \tag{B.16}$$

The properties (B.15), (B.16) and (B.14) ensure that Ψ_T is contracting on $B_T(R)$. Therefore, Ψ_T has a unique fixed point V in $B_T(R)$; V is a continuous and bounded solution to (B.5) such that $\partial_x V$ exists and is continuous on $]0, T[\times\mathbb{R}^N$. Moreover, since $V \in B_T(R)$, we have, for all $a \in]0, T[$ and all $(t, x) \in]a, T[\times\mathbb{R}^N$, $|\partial_x V(t, x)| \leq t^{-1/\lambda}||V||_{E_T} \leq a^{-1/\lambda}R$, which is the estimate on $\partial_x V$ stated in the proposition.
The inequalities (B.15) and (B.14) ensure that Ψ_T is contracting on the ball in $C_b(]0, T[\times\mathbb{R})$ of center 0 and radius R (notice that (B.15) only demands that U and V be bounded by R, nothing on their gradients). Thus, Ψ_T can have only one fixed point in this ball, which is the uniqueness result of the proposition. ■

We prove now the spatial regularity of the weak solutions.

Theorem B.2 *Let $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ and F satisfy (3.2). If u is a weak solution to (3.1) on $[0, T]$, then u is infinitely derivable in space and all its spatial derivatives are in $C(]0, T[\times\mathbb{R}^N)$. Moreover, if $a \in]0, T[$ and $M \geq ||u||_{L^\infty(]0,T[\times\mathbb{R}^N)} + ||\nabla u||_{L^\infty(]0,T[\times\mathbb{R}^N)}$, then all the spatial derivatives of u are bounded on $]a, T[\times\mathbb{R}^N$ by constants only depending on a, M and the constants in Hypothesis (3.2).*

In the following proof, we do not explicitely indicate the dependence on the constants in Hypothesis (3.2), but any real number which depends on M also depends on these constants.

Proof of Theorem B.2
Notice first that u is continuous on $]0, T[\times\mathbb{R}^N$. This can be seen on (3.15) by the same technique as in the first two steps of the proof of Proposition B.1 (with $\nabla K_\lambda(t-s,\cdot)$ replaced by $K_\lambda(t-s,\cdot)$).

We now prove by induction on n that: for all $\alpha \in \mathbb{N}^N$ of length less or equal to n, $\partial_x^\alpha\nabla u$ exists, is continuous on $]0, T[\times\mathbb{R}^N$ and bounded on $]a, T[\times\mathbb{R}^N$ by $C(a, M, n)$ (for all $a \in]0, T[$), that, for all $t_0 \in]0, T[$ and $t \in]0, T - t_0[$,

$$\begin{aligned} \partial_x^\alpha\nabla u(t_0+t,x) \quad = \quad & K_\lambda(t,\cdot) * \partial_x^\alpha u(t_0,\cdot) \\ & + \int_0^t \nabla K_\lambda(t-s,\cdot) * \partial_x^\alpha(F(t_0+s,\cdot,u(t_0+s,\cdot),\nabla u(t_0+s,\cdot)))(x)\,ds \end{aligned} \tag{B.17}$$

and that

$$\begin{aligned} \partial_x^\alpha(F(t,\cdot,u(t,\cdot),\nabla u(t,\cdot)))(x) \quad = \quad & (1-\delta_{n,0})\partial_\xi F(t,x,u(t,x),\nabla u(t,x))\partial_x^\alpha\nabla u(t,x) \\ & +\delta_{n,0}F(t,x,u(t,x),\nabla u(t,x)) + F_n(t,x,(\partial_x^\beta u)_{|\beta|\leq n}) \end{aligned} \tag{B.18}$$

where $\delta_{n,0}$ is Krönecker's symbol, $F_0 = 0$ and, for $n \geq 1$, F_n is regular and has all its derivatives bounded on $]0, T[\times\mathbb{R}^N \times B_R$ for all $R > 0$.

The case $n=0$ is quite simple. Indeed, we easily see that $\nabla(K_\lambda(t,\cdot)*u_0)=K_\lambda(t,\cdot)*\nabla u_0$ (since u_0 is only derivable a.e., we cannot directly use a derivation under the integral sign but, since it is Lipschitz continuous, we can use the definition of the derivative and the dominated convergence theorem); thanks to Lemma 5.1 and (3.16), we can then derivate the second term of (3.15) under the integral sign, and we find (B.17) for $\alpha=0$ and $t_0=0$ (which gives the continuity of ∇u as in Steps 1 and 2 in the proof of Proposition B.1). The deduction of this formula for any $t_0\in]0,T[$ is made exactly as in the proof of Lemma B.1. We also have the simple bound $||\nabla u||_{L^\infty(]0,T[\times\mathbb{R}^N)}\le M$ and (B.18) with $F_0=0$.

We now assume the induction hypothesis up to rank n and we prove it at the rank $n+1$; let α' of length $n+1$ and write $\partial_x^{\alpha'}=\partial_x^\alpha\partial_i$ with α of length n. Let $b_0\in]0,T[$ and $b\in]b_0,T[$; define

$$\begin{aligned}G(t,x,\zeta) &= (1-\delta_{n,0})\partial_\xi F(b+t,x,u(b+t,x),\nabla u(b+t,x))\zeta\\ &\quad+\delta_{n,0}F(b+t,x,u(b+t,x),\zeta)+F_n(b+t,x,(\partial_x^\beta u(b+t,x))_{|\beta|\le n})\end{aligned}$$

By induction hypothesis, G satisfies the hypotheses of Proposition B.1 with ω only depending on b_0, M and n (if $n=0$, we have an estimate on ∇u and, if $n\ge1$, we have estimates on $\partial_x\nabla u$). We also have $||\partial_x^\alpha\nabla u||_{L^\infty(]b_0,T[\times\mathbb{R}^N)}\le R_0$ with R_0 only depending on b_0, M and n. Take T_0 only depending on (R_0,ω) (that is to say on b_0, M and n) given by Proposition B.1.
The function $\partial_x^\alpha\nabla u(b+\cdot,\cdot)$ is continuous, bounded by $R_0\le(2+\mathcal{K})R_0$ and, by (B.17) at rank n, satisfies (B.5) with $V_0=\partial_x^\alpha u(b,\cdot)$ (which is bounded by R_0) and the preceding G. Hence, by Proposition B.1, $\partial_i\partial_x^\alpha\nabla u$ exists, is continuous and bounded by $(2+\mathcal{K})R_0a^{-1/\lambda}$ on $]b+a,\inf(T,b+T_0)[\times\mathbb{R}^N$; this is true for all $b\in]b_0,T[$ and all $a\in]0,\inf(T-b,T_0)[$. Since T_0 does not depend on a or b, taking $t_0\in]0,T[$, $b_0=t_0/2$ and $a=\inf(t_0/2,T_0/2)<T-b_0$, we notice that the intervals $\{]b+a,\inf(T,b+T_0)[,\ b\in]b_0,T-a[\}$ cover $]b_0+a,T[\supset]t_0,T[$ and we deduce that, for all $i\in[1,N]$, $\partial_i\partial_x^\alpha\nabla u$ has the regularity and satisfies the estimates we wanted to obtain.
We deduce then Formula (B.18) at rank $n+1$ by derivating this formula at rank n; we also prove (B.17) at rank $n+1$ by derivating it, under the integral sign, at rank n. These derivations are possible thanks to the regularity we have just proved on $\partial_i\partial_x^\alpha\nabla u$. ■

B.2.2 Temporal regularity

Proposition B.2 *If $\varphi\in C_b^2(\mathbb{R}^N)$ then, for all $x\in\mathbb{R}^N$, $t\in]0,\infty[\to K_\lambda(t,\cdot)*\varphi(x)$ is derivable and, for all $t>0$, we have $\frac{d}{dt}(K_\lambda(\cdot,\cdot)*\varphi(x))(t)=-g_\lambda[K_\lambda(t,\cdot)*\varphi](x)$.*

Proof of Proposition B.2
We notice that $g_\lambda[K_\lambda(t,\cdot)*\varphi]$ makes sense since $K_\lambda(t,\cdot)*\varphi\in C_b^2(\mathbb{R}^N)$.
Suppose first that $\varphi\in\mathcal{S}(\mathbb{R}^N)$. The functions $K_\lambda(t,\cdot)$ and φ are integrable on $\mathbb{R}^N$, so, by definition of K_λ, $K_\lambda(t,\cdot)*\varphi=\mathcal{F}^{-1}(e^{-t|\cdot|^\lambda}\mathcal{F}(\varphi))$. A derivation under the integral sign shows that $t\in]0,\infty[\to K_\lambda(t,\cdot)*\varphi(x)$ is C^1 and that, for all $t>0$ and all $x\in\mathbb{R}^N$,

$$\frac{d}{dt}(K_\lambda(\cdot,\cdot)*\varphi(x))(t)=-\mathcal{F}^{-1}(|\cdot|^\lambda e^{-t|\cdot|^\lambda}\mathcal{F}(\varphi))(x)=-\mathcal{F}^{-1}(|\cdot|^\lambda\mathcal{F}(K_\lambda(t,\cdot)*\varphi))(x)\tag{B.19}$$

Since $\varphi\in\mathcal{S}(\mathbb{R}^N)$ and $K_\lambda(t,\cdot)\in L^1(\mathbb{R}^N)$, we have $K_\lambda(t,\cdot)*\varphi\in W^{\infty,1}(\mathbb{R}^N)$; there exists thus $\psi_n\in\mathcal{S}(\mathbb{R}^N)$ such that $\psi_n\to K_\lambda(t,\cdot)*\varphi$ in $W^{\infty,1}(\mathbb{R}^N)$; it is then obvious that, for all $k\ge0$,

$$|\cdot|^{2k}\mathcal{F}(\psi_n)=(-4\pi^2)^{-k}\mathcal{F}(\Delta^k\psi_n)\longrightarrow(-4\pi^2)^{-k}\mathcal{F}(\Delta^k(K_\lambda(t,\cdot)*\varphi))=|\cdot|^{2k}\mathcal{F}(K_\lambda(t,\cdot)*\varphi)$$

in $L^\infty(\mathbb{R}^N)$. We deduce that $|\cdot|^\lambda\mathcal{F}(\psi_n)\to|\cdot|^\lambda\mathcal{F}(K_\lambda(t,\cdot)*\varphi)$ in $L^1(\mathbb{R}^N)$; thus, applying (1.1) to ψ_n and passing to the limit $n\to\infty$, we see that it also holds with $K_\lambda(t,\cdot)*\varphi$ intead of φ, and (B.19) concludes the proof for $\varphi\in\mathcal{S}(\mathbb{R}^N)$.

Take now $\varphi\in C_b^2(\mathbb{R}^N)$. We can find a sequence $(\varphi_n)_{n\ge1}\in\mathcal{S}(\mathbb{R}^N)$ whose derivatives up to order 2 are bounded in $L^\infty(\mathbb{R}^N)$ and locally uniformly converge to the corresponding derivatives of φ. Fix $x_0\in\mathbb{R}^N$

and define $\gamma_n(t) = K_\lambda(t,\cdot) * \varphi_n(x_0)$. Since $K_\lambda(t,\cdot)$ is integrable, we have $\gamma_n(t) \to K_\lambda(t,\cdot) * \varphi(x_0)$ for all $t > 0$. We know that γ is derivable with $\gamma_n'(t) = -g_\lambda[K_\lambda(t,\cdot) * \varphi_n](x_0)$.
The sequence $(K_\lambda(t,\cdot) * \varphi_n)_{t>0\,,\,n\geq 1}$ is bounded in $L^\infty(\mathbb{R}^N)$ (see (3.16)). Let us study the convergence of $\partial_x^2(K_\lambda(t,\cdot) * \varphi_n) = K_\lambda(t,\cdot) * D^2\varphi_n$ to $\partial_x^2(K_\lambda(t,\cdot) * \varphi) = K_\lambda(t,\cdot) * D^2\varphi$; we write, for $|x| \leq R$ and $r > 0$,

$$\begin{aligned}
&|K_\lambda(t,\cdot) * D^2\varphi_n(x) - K_\lambda(t,\cdot) * D^2\varphi(x)| \\
&\qquad \leq \int_{B_r} K_\lambda(t,y)|D^2\varphi_n(x-y) - D^2\varphi(x-y)|\,dy \\
&\qquad\quad + \left(||D^2\varphi||_{L^\infty(\mathbb{R}^N)} + \sup_{n\geq 1}||D^2\varphi_n||_{L^\infty(\mathbb{R}^N)}\right)\int_{\mathbb{R}^N\setminus B_r} K_\lambda(t,y)\,dy \\
&\qquad \leq ||D^2\varphi_n - D^2\varphi||_{L^\infty(B_{r+R})} \qquad\qquad (B.20)\\
&\qquad\quad + \left(||D^2\varphi||_{L^\infty(\mathbb{R}^N)} + \sup_{n\geq 1}||D^2\varphi_n||_{L^\infty(\mathbb{R}^N)}\right)\int_{\mathbb{R}^N\setminus B_r} K_\lambda(t,y)\,dy. \qquad (B.21)
\end{aligned}$$

For A compact subset of $]0,\infty[$, $(K_\lambda(t,\cdot))_{t\in A}$ is compact in $L^1(\mathbb{R}^N)$ (see (B.4)) and thus (B.21) is small, uniformly for $t \in A$, if r is large; with r fixed, (B.20) is small if n is large. This shows that $\partial_x^2(K_\lambda(t,\cdot) * \varphi_n) \to \partial_x^2(K_\lambda(t,\cdot) * \varphi)$ as $n \to \infty$, locally uniformly with respect to $t \in]0,\infty[$ and $x \in \mathbb{R}^N$. Hence, as in the proof of Proposition 2.1, we see that $g_\lambda[K_\lambda(t,\cdot) * \varphi_n] \to g_\lambda[K_\lambda(t,\cdot) * \varphi]$ locally uniformly with respect to $x \in \mathbb{R}^N$ and $t \in]0,\infty[$ (when reproducing the proof of this proposition with $K_\lambda(t,\cdot) * \varphi_n$, all the bounds do not depend on t provided that it is in a compact subset of $]0,\infty[$).
In particular, this proves that $\gamma_n'(t) \to -g_\lambda[K_\lambda(t,\cdot) * \varphi](x_0)$ locally uniformly for $t \in]0,\infty[$; we deduce that $K_\lambda(t,\cdot) * \varphi(x_0)$ is derivable with respect to $t \in]0,\infty[$, with derivative $-g_\lambda[K_\lambda(t,\cdot) * \varphi](x_0)$. ■

Proposition B.3 *If $\varphi :]0,T[\times\mathbb{R}^N \to \mathbb{R}$ has its spatial derivatives up to order 2 continuous and bounded then, for all $x \in \mathbb{R}^N$,*

$$t \in]0,T[\to \int_0^t K_\lambda(t-s,\cdot) * \varphi(s,\cdot)(x)\,ds$$

is derivable and

$$\frac{d}{dt}\int_0^t K_\lambda(t-s,\cdot) * \varphi(s,\cdot)(x)\,ds = \varphi(t,x) - g_\lambda\left[\int_0^t K_\lambda(t-s,\cdot) * \varphi(s,\cdot)\,ds\right](x).$$

Proof of Proposition B.3
Let $\delta_0 \in]0,T[$ and $\delta \in]0,\delta_0[$. For $t \in]\delta_0,T[$, define $Z_\delta(t) = \int_0^{t-\delta} K_\lambda(t-s,\cdot) * \varphi(s,\cdot)(x)\,ds$. We also define $Z(t) = \int_0^t K_\lambda(t-s,\cdot) * \varphi(s,\cdot)(x)\,ds$. We have $|Z_\delta(t) - Z(t)| \leq \delta||\varphi||_{L^\infty(]0,T[\times\mathbb{R}^N)}\,ds$, so that $Z_\delta \to Z$ uniformly on $]\delta_0,T[$ as $\delta \to 0$.
The function $\phi : (t,s) \in \{(t',s') \in]\delta,T[\times]0,T[\,|\, s' < t' - \delta/2\} \to K_\lambda(t-s,\cdot) * \varphi(s,\cdot)(x)$ is continuous (use (B.4)) and bounded. By Proposition B.2, ϕ is derivable with respect to t and $\partial_t\phi(t,s) = -g_\lambda[K_\lambda(t-s,\cdot) * \varphi(s,\cdot)](x)$. By (3.16) and the hypothesis on φ,

$$\begin{aligned}
&(s,x) \in]0,t[\times\mathbb{R}^N \to K_\lambda(t-s,\cdot) * \varphi(s,\cdot) \text{ and its spatial derivatives up to order 2} \\
&\qquad \text{are continuous and bounded on }]0,t[\times\mathbb{R}^N, \text{ independently of } t \in]0,T[
\end{aligned} \qquad (B.22)$$

(to prove the continuity, we write $||K_\lambda(t'-s',\cdot) * \varphi(s',\cdot) - K_\lambda(t-s,\cdot) * \varphi(s,\cdot)||_{L^\infty(\mathbb{R}^N)} \leq ||K_\lambda(t'-s',\cdot) - K_\lambda(t-s,\cdot)||_{L^1(\mathbb{R}^N)}||\varphi||_\infty + ||K_\lambda(t-s,\cdot) * (\varphi(s',\cdot) - \varphi(s,\cdot))||_{L^\infty(\mathbb{R}^N)}$, then we use (B.4) on the first term and the technique in the proof of Proposition B.2 on the second term). Hence, by continuity under the integral sign on (2.1), $\partial_t\phi$ is continuous and bounded. These properties show that Z_δ is derivable on $]\delta_0,T[$ and that

$$Z_\delta'(t) = K_\lambda(\delta,\cdot) * \varphi(t-\delta,\cdot)(x) - \int_0^{t-\delta} g_\lambda[K_\lambda(t-s,\cdot) * \varphi(s,\cdot)](x)\,ds.$$

Since φ is continuous bounded and $(K_\lambda(\delta,\cdot))_{\delta\to 0}$ is an approximate unit, we see that, as $\delta\to 0$, $K_\lambda(\delta,\cdot)*\varphi(t-\delta,\cdot)(x)\to\varphi(t,x)$ uniformly with respect to $t\in]\delta_0,T[$. (B.22) and (2.1) show that the function $(s,x)\in]0,t[\times\mathbb{R}^N\to g_\lambda[K_\lambda(t-s,\cdot)*\varphi(s,\cdot)](x)$ is bounded independently of $t\in]\delta_0,T[$. We deduce that Z'_δ uniformly converges on $]\delta_0,T[$ to

$$Y : t\in]0,T[\to\varphi(t,x)-\int_0^t g_\lambda[K_\lambda(t-s,\cdot)*\varphi(s,\cdot)](x)\,ds.$$

Since this is true for all $\delta_0\in]0,T[$, this proves that Z is derivable on $]0,T[$ with $Z'=Y$. Thanks to (B.22), Fubini's Theorem gives

$$\int_0^t g_\lambda[K_\lambda(t-s,\cdot)*\varphi(s,\cdot)](x)\,ds = g_\lambda\left[\int_0^t K_\lambda(t-s,\cdot)*\varphi(s,\cdot)\,ds\right](x)\,,$$

which concludes the proof of the proposition. ■

We can now conclude the proof of ii) in Theorem 3.2. We already know that u is regular in space (with estimates on its spatial derivatives far from $t=0$). By Lemma B.1, for all $a\in]0,T[$, we have, on $]0,T-t_0[\times\mathbb{R}^N$,

$$u(a+t,x)=K_\lambda(t,\cdot)*u(a,\cdot)(x)+\int_0^t K_\lambda(t-s,\cdot)*F(a+s,\cdot,u(a+s,\cdot),\nabla u(a+s,\cdot))(x)\,ds.$$

Hence, by the regularity already known on u, Propositions B.2 and B.3 show that u is derivable with respect to t and that

$$\begin{aligned}\partial_t u(a+t,x) &= -g_\lambda[K_\lambda(t,\cdot)*u(a,\cdot)](x)+F(a+t,x,u(a+t,x),\nabla u(a+t,x))\\ &\quad -g_\lambda\left[\int_0^t K_\lambda(t-s,\cdot)*F(a+s,\cdot,u(a+s,\cdot),\nabla u(a+s,\cdot))\,ds\right](x)\\ &= F(a+t,x,u(a+t,x),\nabla u(a+t,x))-g_\lambda[u(a+t,\cdot)](x).\end{aligned}$$

This proves (3.7). Since all the spatial derivatives of $u(a+\cdot,\cdot)$ are continuous and bounded on $]0,T-a[\times\mathbb{R}^N$, some derivations under the integral sign on (2.1) show that the same holds for $g_\lambda[u(a+\cdot,\cdot)]$; an induction allows then to deduce (3.6).

To obtain (3.8), we use (3.15) to write, since u and ∇u are bounded,

$$||u(t,\cdot)-K_\lambda(t,\cdot)*u_0||_{L^\infty(\mathbb{R}^N)}\le t||F(\cdot,\cdot,u,\nabla u)||_{L^\infty(]0,T[\times\mathbb{R}^N)}\to 0\quad\text{as}\quad t\to 0\,,$$

and we conclude by noticing that, since u_0 is bounded and uniformly continuous, $K_\lambda(t,\cdot)*u_0\to u_0$ uniformly on $\mathbb{R}^N$ as $t\to 0$ ($(K_\lambda(t,\cdot))_{t\to 0}$ is an approximate unit).

B.3 iii) : Local existence

Theorem B.3 *Let $u_0\in W^{1,\infty}(\mathbb{R}^N)$ and F satisfy (3.2). If $M\ge||u_0||_{W^{1,\infty}(\mathbb{R}^N)}$ then there exists $T>0$, only depending on M and the constants in Hypothesis (3.2), such that a weak solution to (3.1) on $[0,T]$ exists and satisfies: $||u||_{L^\infty(]0,T[\times\mathbb{R}^N)}+||\nabla u||_{L^\infty(]0,T[\times\mathbb{R}^N)}\le M+1$.*

Proof of Theorem B.3
Let $T\le 1$, $E_T=\{u\in L^\infty(]0,T[\times\mathbb{R}^N)\mid\nabla u\in L^\infty(]0,T[\times\mathbb{R}^N)\}$ and $\Psi_T:E\to E$ defined by

$$\Psi_T(u)(t,x)=K_\lambda(t,\cdot)*u_0(x)+\int_0^t K_\lambda(t-s,\cdot)*F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))(x)\,ds.$$

We simultaneously prove that Ψ_T is well defined and obtain estimates on this operator.

Since $u_0 \in L^\infty(\mathbb{R}^N)$, (3.16) shows that the function $(t,x) \to K_\lambda(t,\cdot) * u_0(x)$ belongs to $L^\infty(]0,T[\times\mathbb{R}^N)$ and that $||K_\lambda(t,\cdot) * u_0||_{L^\infty(\mathbb{R}^N)} \leq ||u_0||_{L^\infty(\mathbb{R}^N)}$. If $u \in E_T$ and $||u||_{E_T} \leq R$, then $F(\cdot,\cdot,u(s,\cdot),\nabla u(s,\cdot))$ is bounded by $C_{1,R,0}$ (the constant appearing in (3.2) — recall that $T \leq 1$); hence, still using (3.16), we see that $\Psi_T(u) \in L^\infty(]0,T[\times\mathbb{R}^N)$ and that, for $u \in E_T$ bounded in this space by R,

$$||\Psi_T(u)||_{L^\infty(]0,T[\times\mathbb{R}^N)} \leq ||u_0||_{L^\infty(\mathbb{R}^N)} + TC_{1,R,0}. \tag{B.23}$$

Since $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, we have $\nabla(K_\lambda(t,\cdot) * u_0) = K_\lambda(t,\cdot) * \nabla u_0 \in L^\infty(\mathbb{R}^N)$ and

$$||\nabla(K_\lambda(t,\cdot) * u_0)||_{L^\infty(\mathbb{R}^N)} \leq ||\nabla u_0||_{L^\infty(\mathbb{R}^N)}.$$

As in the proof of Proposition 3.2, Lemma 5.1 and (3.16) show that

$$\nabla\left(\int_0^t K_\lambda(t-s,\cdot) * F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))\,ds\right) = \int_0^t \nabla K_\lambda(t-s,\cdot) * F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))\,ds.$$

Thus, $\Psi_T(u) \in E_T$ and, if $||u||_{E_T} \leq R$,

$$||\nabla\Psi_T(u)||_{L^\infty(]0,T[\times\mathbb{R}^N)} \leq ||\nabla u_0||_{L^\infty(\mathbb{R}^N)} + \frac{T^{1-\frac{1}{\lambda}}}{1-\frac{1}{\lambda}} C_{1,R,0}. \tag{B.24}$$

If $(u,v) \in E_T$ are bounded in this space by R, we have, on $]0,T[\times\mathbb{R}^N$,

$$|F(\cdot,\cdot,u,\nabla u) - F(\cdot,\cdot,v,\nabla v)| \leq C_{1,R,1}|u-v| + C_{1,R,1}|\nabla u - \nabla v| \leq 2C_{1,R,1}||u-v||_{E_T}$$

Hence, if $||u||_{E_T} \leq R$ and $||v||_{E_T} \leq R$,

$$||\Psi_T(u) - \Psi_T(v)||_{E_T} \leq 2C_{1,R,1}||u-v||_{E_T}T + 2C_{1,R,1}||u-v||_{E_T}\frac{T^{1-\frac{1}{\lambda}}}{1-\frac{1}{\lambda}}. \tag{B.25}$$

Let $M \geq ||u_0||_{W^{1,\infty}(\mathbb{R}^N)}$ and choose $T > 0$ such that

$$M + TC_{1,M+1,0} + \frac{T^{1-\frac{1}{\lambda}}}{1-\frac{1}{\lambda}} C_{1,M+1,0} \leq M+1 \tag{B.26}$$

$$2C_{1,M+1,1}T + 2C_{1,M+1,1}\frac{T^{1-\frac{1}{\lambda}}}{1-\frac{1}{\lambda}} < 1. \tag{B.27}$$

Such a choice of T only depends on M and the constants in (3.2). Let $B_{E_T}(M+1)$ be the ball in E_T of center 0 and radius $M+1$. By (B.23), (B.24) and (B.26), Ψ_T sends $B_{E_T}(M+1)$ into itself; by (B.25) and (B.27), it is strictly contracting on $B_{E_T}(M+1)$. Hence, Ψ_T has a fixed point in $B_{E_T}(M+1)$, which concludes the proof of the theorem. ■

C Construction de la solution globale, une fois les estimations prouvées

Une fois les estimations (3.9) et (3.10) obtenues pour les solutions faibles sur $[0,T]$ de (3.1), la preuve de l'existence d'une solution sur $[0,\infty[$ est standard.
Soit $(u_i)_{i\in I}$ l'ensemble des solutions faibles de (3.1); chaque u_i est solution sur un certain intervalle $[0,T_i]$. Soit $\mathcal{T} = \sup_{i\in I} T_i$. Par i) dans le théorème 3.2, on a $u_i = u_j$ sur $]0,\inf(T_i,T_j)[\times\mathbb{R}^N$; on peut donc définir $u :]0,\mathcal{T}[\times\mathbb{R}^N \to \mathbb{R}$ par: pour tout $i \in I$, $u_{]0,T_i[\times\mathbb{R}^N} = u_i$. La fonction u est solution faible de (3.1) sur

$[0, T_i]$ pour tout $i \in I$ (et vérifie donc (3.6)—(3.8) sur chacun de ces intervalles); il suffit de montrer que $\mathcal{T} = +\infty$ pour conclure.
Supposons $\mathcal{T} < \infty$. On commence par constater que F satisfait (3.3) et (3.4) avec, au lieu de Λ_T et $\Gamma_{T,R}$,

$$\widetilde{\Lambda}_T(a) = \begin{cases} \Lambda_{\mathcal{T}}(a) & \text{si } T \leq \mathcal{T}, \\ \Lambda_T(a) & \text{si } T > \mathcal{T} \end{cases} \quad \text{et} \quad \widetilde{\Gamma}_{T,R}(a) = \begin{cases} \Gamma_{\mathcal{T},R}(a) & \text{si } T \leq \mathcal{T}, \\ \Gamma_{T,R}(a) & \text{si } T > \mathcal{T}. \end{cases}$$

Pour tout $i \in I$ et $t < T_i$, par (3.9) on a donc

$$||u(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \leq (\widetilde{\mathcal{L}}_{T_i})^{-1}\left(t + \widetilde{\mathcal{L}}_{T_i}(||u_0||_{L^\infty(\mathbb{R}^N)})\right)$$

avec $\widetilde{\mathcal{L}}_{T_i}(a) = \int_0^a \frac{1}{\widetilde{\Lambda}_{T_i}(b)}\,db = \int_0^a \frac{1}{\Lambda_{\mathcal{T}}(b)}\,db = \mathcal{L}_T(a)$ puisque $T_i \leq \mathcal{T}$. Ainsi, pour tout $i \in I$ et $t < T_i$,

$$||u(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \leq (\mathcal{L}_{\mathcal{T}})^{-1}\left(t + \mathcal{L}_{\mathcal{T}}(||u_0||_{L^\infty(\mathbb{R}^N)})\right) \leq (\mathcal{L}_{\mathcal{T}})^{-1}\left(\mathcal{T} + \mathcal{L}_{\mathcal{T}}(||u_0||_{L^\infty(\mathbb{R}^N)})\right) =: R_1.$$

Par le même raisonnement en partant de (3.10), et puisque $R_1 \geq ||u||_{L^\infty(]0,T_i[\times\mathbb{R}^N)}$ pour tout $i \in I$, on a, lorsque $i \in I$ et $t < T_i$,

$$||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \leq (\mathcal{G}_{\mathcal{T},R_1})^{-1}\left(\mathcal{T} + \mathcal{G}_{\mathcal{T},R_1}(||Du_0||_{L^\infty(\mathbb{R}^N)})\right) =: R_2.$$

Les fonctions $(F(\tau+\cdot,\cdot,\cdot,\cdot))_{\tau\in[0,\mathcal{T}]}$ vérifient (3.2) avec des constantes indépendantes de $\tau \in [0,\mathcal{T}]$ (il suffit de prendre les constantes correspondant à F et d'ajouter $\mathcal{T}$ à leur premier argument). Soit $T' > 0$ donné par iii) dans le théorème 3.2 pour ces constantes et $M = R_1 + R_2$. Prenons $t_0 \in]\mathcal{T} - T', \mathcal{T}[$; il existe $T_i > t_0$ et donc $||u(t_0,\cdot)||_{W^{1,\infty}(\mathbb{R}^N)} \leq M$. On peut alors construire une solution faible v à (3.1) sur $[0,T']$ avec $u(t_0,\cdot)$ comme condition initiale et $F(t_0+\cdot,\cdot,\cdot,\cdot)$ au lieu de F. Puisque $u(t_0+\cdot,\cdot)$ est solution faible de ce même problème sur $[0,\mathcal{T}-t_0[$ (voir le lemme B.1), on a $u(t_0+\cdot,\cdot) = v$ sur $]0,\mathcal{T}-t_0[$ et on peut donc définir $\widetilde{u} :]0, t_0+T'[\times\mathbb{R}^N \to \mathbb{R}$ par: $\widetilde{u} = u$ sur $]0,\mathcal{T}[\times\mathbb{R}^N$ et $\widetilde{u} = v(\cdot - t_0,\cdot)$ sur $]t_0, t_0+T'[\times\mathbb{R}^N$. Il est aisé de voir que $\widetilde{u}$ est une solution faible de (3.1) sur $[0, t_0+T']$: pour tout $t \in]t_0, t_0+T'[$, on écrit

$$\begin{aligned}
&\int_0^t K_\lambda(t-s,\cdot) * F(s,\cdot,\widetilde{u}(s,\cdot),\nabla\widetilde{u}(s,\cdot))\,ds \\
&\quad= \int_0^{t_0} K_\lambda(t-s,\cdot) * F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))\,ds \\
&\qquad+ \int_{t_0}^t K_\lambda(t-s,\cdot) * F(s,\cdot,v(s-t_0,\cdot),\nabla v(s-t_0,\cdot))\,ds \\
&\quad= \int_0^{t_0} K_\lambda(t-s,\cdot) * F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))\,ds \\
&\qquad+ \int_0^{t-t_0} K_\lambda(t-t_0-s,\cdot) * F(t_0+s,\cdot,v(s,\cdot),\nabla v(s,\cdot))\,ds \\
&\quad= \int_0^{t_0} K_\lambda(t-s,\cdot) * F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))\,ds + v(t-t_0) - K_\lambda(t-t_0,\cdot) * u(t_0,\cdot) \\
&\quad= \int_0^{t_0} K_\lambda(t-s,\cdot) * F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))\,ds + \widetilde{u}(t) - K_\lambda(t-t_0,\cdot) * u(t_0,\cdot)
\end{aligned}$$

et on utilise (B.1). Cette solution est donc un des u_i précédemment considérés, et on doit avoir $t_0+T' \leq \mathcal{T}$, ce qui est la contradiction recherchée.

D Notes sur la preuve du théorème 4.2

D.1 i): Unicité

Le pendant du lemme B.1 pour les solutions faibles de (4.1) est valide et se démontre de la même manière: la présence du terme ∇K_λ ne gène absolument pas, puisqu'il a suffisament de propriétés d'intégrabilité.

L'unicité de la solution faible se prouve alors comme le théorème B.1; c'est même en un sens plus facile (il n'est pas nécessaire d'estimer l'écart entre le gradient des solutions), même si la présence de ∇K_λ complique les choses d'un autre côté (l'estimation (B.2) sur $||u(t,\cdot) - v(t,\cdot)||_\infty$ ne fait pas intervenir qu'un terme en T_1, mais aussi un terme en $T_1^{1-1/\lambda}$).

D.2 ii): Régularité

Il faut commencer par citer un équivalent de la proposition B.1 lorsque l'on rajoute un terme de la forme $\int_0^t K_\lambda(t-s,\cdot) * H(s,\cdot,V(s,\cdot))(x)\,ds$ à (B.5). La preuve n'est pas plus compliquée que celle de la proposition originelle puisque ce terme ajouté est plus sympathique que le terme en ∇K_λ déjà présent dans (B.5).
A partir de ce résultat, on voit immediatement qu'une solution faible u de (4.1) est C_b^1 en espace dès que $t > 0$. On écrit alors (4.7) à partir d'un $t_0 > 0$ (grâce à l'équivalent du lemme B.1) et on a assez de régularité sur u pour transformer cette expression en

$$\begin{aligned} u(t_0+t,x) &= K_\lambda(t,\cdot) * u(t_0,\cdot)(x) \\ &\quad + \int_0^t K_\lambda(t-s,\cdot) * \Bigg[h(t_0+s,\cdot,u(t_0+s,\cdot)) - \sum_{i=1}^N \partial_{x_i} f_i(t_0+s,\cdot,u(t_0+s,\cdot)) \\ &\qquad\qquad - \partial_s f(t_0+s,\cdot,u(t_0+s,\cdot)) \cdot \nabla u(t_0+s,\cdot) \Bigg](x)\,ds \end{aligned}$$

La fonction $u(t_0+\cdot,\cdot)$ est donc solution faible d'un problème (3.1) avec $u_0 = u(t_0,\cdot) \in C_b^1(\mathbb{R}^N)$ et F qui vérifie (3.2). Le théorème 3.2 nous donne alors toute la régularié voulue sur u, et nous montrer qu'elle vérifie son EDP au sens classique.
Il reste à voir (4.6), ce qui est assez évident: on a, par (4.7) et en utilisant le fait que u est bornée,

$$||u(t,\cdot) - K_\lambda(t,\cdot) * u_0||_{L^\infty(\mathbb{R}^N)} \leq Ct^{1-\frac{1}{\lambda}} + Ct \to 0 \quad \text{lorsque } t \to 0\,,$$

et, puisque $(K_\lambda(t,\cdot))_{t\to 0}$ est une approximation de l'unité, $K_\lambda(t,\cdot) * u_0 \to u_0$ dans $L^\infty(\mathbb{R}^N)$ faible-$*$.

D.3 iii): Existence locale

La preuve est similaire à celle du théorème 3.2, sauf que l'on fait un point fixe sur (3.15) dans l'espace $L^\infty(]0,T[\times\mathbb{R}^N)$ pour T assez petit.

E Solutions de Viscosité

Cette partie correspond à l'extraction/adaptation des définitions et résultats de [17] nécessaires ici.

E.1 Sous-gradients

Définition E.1 *Soit $u :]0,T[\times\mathbb{R}^N \to \mathbb{R}$ semi-continue supérieurement. On dit que $(\alpha,p) \in \partial^P u(t,x)$ (le sur-gradient de u en (t,x)) s'il existe $r > 0$ et $\sigma > 0$ tels que, pour $y \in B_r(x)$ et $|s-t| \leq r$,*

$$u(s,y) \leq u(t,x) + \alpha(s-t) + p\cdot(y-x) + \sigma|y-x|^2 + o(s-t).$$

On définit, de manière similaire, le sous-gradient de u semi-continue inférieurement et on a $\partial_P u(t,x) = -\partial^P(-u)(t,x)$. Nous allons prouver un résultat concernant les sur-gradients des semi-limites relaxées; avant cela, un lemme technique est nécessaire.

Lemme E.1 *Soit $f : [-r,r] \to \mathbb{R}^+$ telle que $f(t) = o(t)$ en 0. Il existe $h \in C^1([-r,r])$ telle que $h(0) = h'(0) = 0$ et $h(t) > f(t)$ si $t \neq 0$.*

Preuve du lemme E.1
On étend f en dehors de $[-r,r]$ et on pose $\tilde{f}(t) = \sup_{|s|\leq t} f(s) + |t|^2$. La fonction $\tilde{f}$ est paire, croissante sur $\mathbb{R}^+$, strictement supérieure à f sur $\mathbb{R}\backslash\{0\}$ et c'est un $o(t)$ en 0.
On prend $\rho \in C_c^\infty(]0,1[)$ positive d'intégrale 1, on pose $\rho_t(s) = \frac{1}{t}\rho(\frac{s}{t})$ et on définit, pour $t > 0$,

$$h(t) = \tilde{f} * \rho_t(t) = \int_0^\infty \tilde{f}(s)\rho_t(s-t)\,ds = \frac{1}{t}\int_0^\infty \tilde{f}(s)\rho\left(\frac{s-t}{t}\right)\,ds. \tag{E.1}$$

On voit que h est C^1 sur $]0,\infty[$ (ρ est régulière et $\tilde{f}$ est bornée). De plus, comme $\tilde{f}$ est croissant,

$$h(t) = \frac{1}{t}\int_t^{2t} \tilde{f}(s)\rho\left(\frac{s-t}{t}\right)\,ds \geq \frac{1}{t}\int_t^{2t} \tilde{f}(t)\rho\left(\frac{s-t}{t}\right)\,ds = \tilde{f}(t) > f(t)$$

(rappelons que ρ est d'intégrale 1) et, de même, $h(t) \leq \tilde{f}(2t)$ (ce qui prouve que h se prolonge par continuité en $t = 0$ par 0). Par dérivation sous l'intégrale, on a

$$\frac{d}{dt}\int_0^\infty \tilde{f}(s)\rho\left(\frac{s-t}{t}\right)\,ds = \int_0^\infty \tilde{f}(s)\frac{d}{dt}\rho\left(\frac{s}{t}-1\right)\,ds = \int_0^\infty \tilde{f}(s)\rho'\left(\frac{s}{t}-1\right)\frac{-s}{t^2}\,ds.$$

Or $\rho'(\frac{s}{t}-1) \neq 0$ uniquement lorsque $s \in]t,2t[$, auquel cas $\frac{s}{t^2}$ est majoré par $\frac{2}{t}$; ainsi,

$$\left|\frac{d}{dt}\int_0^\infty \tilde{f}(s)\rho\left(\frac{s-t}{t}\right)\,ds\right| \leq \frac{C}{t}\int_t^{2t} \tilde{f}(s)\,ds \leq C\tilde{f}(2t). \tag{E.2}$$

(E.1) et (E.2) montrent ainsi que h' est majorée par

$$\frac{C}{t^2}\int_0^\infty \tilde{f}(s)\rho\left(\frac{s}{t}-1\right)\,ds + \frac{C}{t}\tilde{f}(2t) \leq \frac{2C}{t}\tilde{f}(2t).$$

On constate donc que $h'(t) \to 0$ lorsque $t \to 0$ (puisque $\tilde{f} = o(t)$ au voisinage de 0), ce qui permet d'étendre h en une fonction C^1 jusqu'en $t = 0$, qui vérifie bien $h'(0) = 0$. On prolonge ensuite h à $\mathbb{R}^-$ par parité (ce qui concerve son caractère C^1 puisque sa dérivée s'annule en 0) et la preuve est achevée. ■

Proposition E.1 *Soit $u_n :]0,T[\times\mathbb{R}^N \to \mathbb{R}$ une suite localement bornée supérieurement de fonctions semi-continues supérieurement et*

$$u(t,x) = (\limsup{}^* u_n)(t,x) = \limsup_{n\to\infty\,,\,(s,y)\to(t,x)} u_n(s,y).$$

Alors u est semi-continue supérieurement et, si $(\alpha,p) \in \partial^P u(t,x)$, il existe $r > 0$, $\sigma > 0$, $k_n \to \infty$ et $(t_n,x_n,\alpha_n,p_n) \to (t,x,\alpha,p)$ tels que $u_{k_n}(t_n,x_n) \to u(t,x)$ et, pour tout $n \geq 1$, $y \in B_r(x_n)$ et $|s-t_n| \leq r$,

$$u_{k_n}(s,y) \leq u_{k_n}(t_n,x_n) + \alpha_n(s-t_n) + p_n\cdot(y-x_n) + \sigma|y-x_n|^2 + o(s-t_n) \tag{E.3}$$

(autrement dit, $(\alpha_n,p_n) \in \partial^P u_{k_n}(t_n,x_n)$ "uniformément pour la partie spatiale").

Preuve de la proposition E.1
Prouvons que u est semi-continue supérieurement. Soit $(t_k,x_k) \to (t,x)$ et $\varepsilon > 0$. Il existe $M \geq 1$ et $\eta > 0$ tel que $u(t,x) \geq -\varepsilon + \sup_{n\geq M}\sup_{|(s,y)-(t,x)|\leq\eta} u_n(s,y)$. Soit $K \geq 1$ tel que $|(t_k,x_k)-(t,x)| \leq \eta/2$ lorsque $k \geq K$; on a alors, dès que $k \geq K$,

$$\begin{aligned}
u(t_k,x_k) &= \lim_{M'\to\infty\,,\,\eta'\to 0}\ \sup_{n\geq M'}\ \sup_{|(s,y)-(t_k,x_k)|\leq\eta'} u_n(s,y)\\
&\leq \sup_{n\geq M}\ \sup_{|(s,y)-(t_k,x_k)|\leq\eta/2} u_n(s,y)\\
&\leq \sup_{n\geq M}\ \sup_{|(s,y)-(t,x)|\leq\eta} u_n(s,y)\\
&\leq \varepsilon + u(t,x)
\end{aligned}$$

ce qui prouve le caractère semi-continu supérieurement de u.

Montrons maintenant la propriété concernant les sur-différentiels. Par définition, il existe $k_n \to \infty$ et $(s_n, y_n) \to (t, x)$ tels que $u_{k_n}(s_n, y_n) \to u(t, x)$; il existe de plus $r' > 0$ et $\sigma' > 0$ tels que, si $y \in B_{r'}(x)$ et $|s - t| \leq r'$,

$$u(s, y) \leq u(t, x) + \alpha(s - t) + p \cdot (y - x) + \sigma'|y - x|^2 + o(s - t). \tag{E.4}$$

Soit $h \in C^1([-r, r])$ strictement supérieure au $o(t)$ de (E.4) sur $[-r', r'] \backslash \{0\}$ et telle que $h(0) = h'(0) = 0$ (voir le lemme E.1). Comme u_{k_n} est semi-continue supérieurement,

$$\sup_{|s-t| \leq r',\ y \in B_{r'}(x)} \left\{ u_{k_n}(s, y) - \alpha(s - t) - p \cdot (y - x) - 2\sigma'|y - x|^2 - h(s - t) \right\} \tag{E.5}$$

est atteint, disons en (t_n, x_n). Dès que $y_n \in B_{r'}(x)$ et $|s_n - t| \leq r'$ (ce qui est vrai pour n grand), on a donc

$$u_{k_n}(s_n, y_n) \leq u_{k_n}(t_n, x_n) + \alpha(s_n - t_n) + p \cdot (y_n - x_n) + 2\sigma'(|y_n - x|^2 - |x_n - x|^2) + h(s_n - t) - h(t_n - t).$$

Quitte à extraire une suite, on suppose que $(t_n, x_n) \to (\overline{t}, \overline{x}) \in [t - r', t + r'] \times B_{r'}(x)$; par choix de (s_n, y_n), la limite inférieure de l'inégalité précédente donne

$$\begin{aligned} u(t, x) &\leq \liminf_{n \to \infty} u_{k_n}(t_n, x_n) + \alpha(t - \overline{t}) + p \cdot (x - \overline{x}) - 2\sigma'|\overline{x} - x|^2 - h(\overline{t} - t) \\ &\leq \limsup_{n \to \infty} u_{k_n}(t_n, x_n) + \alpha(t - \overline{t}) + p \cdot (x - \overline{x}) - 2\sigma'|\overline{x} - x|^2 - h(\overline{t} - t) \\ &\leq u(\overline{t}, \overline{x}) + \alpha(t - \overline{t}) + p \cdot (x - \overline{x}) - 2\sigma'|\overline{x} - x|^2 - h(\overline{t} - t). \end{aligned}$$

Par (E.4), on a

$$u(\overline{t}, \overline{x}) \leq u(t, x) + \alpha(\overline{t} - t) + p \cdot (\overline{x} - x) + \sigma'|\overline{x} - x|^2 + o(\overline{t} - t)$$

et donc

$$\begin{aligned} u(t, x) &\leq \liminf_{n \to \infty} u_{k_n}(t_n, x_n) + \alpha(t - \overline{t}) + p \cdot (x - \overline{x}) - 2\sigma'|\overline{x} - x|^2 - h(\overline{t} - t) \\ &\leq \limsup_{n \to \infty} u_{k_n}(t_n, x_n) + \alpha(t - \overline{t}) + p \cdot (x - \overline{x}) - 2\sigma'|\overline{x} - x|^2 - h(\overline{t} - t) \\ &\leq u(t, x) - \sigma'|\overline{x} - x|^2 + o(\overline{t} - t) - h(\overline{t} - t). \end{aligned} \tag{E.6}$$

Cela prouve que $\overline{x} = x$ et $\overline{t} = t$ (par choix de h); en particulier, toute la suite (t_n, x_n) converge vers (t, x). D'autre part, on a égalité partout dans (E.6), ce qui prouve que $u_{k_n}(t_n, x_n) \to u(t, x)$. Prenons n tel que $|t_n - t| \leq r'/2$ et $x_n \in B_{r'/2}(x)$; par définition de (t_n, x_n) (voir (E.5)) on a, pour tout $|s - t_n| \leq r'/2$ et $y \in B_{r'/2}(x_n)$ (auquel cas $|s - t| \leq r'$ et $y \in B_{r'}(x)$),

$$u_{k_n}(s, y) \leq u_{k_n}(t_n, x_n) + \alpha(s - t_n) + p \cdot (y - x_n) + 2\sigma'(|y - x|^2 - |x_n - x|^2) + h(s - t) - h(t_n - t). \tag{E.7}$$

Or $|y - x|^2 - |x_n - x|^2 = |y - x_n|^2 + 2(y - x_n) \cdot (x_n - x)$ et

$$h(s - t) - h(t_n - t) = h(s - t_n + t_n - t) - h(t_n - t) = (s - t_n)h'(t_n - t) + o(s - t_n).$$

Ainsi, (E.7) montre que (E.3) est vérifiée pour $(s, y) \in [t_n - r'/2, t_n + r'/2] \times B_{r'/2}(x_n)$ avec $\alpha_n = \alpha + h'(t_n - t)$, $p_n = p + 4\sigma'(x_n - x)$, $r = r'/2$ et $\sigma = 2\sigma'$. Puisque $(t_n, x_n) \to (t, x)$ et $h'(0) = 0$, cela conclut la preuve. ■

E.2 Définition des solutions de viscosité

Définition E.2 *On suppose que $F :]0, T[\times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ est continue et que $\lambda \in]0, 2[$. Une fonction $u :]0, T[\times \mathbb{R}^N \to \mathbb{R}$ semi-continue supérieurement et bornée est sous-solution de viscosité de*

$$\partial_t u(t, x) + g_\lambda[u(t, \cdot)](x) = F(t, x, u(t, x), \nabla u(t, x)) \tag{E.8}$$

si, pour tout $(t,x) \in]0,T[\times\mathbb{R}^N$ *et* $(\alpha,p) \in \partial^P u(t,x)$*, on a*

$$\alpha - c_N(\lambda)\int_{B_r} \frac{u(t,x+z)-u(t,x)-p\cdot z}{|z|^{N+\lambda}}\,dz - c_N(\lambda)\int_{\mathbb{R}^N\setminus B_r} \frac{u(t,x+z)-u(t,x)}{|z|^{N+\lambda}}\,dz \le F(t,x,u(t,x),p)$$

où $r>0$ *est donné par la définition E.1 (en fait, on peut prendre n'importe quel* $r>0$*, cela ne change pas la valeur de l'expression).*

Remarque E.1 *Si* $(\alpha,p)\in\partial^P u(t,x)$ *et* $|z|\le r$*, on a* $u(t,x+z)-u(t,x)-p\cdot z \le \sigma|z|^2$*, et* $|z|^2/|z|^{N+\lambda}\in L^1(B_r)$*; ainsi,* $\int_{B_r}\frac{u(t,x+z)-u(t,x)-p\cdot z}{|z|^{N+\lambda}}\,dz$ *est bien définie, éventuellement égale à* $-\infty$ *(mais dans le cas où l'on demande que u soit sous-solution de (E.8), ce cas est exclus par l'inégalité qui doit être vérifiée).*

Remarque E.2 *J'ai un peu changé la définition par rapport à [17] (qui remplace le terme* $u(t,x+z)-u(t,x)-p\cdot z$ *par* $\sigma|z|^2$ *et garde p dans le deuxième terme intégral), ceci afin de coller davantage à la formule (2.1) et de pouvoir traiter le cas* $\lambda\le 1$*.*

De même, une fonction u semi-continue inférieurement et bornée est sur-solution de (E.8) si $-u$ est sous-solution de la même équation avec F remplacé par $-F(\cdot,\cdot,-\cdot,-\cdot)$ (cet hamiltonien vérifie les mêmes propriétés que F); cela revient à prendre des sur-différentiels et à inverser l'inégalité de la définition. On étend de plus sans problème cette définition au cas de combinaisons linéaires positives d'opérateurs du type g_λ.

E.3 Stabilité

Théorème E.1 *Soit* $(\lambda,\mu)\in]0,2[$ *et* $b\ge 0$*. On suppose que* $F_n\in C(]0,T[\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N)$ *converge localement uniformément vers F et on prend* $(b_n)_{n\ge1}$ *une suite de réels positifs qui convergent vers b. Soit* $(u_n)_{n\ge1}$ *une suite bornée sur* $]0,T[\times\mathbb{R}^N$*. Si, pour tout* $n\ge1$*,* u_n *est une sous-solution de viscosité de*

$$\partial_t u_n(t,x) + g_\lambda[u_n(t,\cdot)](x) + b_n g_\mu[u_n(t,\cdot)](x) = F_n(t,x,u_n(t,x),\nabla u_n(t,x))\,,$$

alors $u=\limsup^* u_n$ *est une sous-solution de viscosité de*

$$\partial_t u(t,x) + g_\lambda[u(t,\cdot)](x) + b g_\mu[u(t,\cdot)](x) = F(t,x,u(t,x),\nabla u(t,x)).$$

Remarque E.3 *Ce résultat est aussi valable avec des opérateurs* g_λ *et* g_μ *plus généraux (en particulier, on peut aussi supprimer* g_λ *ou prendre* $\mu=2$ *— mais ce dernier cas demande d'établir le lemme E.3 pour des sous-différentiels d'ordre 2).*

Preuve du théorème E.1
On sait que u est semi-continue supérieurement; comme $(u_n)_{n\ge1}$ est une suite bornée sur $]0,T[\times\mathbb{R}^N$, u est bornée sur $]0,T[\times\mathbb{R}^N$. Soient $(t,x)\in]0,T[\times\mathbb{R}^N$ et $(\alpha,p)\in\partial^P u(t,x)$. Soient $(r,\sigma,k_n,t_n,x_n,\alpha_n,p_n)$ donnés par la proposition E.1. Par (E.3) et l'équation satisfaite par u_{k_n}, on a

$$\begin{aligned}
&\alpha_n - c_N(\lambda)\int_{B_r}\frac{u_{k_n}(t_n,x_n+z)-u_{k_n}(t_n,x_n)-p_n\cdot z}{|z|^{N+\lambda}}\,dz\\
&-c_N(\lambda)\int_{\mathbb{R}^N\setminus B_r}\frac{u_{k_n}(t_n,x_n+z)-u_{k_n}(t_n,x_n)}{|z|^{N+\lambda}}\,dz\\
&-b_n c_N(\mu)\int_{B_r}\frac{u_{k_n}(t_n,x_n+z)-u_{k_n}(t_n,x_n)-p_n\cdot z}{|z|^{N+\mu}}\,dz\\
&-b_n c_N(\mu)\int_{\mathbb{R}^N\setminus B_r}\frac{u_{k_n}(t_n,x_n+z)-u_{k_n}(t_n,x_n)}{|z|^{N+\mu}}\,dz\\
&\qquad\le F_n(t_n,x_n,u_{k_n}(t,x),p_n).
\end{aligned}\tag{E.9}$$

Comme $(t_n, x_n, u_{k_n}(t_n, x_n), p_n) \to (t, x, u(t,x), p)$ et $F_n \to F$ localement uniformément, on a

$$F_n(t_n, x_n, u_{k_n}(t,x), p_n) \to F(t, x, u(t,x), p) \qquad \text{lorsque } n \to \infty.$$

De plus, par définition de u et de (t_n, x_n, k_n, p_n),

$$\limsup_{n\to\infty} (u_{k_n}(t_n, x_n + z) - u_{k_n}(t_n, x_n)) \le u(t, x+z) - u(t,x)$$

et

$$\limsup_{n\to\infty} (u_{k_n}(t_n, x_n + z) - u_{k_n}(t_n, x_n) - p_n \cdot z) \le u(t, x+z) - u(t,x) - p \cdot z.$$

On a, pour $z \in \mathbb{R}^N \setminus B_r$, puisque $(u_n)_{n\ge1}$ est bornée sur $]0,T[\times\mathbb{R}^N$,

$$\frac{u_{k_n}(t_n, x_n + z) - u_{k_n}(t_n, x_n)}{|z|^{N+\lambda}} \le \frac{C}{|z|^{N+\lambda}} \in L^1(\mathbb{R}^N \setminus B_r)$$

et, pour $z \in B_r$, par définition de (t_n, x_n, k_n, p_n),

$$\frac{u_{k_n}(t_n, x_n + z) - u_{k_n}(t_n, x_n) - p_n \cdot z}{|z|^{N+\lambda}} \le \frac{\sigma |z|^2}{|z|^{N+\lambda}} \in L^1(B_r)$$

(remarquons explicitement que σ et r ne dépendent pas de n); le lemme de Fatou donne donc

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_r} \frac{u_{k_n}(t_n, x_n + z) - u_{k_n}(t_n, x_n)}{|z|^{N+\lambda}}\, dz \le \int_{\mathbb{R}^N\setminus B_r} \frac{u(t, x+z) - u(t,x)}{|z|^{N+\lambda}}\, dz < +\infty.$$

et

$$\limsup_{n\to\infty} \int_{B_r} \frac{u_{k_n}(t_n, x_n + z) - u_{k_n}(t_n, x_n) - p_n \cdot z}{|z|^{N+\lambda}}\, dz \le \int_{B_r} \frac{u(t, x+z) - u(t,x) - p \cdot z}{|z|^{N+\lambda}}\, dz < +\infty.$$

En faisant le même raisonnement avec le terme en μ, on peut prendre la $\liminf_{n\to\infty}$ de (E.9); la seule difficulté lors de ce passage à la limite est dans le traitement des termes impliquant b_n, et elle est réglée par le lemme E.2 qui suit. ■

Remarque E.4 *Si λ ou μ sont différents de 1, on peut aussi utiliser, dans la définition E.2, les formules i) ou ii) du théorème 2.1, et le théorème E.1 reste valable.*

Lemme E.2 *Soit $(b_n)_{n\ge1}$ une suite de réels positifs qui converge vers b et $(X_n)_{n\ge1}$ une suite de réels. Si $\limsup_{n\to\infty} X_n = X < +\infty$ (éventuellement $X = -\infty$), alors $\limsup_{n\to\infty}(b_n X_n) \le bX$ (avec la convention usuelle $0 \times (-\infty) = 0$).*

Remarque E.5 *Ce lemme est faux si l'on ne suppose pas $X < +\infty$: considérer $b_n = 1/n$ et $X_n = n$.*

Preuve du lemme E.2
Posons $Y_n = \sup_{k\ge n} X_k$. Comme $b_n \ge 0$, on a $b_n X_n \le b_n Y_n$; de plus, $Y_n \to X$. Si $X > -\infty$ ou $b > 0$, alors $\lim_{n\to\infty} b_n Y_n = bX$ (la forme n'est pas indéterminée) et le lemme est prouvé. Si $X = -\infty$ et $b = 0$ alors, pour n assez grand, $Y_n \le 0$ donc $b_n Y_n \le 0 = bX$ (par convention) et on peut conclure. ■

E.4 Comparaison

Théorème E.2 *Soit $\lambda \in]0,2[$. On suppose que $F \in C([0,T[\times\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ vérifie:*

$$\begin{gathered} \forall R > 0\,,\ \exists \Theta_R \text{ tel que, pour } (t,x,s,s',\xi) \in]0,T[\times\mathbb{R}^N \times [-R,R]^2 \times \mathbb{R}^N\,, \\ \text{si } s \le s' \text{ alors } F(t,x,s',\xi) - F(t,x,s,\xi) \le \Theta_R(s'-s)\,, \end{gathered} \tag{E.10}$$

$$\begin{array}{c}\forall R>0\,,\ \exists C_R \text{ tel que, pour } (t,x,x',s,\xi)\in]0,T[\times(\mathbb{R}^N)^2\times[-R,R]\times\mathbb{R}^N\,,\\ |F(t,x,s,\xi)-F(t,x',s,\xi)|\le C_R(1+|\xi|)|x-x'|\,,\end{array}\tag{E.11}$$

$$\begin{array}{c}\forall R>0\,,\ \exists C'_R \text{ tel que, pour } (t,x,s,\xi,\xi')\in]0,T[\times\mathbb{R}^N\times[-R,R]\times B_R^2,\\ |F(t,x,s,\xi)-F(t,x,s,\xi')|\le C'_R|\xi-\xi'|.\end{array}\tag{E.12}$$

On prend u_0 bornée et uniformément continue. On suppose que $u:[0,T[\times\mathbb{R}^N\to\mathbb{R}$ et $v:[0,T[\times\mathbb{R}^N\to\mathbb{R}$ sont bornées et respectivement sous- et sur-solutions de

$$\partial_t w(t,x)+g_\lambda[w(t,\cdot)](x)=F(t,x,w(t,x),\nabla w(t,x)).$$

Si $u(0,\cdot)\le u_0$ et $v(0,\cdot)\ge u_0$, alors $u\le v$ sur $[0,T[\times\mathbb{R}^N$.

Remarque E.6 *Ici aussi, on peut remplacer g_λ par un opérateur plus général.*

Par soucis de simplicité, nous noterons dans la suite, lorsque ceci a un sens (éventuellement $+\infty$ ou $-\infty$),

$$G_\lambda[w(t,\cdot),p](x)=-c_N(\lambda)\int_{B_r}\frac{w(t,x+z)-w(t,x)-p\cdot z}{|z|^{N+\lambda}}\,dz-c_N(\lambda)\int_{\mathbb{R}^N\setminus B_r}\frac{w(t,x+z)-w(t,x)}{|z|^{N+\lambda}}\,dz.$$

Cette expression ne dépend pas de $r>0$ et est définie lorsque (α,p) est dans le sous- ou sur-différentiel de w en (t,x).

Preuve du théorème E.2
On se ramène pour commencer au cas d'un hamiltonien décroissant en s. Pour cela, on fixe R_0 qui majore $|u|$ et $|v|$ sur $[0,T[\times\mathbb{R}^N$ et on pose

$$\widetilde{F}(t,x,s,\xi)=e^{-\Theta_{R_0}t}F\left(t,x,T_{R_0}(e^{\Theta_{R_0}t}s),e^{\Theta_{R_0}t}\xi\right)-\Theta_{R_0}s,$$

$\widetilde{u}(t,x)=e^{-\Theta_{R_0}t}u(t,x)$ et $\widetilde{v}(t,x)=e^{-\Theta_{R_0}t}v(t,x)$ (T_{R_0} est la troncature usuelle à la hauteur R_0). On constate que $\widetilde{F}$ vérifie (E.11) et (E.12) (avec des constantes qui changent eventuellement); de plus, pour tout $(t,x,s,s',\xi)\in]0,T[\times\mathbb{R}^N\times\mathbb{R}^2\times\mathbb{R}^N$, (E.10) donne, si $s\le s'$,

$$\begin{aligned}\widetilde{F}(t,x,s',\xi)-\widetilde{F}(t,x,s,\xi)&\\ &\le\ e^{-\Theta_{R_0}t}\Theta_{R_0}(T_{R_0}(e^{\Theta_{R_0}t}s')-T_{R_0}(e^{\Theta_{R_0}t}s))-\Theta_{R_0}(s'-s)\\ &\le\ e^{-\Theta_{R_0}t}\Theta_{R_0}(e^{\Theta_{R_0}t}s'-e^{\Theta_{R_0}t}s)-\Theta_{R_0}(s'-s)\\ &=\ 0.\end{aligned}\tag{E.13}$$

Enfin, $\widetilde{u}$ et $\widetilde{v}$ sont sous- et sur-solutions de viscosité de

$$\partial_t w(t,x)+g_\lambda[w(t,\cdot)](x)=\widetilde{F}(t,x,w(t,x),\nabla w(t,x))$$

(car $e^{\Theta_{R_0}t}\widetilde{u}(t,x)=u(t,x)$ et $e^{\Theta_{R_0}t}\widetilde{v}(t,x)=v(t,x)$ sont bornées par R_0).

On suppose que $\sup_{[0,T[\times\mathbb{R}^N}(\widetilde{u}-\widetilde{v})>0$ et on souhaite arriver à une contradiction. Prenons $\phi\in C^2(\mathbb{R}^N)$ positive telle que $\phi(x)=|x|^{\lambda/2}$ au voisinage de l'infini. Soient $(\varepsilon,\alpha,\nu,\gamma)$ dans $]0,1]$ et

$$M_{\varepsilon,\alpha,\nu}=\sup_{[0,T[^2\times(\mathbb{R}^N)^2}\left\{\widetilde{u}(t,x)-\widetilde{v}(s,y)-\frac{|x-y|^2}{2\varepsilon}-\frac{|s-t|^2}{2\nu}-\alpha\phi(x)-\frac{\gamma}{T-t}\right\}.$$

Comme $\widetilde{u}$ est semi-continue supérieurement, $\widetilde{v}$ est semi-continue inférieurement et ces fonctions sont bornées, ce supprémum est atteint en $(t_{\varepsilon,\nu,\alpha},s_{\varepsilon,\nu,\alpha},x_{\varepsilon,\nu,\alpha},y_{\varepsilon,\nu,\alpha})\in[0,T[^2\times(\mathbb{R}^N)^2$ lorsque ν est assez petit (la présence de ϕ et $|x-y|^2$ empèche la fuite à l'infini de (x,y), la présence de $\gamma/(T-t)$ empèche

que t ne s'approche de T — indépendamment de ν — et, lorsque ν est assez petit, le terme $|s-t|^2/(2\nu)$ oblige s à rester proche de t et donc loin de T). On a de plus

$$M_{\varepsilon,\alpha,\nu} \geq \sup_{[0,T[\times\mathbb{R}^N} \left\{ \widetilde{u}(t,x) - \widetilde{v}(t,x) - \alpha\phi(x) - \frac{\gamma}{T-t} \right\} \geq 0$$

lorsque γ et α sont assez petits (il existe un couple (t,x) tel que $\widetilde{u}(t,x) - \widetilde{v}(t,x) > 0$, par définition), ce que l'on suppose à partir de maintenant.

Etape 1: quelques propriétés sur $(t_{\varepsilon,\nu,\alpha}, s_{\varepsilon,\nu,\alpha}, x_{\varepsilon,\nu,\alpha}, y_{\varepsilon,\nu,\alpha})$.
On constate pour commencer, en prenant $t = s = 0$ et $x = y = 0$ que $M_{\varepsilon,\alpha,\nu} \geq -||\widetilde{u}||_\infty - ||\widetilde{v}||_\infty - \alpha\phi(0) - \gamma/T \geq c$ avec c indépendant de (ε,α,ν). En particulier,

$$c \leq ||\widetilde{u}||_\infty + ||\widetilde{v}||_\infty - \frac{|x_{\varepsilon,\alpha,\nu} - y_{\varepsilon,\alpha,\nu}|^2}{2\varepsilon} - \frac{|s_{\varepsilon,\alpha,\nu} - t_{\varepsilon,\alpha,\nu}|^2}{2\nu} - \alpha\phi(x_{\varepsilon,\alpha,\nu}) - \frac{\gamma}{T - t_{\varepsilon,\alpha,\nu}}$$

et on en déduit que

$$\frac{|x_{\varepsilon,\alpha,\nu} - y_{\varepsilon,\alpha,\nu}|^2}{2\varepsilon} \leq c' , \quad \frac{|s_{\varepsilon,\alpha,\nu} - t_{\varepsilon,\alpha,\nu}|^2}{2\nu} \leq c' , \quad \alpha\phi(x_{\varepsilon,\alpha,\nu}) \leq c' \quad \text{et} \quad \frac{\gamma}{T - t_{\varepsilon,\alpha,\nu}} \leq c' \tag{E.14}$$

avec c' indépendant de (ε,α,ν).

La fonction $\nu \to M_{\varepsilon,\alpha,\nu}$ est croissante et minorée; elle converge donc lorsque $\nu \to 0$. On a de plus

$$\begin{aligned} M_{\varepsilon,\alpha,2\nu} - M_{\varepsilon,\alpha,\nu} & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (E.15) \\ \geq\ & \widetilde{u}(t_{\varepsilon,\alpha,\nu}, x_{\varepsilon,\alpha,\nu}) - \widetilde{v}(s_{\varepsilon,\alpha,\nu}, y_{\varepsilon,\alpha,\nu}) - \frac{|x_{\varepsilon,\alpha,\nu} - y_{\varepsilon,\alpha,\nu}|^2}{2\varepsilon} - \frac{|s_{\varepsilon,\alpha,\nu} - t_{\varepsilon,\alpha,\nu}|^2}{2\times(2\nu)} \\ & -\alpha\phi(x_{\varepsilon,\alpha,\nu}) - \frac{\gamma}{T - t_{\varepsilon,\alpha,\nu}} \\ & - \left(\widetilde{u}(t_{\varepsilon,\alpha,\nu}, x_{\varepsilon,\alpha,\nu}) - \widetilde{v}(s_{\varepsilon,\alpha,\nu}, y_{\varepsilon,\alpha,\nu}) - \frac{|x_{\varepsilon,\alpha,\nu} - y_{\varepsilon,\alpha,\nu}|^2}{2\varepsilon} - \frac{|s_{\varepsilon,\alpha,\nu} - t_{\varepsilon,\alpha,\nu}|^2}{2\nu} \right. \\ & \left. -\alpha\phi(x_{\varepsilon,\alpha,\nu}) - \frac{\gamma}{T - t_{\varepsilon,\alpha,\nu}} \right) \\ =\ & \frac{|s_{\varepsilon,\alpha,\nu} - t_{\varepsilon,\alpha,\nu}|^2}{4\nu}. \end{aligned}$$

Comme (E.15) tend vers 0 lorsque $\nu \to 0$ (puisque $M_{\varepsilon,\alpha,\nu}$ converge lorsque $\nu \to 0$), on en déduit

$$\lim_{\nu\to 0} \frac{|s_{\varepsilon,\alpha,\nu} - t_{\varepsilon,\alpha,\nu}|^2}{\nu} = 0. \tag{E.16}$$

Soit

$$M_{\varepsilon,\alpha} = \sup_{[0,T[\times(\mathbb{R}^N)^2} \left\{ \widetilde{u}(t,x) - \widetilde{v}(t,y) - \frac{|x-y|^2}{2\varepsilon} - \alpha\phi(x) - \frac{\gamma}{T-t} \right\} \leq M_{\varepsilon,\alpha,\nu}.$$

Par (E.14), $(t_{\varepsilon,\alpha,\nu}, s_{\varepsilon,\alpha,\nu}, x_{\varepsilon,\alpha,\nu}, y_{\varepsilon,\alpha,\nu})_{\nu>0}$ est bornée et on peut extraire une suite telle que

$$(t_{\varepsilon,\alpha,\nu'}, s_{\varepsilon,\alpha,\nu'}, x_{\varepsilon,\alpha,\nu'}, y_{\varepsilon,\alpha,\nu'}) \to (t_{\varepsilon,\alpha}, t_{\varepsilon,\alpha}, x_{\varepsilon,\alpha}, y_{\varepsilon,\alpha}) \quad \text{lorsque } \nu' \to 0$$

avec $t_{\varepsilon,\alpha} \in [0,T[$. On peut aussi se débrouiller pour que

$$\frac{|x_{\varepsilon,\alpha} - y_{\varepsilon,\alpha}|^2}{2\varepsilon} = \lim_{\nu'\to 0} \frac{|x_{\varepsilon,\alpha,\nu'} - y_{\varepsilon,\alpha,\nu'}|^2}{2\varepsilon} = \limsup_{\nu\to 0} \frac{|x_{\varepsilon,\alpha,\nu} - y_{\varepsilon,\alpha,\nu}|^2}{2\varepsilon}. \tag{E.17}$$

Puisque $\widetilde{u}$ est semi-continue supérieurement et $\widetilde{v}$ est semi-continue inférieurement, (E.16) donne

$$\begin{aligned}
\lim_{\nu\to 0} M_{\varepsilon,\alpha,\nu} &= \limsup_{\nu'\to 0} M_{\varepsilon,\alpha,\nu'} \\
&= \limsup_{\nu'\to 0}\Bigg(\widetilde{u}(t_{\varepsilon,\alpha,\nu'},x_{\varepsilon,\alpha,\nu'}) - \widetilde{v}(s_{\varepsilon,\alpha,\nu'},y_{\varepsilon,\alpha,\nu'}) - \frac{|x_{\varepsilon,\alpha,\nu'}-y_{\varepsilon,\alpha,\nu'}|^2}{2\varepsilon} - \frac{|s_{\varepsilon,\alpha,\nu'}-t_{\varepsilon,\alpha,\nu'}|^2}{2\nu'} \\
&\qquad -\alpha\phi(x_{\varepsilon,\alpha,\nu'}) - \frac{\gamma}{T-t_{\varepsilon,\alpha,\nu'}}\Bigg) \\
&\le \widetilde{u}(t_{\varepsilon,\alpha},x_{\varepsilon,\alpha}) - \widetilde{v}(t_{\varepsilon,\alpha},y_{\varepsilon,\alpha}) - \frac{|x_{\varepsilon,\alpha}-y_{\varepsilon,\alpha}|^2}{2\varepsilon} - \alpha\phi(x_{\varepsilon,\alpha}) - \frac{\gamma}{T-t_{\varepsilon,\alpha}} \\
&\le M_{\varepsilon,\alpha}.
\end{aligned}$$

Cela prouve que $\lim_{\nu\to 0} M_{\varepsilon,\alpha,\nu} = M_{\varepsilon,\alpha}$ (on avait $M_{\varepsilon,\alpha} \le M_{\varepsilon,\alpha,\nu}$), et que ce supremum est atteint en $(t_{\varepsilon,\alpha}, x_{\varepsilon,\alpha}, y_{\varepsilon,\alpha})$.

La fonction $\alpha \to M_{\varepsilon,\alpha}$ est décroissante et majorée (par $||\widetilde{u}||_\infty + ||\widetilde{v}||_\infty$); soit $\overline{M}_\varepsilon = \lim_{\alpha\to 0} M_{\varepsilon,\alpha}$. Comme $M_{\varepsilon,\alpha}$ est bornée indépendamment de (ε,α) et croissante par rapport à ε, il en est de même pour $\overline{M}_\varepsilon$. Ainsi, $\lim_{\varepsilon\to 0}\overline{M}_\varepsilon$ existe dans $\mathbb{R}$. On a de plus

$$\begin{aligned}
M_{2\varepsilon,\alpha} - M_{\varepsilon,\alpha} &\ge \widetilde{u}(t_{\varepsilon,\alpha},x_{\varepsilon,\alpha}) - \widetilde{v}(t_{\varepsilon,\alpha},y_{\varepsilon,\alpha}) - \frac{|x_{\varepsilon,\alpha}-y_{\varepsilon,\alpha}|^2}{2\times(2\varepsilon)} - \alpha\phi(x_{\varepsilon,\alpha}) - \frac{\gamma}{T-t_{\varepsilon,\alpha}} \\
&\quad - \left(\widetilde{u}(t_{\varepsilon,\alpha},x_{\varepsilon,\alpha}) - \widetilde{v}(t_{\varepsilon,\alpha},y_{\varepsilon,\alpha}) - \frac{|x_{\varepsilon,\alpha}-y_{\varepsilon,\alpha}|^2}{2\varepsilon} - \alpha\phi(x_{\varepsilon,\alpha}) - \frac{\gamma}{T-t_{\varepsilon,\alpha}}\right) \\
&= \frac{|x_{\varepsilon,\alpha}-y_{\varepsilon,\alpha}|^2}{4\varepsilon},
\end{aligned}$$

donc, en prenant la $\limsup_{\alpha\to 0}$,

$$\overline{M}_{2\varepsilon} - \overline{M}_\varepsilon \ge \limsup_{\alpha\to 0}\frac{|x_{\varepsilon,\alpha}-y_{\varepsilon,\alpha}|^2}{4\varepsilon}.$$

Puisque $\lim_{\varepsilon\to 0}\overline{M}_\varepsilon$ existe dans $\mathbb{R}$, on obtient finalement, par (E.17),

$$\lim_{\varepsilon\to 0}\limsup_{\alpha\to 0}\frac{|x_{\varepsilon,\alpha}-y_{\varepsilon,\alpha}|^2}{\varepsilon} = \lim_{\varepsilon\to 0}\limsup_{\alpha\to 0}\limsup_{\nu\to 0}\frac{|x_{\varepsilon,\alpha,\nu}-y_{\varepsilon,\alpha,\nu}|^2}{\varepsilon} = 0. \tag{E.18}$$

Finalement, comme $M_{\varepsilon,\alpha,\nu} \ge 0$, on a

$$\begin{aligned}
\widetilde{u}(t_{\varepsilon,\alpha,\nu},x_{\varepsilon,\alpha,\nu}) - \widetilde{v}(s_{\varepsilon,\alpha,\nu},y_{\varepsilon,\alpha,\nu}) &= M_{\varepsilon,\alpha,\nu} + \frac{|x_{\varepsilon,\alpha,\nu}-y_{\varepsilon,\alpha,\nu}|^2}{2\varepsilon} + \frac{|s_{\varepsilon,\alpha,\nu}-t_{\varepsilon,\alpha,\nu}|^2}{2\nu} \\
&\quad +\alpha\phi(x_{\varepsilon,\alpha,\nu}) + \frac{\gamma}{T-t_{\varepsilon,\alpha,\nu}} \ge 0.
\end{aligned} \tag{E.19}$$

Etape 2: on suppose que, pour tout $\varepsilon_0 > 0$, il existe $\varepsilon \le \varepsilon_0$ tel que, pour tout $\alpha_0 > 0$, il existe $\alpha \le \alpha_0$ tel que, pour tout $\nu_0 > 0$, on puisse trouver $\nu \le \nu_0$ satisfaisant $t_{\varepsilon,\alpha,\nu} = 0$ ou $s_{\varepsilon,\alpha,\nu} = 0$.
Si $t_{\varepsilon,\alpha,\nu} = 0$, on a

$$\begin{aligned}
0 &\le M_{\varepsilon,\alpha,\nu} \\
&\le \widetilde{u}(0,x_{\varepsilon,\alpha,\nu}) - \widetilde{v}(s_{\varepsilon,\alpha,\nu},y_{\varepsilon,\alpha,\nu}) - \frac{|x_{\varepsilon,\alpha,\nu}-y_{\varepsilon,\alpha,\nu}|^2}{2\varepsilon} - \frac{\gamma}{T} \\
&\le u_0(x_{\varepsilon,\alpha,\nu}) - u_0(y_{\varepsilon,\alpha,\nu}) - \frac{|x_{\varepsilon,\alpha,\nu}-y_{\varepsilon,\alpha,\nu}|^2}{2\varepsilon} - \widetilde{v}(s_{\varepsilon,\alpha,\nu},y_{\varepsilon,\alpha,\nu}) + u_0(y_{\varepsilon,\alpha,\nu}) - \frac{\gamma}{T} \\
&\le \sup_{(\mathbb{R}^N)^2}\left\{u_0(x) - u_0(y) - \frac{|x-y|^2}{2\varepsilon}\right\} - \widetilde{v}(s_{\varepsilon,\alpha,\nu},y_{\varepsilon,\alpha,\nu}) + u_0(y_{\varepsilon,\alpha,\nu}) - \frac{\gamma}{T} \\
&=: \Upsilon(\varepsilon) - \widetilde{v}(s_{\varepsilon,\alpha,\nu},y_{\varepsilon,\alpha,\nu}) + u_0(y_{\varepsilon,\alpha,\nu}) - \frac{\gamma}{T}
\end{aligned} \tag{E.20}$$

En prenant $x = y$, on voit que $\Upsilon(\varepsilon) \geq 0$; de plus, si $|x-y| \geq \sqrt{4\varepsilon||u_0||_\infty}$, on a $u_0(x) - u_0(y) - \frac{|x-y|^2}{2\varepsilon} \leq 0$ et donc

$$\Upsilon(\varepsilon) = \sup_{|x-y|\leq\sqrt{4\varepsilon||u_0||_\infty}} \left\{ u_0(x) - u_0(y) - \frac{|x-y|^2}{2\varepsilon} \right\} \leq \sup_{|x-y|\leq\sqrt{4\varepsilon||u_0||_\infty}} \{u_0(x) - u_0(y)\} \to 0$$

lorsque $\varepsilon \to 0$, par uniforme continuité de u_0. Par hypothèse, on peut donc trouver ε et α tels que $\Upsilon(\varepsilon) - \gamma/T < 0$ et tels qu'il existe $\nu_n \to 0$ vérifiant, pour tout $n \geq 1$, $t_{\varepsilon,\alpha,\nu_n} = 0$; par (E.14), on a $s_{\varepsilon,\alpha,\nu_n} = s_{\varepsilon,\alpha,\nu_n} - t_{\varepsilon,\alpha,\nu_n} \to 0$ lorsque $n \to \infty$ et, quitte à extraire une suite, $y_{\varepsilon,\alpha,\nu_n} \to y_{\varepsilon,\alpha}$ (voir l'étape 1). Puisque $\widetilde{v}$ est semi-continue inférieurement, on en déduit

$$\begin{aligned} \limsup_{n\to\infty}(-\widetilde{v}(s_{\varepsilon,\alpha,\nu_n}, y_{\varepsilon,\alpha,\nu_n}) + u_0(y_{\varepsilon,\alpha,\nu_n})) &= -\liminf_{n\to\infty} \widetilde{v}(s_{\varepsilon,\alpha,\nu_n}, y_{\varepsilon,\alpha,\nu_n}) + u_0(y_{\varepsilon,\alpha}) \\ &\leq -\widetilde{v}(0, y_{\varepsilon,\alpha}) + u_0(y_{\varepsilon,\alpha}) \leq 0 \end{aligned}$$

et on obtient une contradiction, vu le choix de ε, avec (E.20). On raisonnerait de même si l'on suppose $s_{\varepsilon,\alpha,\nu} = 0$.

Etape 3: on suppose qu'il existe $\varepsilon_0 > 0$ tel que, pour tout $\varepsilon \leq \varepsilon_0$, il existe $\alpha_\varepsilon > 0$ tel que, pour tout $\alpha \leq \alpha_\varepsilon$, il existe $\nu_\alpha > 0$ tel que, dès que $\nu \leq \nu_\alpha$, on a $t_{\varepsilon,\alpha,\nu} > 0$ et $s_{\varepsilon,\alpha,\nu} > 0$. En pratique, cela signifie donc qu'on va pouvoir faire tendre $\nu \to 0$ puis $\alpha \to 0$ puis $\varepsilon \to 0$ dans cet ordre.
Pour simplifier les notations, on pose à partir de maintenant $t_{\varepsilon,\alpha,\nu} = \overline{t}$, $s_{\varepsilon,\alpha,\nu} = \overline{s}$, $x_{\varepsilon,\alpha,\nu} = \overline{x}$ et $y_{\varepsilon,\alpha,\nu} = \overline{y}$.

Par définition de $M_{\varepsilon,\alpha,\nu}$, on a

$$\left(\frac{\gamma}{(T-\overline{t})^2} + \frac{\overline{t}-\overline{s}}{\nu}, \overline{p} + \alpha\nabla\phi(\overline{x}) \right) \in \partial^P \widetilde{u}(\overline{t},\overline{x}) \quad \text{et} \quad \left(\frac{\overline{t}-\overline{s}}{\nu}, \overline{p} \right) \in \partial_P \widetilde{v}(\overline{s},\overline{y}), \quad \text{avec } \overline{p} = \tfrac{\overline{x}-\overline{y}}{\varepsilon}.$$

Ainsi,

$$\frac{\gamma}{(T-\overline{t})^2} + \frac{\overline{t}-\overline{s}}{\nu} + G_\lambda[\widetilde{u}(\overline{t},\cdot), \overline{p} + \alpha\nabla\phi(\overline{x})](\overline{x}) \leq \widetilde{F}(\overline{t},\overline{x},\widetilde{u}(\overline{t},\overline{x}), \overline{p} + \alpha\nabla\phi(\overline{x})) \tag{E.21}$$

et

$$\frac{\overline{t}-\overline{s}}{\nu} + G_\lambda[\widetilde{v}(\overline{s},\cdot), \overline{p}](\overline{y}) \geq \widetilde{F}(\overline{s},\overline{y},\widetilde{v}(\overline{s},\overline{y}), \overline{p}). \tag{E.22}$$

On a

$$\begin{aligned} & G_\lambda[\widetilde{v}(\overline{s},\cdot), \overline{p}](\overline{y}) - G_\lambda[\widetilde{u}(\overline{t},\cdot), \overline{p} + \alpha\nabla\phi(\overline{x})](\overline{x}) \\ = \; & -c_N(\lambda) \int_{B_r} \frac{\widetilde{v}(\overline{s},\overline{y}+z) - \widetilde{v}(\overline{s},\overline{y}) - \overline{p}\cdot z}{|z|^{N+\lambda}} - \frac{\widetilde{u}(\overline{t},\overline{x}+z) - \widetilde{u}(\overline{t},\overline{x}) - (\overline{p} + \alpha\nabla\phi(\overline{x}))\cdot z}{|z|^{N+\lambda}} \, dz \\ & -c_N(\lambda) \int_{\mathbb{R}^N\setminus B_r} \frac{\widetilde{v}(\overline{s},\overline{y}+z) - \widetilde{v}(\overline{s},\overline{y})}{|z|^{N+\lambda}} - \frac{\widetilde{u}(\overline{t},\overline{x}+z) - \widetilde{u}(\overline{t},\overline{x})}{|z|^{N+\lambda}} \, dz \\ = \; & -c_N(\lambda) \int_{B_r} \frac{\widetilde{v}(\overline{s},\overline{y}+z) - \widetilde{v}(\overline{s},\overline{y}) - \widetilde{u}(\overline{t},\overline{x}+z) + \widetilde{u}(\overline{t},\overline{x})}{|z|^{N+\lambda}} - \frac{-\alpha\nabla\phi(\overline{x})\cdot z}{|z|^{N+\lambda}} \, dz \\ & -c_N(\lambda) \int_{\mathbb{R}^N\setminus B_r} \frac{\widetilde{v}(\overline{s},\overline{y}+z) - \widetilde{v}(\overline{s},\overline{y}) - \widetilde{u}(\overline{t},\overline{x}+z) + \widetilde{u}(\overline{t},\overline{x})}{|z|^{N+\lambda}} \, dz. \end{aligned}$$

Or, par définition,

$$\begin{aligned} & \widetilde{u}(\overline{t},\overline{x}) - \widetilde{v}(\overline{s},\overline{y}) - \frac{|\overline{x}-\overline{y}|^2}{2\varepsilon} - \frac{|\overline{s}-\overline{t}|^2}{2\nu} - \alpha\phi(\overline{x}) - \frac{\gamma}{T-\overline{t}} \\ & \quad \geq \; \widetilde{u}(\overline{t},\overline{x}+z) - \widetilde{v}(\overline{s},\overline{y}+z) - \frac{|\overline{x}+z-\overline{y}-z|^2}{2\varepsilon} - \frac{|\overline{s}-\overline{t}|^2}{2\nu} - \alpha\phi(\overline{x}+z) - \frac{\gamma}{T-\overline{t}} \end{aligned}$$

et donc

$$\widetilde{v}(\overline{s},\overline{y}+z)-\widetilde{v}(\overline{s},\overline{y})-\widetilde{u}(\overline{t},\overline{x}+z)+\widetilde{u}(\overline{t},\overline{x})\geq -\alpha(\phi(\overline{x}+z)-\phi(\overline{x})).$$

On en déduit que

$$G_\lambda[\widetilde{v}(\overline{s},\cdot),\overline{p}](\overline{y})-G_\lambda[\widetilde{u}(\overline{t},\cdot),\overline{p}+\alpha\nabla\phi(\overline{x})](\overline{x})\leq -\alpha g_\lambda[\phi](\overline{x})$$

(remarquons que, bien que ϕ ne soit pas dans $C_b^2(\mathbb{R}^N)$, $g_\lambda[\phi]$ est bien défini puisque $\phi(x)=|x|^{\lambda/2}$ à l'infini). En retranchant (E.21) à (E.22), on trouve alors

$$\begin{aligned}-\frac{\gamma}{T^2}-\alpha g_\lambda[\phi](\overline{x}) &\geq -\frac{\gamma}{(T-\overline{t})^2}+G_\lambda[\widetilde{v}(\overline{s},\cdot),\overline{p}](\overline{y})-G_\lambda[\widetilde{u}(\overline{t},\cdot),\overline{p}+\alpha\nabla\phi(\overline{x})](\overline{x})\\ &\geq \widetilde{F}(\overline{s},\overline{y},\widetilde{v}(\overline{s},\overline{y}),\overline{p})-\widetilde{F}(\overline{t},\overline{x},\widetilde{u}(\overline{t},\overline{x}),\overline{p}+\alpha\nabla\phi(\overline{x})).\end{aligned}\tag{E.23}$$

Par (E.14), on a $|\overline{p}|\leq\sqrt{2c'/\varepsilon}$ et, puisque $\phi=|\cdot|^{\lambda/2}$ à l'infini, $\nabla\phi$ est bornée sur $\mathbb{R}^N$. Les propriétés (E.11) et (E.12) pour $\widetilde{F}$ donnent alors

$$\begin{aligned}&|\widetilde{F}(\overline{s},\overline{y},\widetilde{v}(\overline{s},\overline{y}),\overline{p})-\widetilde{F}(\overline{t},\overline{x},\widetilde{v}(\overline{s},\overline{y}),\overline{p}+\alpha\nabla\phi(\overline{x})|\\ &\qquad\leq C|\overline{x}-\overline{y}|(1+|\overline{p}|)+|\widetilde{F}(\overline{s},\overline{x},\widetilde{v}(\overline{s},\overline{y}),\overline{p})-\widetilde{F}(\overline{t},\overline{x},\widetilde{v}(\overline{s},\overline{y}),\overline{p}+\alpha\nabla\phi(\overline{x})|\\ &\qquad\leq C|\overline{x}-\overline{y}|(1+|\overline{p}|)+\alpha C(\varepsilon)+|\widetilde{F}(\overline{s},\overline{x},\widetilde{v}(\overline{s},\overline{y}),\overline{p})-\widetilde{F}(\overline{t},\overline{x},\widetilde{v}(\overline{s},\overline{y}),\overline{p})|.\end{aligned}$$

Par (E.19), on a $\widetilde{u}(\overline{t},\overline{x})\geq\widetilde{v}(\overline{s},\overline{y})$ donc en injectant ceci dans (E.23) et en utilisant (E.13), on obtient

$$-\frac{\gamma}{T^2}+\alpha|g_\lambda[\phi](\overline{x})|\geq -C|\overline{x}-\overline{y}|(1+|\overline{p}|)-\alpha C(\varepsilon)-|\widetilde{F}(\overline{s},\overline{x},\widetilde{v}(\overline{s},\overline{y}),\overline{p})-\widetilde{F}(\overline{t},\overline{x},\widetilde{v}(\overline{s},\overline{y}),\overline{p})|.$$

Lorsque $\nu\to 0$, $\overline{x}$ et $\overline{p}$ restent bornés et $\overline{s}-\overline{t}\to 0$ avec $\overline{t}$ qui reste dans un compact de $[0,T[$ (voir (E.14)); par uniforme continuité de $\widetilde{F}$ sur les compacts de $[0,T[\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N$ (rappelons que $\widetilde{v}$ est bornée), on en déduit que $\lim_{\nu\to 0}|\widetilde{F}(\overline{s},\overline{x},\widetilde{v}(\overline{s},\overline{y}),\overline{p})-\widetilde{F}(\overline{t},\overline{x},\widetilde{v}(\overline{s},\overline{y}),\overline{p})|=0$ et donc que

$$-\frac{\gamma}{T^2}+\alpha\liminf_{\nu\to 0}|g_\lambda[\phi](\overline{x})|\geq -C\limsup_{\nu\to 0}(|\overline{x}-\overline{y}|(1+|\overline{p}|))-\alpha C(\varepsilon).$$

Puisque $\phi=|\cdot|^{\lambda/2}$ à l'infini, il n'est pas dur de voir que $g_\lambda[\phi]$ est bornée sur $\mathbb{R}^N$ ($D^2\phi$ est bornée sur $\mathbb{R}^N$ et, comme $\lambda/2\leq 1$, on a $|x+z|^{\lambda/2}\leq|x|^{\lambda/2}+|z|^{\lambda/2}$ donc, lorsque x est grand et $|z|\leq 1$, $|\phi(x+z)-\phi(x)|\leq|z|^{\lambda/2}$). On peut alors successivement prendre la limite inférieure quand $\alpha\to 0$ puis la limite quand $\varepsilon\to 0$ pour conclure, grâce à (E.18), que $-\frac{\gamma}{T^2}\geq 0$, ce qui donne la contradiction recherchée. ■

Remarque E.7 *Cette démonstration s'adapte directement aux cas où l'on considère des combinaisons linéaires à coefficients positifs d'opérateurs g_λ. Il suffit de prendre un ϕ qui vaut, au voisinage de l'infini, le plus petit des $|\cdot|^{\lambda/2}$ apparaissant.*

E.5 Lemme technique concernant la condition initiale

Nous prouvons ici un fait qui a été énoncé dans l'étape 2 de la preuve du théorème 3.3.

Lemme E.3 *Si $(\lambda,\mu)\in]0,2]$ et $u_0\in W^{1,\infty}(\mathbb{R}^N)$ alors $K_\lambda(t,\cdot)*K_\mu(\varepsilon t,\cdot)*u_0(x)\to u_0(x)$ lorsque $t\to 0$, uniformément par rapport à $x\in\mathbb{R}^N$ et $\varepsilon\in]0,1]$.*

Preuve du lemme E.3
Soit $v\in W^{1,\infty}(\mathbb{R}^N)$ et $\delta\in]0,2]$; on a, pour $\eta>0$,

$$\begin{aligned}|K_\delta(t,\cdot)*v(x)-v(x)| &\leq \int_{\mathbb{R}^N}K_\delta(t,y)|v(x-y)-v(y)|\,dy\\ &\leq ||\nabla v||_{L^\infty(\mathbb{R}^N)}\eta+2||v||_{L^\infty(\mathbb{R}^N)}\int_{|y|\geq\eta}K_\delta(t,y)\,dy.\end{aligned}\tag{E.24}$$

On déduit de ceci que

$$\begin{aligned}|K_\mu(\varepsilon t,\cdot)*u_0(x)-u_0(x)| &\leq ||\nabla u_0||_{L^\infty(\mathbb{R}^N)}\eta+2||u_0||_{L^\infty(\mathbb{R}^N)}\int_{|y|\geq\eta}K_\mu(\varepsilon t,y)\,dy\\ &\leq ||\nabla u_0||_{L^\infty(\mathbb{R}^N)}\eta+2||u_0||_{L^\infty(\mathbb{R}^N)}\sup_{0<s\leq t}\int_{|y|\geq\eta}K_\mu(s,y)\,dy.\end{aligned}$$

De plus, comme $\nabla(K_\mu(\varepsilon t,\cdot)*u_0)=K_\mu(\varepsilon t,\cdot)*\nabla u_0$, (E.24) et (3.16) donnent

$$\begin{aligned}&|K_\lambda(t,\cdot)*(K_\mu(\varepsilon t,\cdot)*u_0)(x)-K_\mu(\varepsilon t,\cdot)*u_0(x)|\\ &\quad\leq ||\nabla(K_\mu(\varepsilon t,\cdot)*u_0)||_{L^\infty(\mathbb{R}^N)}\eta+2||K_\mu(\varepsilon t,\cdot)*u_0||_{L^\infty(\mathbb{R}^N)}\int_{|y|\geq\eta}K_\lambda(t,y)\,dy\\ &\quad\leq ||\nabla u_0||_{L^\infty(\mathbb{R}^N)}\eta+2||u_0||_{L^\infty(\mathbb{R}^N)}\int_{|y|\geq\eta}K_\lambda(t,y)\,dy.\end{aligned}$$

Ainsi,

$$\begin{aligned}|K_\lambda(t,\cdot)*K_\mu(\varepsilon t,\cdot)*u_0(x)-u_0(x)| &\leq 2||\nabla u_0||_{L^\infty(\mathbb{R}^N)}\eta+2||u_0||_{L^\infty(\mathbb{R}^N)}\sup_{0<s\leq t}\int_{|y|\geq\eta}K_\mu(s,y)\,dy\\ &\quad+2||u_0||_{L^\infty(\mathbb{R}^N)}\int_{|y|\geq\eta}K_\lambda(t,y)\,dy.\end{aligned}$$

Comme $(K_\lambda(t,\cdot))_{t\to 0}$ et $(K_\mu(t,\cdot))_{t\to 0}$ sont des approximations de l'unité, cela conclut la preuve. ■

Remarque E.8 *Ce lemme est aussi valable si on supprime K_λ, donc la preuve du théorème 3.3 marche aussi quand on considère une pure équation de Hamilton-Jacobi.*

E.6 Estimation d'erreur

E.6.1 Preuve du théorème 3.4

Par la transformation usuelle (voir le début de la preuve du théorème E.2: on a des estimations sur u^ε indépendantes de ε), on se ramène au cas où F est décroissant en s.
Posons

$$M=\sup_{[0,T[\times(\mathbb{R}^N)^2}\left\{u(t,x)-u^\varepsilon(t,y)-\frac{|x-y|^2}{2\alpha}-\frac{\beta|x|^2}{2}-\eta t-\frac{\gamma}{T-t}\right\}$$

(avec α, β, η strictement positifs et γ dans $]0,1]$). Comme u et u^ε sont continues et bornées, ce supremum est atteint.

Etape 1: nous montrons qu'on peut choisir η et γ tels que M soit atteint en $t=0$.
Soit

$$M_\nu=\sup_{[0,T[^2\times(\mathbb{R}^N)^2}\left\{u(t,x)-u^\varepsilon(s,y)-\frac{|x-y|^2}{2\alpha}-\frac{|t-s|^2}{2\nu}-\frac{\beta|x|^2}{2}-\eta t-\frac{\gamma}{T-t}\right\}\geq M.$$

Pour ν assez petit, ce supremum est atteint en $(t_\nu,s_\nu,x_\nu,y_\nu)\in[0,T[^2\times(\mathbb{R}^N)^2$. En prenant $t=s=0$ et $x=y=0$, on voit que M_ν est minoré par $-\frac{\gamma}{T}\geq-\frac{1}{T}$, donc

$$-\frac{1}{T}\leq||u||_\infty+||u^\varepsilon||_\infty-\frac{|x_\nu-y_\nu|^2}{2\alpha}-\frac{|t_\nu-s_\nu|^2}{2\nu}-\frac{\beta|x_\nu|^2}{2}-\eta t_\nu-\frac{\gamma}{T-t_\nu}$$

et

$$\frac{|x_\nu-y_\nu|^2}{2\alpha}\leq c\,,\quad\frac{|t_\nu-s_\nu|^2}{2\nu}\leq c\,,\quad\frac{\beta|x_\nu|^2}{2}\leq c\quad\text{et}\quad\frac{\gamma}{T-t_\nu}\leq c\tag{E.25}$$

avec $c = \frac{1}{T} + ||u||_\infty + ||u^\varepsilon||_\infty$. La fonction $\nu \to M_\nu$ est croissante minorée: elle converge donc en 0; on a de plus

$$\begin{aligned} M_{2\nu} - M_\nu &\geq u(t_\nu, x_\nu) - u^\varepsilon(s_\nu, y_\nu) - \frac{|x_\nu - y_\nu|^2}{2\alpha} - \frac{|t_\nu - s_\nu|^2}{2 \times (2\nu)} - \frac{\beta|x_\nu|^2}{2} - \eta t_\nu - \frac{\gamma}{T - t_\nu} \\ &\quad - \left(u(t_\nu, x_\nu) - u^\varepsilon(s_\nu, y_\nu) - \frac{|x_\nu - y_\nu|^2}{2\alpha} - \frac{|t_\nu - s_\nu|^2}{2\nu} - \frac{\beta|x_\nu|^2}{2} - \eta t_\nu - \frac{\gamma}{T - t_\nu} \right) \\ &= \frac{|t_\nu - s_\nu|^2}{4\nu} \end{aligned}$$

et donc $\frac{|t_\nu - s_\nu|^2}{\nu} \to 0$ lorsque $\nu \to 0$. Par (E.25), quitte à extraire une suite on peut supposer que $(t_\nu, s_\nu, x_\nu, y_\nu) \to (\overline{t}, \overline{t}, \overline{x}, \overline{y})$ avec $\overline{t} \in [0, T[$ et on a alors, par continuité de u et u^ε,

$$\begin{aligned} \lim_{\nu \to 0} M_\nu &= \lim_{\nu \to 0} \left(u(t_\nu, x_\nu) - u^\varepsilon(s_\nu, y_\nu) - \frac{|x_\nu - y_\nu|^2}{2\alpha} - \frac{|t_\nu - s_\nu|^2}{2\nu} - \frac{\beta|x_\nu|^2}{2} - \eta t_\nu - \frac{\gamma}{T - t_\nu} \right) \\ &= u(\overline{t}, \overline{x}) - u^\varepsilon(\overline{t}, \overline{y}) - \frac{|\overline{x} - \overline{y}|^2}{2\alpha} - \frac{\beta|\overline{x}|^2}{2} - \eta\overline{t} - \frac{\gamma}{T - \overline{t}} \\ &\leq M. \end{aligned}$$

Comme on avait $M_\nu \geq M$, cela prouve que $M_\nu \to M$ lorsque $\nu \to 0$ (en fait à une sous-suite près, dont on peut se débarasser puisque l'on sait que $(M_\nu)_{\nu > 0}$ converge en 0) et que M est atteint en $(\overline{t}, \overline{x}, \overline{y})$.

Si l'on suppose que $\overline{t} > 0$, alors pour ν assez petit on a $t_\nu > 0$ et $s_\nu > 0$. La définition de M_ν implique que

$$\left(\frac{\gamma}{(T - t_\nu)^2} + \frac{t_\nu - s_\nu}{\nu} + \eta, p_\nu + \beta x_\nu \right) \in \partial^P u(t_\nu, x_\nu) \quad \text{et} \quad \left(\frac{t_\nu - s_\nu}{\nu}, p_\nu \right) \in \partial_P u^\varepsilon(s_\nu, y_\nu),$$

avec $p_\nu = \frac{x_\nu - y_\nu}{\alpha}$. On a donc

$$\begin{aligned} &\frac{\gamma}{(T - t_\nu)^2} + \frac{t_\nu - s_\nu}{\nu} + \eta \\ &- c_N(\lambda) \int_{B_r} \frac{u(t_\nu, x_\nu + z) - u(t_\nu, x_\nu) - (p_\nu + \beta x_\nu) \cdot z}{|z|^{N+\lambda}} \, dz \\ &- c_N(\lambda) \int_{\mathbb{R}^N \setminus B_r} \frac{u(t_\nu, x_\nu + z) - u(t_\nu, x_\nu)}{|z|^{N+\lambda}} \, dz \\ &\qquad \leq F(t_\nu, x_\nu, u(t_\nu, x_\nu), p_\nu + \beta x_\nu) \end{aligned}$$

et

$$\begin{aligned} &\frac{t_\nu - s_\nu}{\nu} \\ &- c_N(\lambda) \int_{B_r} \frac{u^\varepsilon(s_\nu, y_\nu + z) - u^\varepsilon(s_\nu, y_\nu) - p_\nu \cdot z}{|z|^{N+\lambda}} \, dz \\ &- c_N(\lambda) \int_{\mathbb{R}^N \setminus B_r} \frac{u^\varepsilon(s_\nu, y_\nu + z) - u^\varepsilon(s_\nu, y_\nu)}{|z|^{N+\lambda}} \, dz \\ &- \varepsilon c_N(\mu) \int_{B_R} \frac{u^\varepsilon(s_\nu, y_\nu + z) - u^\varepsilon(s_\nu, y_\nu) - p_\nu \cdot z}{|z|^{N+\mu}} \, dz \\ &- \varepsilon c_N(\mu) \int_{\mathbb{R}^N \setminus B_R} \frac{u^\varepsilon(s_\nu, y_\nu + z) - u^\varepsilon(s_\nu, y_\nu)}{|z|^{N+\mu}} \, dz \\ &\qquad \geq F(s_\nu, y_\nu, u^\varepsilon(s_\nu, y_\nu), p_\nu). \end{aligned}$$

En retranchant ces deux équations, on trouve

$$
\begin{aligned}
&-\frac{\gamma}{(T-t_\nu)^2}-\eta\\
&-c_N(\lambda)\int_{B_r}\frac{u^\varepsilon(s_\nu,y_\nu+z)-u^\varepsilon(s_\nu,y_\nu)-u(t_\nu,x_\nu+z)+u(t_\nu,x_\nu)+\beta x_\nu\cdot z}{|z|^{N+\lambda}}\,dz\\
&-c_N(\lambda)\int_{\mathbb{R}^N\setminus B_r}\frac{u^\varepsilon(s_\nu,y_\nu+z)-u^\varepsilon(s_\nu,y_\nu)-u(t_\nu,x_\nu+z)+u(t_\nu,x_\nu)}{|z|^{N+\lambda}}\,dz\\
&-\varepsilon c_N(\mu)\int_{B_R}\frac{u^\varepsilon(s_\nu,y_\nu+z)-u^\varepsilon(s_\nu,y_\nu)-p_\nu\cdot z}{|z|^{N+\mu}}\,dz\\
&-\varepsilon c_N(\mu)\int_{B_1\setminus B_R}\frac{u^\varepsilon(s_\nu,y_\nu+z)-u^\varepsilon(s_\nu,y_\nu)}{|z|^{N+\mu}}\,dz-\varepsilon c_N(\mu)\int_{\mathbb{R}^N\setminus B_1}\frac{u^\varepsilon(s_\nu,y_\nu+z)-u^\varepsilon(s_\nu,y_\nu)}{|z|^{N+\mu}}\,dz\\
&\qquad\geq F(s_\nu,y_\nu,u^\varepsilon(s_\nu,y_\nu),p_\nu)-F(t_\nu,x_\nu,u(t_\nu,x_\nu),p_\nu+\beta x_\nu)
\end{aligned}
\tag{E.26}
$$

(on prend $R\leq 1$). Par definition de M_ν, on a

$$
\begin{aligned}
&u(t_\nu,x_\nu)-u^\varepsilon(s_\nu,y_\nu)-\frac{|x_\nu-y_\nu|^2}{2\alpha}-\frac{|t_\nu-s_\nu|^2}{2\nu}-\frac{\beta|x_\nu|^2}{2}-\eta t_\nu-\frac{\gamma}{T-t_\nu}\\
&\quad\geq u(t_\nu,x_\nu+z)-u^\varepsilon(s_\nu,y_\nu+z)-\frac{|x_\nu+z-y_\nu-z|^2}{2\alpha}-\frac{|t_\nu-s_\nu|^2}{2\nu}-\frac{\beta|x_\nu+z|^2}{2}-\eta t_\nu-\frac{\gamma}{T-t_\nu}.
\end{aligned}
$$

Ainsi,

$$
u^\varepsilon(s_\nu,y_\nu+z)-u^\varepsilon(s_\nu,y_\nu)+u(t_\nu,x_\nu)-u(t_\nu,x_\nu+z)\geq\frac{\beta|x_\nu|^2}{2}-\frac{\beta|x_\nu+z|^2}{2}=-\beta x_\nu\cdot z-\frac{\beta|z|^2}{2}
$$

et on déduit de (E.26), en utilisant le fait que u et u^ε sont bornées (indépendamment de ε) et que u^ε est lipschitzienne en espace avec une constante uniforme en temps et indépendante de ε, que

$$
\begin{aligned}
&-\frac{\gamma}{(T-t_\nu)^2}-\eta\\
&+c_N(\lambda)\frac{\beta}{2}\int_{B_r}\frac{|z|^2}{|z|^{N+\lambda}}\,dz\\
&+C\int_{\mathbb{R}^N\setminus B_r}\frac{1}{|z|^{N+\lambda}}\,dz\\
&-\varepsilon c_N(\mu)\int_{B_R}\frac{u^\varepsilon(s_\nu,y_\nu+z)-u^\varepsilon(s_\nu,y_\nu)-p_\nu\cdot z}{|z|^{N+\mu}}\,dz\\
&+C\varepsilon\int_{B_1\setminus B_R}\frac{|z|}{|z|^{N+\mu}}\,dz+C\varepsilon\int_{\mathbb{R}^N\setminus B_1}\frac{1}{|z|^{N+\mu}}\,dz\\
&\qquad\geq F(s_\nu,y_\nu,u^\varepsilon(s_\nu,y_\nu),p_\nu)-F(t_\nu,x_\nu,u(t_\nu,x_\nu),p_\nu+\beta x_\nu)
\end{aligned}
\tag{E.27}
$$

(ici et dans la suite, C désigne des constantes, qui peuvent changer d'une ligne à l'autre, qui ne dépendent pas de ε, r, R, ν, α, β, η et γ). La définition de M_ν donne encore, pour $z\in\mathbb{R}^N$,

$$
\begin{aligned}
&u(t_\nu,x_\nu)-u^\varepsilon(s_\nu,y_\nu)-\frac{|x_\nu-y_\nu|^2}{2\alpha}-\frac{|t_\nu-s_\nu|^2}{2\nu}-\frac{\beta|x_\nu|^2}{2}-\eta t_\nu-\frac{\gamma}{T-t_\nu}\\
&\quad\geq u(t_\nu,x_\nu)-u^\varepsilon(s_\nu,y_\nu+z)-\frac{|x_\nu-y_\nu-z|^2}{2\alpha}-\frac{|t_\nu-s_\nu|^2}{2\nu}-\frac{\beta|x_\nu|^2}{2}-\eta t_\nu-\frac{\gamma}{T-t_\nu}
\end{aligned}
$$

et donc

$$
u^\varepsilon(s_\nu,y_\nu+z)-u^\varepsilon(s_\nu,y_\nu)-p_\nu\cdot z\geq\frac{|x_\nu-y_\nu|^2-|x_\nu-y_\nu-z|^2}{2\alpha}-\frac{2(x_\nu-y_\nu)\cdot z}{2\alpha}=-\frac{|z|^2}{2\alpha}.
$$

Ainsi, on peut borner le terme en u^ε de (E.27) et passer à la limite $\nu \to 0$ pour trouver, en posant $\overline{p} = \frac{\overline{x}-\overline{y}}{\alpha} = \lim_{\nu\to 0} p_\nu$,

$$\begin{aligned}
&-\frac{\gamma}{(T-\overline{t})^2} - \eta + C\beta \int_{B_r} \frac{|z|^2}{|z|^{N+\lambda}}\, dz + C \int_{\mathbb{R}^N\setminus B_r} \frac{1}{|z|^{N+\lambda}}\, dz \\
&+\varepsilon \frac{c_N(\mu)}{2\alpha} \int_{B_R} \frac{|z|^2}{|z|^{N+\mu}}\, dz + C\varepsilon \int_{B_1\setminus B_R} \frac{|z|}{|z|^{N+\mu}}\, dz + C\varepsilon \int_{\mathbb{R}^N\setminus B_1} \frac{1}{|z|^{N+\mu}}\, dz \\
&\qquad \geq F(\overline{t},\overline{y},u^\varepsilon(\overline{t},\overline{y}),\overline{p}) - F(\overline{t},\overline{x},u(\overline{t},\overline{x}),\overline{p}+\beta\overline{x}).
\end{aligned}$$

En prenant $t = 0$ et $x = y = 0$ dans la définition de M, on a $M \geq -\frac{\gamma}{T}$ et donc $u(\overline{t},\overline{x}) - u^\varepsilon(\overline{t},\overline{y}) = M + \frac{|\overline{x}-\overline{y}|^2}{2\alpha} + \frac{\beta|\overline{x}|^2}{2} + \eta\overline{t} + \frac{\gamma}{T-\overline{t}} \geq \frac{\gamma}{T-\overline{t}} - \frac{\gamma}{T} \geq 0$; la décroissance de F par rapport à sa troisième variable implique alors

$$\begin{aligned}
&-\frac{\gamma}{(T-\overline{t})^2} - \eta + C\beta \int_{B_r} \frac{|z|^2}{|z|^{N+\lambda}}\, dz + C \int_{\mathbb{R}^N\setminus B_r} \frac{1}{|z|^{N+\lambda}}\, dz \\
&+\frac{C\varepsilon}{\alpha} \int_{B_R} \frac{|z|^2}{|z|^{N+\mu}}\, dz + C\varepsilon \int_{B_1\setminus B_R} \frac{|z|}{|z|^{N+\mu}}\, dz + C\varepsilon \\
&\qquad \geq F(\overline{t},\overline{y},u(\overline{t},\overline{x}),\overline{p}) - F(\overline{t},\overline{x},u(\overline{t},\overline{x}),\overline{p}+\beta\overline{x}). \qquad \text{(E.28)}
\end{aligned}$$

Par (E.25), on a $\beta|\overline{x}| \leq C\sqrt{\beta}$; de plus, comme $(\frac{t_\nu - s_\nu}{\nu}, p_\nu) \in \partial_P u^\varepsilon(s_\nu, y_\nu)$ et u^ε est lipschitzienne en espace (uniformément en temps et en $\varepsilon \in]0,1]$), on a $|p_\nu| \leq C$, donc $|\overline{p}| \leq C$ et $|\overline{x}-\overline{y}| \leq C\alpha$. Par (3.21) et (3.22), on obtient donc (en prenant $\beta \leq 1$)

$$\begin{aligned}
F(\overline{t},\overline{y},u(\overline{t},\overline{x}),\overline{p}) - F(\overline{t},\overline{x},u(\overline{t},\overline{x}),\overline{p}+\beta\overline{x}) &\geq -C|\overline{x}-\overline{y}| + F(\overline{t},\overline{x},u(\overline{t},\overline{x}),\overline{p}) - F(\overline{t},\overline{x},u(\overline{t},\overline{x}),\overline{p}+\beta\overline{x}) \\
&\geq -C\alpha - C\sqrt{\beta}.
\end{aligned}$$

Ré-injecté dans (E.28), cela conduit à

$$\begin{aligned}
&C\alpha + C\sqrt{\beta} + C\beta \int_{B_r} \frac{|z|^2}{|z|^{N+\lambda}}\, dz + C \int_{\mathbb{R}^N\setminus B_r} \frac{1}{|z|^{N+\lambda}}\, dz \\
&+\frac{C\varepsilon}{\alpha} \int_{B_R} \frac{|z|^2}{|z|^{N+\mu}}\, dz + C\varepsilon \int_{B_1\setminus B_R} \frac{|z|}{|z|^{N+\mu}}\, dz + C\varepsilon \\
&\qquad \geq \frac{\gamma}{(T-\overline{t})^2} + \eta \geq \frac{\gamma}{T^2} + \eta.
\end{aligned}$$

On a

$$\int_{B_R} \frac{|z|^2}{|z|^{N+\mu}}\, dz = \int_{B_1} |w|^{2-N-\mu} R^{2-N-\mu} R^N\, dw = CR^{2-\mu}\,,$$

$$\int_{\mathbb{R}^N\setminus B_r} \frac{1}{|z|^{N+\lambda}}\, dz = \int_{B_1} |w|^{-N-\lambda} r^{-N-\lambda} r^N\, dw = Cr^{-\lambda}$$

et

$$\int_{B_1\setminus B_R} \frac{|z|}{|z|^{N+\mu}}\, dz = C\int_R^1 \rho^{-\mu}\, d\rho \leq A_\mu(R) := \begin{cases} C & \text{si } \mu < 1\,, \\ C|\ln(R)| & \text{si } \mu = 1\,, \\ CR^{1-\mu} & \text{si } \mu > 1 \end{cases}$$

donc on arrive à

$$C\sqrt{\beta} + C\beta r^{2-\lambda} + Cr^{-\lambda} + C\alpha + \frac{C\varepsilon}{\alpha} R^{2-\mu} + C\varepsilon A_\mu(R) + C\varepsilon \geq \frac{\gamma}{T^2} + \eta.$$

On prend alors r tel que $\beta r^{2-\lambda} = \sqrt{\beta}$ (par exemple), c'est à dire $r = \beta^{-\frac{1}{2(2-\lambda)}}$, et on trouve

$$C\sqrt{\beta} + C\beta^{\frac{\lambda}{2(2-\lambda)}} + C\alpha + \frac{C\varepsilon}{\alpha} R^{2-\mu} + C\varepsilon A_\mu(R) + C\varepsilon \geq \frac{\gamma}{T^2} + \eta.$$

En prenant $\eta = C\alpha + \frac{C\varepsilon}{\alpha}R^{2-\mu} + C\varepsilon A_\mu(R) + C\varepsilon$ et $\gamma = (C\sqrt{\beta} + C\beta^{\frac{\lambda}{2(2-\lambda)}} + \beta)T^2$, on aboutit à $0 \geq \beta$, ce qui est une contradiction.

Etape 2: conclusion.
On a montré ci-dessus que, pour tout $(\varepsilon, \alpha, \beta, R) \in]0,1]^4$, en posant $\eta = C\alpha + \frac{C\varepsilon}{\alpha}R^{2-\mu} + C\varepsilon A_\mu(R) + C\varepsilon$ et $\gamma = (C\sqrt{\beta} + C\beta^{\frac{\lambda}{2(2-\lambda)}} + \beta)T^2$ (si β est assez petit, γ est bien inférieur à 1), M est atteint en $(0, \overline{x}, \overline{y})$. Par définition de M cela signifie que, pour tout $(t,x) \in [0,T[\times\mathbb{R}^N$,

$$u(t,x) - u^\varepsilon(t,x) - \frac{\beta|x|^2}{2} - \eta t - \frac{\gamma}{T-t} \leq u_0(\overline{x}) - u_0(\overline{y}) - \frac{|\overline{x}-\overline{y}|^2}{2\alpha} - \frac{\beta|\overline{x}|^2}{2} - \frac{\gamma}{T} \leq u_0(\overline{x}) - u_0(\overline{y}) - \frac{|\overline{x}-\overline{y}|^2}{2\alpha}.$$

Comme u_0 est lipschitzienne, on a $u_0(\overline{x}) - u_0(\overline{y}) - \frac{|\overline{x}-\overline{y}|^2}{2\alpha} \leq C|\overline{x}-\overline{y}| - \frac{|\overline{x}-\overline{y}|^2}{2\alpha}$; le maximum de la parabole $X \to CX - \frac{X^2}{2\alpha}$ est $\frac{C^2\alpha}{2}$, et donc $u_0(\overline{x}) - u_0(\overline{y}) - \frac{|\overline{x}-\overline{y}|^2}{2\alpha} \leq \frac{C^2\alpha}{2}$. Ainsi, pour tout $(t,x) \in [0,T[\times\mathbb{R}^N$,

$$u(t,x) \leq u^\varepsilon(t,x) + \frac{\beta|x|^2}{2} + \left(C\alpha + \frac{C\varepsilon}{\alpha}R^{2-\mu} + C\varepsilon A_\mu(R) + C\varepsilon\right)T + \frac{(C\sqrt{\beta} + C\beta^{\frac{\lambda}{2(2-\lambda)}} + \beta)T^2}{T-t} + C\alpha.$$

On fait maintenant tendre β vers 0:

$$u(t,x) \leq u^\varepsilon(t,x) + C\alpha + \frac{C\varepsilon}{\alpha}R^{2-\mu} + C\varepsilon A_\mu(R) + C\varepsilon$$

et il faut ensuite étudier la situation selon la position de μ par rapport à 1.

Si $\mu < 1$ alors $A_\mu(R) = C$ et $u(t,x) \leq u^\varepsilon(t,x) + C\alpha + \frac{C\varepsilon}{\alpha}R^{2-\mu} + C\varepsilon$. On fait alors tendre R puis α vers 0 et on trouve $u(t,x) \leq u^\varepsilon(t,x) + C\varepsilon$.

Si $\mu = 1$ alors $A_\mu(R) = C|\ln(R)|$ et $u(t,x) \leq u^\varepsilon(t,x) + C\alpha + \frac{C\varepsilon}{\alpha}R + C\varepsilon|\ln(R)|$ (on a pris $R \leq 1/2$ par exemple). En optimisant en R on va prendre $R = \alpha$ et obtenir $u(t,x) \leq u^\varepsilon(t,x) + C\alpha + C\varepsilon + C\varepsilon|\ln(\alpha)| \leq u^\varepsilon(t,x) + C\alpha + C\varepsilon|\ln(\alpha)|$ (on prend $\alpha \leq 1/2$). Le choix optimal est alors $\alpha = \varepsilon$, ce qui conduit à $u(t,x) \leq u^\varepsilon(t,x) + C\varepsilon|\ln(\varepsilon)|$.

Si $\mu > 1$ alors $A_\mu(R) = CR^{1-\mu}$ et $u(t,x) \leq u^\varepsilon(t,x) + C\alpha + \frac{C\varepsilon}{\alpha}R^{2-\mu} + C\varepsilon R^{1-\mu}$ (on prend toujours $R \leq 1/2$). Une optimisation en R donne comme choix $R = \alpha$ (à une constante multiplicative près), d'où $u(t,x) \leq u^\varepsilon(t,x) + C\alpha + C\varepsilon\alpha^{1-\mu}$; optimisant en α, on est amené à prendre $\alpha = \varepsilon^{1/\mu}$ et on trouve $u(t,x) \leq u^\varepsilon(t,x) + C\varepsilon^{1/\mu}$.

Dans tous les cas, on obtient bien la moitié de l'inégalité souhaitée et on conclut en intervertissant, dès le début, les rôles de u et u^ε. ■

E.6.2 Optimalité de l'erreur

On commence par prouver la chose suivante.

Lemme E.4 *Soit $\mu \in]0,2[$. Il existe $C_\mu > 0$ tel que, au voisinage de l'infini, on a $K_\mu(1,x) \sim C_\mu|x|^{-N-\mu}$.*

Preuve du lemme E.4
Soit $\theta \in C_c^\infty(\mathbb{R}^N)$ paire d'intégrale sur $\mathbb{R}^N$ égale à 1. On pose $\chi = \mathcal{F}(\theta) \in \mathcal{S}(\mathbb{R}^N)$; puisque θ est d'intégrale 1, on a $\chi(0) = 1$ et, puisque θ est paire, $\nabla\chi(0) = -2i\pi\int_{\mathbb{R}^N} x\theta(x)\,dx = 0$. Ainsi, $\chi(\xi) - 1 = \mathcal{O}(|\xi|^2)$ au voisinage de 0.
Soit $n \geq 1$ et

$$f(\xi) = e^{-|\xi|^\mu} - \chi(\xi)\left(1 - |\xi|^\mu + \frac{|\xi|^{2\mu}}{2!} + \cdots + \frac{(-1)^n|\xi|^{n\mu}}{n!}\right).$$

Clairement, toutes les dérivées de f sont intégrables à l'infini. Sur $B(0,1)$, on a

$$f(\xi) = (1-\chi(\xi))\left(1 - |\xi|^\mu + \frac{|\xi|^{2\mu}}{2!} + \cdots + \frac{(-1)^n|\xi|^{n\mu}}{n!}\right) + G(\xi)$$

avec $G(\xi) = \sum_{k\geq n+1} \frac{(-1)^k|\xi|^{k\mu}}{k!}$. Comme les séries dérivées convergent uniformément sur $0 < a \leq |\xi| \leq 1$, on a $\partial^\alpha G(\xi) = \sum_{k\geq n+1} \frac{(-1)^k\partial^\alpha|\xi|^{k\mu}}{k!}$ sur $B(0,1)\backslash\{0\}$; mais, pour $k \geq n+1$, $|\partial^\alpha|\xi|^{k\mu}| \leq (k\mu)^{|\alpha|}|\xi|^{k\mu-|\alpha|} \leq (k\mu)^{|\alpha|}|\xi|^{(n+1)\mu-|\alpha|}$ sur $B(0,1)$. Pourvu que $(n+1)\mu-|\alpha| > -N$, cette expression de $\partial^\alpha G$ sur $B(0,1)\backslash\{0\}$ est intégrable sur $B(0,1)$, et coincide donc avec $\partial^\alpha G$ sur $B(0,1)$ en entier. Comme $1-\chi(\xi) = \mathcal{O}(|\xi|^2)$ et $\nabla\chi(\xi) = \mathcal{O}(|\xi|)$ au voisinage de 0, la formule de Leibniz montre que $\partial^\alpha((1-\chi(\xi))|\xi|^{p\mu})$ est intégrable sur $B(0,1)$ pourvu que $p\mu - |\alpha| + 2 > -N$.
Si on prend donc $n \geq 1$ tel que $(n+1)\mu > [\mu]+1$, alors $\partial^\alpha f \in L^1(\mathbb{R}^N)$ pourvu que $|\alpha| \leq N+[\mu]+1$ (car on a alors $(n+1)\mu-|\alpha| > [\mu]+1-N-[\mu]-1 = -N$ et, pour $p \in [1,n]$, $p\mu-|\alpha|+2 \geq \mu-N-[\mu]-1+2 \geq -N+1 > -N$); en particulier, cela signifie que

$$|\mathcal{F}(f)(x)| \leq \frac{C_0}{1+|x|^{N+[\mu]+1}} = o(|x|^{-N-\mu}) \quad \text{à l'infini} \tag{E.29}$$

(puisque $[\mu]+1 > \mu$).

Pour tout $p \in [1,n]$, on a

$$\mathcal{F}(\chi|\cdot|^{p\mu})(x) = \theta * \mathcal{F}(|\cdot|^{p\mu})(x) = \langle \mathcal{F}(|\cdot|^{p\mu}), \theta(x-\cdot)\rangle_{\mathcal{S}'(\mathbb{R}^N),\mathcal{S}(\mathbb{R}^N)}; \tag{E.30}$$

pour voir cela, il suffit de prendre $\varphi \in C_c^\infty(\mathbb{R}^N)$, de noter que

$$\langle \mathcal{F}(\chi|\cdot|^{p\mu}), \varphi\rangle = \langle |\cdot|^{p\mu}, \chi\mathcal{F}(\varphi)\rangle = \langle |\cdot|^{p\mu}, \mathcal{F}(\mathcal{F}^{-1}(\chi) * \varphi)\rangle = \langle \mathcal{F}(|\cdot|^{p\mu}), \theta^\vee * \varphi\rangle$$

puis d'utiliser la définition de la convolution d'une distribution $\mathcal{F}(|\cdot|^{p\mu})$ par une fonction $\theta \in C_c^\infty(\mathbb{R}^N)$. On sait que $\mathcal{F}(|\cdot|^{p\mu}) = M_{p\mu}|\cdot|^{-N-p\mu}$ dans $\mathcal{D}'(\mathbb{R}^N\backslash\{0\})$. Soit B_R qui contient le support de θ; pour $|x| > R$, $\theta(x-\cdot) \in C_c^\infty(\mathbb{R}^N\backslash\{0\})$ et (E.30) donne donc

$$\begin{aligned}
\mathcal{F}(\chi|\cdot|^{p\mu})(x) &= M_{p\mu}\int_{\mathbb{R}^N} \theta(x-y)|y|^{-N-p\mu}\,dy \\
&= M_{p\mu}\int_{B_R} \theta(y)|x-y|^{-N-p\mu}\,dy \\
&= M_{p\mu}|x|^{-N-p\mu}\int_{B_R} \theta(y)\left|\frac{x}{|x|} - \frac{y}{|x|}\right|^{-N-p\mu} dy.
\end{aligned} \tag{E.31}$$

La fonction $|\cdot|^{-N-\mu}$ est régulière sur la couronne $\{1/2 \leq |z| \leq 3/2\}$, qui contient le segment $[\frac{x}{|x|}, \frac{x}{|x|} - \frac{y}{|x|}]$ dès que $|x| \geq 2R$ et $|y| \leq R$; dans cette situation, le théorème des accroissements finis donne donc

$$\left|\left|\frac{x}{|x|} - \frac{y}{|x|}\right|^{-N-p\mu} - 1\right| = \left|\left|\frac{x}{|x|} - \frac{y}{|x|}\right|^{-N-p\mu} - \left|\frac{x}{|x|}\right|^{-N-p\mu}\right| \leq A\frac{|y|}{|x|} \leq \frac{AR}{|x|}.$$

On en déduit que la fonction $y \to \left|\frac{x}{|x|} - \frac{y}{|x|}\right|^{-N-p\mu}$ converge uniformément sur B_R vers 1 lorsque $|x| \to \infty$ et (E.31) montre que

$$|x|^{N+p\mu}\mathcal{F}(\chi|\cdot|^{p\mu})(x) \to M_{p\mu}\int_{B_R} \theta(y)\,dy = M_{p\mu} \quad \text{lorsque } |x| \to \infty. \tag{E.32}$$

Prouvons $M_\mu \neq 0$. Si ce n'était pas le cas, $\mathcal{F}(|\cdot|^\mu)$ serait une distribution portée par 0, donc une combinaison de dérivées de masses de Dirac: $\mathcal{F}(|\cdot|^\mu) = \sum_{l=1}^m \beta_l\partial^{\alpha_l}\delta_0$. Comme $\mathcal{F}^{-1}(\partial^{\alpha_l}\delta_0)(\xi) = (-2i\pi)^{|\alpha_l|}\xi^{\alpha_l}$,

on en déduirait que $|\xi|^\mu = \sum_{l=1}^m \beta_l(-2i\pi)^{|\alpha_l|}\xi^{\alpha_l}$, donc que $|\cdot|^\mu$ est une fonction C^∞ en 0, ce qui n'est pas le cas puisque $\mu \in]0,2[$.

Comme

$$K_\mu(1,\cdot) = \mathcal{F}(e^{-|\cdot|^\mu}) = \mathcal{F}(f) + \mathcal{F}(\chi) + \sum_{p=1}^n \frac{(-1)^p}{p!}\mathcal{F}(\chi|\cdot|^{p\mu})$$

et $M_\mu \neq 0$, (E.29) et (E.32) donnent $C_\mu \neq 0$ tel que $K_\mu(1,\cdot) \sim C_\mu|\cdot|^{-N-\mu}$ (le terme dominant à l'infini est celui correspondant à $p=1$ dans la somme ci-dessus, et (E.32) exprime que ce terme est équivalent à M_μ puisque ce réel est non-nul), et la positivité de C_μ vient du fait que K_μ est lui-même positif. ■

On considère maintenant l'équation $\partial_t u^\varepsilon + \varepsilon g_\mu[u^\varepsilon] = 0$ (*i.e.* $F=0$ et sans g_λ), avec $u_0(x) = \max(1-|x|,0)$. L'équation limite est $\partial_t u = 0$, donc $u(t,x) = u_0(x)$. On a $u^\varepsilon(1,x) = K_\mu(\varepsilon,\cdot) * u_0(x)$ avec $K_\mu(\varepsilon,x) = \varepsilon^{-N/\mu}K_\mu(1,\varepsilon^{-1/\mu}x)$, donc

$$\begin{aligned} u^\varepsilon(1,0) &= \int_{B_1} \varepsilon^{-N/\mu}K_\mu(1,\varepsilon^{-1/\mu}y)(1-|y|)\,dy \\ &= \int_{B_{\varepsilon^{-1/\mu}}} K_\mu(1,z)(1-\varepsilon^{1/\mu}|z|)\,dz. \end{aligned}$$

Mais $u(1,0) = u_0(0) = 1 = \int_{\mathbb{R}^N} K_\mu(1,z)\,dz$, d'où

$$u^\varepsilon(1,0) - u(1,0) = -\varepsilon^{1/\mu}\int_{B_{\varepsilon^{-1/\mu}}} K_\mu(1,z)|z|\,dz - \int_{\mathbb{R}^N\setminus B_{\varepsilon^{-1/\mu}}} K_\mu(1,z)\,dz. \tag{E.33}$$

Si $\mu < 1$, on ne garde dans le membre de droite de (E.33) que le deuxième terme et on écrit, pour ε assez petit, en utilisant le lemme E.4,

$$u^\varepsilon(1,0) - u(1,0) \leq -\int_{\mathbb{R}^N\setminus B_{\varepsilon^{-1/\mu}}} K_\mu(1,z)\,dz \leq -\frac{C_\mu}{2}\int_{\{|z|\geq\varepsilon^{-1/\mu}\}} |z|^{-N-\mu}\,dz \leq -C(\varepsilon^{-1/\mu})^{-\mu} = -C\varepsilon$$

(avec $C>0$ indépendant de ε), ce qui prouve que l'estimation en $\mathcal{O}(\varepsilon)$ est optimale.
Si $\mu = 1$, alors c'est le premier terme du membre de droite de (E.33) qui va être dominant: pour ε assez petit et R assez grand (indépendant de ε), par le lemme E.4,

$$\begin{aligned} u^\varepsilon(1,0) - u(1,0) &\leq -\varepsilon\int_{B_{\varepsilon^{-1}}} K_1(1,z)|z|\,dz \\ &\leq -\varepsilon\int_{B_{\varepsilon^{-1}}\setminus B_R} K_1(1,z)|z|\,dz \\ &\leq -\frac{C_1}{2}\varepsilon\int_{B_{\varepsilon^{-1}}\setminus B_R} |z|^{-N}\,dz \\ &\leq -C\varepsilon\int_R^{\varepsilon^{-1}} r^{-1}\,dr \\ &= -C\varepsilon(\ln(\varepsilon^{-1}) - \ln(R)) \end{aligned}$$

($C>0$ ne dépend pas de ε), ce qui prouve que l'estimation en $\varepsilon|\ln(\varepsilon)|$ est optimale dans ce cas.
Si $\mu > 1$, on conserve toujours le premier terme du membre de droite de (E.33):

$$u^\varepsilon(1,0) - u(1,0) \leq -\varepsilon^{1/\mu}\int_{B_{\varepsilon^{-1/\mu}}} K_\mu(1,z)|z|\,dz.$$

Par le lemme E.4, $K(1,z)|z|$ est intégrable sur $\mathbb{R}^N$ (puisque $\mu > 1$) donc, lorsque $\varepsilon \to 0$,

$$\int_{B_{\varepsilon^{-1/\mu}}} K_\mu(1,z)|z|\,dz \to \int_{\mathbb{R}^N} K_\mu(1,z)|z|\,dz > 0,$$

ce qui prouve que $u^\varepsilon(1,0) - u(1,0) \leq -C\varepsilon^{1/\mu}$ pour ε petit. Ce raisonnement est aussi valable lorsque $\mu = 2$ (le lemme E.4 ne s'applique pas, mais il est bien connu que $K_2(1,z)|z|$ est intégrable puisque $K_2(1,\cdot)$ est une gaussienne).

F Quelques notes sur la sous-section 4.2

Lemme F.1 *Soit $\lambda \in]0,2[$, $u \in C_b^2(\mathbb{R}^N)$ et $\varphi \in C_c^\infty(\mathbb{R}^N)$. Alors*

$$\int_{\mathbb{R}^N} g_\lambda[u](x)\varphi(x)\,dx = \int_{\mathbb{R}^N} u(x) g_\lambda[\varphi](x)\,dx.$$

Preuve du lemme F.1
Comme $\varphi \in C_c^\infty(\mathbb{R}^N)$, on peut Fubiniser (car $|u(x+z) - u(x) - \nabla u(x)\cdot z| \leq C|z|^2$ avec C indépendant de x et z) pour écrire

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{B_r} &\frac{u(x+z) - u(x) - \nabla u(x)\cdot z}{|z|^{N+\lambda}}\,dz\varphi(x)\,dx \\ &= \int_{B_r} \frac{1}{|z|^{N+\lambda}} \int_{\mathbb{R}^N} u(x+z)\varphi(x) - u(x)\varphi(x) - \nabla u(x)\varphi(x)\cdot z\,dx\,dz. \qquad \text{(F.1)}\end{aligned}$$

Or, pour tout $z \in \mathbb{R}^N$, puisque $u \in C^1(\mathbb{R}^N)$ et $\varphi \in C_c^\infty(\mathbb{R}^N)$,

$$\begin{aligned}\int_{\mathbb{R}^N} u(x+z)\varphi(x) &- u(x)\varphi(x) - \nabla u(x)\varphi(x)\cdot z\,dx \\ &= \int_{\mathbb{R}^N} u(x+z)\varphi(x)\,dx - \int_{\mathbb{R}^N} u(x)\varphi(x)\,dx - z\cdot\int_{\mathbb{R}^N} \nabla u(x)\varphi(x)\,dx \\ &= \int_{\mathbb{R}^N} u(x)\varphi(x-z)\,dx - \int_{\mathbb{R}^N} u(x)\varphi(x)\,dx + z\cdot\int_{\mathbb{R}^N} \nabla\varphi(x)u(x)\,dx \\ &= \int_{\mathbb{R}^N} u(x)(\varphi(x-z) - \varphi(x) - \nabla\varphi(x)\cdot(-z))\,dx\end{aligned}$$

et (F.1) donne donc, toujours par Fubini (on utilise le fait que φ a un support compact),

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{B_r} &\frac{u(x+z) - u(x) - \nabla u(x)\cdot z}{|z|^{N+\lambda}}\,dz\varphi(x)\,dx \\ &= \int_{B_r} \frac{1}{|z|^{N+\lambda}} \int_{\mathbb{R}^N} u(x)(\varphi(x-z) - \varphi(x) - \nabla\varphi(x)\cdot(-z))\,dx\,dz \\ &= \int_{\mathbb{R}^N} u(x) \int_{B_r} \frac{\varphi(x-z) - \varphi(x) - \nabla\varphi(x)\cdot(-z)}{|z|^{N+\lambda}}\,dz\,dx.\end{aligned}$$

On effectue ensuite le changement de variable $z \to -z$ puis la même manipulation sur le deuxième terme de (2.1) pout conclure. ■

Grâce à ce lemme on peut, lorsque $\varphi \in C_c^\infty([0,\infty[\times\mathbb{R}^N)$ est positive, multiplier (4.12) par φ puis intégrer sur $x \in \mathbb{R}^N$ pour trouver

$$\begin{aligned}
&\int_{\mathbb{R}^N} \partial_t(\eta(u^\varepsilon))(t,x)\varphi(t,x)\,dx + \varepsilon \int_{\mathbb{R}^N} \eta(u^\varepsilon(t,x)) g_\lambda[\varphi(t,\cdot)](x)\,dx \\
&\leq \int_{\mathbb{R}^N} [\eta'(u^\varepsilon(t,x))H(t,x,u^\varepsilon(t,x)) + \Phi(t,x,u^\varepsilon(t,x))]\,\varphi(t,x)\,dx \\
&\quad + \int_{\mathbb{R}^N} \phi(t,x,u^\varepsilon(t,x)) \cdot \nabla\varphi(t,x)\,dx
\end{aligned} \tag{F.2}$$

où $H(t,x,s) = h(t,x,s) - \sum_{i=1}^N \partial_{x_i} f_i(t,x,s)$ et $\Phi(t,x,s) = \sum_{i=1}^N \partial_{x_i}\phi_i(t,x,s)$. On a, pour $t_0 > 0$,

$$\int_{t_0}^\infty \int_{\mathbb{R}^N} \partial_t(\eta(u^\varepsilon))(t,x)\varphi(t,x)\,dx = -\int_{\mathbb{R}^N} \eta(u^\varepsilon(t_0,x))\varphi(t_0,x)\,dx - \int_{t_0}^\infty \int_{\mathbb{R}^N} \eta(u^\varepsilon(t,x))\partial_t\varphi(t,x)\,dx$$

et, par la Remarque 4.1, puisque $(u^\varepsilon)_{\varepsilon>0}$ est bornée dans $L^\infty(\mathbb{R}^N)$, $\eta(u^\varepsilon(t_0,\cdot)) \to \eta(u_0)$ dans $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ lorsque $t_0 \to 0$. On peut donc intégrer (F.2) sur $]t_0,\infty[$ et passer à la limite $t_0 \to 0$ pour trouver

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^N} \eta(u^\varepsilon(t,x))\partial_t\varphi(t,x)\,dtdx + \int_0^\infty \int_{\mathbb{R}^N} \phi(t,x,u^\varepsilon(t,x)) \cdot \nabla\varphi(t,x)\,dtdx \\
&+ \int_{\mathbb{R}^N} \eta(u_0(x))\varphi(0,x)\,dx + \int_0^\infty \int_{\mathbb{R}^N} [\eta'(u^\varepsilon(t,x))H(t,x,u^\varepsilon(t,x)) + \Phi(t,x,u^\varepsilon(t,x))]\,\varphi(t,x)\,dtdx \\
&\geq \varepsilon \int_{\mathbb{R}^N} \eta(u^\varepsilon(t,x)) g_\lambda[\varphi(t,\cdot)](x)\,dx\,,
\end{aligned} \tag{F.3}$$

ce qui est la formulation entropique intégrale de (4.10). Si $\kappa \in \mathbb{R}$, il existe une suite $(\eta_n)_{n\geq 1} \in C^2(\mathbb{R})$ de fonctions convexes qui converge localement uniformément vers $s \to |s-\kappa|$ et dont la dérivée converge partout vers $\mathrm{sgn}(s-\kappa)$ $(= -\mathbf{1}_{\{s<\kappa\}} + \mathbf{1}_{\{s>\kappa\}})$ en étant bornée par 1; les flux $\phi_n(t,x,s)$ correspondants convergent vers $f(t,x,s\top\kappa) - f(t,x,s\bot\kappa)$. En utilisant ces fonctions dans (F.3), on peut passer à la limite $n \to \infty$ pour trouver

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^N} |u^\varepsilon(t,x) - \kappa|\partial_t\varphi(t,x)\,dtdx \\
&+ \int_0^\infty \int_{\mathbb{R}^N} (f(t,x,u^\varepsilon(t,x)\top\kappa) - f(t,x,u^\varepsilon(t,x)\bot\kappa)) \cdot \nabla\varphi(t,x)\,dtdx \\
&+ \int_{\mathbb{R}^N} |u_0(x) - \kappa|\varphi(0,x)\,dx \\
&+ \int_0^\infty \int_{\mathbb{R}^N} [\mathrm{sgn}(u^\varepsilon(t,x) - \kappa)H(t,x,u^\varepsilon(t,x)) + \Psi_\kappa(t,x,u^\varepsilon(t,x))]\,\varphi(t,x)\,dtdx \\
&\geq \varepsilon \int_{\mathbb{R}^N} |u^\varepsilon(t,x) - \kappa| g_\lambda[\varphi(t,\cdot)](x)\,dx\,,
\end{aligned} \tag{F.4}$$

où $\Psi_\kappa(t,x,s) = \sum_{i=1}^N \partial_{x_i} f_i(t,x,s\top\kappa) - \partial_{x_i} f_i(t,x,s\bot\kappa)$ vérifie, grâce à (4.2),

$$\begin{gathered}
\forall R > 0\,,\ \forall T > 0\,,\ |\Psi_\kappa(t,x,s)| \leq M_{T,R}|s-\kappa| \\
\text{avec } M_{T,R} \text{ indépendant de } (t,x,s,\kappa) \in [0,T] \times \mathbb{R}^N \times [-R,R]^2.
\end{gathered} \tag{F.5}$$

On peut alors raisonner comme dans [12], en combinant (F.4) avec l'inégalité entropique pour (4.11). Les termes nouveaux qui apparaissent à droite sont alors:

- En facteur de $\nabla\varphi$: $f(t,x,u^\varepsilon(t,x)\top u(s,y)) - f(t,x,u^\varepsilon(t,x)\bot u(s,y))$ et $f(s,y,u(s,y)\top u^\varepsilon(t,x)) - f(s,y,u(s,y)\bot u^\varepsilon(t,x))$, qui ne pose pas de problème puisque chaque terme s'estime en $C|u^\varepsilon(t,x) - u(s,y)|$.

- En facteur de φ: $\Psi_{u(s,y)}(t,x,u^\varepsilon(t,x))$, $\Psi_{u^\varepsilon(t,x)}(s,y,u(s,y))$, $\mathrm{sgn}(u^\varepsilon(t,x)-u(s,y))(H(t,x,u^\varepsilon(t,x))-H(s,y,u(s,y))$; les deux premiers termes s'estiment, par (F.5), en $C|u^\varepsilon(t,x)-u(s,y)|$; le dernier terme, par (4.2), en $C(|t-s|+|x-y|+|u^\varepsilon(t,x)-u(s,y)|)$. Comme φ contient $\rho_\nu(y-x)$ et $\theta_\mu(s-t)$, le terme $|x-y|+|t-s|$ donne en fin de compte un $\mathcal{O}(\nu+\mu)$ et il reste finalement des $|u^\varepsilon(t,x)-u(s,y)|$.

En utilisant ensuite les propriétés de continuité de u, on transforme comme dans [12] les $u(s,y)$ en $u(t,x)$ et on obtient finalement

$$\int_0^\infty \int_{\mathbb{R}^N} |u^\varepsilon(t,x)-u(t,x)|(\partial_t\psi(t,x)+C|\nabla\psi(t,x)|+C|\psi(t,x)|)\,dtdx \geq A$$

avec A bien estimé. En prenant alors $\psi(t,x)=w_M(|x|+Ct)\Theta(t)$ (avec w_M et Θ décroissants et à valeurs dans $[0,1]$), on a

$$\begin{aligned}\partial_t\psi(t,x)+C|\nabla\psi(t,x)|+C|\psi(t,x)| &= Cw_M'(|x|+Ct)\Theta(t)+w_M(|x|+Ct)\Theta'(t)\\ &\quad -Cw_M'(|x|+Ct)\Theta(t)+Cw_M(|x|+Ct)\Theta(t)\\ &= w_M(|x|+Ct)\Theta'(t)+Cw_M(|x|+Ct)\Theta(t)\end{aligned}$$

et on arrive, une fois que l'on a fait tendre $-\Theta'$ vers une masse de dirac en t_0 (et donc Θ vers $\mathbf{1}_{[0,t_0]}$), à quelque chose comme

$$\int_{\mathbb{R}^N} |u^\varepsilon(t_0,x)-u(t_0,x)|w_M(|x|+Ct_0)\,dx \leq A'+C\int_0^{t_0}\int_{\mathbb{R}^N}|u^\varepsilon(t,x)-u(t,x)|w_M(|x|+Ct)\,dtdx$$

avec A' qui est bien estimé. Ceci étant vrai pour tout t_0, Gronwall sur $t\to\int_{\mathbb{R}^N}|u^\varepsilon(t,x)-u(t,x)|w_M(|x|+Ct)\,dt$ donne alors

$$\int_{\mathbb{R}^N} |u^\varepsilon(t_0,x)-u(t_0,x)|w_M(|x|+Ct_0)\,dx \leq A'e^{Ct_0}$$

(rappelons que $u^\varepsilon(0,\cdot)=u(0,\cdot)$), ce qui conclut la preuve.